QUESTION BANK

UNIT -1: PARTIAL DIFFERENTIAL EQUATIONS

PART – A

1. Form the PDE of $(x-a)^2+(y-b)^2+z^2=r^2$.

   Sol:

   Given eqn is $(x-a)^2+(y-b)^2+z^2=r^2$ .....(1)

   Diff partially w.r.to x and y

   $$2(x-a)+2zp=0$$

   $$x-a=-zp \qquad .....(2)$$

   $$2(y-b)+2zq=0$$

   $$y-b=-zq \qquad .....(3)$$

   Sub (2) and (3) in (1)

   $$z^2p^2+z^2q^2+z^2=r^2$$

   $$z^2(p^2+q^2+1)=r^2$$

2. Find the complete integral of $p+q=pq$.

   Sol:

   Given $p+q=pq$ .....(1)

   Let $z=ax+by+c$ .....(1) be the solution.

   Diff partially w.r.to x and y

   $$p=a \text{ and } q=b \qquad .....(2)$$

   Sub (2) in (1)

   $$a+b=ab$$

   $$\Rightarrow b=\frac{a}{a-1} \qquad .....(3)$$

   Sub (3) in (1)

   $$z=ax+\left(\frac{a}{a-1}\right)y+c.$$

3. Find the complete integral of $p-y^2=q+x^2$.

   Sol:

   Given $p-y^2=q+x^2$ .....(1)

   $$p-x^2=q+y^2=k$$

   $$p-x^2=k\ ,\ q+y^2=k$$

   $$p=k+x^2\ ,\ q=k-y^2$$

   We know that

$$z=\int pdx+\int qdy$$
$$=\int (k+x^2)dx+\int (k-y^2)dy$$
$$=\frac{1}{3}\left(x^3+y^3\right)+k(x+y)+a.$$

4. Form the PDE by eliminating a and b from $z=(x^2+a^2)(y^2+b^2)$.

Sol:

Given $z=(x^2+a^2)(y^2+b^2)$ .....(1)

Diff partially w.r.to x and y

$$p=2x(y^2+b^2) \quad .....(2)$$
$$q=2y\left(x^2+a^2\right) \quad .....(3)$$
$$(2)\Rightarrow \frac{p}{2x}=y^2+b^2 \quad .....(4)$$
$$(3)\Rightarrow \frac{q}{2y}=x^2+a^2 \quad .....(5)$$

Sub (4) and (5) in (1)

$$z=\frac{p}{2x}\frac{q}{2y}$$
$$pq=4xyz.$$

5. Form the PDE by eliminating f from $z=x^2+2f\left(\frac{1}{y}+\log x\right)$.

Sol:

Given $z=x^2+2f\left(\frac{1}{y}+\log x\right)$ .....(1)

Diff partially (1) w.r.to x and y

$$p=2x+2f'\left(\frac{1}{y}+\log x\right)\frac{1}{x} \quad .....(2)$$
$$q=2f'\left(\frac{1}{y}+\log x\right)\frac{-1}{y^2}$$
$$=\frac{-2}{y^2}f'\left(\frac{1}{y}+\log x\right)$$
$$2f'\left(\frac{1}{y}+\log x\right)=-qy^2 \quad .....(3)$$

Sub (3) in (2)

$$p=2x-\frac{qy^2}{x}$$
$$px+qy^2=2x^2$$

6. Find the PDE by eliminating the arbitrary function $\phi\left[z^2-xy,\frac{x}{z}\right]$.

   Given $f(z^2-xy,\frac{x}{z})=0$.

   $\therefore$ (1) is of the form $\phi(u,v)=0$

   Let $u=z^2-xy \ \ and \ \ v=\frac{x}{z}$

   $$\frac{\partial u}{\partial x}=2zp-y \qquad \frac{\partial v}{\partial x}=\frac{z-px}{z^2}$$
   $$\frac{\partial u}{\partial y}=2zq-x \qquad \frac{\partial v}{\partial y}=\frac{-xq}{z^2}$$

   Sub the above derivatives in $\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{vmatrix}=0$

   $$\begin{vmatrix}2zp-y & \frac{z-px}{z^2}\\ 2zq-x & \frac{-xq}{z^2}\end{vmatrix}=0$$
   $$(2zp-y)\left(\frac{-xq}{z^2}\right)-\left(\frac{z-px}{z^2}\right)(2zq-x)=0$$
   $$multiplying \ by \ z^2$$
   $$(2zp-y)(-xq)-(z-px)(2zq-x)=0$$
   $$px^2-q(xy-2z^2)=zx.$$

7. Obtain the complete integral of $z=px+qy+p^2+q^2$.

   Sol:

   This is of the form $z=px+qy+f(p,q)$

   Replace p = a and q = b.

   The complete solution is $z=ax+by+a^2+b^2$.

8. Obtain the complete integral of $\frac{z}{pq}=\frac{x}{q}+\frac{y}{p}+\sqrt{pq}$

   Sol:

   Given $\frac{z}{pq}=\frac{x}{q}+\frac{y}{p}+\sqrt{pq}$

Multiplying by pq.

$$z = px + qy + (pq)^{\frac{3}{2}}$$

$$z = ax + by + (ab)^{\frac{3}{2}}$$

9. Solve $(4D^2 - 12DD' + 9D'^2)z = 0$.

   Sol:

   The auxiliary eqn is

$$4m^2 - 12m + 9 = 0$$

$$4m^2 - 6m - 6m + 9 = 0$$

$$2m(2m-3) - 3(2m-3) = 0$$

$$(2m-3)(2m-3) = 0$$

$$m = \frac{3}{2}, \frac{3}{2}$$

   Sol is $y = f_1\left(y + \frac{3}{2}x\right) + xf_2\left(y + \frac{3}{2}x\right)$

10. Solve $(D^2 + DD')z = 0$.

   Sol:

   The auxiliary eqn is

$$m^2 + m = 0$$

$$m(m+1) = 0$$

$$m = 0, -1$$

   Sol is $y = f_1(y + 0x) + f_2(y - x)$

PART – B

1. Form the PDE by eliminating the arbitrary functions $f \; and \; \phi$ from $z = f(y) + \phi(x + y + z)$.

   Sol:

   Given $z = f(y) + \phi(x + y + z) \qquad .....(1)$

$$p = \phi'(x+y+z)(1+p) \qquad .....(2)$$

$$q = f'(y) + \phi'(x+y+z)(1+q) \qquad .....(3)$$

$$r = \phi'(x+y+z)(r) + \phi''(x+y+z)(1+p)^2$$

$$r - \phi'(x+y+z)(r) = \phi''(x+y+z)(1+p)^2$$

$$r[1-\phi'(x+y+z)] = \phi''(x+y+z)(1+p)^2 \qquad .....(4)$$

$$s = \phi'(x+y+z)(s) + \phi''(x+y+z)(1+p)(1+q)$$

$$s - \phi'(x+y+z)(s) + \phi''(x+y+z)(1+p)(1+q)$$

$$s[1-\phi'(x+y+z)] = \phi''(x+y+z)(1+p)(1+q) \qquad .....(5)$$

$$t = f''(y) + \phi'(x+y+z)(t) + \phi''(x+y+z)(1+q)^2 \qquad .....(6)$$

$$\frac{(7)}{(8)} \Rightarrow \frac{r}{s} = \frac{1+p}{1+q}.$$

2. Solve $z = px + qy + \sqrt{1+p^2+q^2}$.

Sol:

This is of Clairaut's form .

The complete integral is $z = ax + by + \sqrt{1+a^2+b^2}$ .....(1)

Diff partially w.r.to a and b

$$\frac{\partial z}{\partial a} = x + \frac{a}{\sqrt{1+a^2+b^2}} = 0 \qquad .....(2)$$

$$x = \frac{-a}{\sqrt{1+a^2+b^2}}$$

$$x^2 = \frac{a^2}{1+a^2+b^2}$$

$$\frac{\partial z}{\partial a} = y + \frac{b}{\sqrt{1+a^2+b^2}} = 0 \qquad .....(3)$$

$$y = \frac{-b}{\sqrt{1+a^2+b^2}}$$

$$y^2 = \frac{b^2}{1+a^2+b^2}$$

$$x^2 + y^2 = \frac{a^2+b^2}{1+a^2+b^2}$$

$$1-(x^2+y^2) = 1-\left(\frac{a^2+b^2}{1+a^2+b^2}\right)$$

$$1-x^2-y^2 = \frac{1}{1+a^2+b^2}$$

$$\therefore \sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}} \qquad .....(4)$$

Sub (4) in (2) and (3)

$$a = \frac{-x}{\sqrt{1-x^2-y^2}}\ ,\ b = \frac{-y}{\sqrt{1-x^2-y^2}} \qquad .....(5)$$

Sub (4) and (5) in (1)

$$z = \frac{x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} - \frac{1}{\sqrt{1-x^2-y^2}}$$

$$= \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$$

$$z = \sqrt{1-x^2-y^2}$$

$$z^2 = 1-x^2-y^2.$$

3. Solve $(D^2+DD'-6D'^2)z = x^2y + e^{3x+y}$.

Sol:

The auxiliary eqn is

$$m^2 - m - 6 = 0$$

$$(m-2)(m+3) = 0$$

$$m = 2, -3$$

C.F. is $y = f_1(y+2x) + f_2(y-3x)$

$$P.I. = P.I_1 + P.I_2$$

$$P.I_1 = \frac{1}{D^2+DD'-6D'^2}\ x^2y$$

$$= \frac{1}{D^2\left(1+\left(\frac{D'}{D}-\frac{6D'^2}{D^2}\right)\right)}\ x^2y$$

$$= \frac{1}{D^2}\left(1+\left(\frac{D'}{D}-\frac{6D'^2}{D^2}\right)\right)^{-1} x^2y$$

$$= \frac{1}{D^2}\left(1-\frac{D'}{D}+\ldots\right)x^2y$$

$$= \frac{1}{D^2}\left(x^2y - \frac{1}{D}(x^2)\right)$$

$$= \frac{1}{D^2}(x^2y) - \frac{1}{D^3}(x^2)$$

$$= y.\frac{x^4}{12} - \frac{x^5}{60}$$

$$P.I_2 = \frac{1}{D^2+DD'-6D'^2}\, e^{3x+y} \qquad (D=3\ ,D'=1)$$

$$=\frac{1}{9+3-6}\, e^{3x+y}$$

$$=\frac{1}{6}e^{3x+y}$$

$$\therefore z = f_1(y+2x)+f_2(y-3x)+\frac{x^4y}{12}-\frac{x^5}{60}+\frac{1}{6}e^{3x+y}$$

4. Solve $r+s-6t=y\cos x$.

Sol:

The auxiliary eqn is

$$(D^2+DD'-6D'^2)z=y\cos x$$

$$m^2+m-6=0$$

$$m=-3\ ,2$$

$$C.F.=f_1(y-3x)+f_2(y+2x)$$

$$P.I.=\frac{1}{D^2+DD'-6D'^2}\,y\cos x$$

$$=\frac{1}{(D-2D')(D+3D')}\,y\cos x$$

$$=\frac{1}{(D-2D')}\int (c+3x)\cos x dx \qquad (y=c+3x)$$

$$=\frac{1}{(D-2D')}\left[(c+3x)\sin x-3(-\cos x)\right]$$

$$=\frac{1}{(D-2D')}\left[(c+3x)\sin x+3\cos x\right]$$

$$=\frac{1}{(D-2D')}\left[y\sin x+3\cos x\right]$$

$$=\int (c_1-2x)\sin x+3\cos x dx \qquad (y=c_1-2x)$$

$$=-c_1\cos x+3\sin x-2\int x\sin x dx$$

$$=-c_1\cos x+3\sin x-2\left(-x\cos x+\int \cos x\, dx\right)$$

$$=-c_1\cos x+3\sin x-2x\cos x-2\sin x$$

$$=-y\cos x-2x\cos x+3\sin x+2x\cos x-2\sin x$$

$$=\sin x-y\cos x$$

$$z=f_1(y-3x)+f_2(y+2x)+\sin x-y\cos x$$

FOURIER SERIES
Question Bank
Part A

1.State Drichlet's conditions :

A function f(x) can be expanded as a fourier series in an interval $c \le x \le c+2l$

If the following conditions are satisfied

(i) f(x) is periodic with period 2l in (c, c + 2l) and f(x) is bounded.

(ii)The function f(x) must have finite number of maxima and minima.

(iii)The function f(x) must be piecewise continuous and has a finite number of finite discontinuities.

2.**Find the mean square value of the function *f(x) = x* in the interval *(0,l)*.**

Solution:

$$\text{Mean Square value} = \frac{\int_a^b [f(x)]^2\, dx}{(b-a)}$$

$$= \frac{1}{l}\int_0^l x^2\, dx$$

$$= \frac{1}{l}\left[\frac{x^3}{3}\right]_0^l$$

$$= \frac{l^2}{3}$$

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3. Find the value of a_n in the cosine series expansion of *f(x)* = 10 in the interval (0,10).

Solution:

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{10}\int_0^{10} 10\cos\left(\frac{n\pi x}{10}\right) dx$$

$$= \frac{2}{10}\,10\left[\sin\left(\frac{n\pi x}{10}\right)\frac{10}{n\pi}\right]_0^{10}$$

$$= 2\,\frac{10}{n\pi}\left[\sin n\pi - \sin 0\right]$$

$$a_n = 0$$

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4. What do you mean by Harmonic Analysis.

Solution:_ When a function is unknown but the values of that function at certain points are known, then the Fourier series of that function can be obtained numerically and the process is called Harmonic Analysis.

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5. In the Fourier series expansion of $f(x) = \begin{cases} 1+\dfrac{2x}{\pi}, & -\pi < x < 0 \\ 1-\dfrac{2x}{\pi}, & 0 < x < \pi \end{cases}$ **in (-π,π), find the coefficient of *sin nx*.**

Solution: Since the interval is (-π,π), let us verify whether the function is odd or even

$$f(-x) = \begin{cases} 1+\dfrac{2(-x)}{\pi}, & -\pi < -x < 0 \\ 1-\dfrac{2(-x)}{\pi}, & 0 < -x < \pi \end{cases}$$

$$= \begin{cases} 1-\dfrac{2(x)}{\pi}, & 0 < x < \pi \\ 1+\dfrac{2(x)}{\pi}, & -\pi < x < 0 \end{cases}$$

$$= f(x)$$

Hence the function is even. So, the coefficient of *sin nx* that is $b_n = 0$.

6. Find a_n in expanding e^{-ax} as a Fourier series in (-π,π).

Solution:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\left(\frac{e^{-ax}}{(a^2+b^2)}\right)(-a\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{e^{-a\pi}}{a^2+b^2}\right)(-a\cos n\pi + 0) - \left(\frac{e^{a\pi}}{a^2+b^2}\right)(-a\cos n\pi + 0)\right]$$

$$= \frac{1}{\pi}\frac{a}{a^2+b^2}\cos n\pi\,(e^{a\pi} - e^{-a\pi})$$

$$a_n = \frac{2}{\pi}\frac{a}{a^2+b^2}(-1)^n \sinh a\pi$$

7. What is the constant term a_0 and the coefficient a_n in the Fourier series expansion of $f(x) = x - x^3$ in (-π,π).

Solution: Since the interval is (-π,π), let us verify whether the function is odd or even

$f(-x) = -x - (-x^3) = -x + x^3 = -(x - x^3) = -f(x)$. The given function is odd.

Hence, the coefficients a_0 and a_n are zero.

8. State the Parseval's identity for Fourier series.

Solution:

The Parseval's Identity for Fourier series in the interval $(c, c+2l)$

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) = \frac{1}{l}\int_{c}^{c+2l}[f(x)]^2\, dx$$

9. Find the constant term in the Fourier series corresponding to $f(x) = \cos^2 x$ expanded in the interval (-π,π).

Solution: Since the interval is (-π,π), let us verify whether the function is odd or even.

$f(-x) = \cos^2(-x) = \cos^2 x = f(x)$. Hence the function is even.

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\cos^2 x\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\frac{1+\cos 2x}{2}dx$$

$$= \frac{1}{\pi}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}[\pi]$$

$$a_0 = 1$$

Hence the constant term in the Fourier expansion is $\dfrac{a_0}{2} = \dfrac{1}{2}$

10. To which value the half range sine series corresponding to $f(x) = x^2$ expressed in the interval (0,2) converges at $x = 2$?

Solution: In order to expand in a sine series the function must be defined as an odd function in the interval (-2,2). Hence in the interval (-2,0) it should be defined in the form of $-f(-x) = -(-x)^2 = -x^2$.

$$\begin{array}{ccc} & -x^2 & & x^2 & \\ (---------- & &)(& & ----------) \\ -2 & & 0 & & 2 \end{array}$$

Since at $x = 2$ the function is discontinuous (end point discontinuity) the Fourier (sine) series converges to

$$\frac{f(2-)+f(2+)}{2} = \frac{f(2)-f(-2)}{2}$$
$$= \frac{(2)^2 + (-(-2)^2)}{2}$$
$$= 0$$

11.If the Fourier series of the function $f(x) = x + x^2$, in the interval $(-\pi,\pi)$ is $\frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{n^2}\cos nx - \frac{2}{n}\sin nx\right]$, then find the value of the infinite series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+....$

Solution: Put $x = \pi$, which is an end point discontinuity. So,

$$\frac{f(-\pi)+f(\pi)}{2} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{n^2}\cos n\pi - \frac{2}{n}\sin n\pi\right]$$
$$\Rightarrow \frac{-\pi+\pi^2+\pi+\pi^2}{2} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{n^2}(-1)^n\right]$$
$$\Rightarrow \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}\right]$$
$$\Rightarrow 4\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\right] = -\frac{\pi^2}{3} + \pi^2$$
$$\Rightarrow 4\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\right] = \frac{2\pi^2}{3}$$
$$\Rightarrow \sum_{n=1}^{\infty}\left[\frac{1}{n^2}\right] = \frac{\pi^2}{6}$$
$$\Rightarrow \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+.... = \frac{\pi^2}{6}$$

12.Find a_0 if $f(x) = |x|$, expanded as a Fourier series in $(-\pi,\pi)$.

Solution: Since $f(-x) = |-x| = |x| = f(x)$, the function is even.

$$a_0 = \frac{2}{\pi}\int_0^{\pi}|x|\,dx = \frac{2}{\pi}\int_0^{\pi}x\,dx$$
$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[\frac{\pi^2}{2}\right]$$
$$a_0 = \pi$$

13.Find the Fourier constant b_n for $f(x) = x \sin x$ in $(-\pi,\pi)$.

Solution: $f(-x) = (-x)\sin(-x) = x\sin x = f(x)$ the function is even. Therefore, the coefficient $b_n = 0$.

14.Find the Fourier constant b_n for $f(x) = x^2$ in $(-\pi,\pi)$.

Solution: $f(-x) = (-x)^2 = x^2 = f(x)$ the function is even. Therefore, the coefficient $b_n = 0$.

15.Find the constant term in the Fourier expansion of $f(x) = x^2 - 2$ in $-2 < x < 2$

Solution: $f(-x) = (-x)^2 - 2 = x^2 - 2 = f(x)$ the function is even. So,

$$a_0 = \frac{2}{2}\int_0^2 (x^2 - 2)\, dx$$

$$= \left[\frac{x^3}{3} - 2x\right]_0^2 = \frac{8}{3} - 4$$

$$a_0 = -\frac{4}{3}$$

Hence the constant term in the Fourier expansion is $\frac{a_0}{2} = -\frac{2}{3}$

16.Find a sine series for *f(x)* = *x*, in (0,π).

Solution:

$$b_n = \frac{2}{\pi}\int_0^\pi x \sin nx\, dx$$

$$= \frac{2}{\pi}\left\{(x)\left(-\frac{\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right)\right\}_0^\pi$$

$$= \frac{2}{\pi}\left[-\pi\frac{\cos n\pi}{n} + 0 + 0 - 0\right]$$

$$= -2\frac{(-1)^n}{n}$$

$$b_n = 2\frac{(-1)^{n+1}}{n}$$

$$x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

is the half range sine series .

17.Find the half range sine series for *f(x)* = 2 in 0 < *x* < π.

Solution:

$$b_n = \frac{2}{\pi}\int_0^\pi 2 \sin nx\, dx$$

$$= \frac{2}{\pi}\left\{(2)\left(-\frac{\cos nx}{n}\right)\right\}_0^\pi$$

$$= -\frac{4}{\pi}\left[\frac{\cos n\pi}{n} - \frac{1}{n}\right] = -\frac{4}{\pi}\left[\frac{(-1)^n}{n} - \frac{1}{n}\right]$$

$$= -\frac{4}{\pi}\begin{cases} -\frac{2}{n}, & if\ n\ is\ odd \\ 0, & if\ n\ is\ even \end{cases}$$

$$b_n = \begin{cases} \frac{8}{n\pi}, & if\ n\ is\ odd \\ 0, & if\ n\ is\ even \end{cases}$$

Hence the half range sine series is $f(x) = \frac{8}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n}\sin nx$

18.The cosine series for *f(x)* = *x sin x* in 0 < *x* < π is given as

$x \sin x = 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2 - 1}\cos nx$. **Deduce that** $1 + 2\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\right] = \frac{\pi}{2}$.

Solution: As $n^2 - 1 = (n - 1)(n + 1)$

$$x \sin x = 1 - \frac{1}{2}\cos x - 2\left[\frac{1}{1.3}\cos 2x - \frac{1}{2.4}\cos 3x + \frac{1}{3.5}\cos 4x - \frac{1}{4.6}\cos 5x + \frac{1}{5.7}\cos 6x - \ldots\right]$$

Put $x = \pi/2$ in the above series we get

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1 - \frac{1}{2}\cos\frac{\pi}{2} - 2\left[\frac{1}{1.3}\cos 2\frac{\pi}{2} - \frac{1}{2.4}\cos 3\frac{\pi}{2} + \frac{1}{3.5}\cos 4\frac{\pi}{2} - \frac{1}{4.6}\cos 5\frac{\pi}{2} + \frac{1}{5.7}\cos 6\frac{\pi}{2} - \dots\right]$$

$$\Rightarrow \frac{\pi}{2} = 1 - 2\left[-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \dots\right]$$

$$\Rightarrow \frac{\pi}{2} = 1 + 2\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots\right]$$

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19. Define RMS value of a function.

Solution: The RMS value of a function $f(x)$ in (a,b) is defined by

$$\bar{y} = \sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2\,dx}$$

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20. If $f(x)$ is an odd function in the interval $(-l, l)$, write the formula to find the Fourier coefficients.

Solution: $a_0 = a_n = 0$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

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21. If $f(x)$ is an even function in the interval $(-l, l)$, write the formula to find the Fourier coefficients.

Solution: $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$b_n = 0$

22. Does $f(x) = \tan x$ posses a Fourier series? Justify your answer.

Solution: For a function $f(x)$ to have Fourier series expansion it must satisfy all the three criteria in Drichlet's conditions. But $f(x) = \tan x$ has value ∞ at $x = \frac{\pi}{2}$ and so it is a discontinuous point and moreover it is an infinite discontinuity. So, it doesn't have a Fourier series expansion.

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23. Does $f(x) = \sin(1/x)$ posses a Fourier series? Justify your answer.

Solution: For a function $f(x)$ to have Fourier series expansion it must satisfy all the three criteria in Drichlet's conditions. But $f(x) = \sin(1/x)$ has minimum or maximum value at (odd multiple of $\pi/2$)

That is when $1/x = (2n-1)\pi/2$, Hence $x = 2/(2n-1)\pi$ As n tend to ∞, $x = 0$ So, the function doesn't have Fourier series expansion.

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24. Write the formula for finding Fourier coefficients.

Solution:

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

PART B

1. Find the Fourier series Expansion of $f(x) = x^2, 0 < x < 2\pi$.

Hence deduce that, (i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...$

(ii) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ...$

(iii) $\frac{\pi^2}{8} = \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right)$

Solution :

We need to find a_0, a_n, b_n where the formulas are given by

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx,\quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,\quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx$$

applying Bernoulli's formula

$$= \frac{1}{\pi}\left[\left(x^2\right)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{\begin{array}{l}\left[\left(4\pi^2\right)\left(\frac{\sin n2\pi}{n}\right) - (4\pi)\left(\frac{-\cos n2\pi}{n^2}\right) + (2)\left(\frac{-\sin n2\pi}{n^3}\right)\right]_0^{2\pi} - \\ \qquad \frac{1}{\pi}\left[\left(0^2\right)\left(\frac{\sin n0}{n}\right) - (0)\left(\frac{-\cos n0}{n^2}\right) + (2)\left(\frac{-\sin n0}{n^3}\right)\right]_0^{2\pi}\end{array}\right\}$$

$$= \frac{1}{\pi}4\pi\frac{1}{n^2}$$

$$= \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\sin nx\,dx$$

applying Bernoulli's formula

$$= \frac{1}{\pi}\left[\left(x^2\right)\left(\frac{-\cos nx}{n}\right) - (2x)\left(\frac{-\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{\left[\left(4\pi^2\right)\left(\frac{-\cos n2\pi}{n}\right) - (4\pi)\left(\frac{-\sin n2\pi}{n^2}\right) + (2)\left(\frac{\cos n2\pi}{n^3}\right)\right] - \left[\left(0^2\right)\left(\frac{-\cos n0}{n}\right) - (0)\left(\frac{-\sin n0}{n^2}\right) + (2)\left(\frac{\cos n0}{n^3}\right)\right]\right\}$$

$$= \frac{1}{\pi}\left[\left(\frac{-4\pi^2}{n} + \frac{2}{n^3}\right) - \left(\frac{2}{n^3}\right)\right] \quad = -\frac{4\pi}{n}$$

Hence the Fourier series expansion is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx + \sum_{n=1}^{\infty} -\frac{4\pi}{n}\sin nx$$

Deduction 1:

Put $x = 0$ (is a point of discontinuity at end point) in the above Fourier series

$$\frac{f(0)+f(2\pi)}{2} = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos n0 + \sum_{n=1}^{\infty} -\frac{4\pi}{n}\sin n0$$

$$\Rightarrow \frac{0+4\pi^2}{2} = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}$$

$$\Rightarrow 2\pi^2 - \frac{4\pi^2}{3} = \sum_{n=1}^{\infty}\frac{4}{n^2}$$

$$\Rightarrow \frac{2\pi^2}{3} = \sum_{n=1}^{\infty}\frac{4}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Hence, $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$ i.e., $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...$

Deduction 2:

Put $x = \pi$ (is a point of continuity) in the above Fourier series

$$\pi^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos n\pi + \sum_{n=1}^{\infty} -\frac{4\pi}{n}\sin n\pi$$

$$\Rightarrow \pi^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n$$

$$\Rightarrow \pi^2 - \frac{4\pi^2}{3} = \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n$$

$$\Rightarrow \frac{-\pi^2}{3} = \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n$$

$$\Rightarrow \frac{-\pi^2}{12} = \sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n$$

$$\Rightarrow \frac{-\pi^2}{12} = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - ....$$

$$\Rightarrow \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ...$$

Deduction 3:

x value substitution (such as 0, $\frac{\pi}{2}$, π, 2π) will not give the deduction.

So, let us add the above two series (given in the above two deductions),

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ...\right) + \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...\right)$$

$$\frac{\pi^2}{4} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right)$$

$$\Rightarrow \frac{\pi^2}{8} = \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right)$$

Hence $\frac{\pi^2}{8} = \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right)$

2. Find the Fourier series expansion of $(\pi - x)^2$, in $-\pi < x < \pi$.

and hence deduce that (i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - .... = \frac{\pi^2}{12}$, **(ii)** $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{\pi^2}{6}$

Solution: Since the range is $-\pi < x < \pi$.

So, let us first verify whether the function is odd or even.

$f(-x) = (\pi - (-x))^2 = (\pi + x)^2$ which is entirely different function. So, we have to find all the three fourier coefficients.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx,\; a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx,\; b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)^2\, dx$$

$$= \frac{1}{\pi}\left[\frac{(\pi - x)^3}{-3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[0 - \frac{8\pi^3}{-3}\right]$$

$$= \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)^2 \cos nx\, dx$$

$$= \frac{1}{\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right) - 2(\pi - x)(-1)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{[0 - 0 + 0] - \left[(4\pi^2)(0) + 4\pi\left(\frac{-(-1)^n}{n^2}\right) + 2(0)\right]\right\}$$

$$= \frac{1}{\pi}4\pi\frac{(-1)^n}{n^2}$$

$$a_n = 4\frac{(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)^2 \sin nx\, dx$$

$$= \frac{1}{\pi}\left[(\pi - x)^2\left(\frac{-\cos nx}{n}\right) - 2(\pi - x)(-1)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[0 - 0 - 2\frac{\cos n\pi}{n^3}\right] - \left[4\pi^2\left(\frac{-\cos n\pi}{n}\right) - 2(2\pi)\frac{\sin n\pi}{n^2} + 2\frac{\cos n\pi}{n^3}\right]\right\}$$

$$b_n = 4\pi\frac{(-1)^n}{n}$$

Hence the Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$(\pi - x)^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} 4\frac{(-1)^n}{n^2}\cos nx - \sum_{n=1}^{\infty} 4\pi\frac{(-1)^n}{n}\sin nx$$

<u>Deduction (i)</u>:

Put $x = 0$ (which is a continuous point)

$$\pi^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} 4\frac{(-1)^n}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} 4\frac{(-1)^n}{n^2} = -\frac{4\pi^2}{3} + \pi^2$$

$$\Rightarrow \sum_{n=1}^{\infty} 4\frac{(-1)^n}{n^2} = -\frac{\pi^2}{3}$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - .... = \frac{\pi^2}{12}$$

Deduction (ii):

Put $x = \pi$ (which is a discontinuous point)

$$\frac{(\pi-\pi)^2 + ((\pi-(-\pi))^2}{2} = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} 4\frac{(-1)^n}{n^2}(-1)^n$$

$$\Rightarrow 2\pi^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} 4\frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} 4\frac{1}{n^2} = 2\pi^2 - \frac{4\pi^2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty} 4\frac{1}{n^2} = \frac{2\pi^2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

3. Find the Fourier series expansion of *f(x)* = *x sin x* in (0, 2π).

Solution:

Since the interval is 0 to 2π we need to find all the three fourier constants.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx,\ a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,\ b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx = \frac{1}{\pi}\Big[(x)(-\cos x) - 1(-\sin x)\Big]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[2\pi(-1) - 0 - 0 + 0\right]$$

$$a_0 = -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\frac{\sin\,(n+1)x - \sin\,(n-1)x}{2}\,dx \text{ , Provided } n \neq 1$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\sin(n+1)x\,dx - \int_0^{2\pi} x\sin(n-1)x\,dx\right]$$

$$= \frac{1}{2\pi}\left\{\left[(x)\left(-\frac{\cos(n+1)x}{n+1}\right) - 1\left(-\frac{\sin(n+1)x}{(n+1)^2}\right)\right] - \left[(x)\left(-\frac{\cos(n-1)x}{n-1}\right) - 1\left(-\frac{\sin(n-1)x}{(n-1)^2}\right)\right]\right\}_0^{2\pi}$$

provided $n \neq 1$

$$= \frac{1}{2\pi}\left[-\frac{2\pi}{n+1} - \left(-\frac{2\pi}{n-1}\right)\right],\ \textit{provided}\ n \neq 1$$

$$= \frac{1}{2\pi} 2\pi \left(\frac{1}{n+1} - \frac{1}{n-1} \right), \text{ provided } n \neq 1$$

$$a_n = \frac{2}{n^2 - 1}, \text{ provided } n \neq 1$$

When $n = 1$, *we have*

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} (x \sin x) \cos 1x \, dx = \frac{1}{2\pi} \int_0^{2\pi} (x \sin 2x) \, dx$$

$$= \frac{1}{2\pi} \left[(x)\left(-\frac{\cos 2x}{2} \right) - 1\left(-\frac{\sin 2x}{4} \right) \right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \left(-\frac{2\pi}{2} \right)$$

$$a_1 = -\frac{1}{2}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \sin x \sin nx \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \frac{\cos(n-1)x - \cos(n+1)x}{2} dx$$

$$= \frac{1}{2\pi} \left[\int_0^{2\pi} x \cos(n-1)x \, dx - \int_0^{2\pi} x \cos(n+1)x \, dx \right]$$

$$= \frac{1}{2\pi} \left\{ \left[(x)\left(\frac{\sin(n-1)x}{n-1} \right) - 1\left(-\frac{\cos(n-1)x}{(n-1)^2} \right) \right] - \left[(x)\left(\frac{\sin(n-+1)x}{n+1} \right) - 1\left(-\frac{\cos(n+1)x}{(n+1)^2} \right) \right] \right\}_0^{2\pi},$$

provided $n \neq 1$

$$= \frac{1}{2\pi} \left\{ \left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} \right] - \left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} \right] \right\}, \text{ provided } n \neq 1$$

$$= 0, \text{ provided } n \neq 1$$

$$b_1 = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \sin x \sin 1x \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \frac{1 - \cos 2x}{2} dx$$

$$= \frac{1}{2\pi} \left[\int_0^{2\pi} x \, dx - \int_0^{2\pi} x \cos 2x \, dx \right]$$

$$= \frac{1}{2\pi} \left\{ \left[\frac{x^2}{2} \right]_0^{2\pi} - \left[(x)\left(\frac{\sin 2x}{2} \right) - 1\left(-\frac{\cos 2x}{4} \right) \right]_0^{2\pi} \right\}$$

$$= \frac{1}{2\pi} \left[2\pi^2 - 0 + \frac{1}{4} - 0 + 0 - \frac{1}{4} \right] \qquad = \pi$$

Hence the Fourier series is

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$x \sin x = -1 - \frac{1}{2} \cos x + \sum_{n=2}^{\infty} \frac{2}{n^2 - 1} \cos nx + \pi \sin x$$

4. Find the Fourier series of $f(x) = \begin{cases} 1, & 0 < x < \pi \\ 2, & \pi < x < 2\pi \end{cases}$**. Hence evaluate the value of the series** $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots.$

Solution:

Here the interval given is 0 to 2π. So, let us find all the three fourier constants.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx,\quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,\quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} 1\,dx + \int_{\pi}^{2\pi} 2\,dx\right]$$

$$= \frac{1}{\pi}\left[(x)_0^{\pi} + (2x)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}[\pi + 2\pi]$$

$$= 3$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi}\cos nx\,dx + \int_{\pi}^{2\pi} 2\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\sin nx}{n}\right)_0^{\pi} + \left(2\frac{\sin nx}{n}\right)_0^{2\pi}\right]$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi}\sin nx\,dx + \int_{\pi}^{2\pi} 2\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{\cos nx}{n}\right)_0^{\pi} + \left(-2\frac{\cos nx}{n}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{(-1)^n}{n} + \frac{1}{n}\right) + \left(-\frac{2}{n} + 2\frac{(-1)^n}{n}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n} - \frac{1}{n}\right]$$

$$b_n = \frac{1}{\pi}\begin{cases} \dfrac{-2}{n}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Hence the Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$f(x) = \frac{3}{2} + \sum_{n=1,3,5,\ldots}^{\infty} -\frac{2}{n\pi}\sin nx$$

Deduction: When we put $x = 0, \frac{\pi}{2}, \pi, 2\pi$ we don't get the series (As the denominator of the series is n^2 and the denominator of the Fourier series is only n). so, let us apply Parseval's identity for full range series.

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) = \frac{1}{\pi}\int_0^{2\pi}[f(x)]^2\,dx$$

$$\Rightarrow \frac{9}{2} + \sum_{n=1,3,5,..}^{\infty}\frac{4}{n^2\pi^2} = \frac{1}{\pi}\left[\int_0^{\pi}1^2\,dx + \int_{\pi}^{2\pi}2^2\,dx\right]$$

$$\Rightarrow \frac{9}{2} + \frac{4}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+....\right) = \frac{1}{\pi}\left[(x)_0^{\pi} + (4x)_{\pi}^{2\pi}\right]$$

$$\Rightarrow \frac{9}{2} + \frac{4}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+....\right) = 5$$

$$\Rightarrow \frac{4}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+....\right) = 5 - \frac{9}{2} = \frac{1}{2}$$

$$\Rightarrow \left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+....\right) = \frac{\pi^2}{8}$$

5. Find the Fourier series expansion of $f(x) = |\sin x|$ in $-\pi < x < \pi$.

Solution: Here the interval given is $-\pi$ to π. So, let us verify whether the given function is odd or even.

$f(-x) = |\sin(-x)| = |-\sin x| = \sin x = |\sin x| = f(x)$. Hence the function is even. So, let us find the fourier constants a_0, a_n. ($b_n = 0$).

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}|\sin x|\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\sin x\,dx$$

$$= \frac{2}{\pi}[-\cos x]_0^{\pi}$$

$$= \frac{2}{\pi}[1+1]$$

$$= \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}|\sin x|\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\frac{\sin(n+1)x - \sin(n-1)x}{2}\,dx$$

$$=\frac{1}{\pi}\left\{\left[\left(-\frac{\cos(n+1)x}{n+1}\right)-\left(-\frac{\cos\,(n-1)x}{n-1}\right)\right]_0^{\pi}\right\},\ provided\ n\neq 1$$

$$=\frac{1}{\pi}\left\{\left[-\frac{(-1)^{n+1}}{n+1}+\frac{1}{n+1}\right]-\left[-\frac{(-1)^{n-1}}{n-1}+\frac{1}{n-1}\right]\right\},\ provided\ n\neq 1$$

$$=\frac{1}{\pi}\left\{\left[-\frac{-(-1)^{n}}{n+1}+\frac{1}{n+1}\right]-\left[-\frac{-(-1)^{n}}{n-1}+\frac{1}{n-1}\right]\right\},\ provided\ n\neq 1$$

$$=\frac{1}{\pi}\left\{\left[(-1)^n\left(\frac{1}{n+1}-\frac{1}{n-1}\right)+\left(\frac{1}{n+1}-\frac{1}{n-1}\right)\right]\right\},\ provided\ n\neq 1$$

$$=\frac{1}{\pi}\left(\frac{1}{n+1}-\frac{1}{n-1}\right)\left((-1)^n+1\right),\ provided\ n\neq 1$$

$$=\frac{1}{\pi}\left(-\frac{2}{n^2-1}\right)\begin{cases}2, & if\ n\ is\ even\\ 0, & if\ n\ is\ odd\end{cases}$$

When n = 1, we have

$$a_1=\frac{2}{\pi}\int_0^{\pi} f(x)\cos 1x\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}|\sin x|\cos 1x\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}\sin x\cos 1x\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}\frac{\sin 2x}{2}\,dx$$

$$=\frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi}$$

$$=\frac{1}{2\pi}[-1+1]$$

$$=0$$

Hence the Fourier series is

$$|\sin x| = \frac{2}{\pi}-\frac{4}{\pi}\left[\sum_{n=2,4,6,..}^{\infty}\frac{1}{n^2-1}\cos nx\right].$$

6. Find the Fourier series expansion of $f(x)=\begin{cases}x, & 0<x<l/2\\ l-x, & l/2<x<l\end{cases}$**. Hence deduce, the value of** $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$

<u>Solution</u>

Since the interval for fourier series is 2L

$$put\ 2L=l\Rightarrow L=\frac{l}{2}$$

$$Hence\ f(x)=\begin{cases}x, & 0<x<L\\ 2L-x, & L<x<2L\end{cases}$$

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)\, dx$$

$$= \frac{1}{L}\int_0^{2L} f(x)\, dx$$

$$= \frac{1}{L}\left\{\int_0^{L} x\, dx + \int_L^{2L} (2L-x)\, dx\right\}$$

$$= \frac{1}{L}\left\{\left[\frac{x^2}{2}\right]_0^L + \left[\frac{(2L-x)^2}{2(-1)}\right]_L^{2L}\right\}$$

$$= \frac{1}{L}\left[\frac{L^2}{2} - 0 - 0 + \frac{L^2}{2}\right]$$

$$a_0 = \frac{1}{L}\left(L^2\right) = L$$

$$L = \frac{l}{2} \Rightarrow a_0 = \frac{l}{2}$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{L}\int_0^{2L} f(x)\ \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{L}\left\{\int_0^{L} x\ \cos\left(\frac{n\pi x}{L}\right) dx + \int_L^{2L} (2L-x)\ \cos\left(\frac{n\pi x}{L}\right) dx\right\}$$

$$= \frac{1}{L}\left\{\begin{array}{l} \left[(x)\left(\sin\left(\frac{n\pi x}{L}\right)\left(\frac{L}{n\pi}\right)\right) - 1\left(-\cos\left(\frac{n\pi x}{L}\right)\left(\frac{L}{n\pi}\right)^2\right)\right]_0^L \\ + \left[(2L-x)\left(\sin\left(\frac{n\pi x}{L}\right)\left(\frac{L}{n\pi}\right)\right) - (-1)\left(-\cos\left(\frac{n\pi x}{L}\right)\left(\frac{L}{n\pi}\right)^2\right)\right]_L^{2L} \end{array}\right\}$$

$$= \frac{1}{L}\left\{\begin{array}{l} \left[L\sin n\pi \frac{L}{n\pi} + \cos n\pi\left(\frac{L}{n\pi}\right)^2 - 0 - \left(\frac{L}{n\pi}\right)^2\right] + \\ \left[0 - \left(\frac{L}{n\pi}\right)^2 - L\sin n\pi \frac{L}{n\pi} + \cos n\pi\left(\frac{L}{n\pi}\right)^2\right] \end{array}\right\}$$

$$= \frac{1}{L}\left(\frac{L}{n\pi}\right)^2 \left[2\cos n\pi - 2\right]$$

$$= \frac{2}{L}\left(\frac{L}{n\pi}\right)^2 \left[(-1)^n - 1\right]$$

$$= \frac{2}{L}\frac{L^2}{n^2\pi^2}\begin{cases} -2, & if\ \ n\ is\ odd \\ 0, & if\ \ n\ is\ even \end{cases}$$

$$= \frac{2L}{n^2\pi^2}\begin{cases} -2, & if\ \ n\ is\ odd \\ 0, & if\ \ n\ is\ even \end{cases}$$

$$L = \frac{l}{2} \Rightarrow a_n = \frac{l}{n^2\pi^2}\begin{cases} -2, & if\ \ n\ is\ odd \\ 0, & if\ \ n\ is\ even \end{cases}$$

$$\Rightarrow a_n = \begin{cases} \frac{-2l}{n^2\pi^2} & if\ n\, is\, odd \\ 0, & if\ n\, is\, even \end{cases}$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$
$$= \frac{1}{L}\int_0^{2L} f(x)\ \sin\left(\frac{n\pi x}{L}\right)dx$$
$$= \frac{1}{L}\left\{\int_0^{L} x\sin\left(\frac{n\pi x}{L}\right)\ dx + \int_L^{2L}(2L-x)\ \sin\left(\frac{n\pi x}{L}\right)dx\right\}$$

$$= \frac{1}{L}\left\{\begin{array}{l}\left[(x)\left(-\cos\left(\frac{n\pi x}{L}\right)\left(\frac{L}{n\pi}\right)\right)-1\left(-\sin\left(\frac{n\pi x}{L}\right)\left(\frac{L}{n\pi}\right)^2\right)\right]_0^L \\ +\left[(2L-x)\left(-\cos\left(\frac{n\pi x}{L}\right)\left(\frac{L}{n\pi}\right)\right)-(-1)\left(-\sin\left(\frac{n\pi x}{L}\right)\left(\frac{L}{n\pi}\right)^2\right)\right]_L^{2L}\end{array}\right\}$$

$$=\frac{1}{L}\left\{\begin{array}{l}\left[L(-\cos n\pi)\left(\frac{L}{n\pi}\right)+\sin n\pi\left(\frac{L}{n\pi}\right)^2-0+0\right] \\ +\left[0-\sin n\pi\left(\frac{L}{n\pi}\right)^2+L(\cos n\pi)\left(\frac{L}{n\pi}\right)+\sin n\pi\left(\frac{L}{n\pi}\right)^2\right]\end{array}\right\}$$
$$b_n = 0$$

Hence the Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad where\ L = \frac{l}{2}$$
$$f(x) = \frac{l}{4} + \sum_{n=1,3,5,..}^{\infty} \frac{-2l}{n^2\pi^2}\cos\left(\frac{2n\pi x}{l}\right)$$

Since the denominator of the series is of the form $\frac{1}{n^4}$ and the denominator in the Fourier series is of the form $\frac{1}{n^2}$ let us use the Parseval's identity.

$$\frac{a_0^2}{2}+\left(a_n^2+b_n^2\right)=\frac{1}{L}\int_0^{2L}[f(x)]^2\,dx$$
$$\Rightarrow \frac{l^2}{8}+\frac{4l^2}{\pi^4}\sum_{n=1,3,5,..}^{\infty}\frac{1}{n^4}=\frac{1}{L}\int_0^{2L}[f(x)]^2\,dx$$
$$\Rightarrow \frac{l^2}{8}+\frac{4l^2}{\pi^4}\sum_{n=1,3,5,..}^{\infty}\frac{1}{n^4}=\frac{1}{L}\left\{\int_0^{L}x^2\ dx+\int_L^{2L}(l-x)^2\,dx\right\}$$
$$\Rightarrow \frac{l^2}{8}+\frac{4l^2}{\pi^4}\sum_{n=1,3,5,..}^{\infty}\frac{1}{n^4}=\frac{1}{L}\left\{\left[\frac{x^3}{3}\right]_0^L+\left[\frac{(2L-x)^3}{3(-1)}\right]_L^{2L}\right\}$$
$$\Rightarrow \frac{l^2}{8}+\frac{4l^2}{\pi^4}\sum_{n=1,3,5,..}^{\infty}\frac{1}{n^4}=\frac{1}{L}\left[\frac{L^3}{3}-0+\frac{L^3}{3}\right]$$

$$\Rightarrow \frac{l^2}{8}+\frac{4l^2}{\pi^4}\sum_{n=1,3,5,..}^{\infty}\frac{1}{n^4}=\frac{2}{l}\left[\frac{l^3}{24}-0-0+\frac{l^3}{24}\right](\because L=\frac{l}{2})$$
$$\Rightarrow \frac{l^2}{8}+\frac{4l^2}{\pi^4}\sum_{n=1,3,5,..}^{\infty}\frac{1}{n^4}=\frac{2}{l}\frac{l^3}{12}$$
$$\Rightarrow \frac{l^2}{8}+\frac{4l^2}{\pi^4}\sum_{n=1,3,5,..}^{\infty}\frac{1}{n^4}=\frac{l^2}{6}$$

$$\Rightarrow \frac{4l^2}{\pi^4} \sum_{n=1,3,5,..}^{\infty} \frac{1}{n^4} = \frac{l^2}{6} - \frac{l^2}{8}$$

$$\Rightarrow \frac{4l^2}{\pi^4} \sum_{n=1,3,5,..}^{\infty} \frac{1}{n^4} = \frac{l^2}{24}$$

$$\Rightarrow \sum_{n=1,3,5,..}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{4l^2} \frac{l^2}{24}$$

$$\Rightarrow \sum_{n=1,3,5,..}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96}$$

7. Find the Fourier series expansion of the function $f(x) = \begin{cases} l + x, & -l \le x \le 0 \\ l - x, & 0 \le x \le l \end{cases}$**. Hence deduce that** $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

<u>**Solution:**</u>

Here the interval is – l to l let us first check whether the function is odd or even.

$$f(-x) = \begin{cases} l + (-x), & -l \le -x \le 0 \\ l - (-x), & 0 \le -x \le l \end{cases}$$

$$= \begin{cases} l - x, & 0 \le x \le l \\ l + x, & -l \le x \le l \end{cases}$$ So, the function is even.

$$= f(x)$$

(Note that when an inequality is multiplied by -1 the inequality is reversed.)

So, let us find only the two Fourier coefficients a_0, , a_n.

$$a_0 = \frac{2}{l} \int_0^l (l - x)\, dx$$

$$= \frac{2}{l} \left[\frac{(l-x)^2}{2(-1)} \right]_0^l$$

$$= \frac{2}{l} \left[0 + \frac{l^2}{2} \right]$$

$$= l$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l (l-x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[(l-x)\left(\sin\left(\frac{n\pi x}{l}\right)\left(\frac{l}{n\pi}\right)\right) - (-1)\left(-\cos\left(\frac{n\pi x}{l}\right)\left(\frac{l}{n\pi}\right)^2\right)\right]_0^l$$

$$= \frac{2}{l}\left[0 - \cos n\pi\left(\frac{l}{n\pi}\right)^2 - 0 + \left(\frac{l}{n\pi}\right)^2\right]$$

$$= \frac{2}{l}\left(\frac{l}{n\pi}\right)^2[1-\cos n\pi]$$

$$= \frac{2l}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$= \frac{2l}{n^2\pi^2}\begin{cases} 2, \text{ if } n \text{ is odd} \\ 0, \text{ if } n \text{ is even}\end{cases}$$

$$a_n = \begin{cases} \dfrac{4l}{n^2\pi^2}, \text{ if } n \text{ is odd} \\ 0, \text{ if } n \text{ is even}\end{cases}$$

Hence the Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$f(x) = \frac{l}{2} + \frac{4l}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\cos\left(\frac{n\pi x}{l}\right)$$

Deduction: To get the deduction put $x = 0$(which is a continuous point)

$$l = \frac{l}{2} + \frac{4l}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow l - \frac{l}{2} = \frac{4l}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \frac{l}{2} = \frac{4l}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{8}$$

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8. Find the half range sine series of *f(x) = x cos x* in *(0, π)*.

Solution:

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \cos x \sin(nx)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\, \frac{\sin(n+1)x + \sin(n-1)x}{2}\, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x \sin(n+1)x\, dx + \int_0^{\pi} x \sin(n-1)x\, dx\right]$$

$$= \frac{1}{\pi}\left\{\begin{matrix}\left[(x)\left(-\frac{\cos(n+1)x}{n+1}\right) - 1\left(-\frac{\sin(n+1)x}{(n+1)^2}\right)\right] + \\ \left[(x)\left(-\frac{\cos(n-1)x}{n-1}\right) - 1\left(-\frac{\sin(n-1)x}{(n-1)^2}\right)\right]\end{matrix}\right\}_0^{\pi}$$

provided $n \neq 1$

$$= \frac{1}{\pi}\left[-\pi\frac{(-1)^{n+1}}{n+1} - \pi\frac{(-1)^{n-1}}{n-1}\right]$$

$$= \frac{1}{\pi}\left\{\left[\pi\frac{(-1)^n}{n+1}\right] + \left[\pi\frac{(-1)^n}{n-1}\right]\right\} \quad \textit{provided } n \neq 1$$

$$= (-1)^n\left[\frac{1}{n+1} + \frac{1}{n-1}\right] \quad \textit{provided } n \neq 1$$

$$b_n = (-1)^n \frac{2}{n^2-1} \quad \textit{provided } n \neq 1$$

When $n = 1$ $b_1 = \frac{2}{\pi}\int_0^{\pi} x \cos x \sin(x)\, dx$

$$= \frac{2}{\pi}\int_0^{\pi} x\, \frac{\sin 2x}{2}\, dx$$

$$= \frac{1}{\pi}\left[(x)\left(\frac{-\cos 2x}{2}\right) - 1\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\pi\left(-\frac{1}{2}\right)$$

$$b_1 = -\frac{1}{2}$$

Hence the half range sine series is

$$f(x) = b_1 \sin x + \sum_{n21}^{\infty} b_n \sin nx$$

$$x \cos x = -\frac{1}{2}\sin x + \sum_{n=2}^{\infty}(-1)^n \frac{2}{n^2-1}\sin nx$$

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9. Obtain the sine series for the function $f(x) = \begin{cases} x, & 0 < x < l/2 \\ l - x, & l/2 < x < l \end{cases}$.

<u>Solution:</u>

$$b_n = \frac{2}{l}\int_0^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l}\left\{\int_0^{l/2} x \sin\left(\frac{n\pi x}{l}\right) dx + \int_{l/2}^{l} (l-x) \sin\left(\frac{n\pi x}{l}\right) dx\right\}$$

$$= \frac{2}{l}\left\{\begin{matrix}\left[(x)\left(-\cos\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi}\right) - 1\left(-\sin\left(\frac{n\pi x}{l}\right)\right)\left(\frac{l}{n\pi}\right)^2\right]_0^{l/2} \\ + \left[(l-x)\left(-\cos\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi}\right) - (-1)\left(-\sin\left(\frac{n\pi x}{l}\right)\right)\left(\frac{l}{n\pi}\right)^2\right]_{l/2}^{l}\end{matrix}\right\}$$

$$= \frac{2}{l}\left\{\begin{array}{l}\left[-\frac{l}{2}\frac{l}{n\pi}\cos\left(\frac{n\pi}{2}\right)+\sin\left(\frac{n\pi}{2}\right)\frac{l^2}{n^2\pi^2}-0+0\right] \\ +\left[0-0+\frac{l}{2}\frac{l}{n\pi}\cos\left(\frac{n\pi}{2}\right)+\sin\left(\frac{n\pi}{2}\right)\frac{l^2}{n^2\pi^2}\right]\end{array}\right\}$$

$$= \frac{2}{l}\left[2\sin\left(\frac{n\pi}{2}\right)\frac{l^2}{n^2\pi^2}\right]$$

$$= \frac{4l}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

Hence the half range sine series is $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

$$f(x) = \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{l}\right)$$

10. Find the Fourier series expansion of $f(x) = \begin{cases} l-x, & 0<x<l \\ 0, & l<x<2l \end{cases}$**. Hence deduce the value of the series (i)** $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots.$

(ii) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots.$

Solution: We need to find all the three fourier constants.

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{l}\left\{\int_0^{l} l-x\,dx + \int_l^{2l} 0\,dx\right\}$$

$$= \frac{1}{l}\left[\frac{(l-x)^2}{2(-1)}\right]_0^l$$

$$= \frac{1}{l}\frac{l^2}{2}$$

$$= \frac{l}{2}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left\{\int_0^{l}(l-x)\cos\left(\frac{n\pi x}{l}\right)dx + \int_l^{2l} 0\cos\left(\frac{n\pi x}{l}\right)dx\right\}$$

$$= \frac{1}{l}\left[(l-x)\sin\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi} - (-1)\left(-\cos\left(\frac{n\pi x}{l}\right)\right)\left(\frac{l}{n\pi}\right)^2\right]_0^l$$

$$= \frac{1}{l}\left[0 - \cos n\pi\left(\frac{l}{n\pi}\right)^2 - 0 + \left(\frac{l}{n\pi}\right)^2\right]$$

$$= \frac{1}{l}\left(\frac{l}{n\pi}\right)^2\left[1-\cos n\pi\right]$$

$$= \frac{l}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$a_n = \frac{l}{n^2\pi^2}\begin{cases}2, \textit{ if } n \textit{ is odd}\\ 0, \textit{ if } n \textit{ is even}\end{cases}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left\{\int_0^{l}(l-x)\sin\left(\frac{n\pi x}{l}\right)dx + \int_l^{2l} 0\sin\left(\frac{n\pi x}{l}\right)dx\right\}$$

$$= \frac{1}{l}\left[(l-x)\left(-\cos\left(\frac{n\pi x}{l}\right)\right)\frac{l}{n\pi} - (-1)\left(-\sin\left(\frac{n\pi x}{l}\right)\left(\frac{l}{n\pi}\right)^2\right)\right]_0^l$$

$$= \frac{1}{l}\left[0 - \sin n\pi\left(\frac{l}{n\pi}\right)^2 + l\,\frac{l}{n\pi} - 0\right]$$

$$b_n = \frac{l}{n\pi}$$

Hence the Fourier series is $f(x) = \frac{l}{4} + \frac{2l}{\pi^2}\sum_{n=1,3,5,..}^{\infty}\frac{1}{n^2}\cos\left(\frac{n\pi x}{l}\right) + \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{l}\right)$

<u>Deduction(i)</u> The denominator of cosine terms are in the form $\frac{1}{n^2}$ and the denominator of sine terms are in the form $\frac{1}{n}$. So, to get deduction (i) , let us make all the cosine terms vanish. This can be done by taking $x = \frac{\pi}{2}, 3\frac{\pi}{2},....$ Let us put $x = \frac{l}{2}$ (which is a continuous point)

$$l - \frac{l}{2} = \frac{l}{4} + 0 + \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \frac{l}{2} = \frac{l}{4} + \frac{l}{\pi}\left[\frac{1}{1}\sin\left(\frac{\pi}{2}\right) + \frac{1}{2}\sin\pi + \frac{1}{3}\sin\left(3\frac{\pi}{2}\right) + \frac{1}{4}\sin 2\pi + \frac{1}{5}\sin\left(5\frac{\pi}{2}\right) + ...\right]$$

$$\Rightarrow \frac{l}{2} = \frac{l}{4} + \frac{l}{\pi}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5}...\right]$$

$$\Rightarrow \frac{l}{2} - \frac{l}{4} = \frac{l}{\pi}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5}...\right]$$

$$\Rightarrow \frac{l}{4} = \frac{l}{\pi}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5}...\right]$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5}...$$

<u>Deduction(ii):</u> To get the deduction all the sine terms must vanish and this can be done by taking $x = 0, 2\pi, ...$ So, let us put $x = 0$ (which is a discontinuous point).

$$\frac{f(0)+f(2l)}{2}=\frac{l}{4}+\frac{2l}{\pi^2}\sum_{n=1,3,5..}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow\frac{l}{2}=\frac{l}{4}+\frac{2l}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...\right]$$

$$\Rightarrow\frac{l}{2}=\frac{l}{4}+\frac{2l}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...\right]$$

$$\Rightarrow\frac{l}{2}-\frac{l}{4}=\frac{2l}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...\right]$$

$$\Rightarrow\frac{l}{4}=\frac{2l}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...\right]$$

$$\Rightarrow\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...$$

11.Find the Fourier series expansion for the function *f(x)* = $x+x^2$ in $-\pi<x<\pi$

Solution: Here the interval is $-\pi<x<\pi$. So, let us verify whether the function is odd or even. $f(-x)=(-x)+(-x)^2=-x+x^2\neq\begin{cases} f(x)=x+x^2\\ -f(x)=-(x+x^2)\end{cases}$ which is neither odd nor even. So, Let us find all the three constants.

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx$$

$$=\frac{1}{\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left[\left(\frac{\pi^2}{2}+\frac{\pi^3}{3}\right)-\left(\frac{\pi^2}{2}-\frac{\pi^3}{3}\right)\right]$$

$$=\frac{1}{\pi}\frac{2\pi^3}{3}$$

$$a_0=\frac{2\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx$$

$$=\frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n}-(1+2x)\left(-\frac{\cos nx}{n^2}\right)+2\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left\{\left[0-(1+2\pi)\left(-\frac{\cos n\pi}{n^2}\right)+0\right]-\left[0-(1)\left(-\frac{\cos n\pi}{n^2}\right)+0\right]\right\}$$

$$=\frac{1}{\pi}\left(-\frac{\cos n\pi}{n^2}\right)(-1-2\pi+1-2\pi)$$

$$=\frac{1}{\pi}\left(\frac{(-1)^n}{n^2}\right)4\pi$$

$$a_n = 4\frac{(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\, dx$$

$$=\frac{1}{\pi}\left[(x+x^2)\left(-\frac{\cos nx}{n}\right)-(1+2x)\left(-\frac{\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left\{\begin{array}{l}\left[(\pi+\pi^2)\left(-\frac{\cos n\pi}{n}\right)-(1+2\pi)\left(-\frac{\sin n\pi}{n^2}\right)+2\left(\frac{\cos n\pi}{n^3}\right)\right]- \\ \left[(-\pi+\pi^2)\left(-\frac{\cos n\pi}{n}\right)-(1-2\pi)\left(+\frac{\sin n\pi}{n^2}\right)+2\left(\frac{\cos n\pi}{n^3}\right)\right]\end{array}\right\}$$

$$=\frac{1}{\pi}\left[\frac{\cos n\pi}{n}\left(-\pi-\pi^2-\pi+\pi^2\right)\right]$$

$$=\frac{1}{\pi}\frac{(-1)^n}{n}(-2\pi)$$

$$b_n = -2\frac{(-1)^n}{n}$$

Hence the Fourier series is of the form $x+x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$

12.Prove that $1=\frac{4}{\pi}\left[\sin\frac{\pi x}{l}+\frac{1}{3}\sin\frac{3\pi x}{l}+\frac{1}{5}\sin\frac{5\pi x}{l}+....\right]$ **in the interval 0< x< l**

Solution:

As per the RHS we need to find the Fourier sine series expansion of the function in $0 < x < l$
Here *f(x) = 1* and the interval shall be taken as *(0,l)*.

$$b_n = \frac{2}{l}\int_0^l 1\sin\left(\frac{n\pi x}{l}\right)dx$$

$$=\frac{2}{l}\left(-\cos\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi}\right)_0^l$$

$$=\frac{2}{l}\frac{l}{n\pi}\left[-\cos n\pi+1\right]$$

$$=\frac{2}{n\pi}\left[-(-1)^n+1\right]$$

$$=\frac{2}{n\pi}\begin{cases}2, & if\ n\ is\ odd\\ 0, & if\ n\ is\ even\end{cases}$$

$$b_n=\begin{cases}\frac{4}{n\pi}, & if\ n\ is\ odd\\ 0, & if\ n\ is\ even\end{cases}$$

Hence the Fourie siner series is

$$1 = \sum_{n=1,3,5,....}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow 1 = \frac{4}{\pi}\left[\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + ...\right]$$

13. **Find the half range cosine series of** $f(x) = x^2,\ in\ 0 < x < \pi$. **Hence deduce the value of** $\sum_{n=1}^{\infty}\frac{1}{n^4}$.

Solution:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

applying Bernoulli's formula

$$= \frac{2}{\pi}\left[\left(x^2\right)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[\left(\pi^2\right)\left(\frac{\sin n\pi}{n}\right) - (2\pi)\left(\frac{-\cos n\pi}{n^2}\right) + (2)\left(\frac{-\sin n\pi}{n^3}\right)\right]_0^{2\pi} - \frac{1}{\pi}\left[\left(0^2\right)\left(\frac{\sin n0}{n}\right) - (0)\left(\frac{-\cos n0}{n^2}\right) + (2)\left(\frac{-\sin n0}{n^3}\right)\right]_0^{2\pi}\right\}$$

$$= \frac{1}{\pi}4\pi\frac{(-1)^n}{n^2}$$

$$a_n = 4\frac{(-1)^n}{n^2}$$

Hence the half range cosine series is $x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} 4\frac{(-1)^n}{n^2}$

Deduction: Since the denominator of the series is n^4 and that of the cosine series is only n^2 let us apply Parseval's identity for fourier cosine series is

$$\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)=\frac{2}{L}\int_0^L [f(x)]^2\,dx$$

$$\Rightarrow \frac{\left(\frac{4\pi^4}{9}\right)}{2}+\sum_{n=1}^{\infty}\frac{16}{n^4}=\frac{2}{\pi}\int_0^{\pi} x^4\,dx$$

$$\Rightarrow \frac{2\pi^4}{9}+16\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{2}{\pi}\left[\frac{x^5}{5}\right]_0^{\pi}$$

$$\Rightarrow \frac{2\pi^4}{9}+16\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{2}{\pi}\left[\frac{\pi^5}{5}\right]$$

$$\Rightarrow \frac{2\pi^4}{9}+16\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{2\pi^4}{5}$$

$$\Rightarrow 16\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{2\pi^4}{5}-\frac{2\pi^4}{9}$$

$$\Rightarrow 16\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{8\pi^4}{45}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$$

14. Find the half range cosine series for the function $f(x)=x,\ in\ 0<x<l$. Hence deduce the value of the series $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$.

Solution:

$$a_0=\frac{2}{l}\int_0^l x\,dx$$

$$=\frac{2}{l}\left[\frac{x^2}{2}\right]_0^l$$

$$=\frac{2}{l}\left[\frac{l^2}{2}\right]$$

$$a_0=l$$

$$a_n = \frac{2}{l}\int_0^l x\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[x\ \sin\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi} - 1\left(-\cos\left(\frac{n\pi x}{l}\right)\left(\frac{l}{n\pi}\right)^2\right)\right]_0^l$$

$$= \frac{2}{l}\left[l\ \sin\left(\frac{n\pi l}{l}\right)\frac{l}{n\pi} - 1\left(-\cos\left(\frac{n\pi l}{l}\right)\left(\frac{l}{n\pi}\right)^2\right) - 0 - \left(\frac{l}{n\pi}\right)^2\right]$$

$$= \frac{2}{l}\left(\frac{l}{n\pi}\right)^2[\cos n\pi - 1]$$

$$= \frac{2l}{n^2\pi^2}[(-1)^n - 1]$$

$$= \frac{2l}{n^2\pi^2}\begin{cases}(-2), & if\ n\ is\ odd\\ 0\ , & if\ n\ is\ even\end{cases}$$

$$a_n = \begin{cases}-\dfrac{4l}{n^2\pi^2}, & if\ n\ is\ odd\\ 0\ , & if\ n\ is\ even\end{cases}$$

Hence the half range series is $x = \frac{l}{2} + \sum_{n=1,3,5,...}^{\infty} -\frac{4l}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right)$

<u>Deduction:</u> Since the denominator of the series is n^4 and that of the cosine series is only n^2 let us apply Parseval's identity

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) = \frac{2}{L}\int_0^L [f(x)]^2\ dx$$

$$\Rightarrow \frac{l^2}{2} + \sum_{n=1,3,5,...}^{\infty}\frac{16l^2}{n^4\pi^4} = \frac{2}{l}\int_0^l x^2\ dx$$

$$\Rightarrow \frac{l^2}{2} + \sum_{n=1,3,5,...}^{\infty}\frac{16l^2}{n^4\pi^4} = \frac{2}{l}\left[\frac{x^3}{3}\right]_0^l$$

$$\Rightarrow \frac{l^2}{2} + \sum_{n=1,3,5,...}^{\infty}\frac{16l^2}{n^4\pi^4} = \frac{2l^2}{3}$$

$$\Rightarrow \sum_{n=,3,5,...1}^{\infty}\frac{16l^2}{n^4\pi^4} = \frac{2l^2}{3} - \frac{l^2}{2}$$

$$\Rightarrow \sum_{n=1,3,5,..}^{\infty}\frac{16l^2}{n^4\pi^4} = \frac{l^2}{6}$$

$$\Rightarrow \sum_{n=1,3,5,...}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{96}$$

15.Find the Fourier series expansion upto third harmonic from the following data:

x:	**0**	**1**	**2**	**3**	**4**	**5**
f(x):	**9**	**18**	**24**	**28**	**26**	**20**

<u>Solution:</u>

x	y	θ =πx/3	θ	Cosθ	y Cosθ	Cos 2θ	yCos 2θ	Cos 3θ	yCos 3θ
0	9	0	0	1	9	1	9	1	9
1	18	π/3	60	0.5	9	-0.5	-9	-1	-18
2	24	2π/3	120	-0.5	-12	-0.5	-12	1	24
3	28	3π/3	180	-1	-28	1	28	-1	-28
4	26	4π/3	240	-0.5	-13	-0.5	-13	1	26
5	20	5π/3	300	0.5	10	-0.5	-10	-1	-20
	$\sum y=$ 125				$\sum y\cos\theta$ = - 25		$\sum y\cos 2\theta$ = - 7		$\sum y\cos 3\theta$ = - 7

sinθ	ysinθ	sin 2θ	y sin 2θ	sin 3θ	y sin 3θ
0	0	0	0	0	0
0.866	15.588	0.866	15.588	0	0
0.866	20.784	-0.866	-20.784	0	0
0	0	0	0	0	0
-0.866	-22.516	0.866	22.516	0	0
-0.866	-17.32	-0.866	-17.32	0	0
	$\sum y\sin\theta$ = - 3.464		$\sum y\sin 2\theta$ = 0.0		$\sum y\sin 3\theta$ = 0

$$a_0 = \frac{2}{q}\sum y = \frac{2}{6} * 125 = 41.6667$$

$$a_1 = \frac{2}{q}\sum y\cos\theta = \frac{2}{6} * (-25) = -8.3333$$

$$a_2 = \frac{2}{q}\sum y\cos 2\theta = \frac{2}{6} * (-7) = -2.3333$$

$$a_3 = \frac{2}{q}\sum y\cos 3\theta = \frac{2}{6} * (-7) = -2.3333$$

$$b_1 = \frac{2}{q}\sum y\sin\theta = \frac{2}{6} * (-3.464) = -1.15$$

$$b_2 = \frac{2}{q}\sum y\sin 2\theta = \frac{2}{6} * (0) = 0$$

$$b_3 = \frac{2}{q}\sum y\sin 3\theta = \frac{2}{6} * (0) = 0$$

Hence the Fourier series expansion

$$f(x) = \frac{a_0}{2} + (a_1 \cos\theta + b_1 \sin\theta) + (a_2 \cos 2\theta + b_2 \sin 2\theta) + (a_3 \cos 3\theta + b_3 \sin 3\theta)$$

$$\Rightarrow f(x) = 20.8334 + (-8.3333 \cos\theta - 1.15 \sin\theta) + (-2.3333 \cos 2\theta + 0 \sin 2\theta) + (-2.3333 \cos 3\theta)$$

$$\Rightarrow f(x) = 20.8334 + \left(-8.3333 \cos\frac{\pi x}{3} - 1.15 \sin\frac{\pi x}{3}\right) + \left(-2.3333 \cos 2\frac{\pi x}{3}\right) + \left(-2.3333 \cos 3\frac{\pi x}{3}\right)$$

16.Express the function $f(x) = e^{-x}, in -1 < x < 1$ in the complex form of the Fourier series.

Solution: The Complex form of the Fourier series is $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{-in\pi x}$, where

$$C_n = \frac{1}{2L} \int_c^{c+2L} f(x) e^{in\pi x} dx$$

$$C_n = \frac{1}{2} \int_{-1}^{1} e^{-x} e^{in\pi x} dx$$

$$= \frac{1}{2} \int_{-1}^{1} e^{-(1-in\pi)x} dx$$

$$= \frac{1}{2} \left[\frac{e^{-(1-in\pi)x}}{-(1-in\pi)} \right]_{-1}^{1}$$

$$= \frac{1}{2} \left[\frac{e^{-(1-in\pi)}}{-(1-in\pi)} - \frac{e^{(1-in\pi)}}{-(1-in\pi)} \right]$$

$$= \frac{1}{2} \frac{1}{1-in\pi} \left[-e^{-1} e^{in\pi} + e\, e^{-in\pi} \right]$$

$$= \frac{1}{2} \frac{1}{1-in\pi} \frac{1+in\pi}{1+in\pi} \left[-\frac{1}{e}(\cos n\pi + i \sin n\pi) + e(\cos n\pi - i \sin n\pi) \right]$$

$$= \frac{1}{2} \frac{1+in\pi}{1+n^2\pi^2} (-1)^n \left(e - \frac{1}{e} \right)$$

$$C_n = \frac{e^2 - 1}{2e} \frac{1+in\pi}{1+n^2\pi^2} (-1)^n$$

Hence the Complex form of the Fourier series of the given function is

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{-\frac{in\pi x}{L}}$$

$$e^{-x} = \sum_{n=-\infty}^{\infty} \frac{e^2 - 1}{2e} \frac{1+in\pi}{1+n^2\pi^2} (-1)^n e^{-in\pi x}$$

17.Obtain the fourier series for the function $f(x) = |x| \quad in \quad -\pi < x < \pi$ also deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$.

Sol :

$f(x) = |x|$

$f(-x) = |-x|$

$\therefore f(x) = f(-x)$ *it is an even function.*

Hence $b_n = 0$

Fourier series becomes

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx$$

$$=\frac{2}{\pi}\int_0^{\pi}|x|dx \qquad \left(\because |x|=\begin{cases}-x & :-\pi<x<0\\ x & :0<x<\pi\end{cases}\right)$$

$$=\frac{2}{\pi}\int_0^{\pi}xdx=\frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi}$$

$$=\frac{2}{\pi}\left(\frac{\pi^2}{2}-0\right)=\pi$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nxdx$$

$$=\frac{2}{\pi}\int_0^{\pi}|x|\cos nxdx=\frac{2}{\pi}\int_0^{\pi}x\cos nxdx$$

$$=\frac{2}{\pi}\left[\frac{x\sin nx}{n}+\frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$a_n=\frac{2}{\pi}\left[\frac{(-1)^n}{n^2}-\frac{1}{n^2}\right]$$

$$\therefore a_n=\begin{cases}-\frac{4}{\pi n^2} & :n\,is\,odd\\ 0 & :n\,is\,even\end{cases}$$

Hence Fourier series becomes

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1,3,5,...}^{\infty}\frac{\cos nx}{n^2}$$

$Put\ x=0\,(continuous\ point)$

$\therefore f(0)=0$

$$\Rightarrow f(0)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1,3,5,...}^{\infty}\frac{1}{n^2}$$

$$0-\frac{\pi}{2}=-\frac{4}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...\right)$$

$$\Rightarrow\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...\right)=\frac{\pi^2}{8}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$$

18.Find the Fourier series of $f(x)=|\cos x|\ in\ (-\pi,\pi)$

Sol:

$Here\ f(x)=f(-x)$

$\therefore f(x)=|\cos x|\ is\ even\ function.\ Hence\ b_n=0$

Fourier series is $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx$$

$$=\frac{2}{\pi}\int_0^{\pi}|\cos x|dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \cos x dx + \int_{\frac{\pi}{2}}^{\pi} -\cos x dx\right]$$

$$= \frac{2}{\pi}\left[\left(\sin x\right)_0^{\pi/2} - \left(\sin x\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[(1-0)-(0-1)\right]$$

$$a_0 = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\cos x|\cos nx dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos nx.\cos x dx - \int_{\pi/2}^{\pi} \cos nx \cos x dx\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{2}\left(\int_0^{\pi/2} \cos(n+1)x + \cos(n-1)dx - \int_{\pi/2}^{\pi} \cos(n+1)x + \cos(n-1)x dx\right)\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right)_0^{\pi/2} - \left(\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1} + \frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos\frac{n\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}}{n-1} + \frac{\cos\frac{n\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}}{n-1}\right]$$

$$= \frac{2}{\pi}\cos\frac{n\pi}{2}\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$a_n = \frac{2}{\pi}\cos\frac{n\pi}{2}\left[\frac{-2}{n^2-1}\right] \quad n \neq 1$$

$$= \frac{-4}{\pi(n^2-1)}\cos\frac{n\pi}{2}, \quad n \neq 1$$

When n = 1 we have

$$a_1 = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cos x dx - \int_{\pi/2}^{\pi} \cos x \cos x dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x dx - \int_{\pi/2}^{\pi} \cos^2 x dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\left(\frac{1+\cos 2x}{2}\right)dx - \int_{\pi/2}^{\pi}\left(\frac{1+\cos 2x}{2}\right)dx\right]$$

$$= \frac{1}{\pi}\left[\left(x + \frac{\sin 2x}{2}\right)_0^{\pi/2} - \left(x + \frac{\sin 2x}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \pi + \frac{\pi}{2}\right] = 0$$

$$f(x) = \frac{2}{\pi} + \sum_{n=2,4,\ldots}^{\infty} \frac{-4}{\pi(n^2-1)}\cos\frac{n\pi}{2}\cos nx$$

UNIT – 4 : APPLICATION OF PARTIAL DIFFERENTIAL EQUATIONS

Question Bank

PART – A

1. **Classify the following partial differential equations**

   (a) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$

   (b) $\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial u}{\partial y}\right) + xy$

Solution:

(a) A=1, B=0, C=-1

$B^2 - 4AC = 0 + 4 = 4 > 0$

Equation is hyperbolic

(b) A=0, B=1, C=0

$B^2 - 4AC = 1 - 0 = 1 > 0$

Equation is hyperbolic

2. **Classify the following partial differential equations**

   (a) $4\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} - 4\frac{\partial u}{\partial x} - 8\frac{\partial u}{\partial y} - 16u = 0$

   (b) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$

Solution:

(a) A=4, B=4, C=1

$B^2 - 4AC = 16 - 16 = 0$

Equation is parabolic

(b) A=1, B=0, C=1

$B^2 - 4AC = -4 < 0$

Equation is Elliptic.

3. **Classify the following partial differential equations**

   $x^2 f_{xx} + (1 - y^2) f_{xx} = 0 \; for -1 < y < 1, -\infty < x < \infty$

   Solution:

   $A = x^2, B = 0, C = 1 - y^2$

$$B^2 - 4AC = -4x^2(1-y^2)$$
$$= 4x^2(y^2-1)$$

x^2 is always $+ve$ in $-\infty < x < \infty,\ \ x \neq 0$

In $-1 < y < 1,\ \ y^2 - 1$ is $-ve$

Equation is Elliptic

If x = 0, B^2-4AC=0, the equation is **Parabolic.**

4. Classify the following partial differential equations:

$(a)\ y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} + 2u_x - 3u = 0$

$(b)\ y^2u_{xx} + u_{yy} + u_x^{\,2} + u_y^{\,2} + 7 = 0$

Solution:

(a) Here A = y^2 ,B = $-2xy$, C = x^2

B^2 - 4AC = 4 x^2 y^2 - 4 x^2 y^2 = 0

Equation is parabolic

(b) Here A = y^2 ,B =0, C = 1

B^2 - 4AC = 0 - 4 y^2 = - 4 y^2 < 0

Here y^2 is always positive

Equation is elliptic.

5. Classify $(1+x)^2 u_{xx} - 4xu_{xy} + u_{yy} = x$

Solution:

Here A = $(1+x)^2$,B = $-4x$, C = 1

B^2 - 4AC = 16 x^2 - 4 $(1+x)^2$

= 16 x^2 - 4 - 8 x - 4 x^2

=4(3 x^2 -2 x -1)

If x =1 ,then B^2 - 4AC = 0

Given PDE is Parabolic.

If x <0 (or) x >0 Then B^2 – 4AC >0

Given PDE is Hyperbolic.

6. What are the possible solution of one dimensional wave equation?

Solution:

$(i)\ y(x,t) = \left(c_1 e^{px} + c_2 e^{-px}\right)\left(c_3 e^{apt} + c_4 e^{-apt}\right)$

$(ii)\ y(x,t) = \left(c_5 \cos px + c_6 \sin px\right)\left(c_7 \cos apt + c_8 \sin apt\right)$

$(iii)\ y(x,t) = \left(c_9 x + c_{10}\right)\left(c_{11} t + c_{12}\right)$

7. What is the constant a^2 in wave equation $u_{tt} = a^2 u_{xx}$?

Solution:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

$$where\ \ a^2 = \frac{tension}{mass\ per\ unit\ length\ of\ the\ string}$$

8. A tightly stretched string of length 2l is fastened at both ends. The midpoint of the string is displaced to a distance b and released from rest in this position. Write the initial conditions.

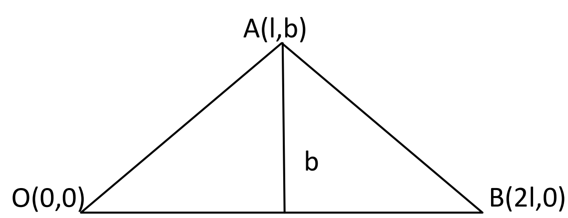
Solution:

The one dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

The equation of OA is

$$\frac{y-0}{b} = \frac{x-0}{l-0}$$

$$\Rightarrow y = \frac{bx}{l}, 0 < x < l$$



$The\ equation\ of\ \ AB\ is\ B(2l,0)$

$$\frac{y-b}{-b} = \frac{x-l}{l}$$

$$\Rightarrow y - b = \frac{lb - bx}{l}$$

$$\Rightarrow y = \frac{lb - bx + lb}{l} = \frac{b}{l}(2l - x),\ l < x < 2l$$

The initial boundary conditions are

$(i)\, y(0,t) = 0$

$(ii)\, y(l,t) = 0$

$(iii)\, \frac{\partial y(x,0)}{\partial t} = 0$

$$(iv)\, y(x,0) = \begin{cases} \frac{bx}{l},\ 0 < x < l \\ \frac{2}{l}(2l - x), l < x < 2l \end{cases}$$

9. Write the boundary conditions for the following boundary value problem "If a string of length l initially at rest in its equilibrium position and each of its point is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3 \frac{\pi x}{l}, 0 < x < l$ Determine the displacement function y(x,t)"

Solution:

The boundary conditions are

(i) $y(0,t) = 0, t > 0,$

(ii) $y(l,t) = 0, t > 0,$

(iii) $y(x,0) = 0, 0 < x < l$,

(iv) $\frac{\partial y}{\partial t}(x,0) = v_0 \sin^3 \frac{\pi x}{l}, 0 < x < l$

10. Write the boundary conditions for solving the string equation if the string is subjected to initial displacement f(x) and initial velocity g(x).

Solution:

(i) $y(0,t) = 0$

(ii) $y(l,t) = 0$

(iii) $\frac{\partial y}{\partial t}(x,0) = g(x), 0 < x < l$

(iv) $y(x,0) = f(x), 0 < x < l$

11. In one dimensional heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ what does α^2 refer to?

Solution:

$\alpha^2 = \frac{k}{s\rho}$ *is diffusivity of the material*

$where\, k - thermal\ conductivity$

$S - specific\ heat\ capacity$

$\rho - Density.$

12. What are the possible solution of one dimensional heat equation?

Solution:

The possible solutions are

$(i)\, u(x,t) = Ae^{-\alpha^2 p^2 t}\left(B\cos px + C\sin px\right)$

$(ii)\, u(x,t) = Ae^{-\alpha^2 p^2 t}\left(Be^{px} + Ce^{-px}\right)$

$(iii)\, u(x,t) = Ae^{-\alpha^2 p^2 t}\left(Bx + c\right)$

13. What is the steady state temperature of a rod of length l whose ends are kept at 30^0 and 40^0

Solution:

The heat flow equation is $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$

When the steady state condition exist, $\dfrac{\partial u}{\partial t} = 0$ ($\because$ u is independent of t)

Then the heat flow equation becomes $\dfrac{\partial^2 u}{\partial x^2} = 0$

$$u = ax + b \qquad \rightarrow (1)$$

$$x = 0, u = 30 \Rightarrow b = 30$$

$$x = l, u = 40 \Rightarrow 40 = al + 30$$

$$a = \frac{10}{l}$$

$$(1) \Rightarrow u = \frac{10}{l}x + 30$$

14. The bar of length 50 cm has its ends kept at 20 C and 100 C until steady state conditions prevails. Find the steady state temperature of the rod.

Solution:

The steady state equation of the one dimensional heat equation is

$$\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow u(x) = ax + b \rightarrow (1)$$

$$The\,boundary\,conditions\,are\,(a)\,u(0) = 20\,\&\,(b)\,u(l) = 100$$

$$Applying\ (a)\,in\,(1)$$

$$u(0) = 20 \Rightarrow b = 20$$

$$substitute\,this\,value\,in\,(1)\,we\,get$$

$$u(x) = ax + 20 \rightarrow (2)$$

$$Applying\ (b)\,in(2)$$

$$u(l) = 100 \Rightarrow al + 20 = 100$$

$$\Rightarrow al = 80$$

$$\Rightarrow a = \frac{80}{l}$$

Substitute this value in (2) we get

$$u(x) = \frac{80x}{l} + 20$$

$$l = 50 \Rightarrow u(x) = \frac{80x}{50} + 20$$

$$u(x) = \frac{8x}{5} + 20$$

15. Derive the steady state solution of one dimensional heat equation

Solution:

The one dimensional heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

$$u(x,t) = u(x)$$

$$\Rightarrow \frac{\partial u}{\partial t} = 0$$

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} = 0$$

$$\Rightarrow \frac{d^2 u}{dx^2} = 0$$

$$u(x) = ax + b$$

16. What are the possible solutions of two dimensional heat equation or laplace equation?

Solution:

$(i)\ y(x,t)=\left(c_1e^{px}+c_2e^{-px}\right)\left(c_3\cos py+c_4\sin py\right)$

$(ii)\ y(x,t)=\left(c_5\cos px+c_6\sin px\right)\left(c_7e^{py}+c_8e^{-py}\right)$

$(iii)\ y(x,t)=\left(c_9x+c_{10}\right)\left(c_{11}y+c_{12}\right)$

PART - B

Problems on vibrating on strings with initial velocity zero

1. A string is stretched and fastened to two points x = 0 and x = l apart motion is started by displacing the string into the form $y=k(lx-x^2)$ from which it is released at time t = 0. Find the displacement of any point on the string at a distance of x from one end at time t.

Solution:

The wave equation is $\dfrac{\partial^2 y}{\partial t^2}=a^2\dfrac{\partial^2 y}{\partial x^2}$

From the given problem, we get the following boundary conditions.

$(i)\, y(0,t)=0\ \forall t>0$

$(ii)\, y(l,t)=0\ \forall t>0$

$(iii)\, \dfrac{\partial y(x,0)}{\partial t}=0$

$(iv)\, y(x,0)=k(lx-x^2)=f(x)$

The correct solution which satisfies our boundary conditions is given by

$y(x,t)=\left(A\cos px+B\sin px\right)(C\cos apt+D\sin apt)\rightarrow(1)$

Apply (i) in (1) we get,

$y(0,t)=A(C\cos apt+D\sin apt)$

$\Rightarrow 0=A(C\cos apt+D\sin apt)$

$\Rightarrow A=0,\ C\cos apt+D\sin apt)\neq 0$

Putting A= 0 in (1) we get

$y(x,t)=\left(B\sin px\right)(C\cos apt+D\sin apt)\rightarrow(2)$

Applying (ii) in (2) we get

$$y(l,t) = (B\sin pl)(C\cos apt + D\sin apt)$$
$$0 = (B\sin pl)(C\cos apt + D\sin apt)$$
$$If\ B = 0\, then\ we\ get\, trivial\ solution$$
$$B \neq 0,\ (C\cos apt + D\sin apt) \neq 0$$
$$\therefore \sin pl = 0$$
$$\sin pl = \sin n\pi$$
$$\Rightarrow p = \frac{n\pi}{l}$$
$$(2) \Rightarrow y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \rightarrow (3)$$

Before applying condition (iii) differentiate (3) partially with respect to t

$$\frac{\partial y(x,t)}{\partial t} = B\sin\frac{n\pi x}{l}\left(-C\sin\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \rightarrow (I)$$

Apply (iii) in (I) we get

$$\frac{\partial y(x,0)}{\partial t} = B\sin\frac{n\pi x}{l}\left(D\frac{n\pi a}{l}\right)$$
$$\Rightarrow 0 = B\sin\frac{n\pi x}{l}\left(D\frac{n\pi a}{l}\right)$$
$$Here\ B \neq 0,\ \sin\frac{n\pi x}{l} \neq 0,\ \frac{n\pi a}{l} \neq 0$$
$$\therefore D = 0$$
$$(3) \Rightarrow y(x,t) = BC\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$
$$y(x,t) = B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \rightarrow (II),\ where\ B_n = BC$$

Since the partial differential equation is linear, any linear combination of solutions of the form (4) with n = 1, 2, 3, … is also a solution of the equation.
The solution (II) can be written as

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \rightarrow (4)$$

Apply (iv) in (4)

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$f(x)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \to (5)$$

(5) represents half range fourier sine series in the interval $(0,l)$

$$\therefore B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{2}{l}\int_0^l k(lx-x^2)\sin\frac{n\pi x}{l}$$

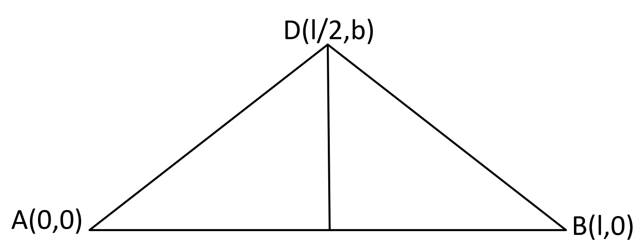
$$=\frac{2k}{l}\left[\left(lx-x^2\right)\frac{l}{n\pi}\left(-\cos\frac{n\pi x}{l}\right)+\left(l-2x\right)\sin\frac{n\pi x}{l}.\frac{l^2}{n^2\pi^2}-2\cos\frac{n\pi x}{l}\frac{l^3}{n^3\pi^3}\right]_0^l$$

$$=\frac{2k}{l}\left[-2(-1)^n\frac{l^3}{n^3\pi^3}+\frac{2l^3}{n^3\pi^3}\right]=\frac{4kl^2}{n^3\pi^3}\left[1-(-1)^n\right]=\begin{cases}\dfrac{8kl^2}{n^3\pi^3}, & n \text{ is odd}\\ 0, & n \text{ is even}\end{cases}$$

$$(4)\Rightarrow y(x,t)=\sum_{n=1,3,5,...}^{\infty}\frac{8kl^2}{n^3\pi^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

2. **A string is stretched and its ends are fastened at two points x = 0 and x = l. the midpoint of the string is displaced transversely through a small distance b and string is released from the rest in that position. Find an expression for the transverse displacement of the string at anytime during the subsequent motion.**
Solution:

To find the equation of the string in its initial position.
The equation of the string AD is



$$\frac{x-0}{l/2-0}=\frac{y-0}{b-0}$$

$$\frac{2x}{l}=\frac{y}{b}$$

$$y=\frac{2bx}{l}\ ,\ 0<x<l/2$$

The equation of the string DB is

$$\frac{x-l/2}{l-l/2}=\frac{y-b}{0-b}$$

$$\frac{2x-l}{l}=\frac{y-b}{-b}$$

$$y=\frac{2b}{l}(l-x)\,,\; l/2<x<l$$

Hence initially the displacement of the string is in the form

$$y(x,0)=\begin{cases}\dfrac{2bx}{l}\,,\; 0<x<l/2\\ \dfrac{2b}{l}(l-x)\,,\; l/2<x<l\end{cases}=f(x)$$

The wave equation is $\dfrac{\partial^2 y}{\partial t^2}=a^2\dfrac{\partial^2 y}{\partial x^2}$

From the given problem, we get the following boundary conditions.

$(i)\, y(0,t)=0\,\forall t>0$

$(ii)\, y(l,t)=0\,\forall t>0$

$(iii)\, \dfrac{\partial y(x,0)}{\partial t}=0$

$$(iv)\, y(x,0)=\begin{cases}\dfrac{2bx}{l}\,,\; 0<x<l/2\\ \dfrac{2b}{l}(l-x)\,,\; l/2<x<l\end{cases}=f(x)$$

The correct solution which satisfies our boundary conditions is given by

$y(x,t)=(A\cos px+B\sin px)(C\cos apt+D\sin apt)\rightarrow(1)$

Apply (i) in (1) we get,

$y(0,t)=A(C\cos apt+D\sin apt)$

$\Rightarrow 0=A(C\cos apt+D\sin apt)$

$\Rightarrow A=0,\; C\cos apt+D\sin apt)\neq 0$

Putting A= 0 in (1) we get

$y(x,t)=(B\sin px)(C\cos apt+D\sin apt)\rightarrow(2)$

Applying (ii) in (2) we get

$y(l,t)=(B\sin pl)(C\cos apt+D\sin apt)$

$0=(B\sin pl)(C\cos apt+D\sin apt)$

$If\;\; B=0\, then\;\; we\; get\; trivial\; solution$

$B\neq 0,\;(C\cos apt+D\sin apt)\neq 0$

$\therefore \sin pl=0$

$\sin pl=\sin n\pi$

$$\Rightarrow p = \frac{n\pi}{l}$$

$$(2) \Rightarrow y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \to (3)$$

Before applying condition (iii) differentiate (3) partially with respect to t

$$\frac{\partial y(x,t)}{\partial t} = B\sin\frac{n\pi x}{l}\left(-C\sin\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \to (I)$$

Apply (iii) in (I) we get

$$\frac{\partial y(x,0)}{\partial t} = B\sin\frac{n\pi x}{l}\left(D\frac{n\pi a}{l}\right)$$

$$\Rightarrow 0 = B\sin\frac{n\pi x}{l}\left(D\frac{n\pi a}{l}\right)$$

$$Here\ B \neq 0,\ \sin\frac{n\pi x}{l} \neq 0,\ \frac{n\pi a}{l} \neq 0$$

$$\therefore D = 0$$

$$(3) \Rightarrow y(x,t) = BC\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$y(x,t) = B_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \to (II),\ where\ B_n = BC$$

Since the partial differential equation is linear, any linear combinationof solutions of the form (4) with n = 1, 2, 3, … is also a solution of the equation.

The solution (II) can be written as

$$y(x,t) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \to (4)$$

Apply (iv) in (4)

$$y(x,0) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}$$

$$f(x) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l} \to (5)$$

(5) represents half range fourier sine series in the interval $(0,l)$

$$\therefore B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\frac{2b}{l}\left\{\int_0^{l/2} x\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l}(l-x)\sin\left(\frac{n\pi x}{l}\right)dx\right\}$$

$$= \frac{4b}{l^2}\left\{\begin{array}{l}\left[(x)\left(-\cos\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi}\right) - 1\left(-\sin\left(\frac{n\pi x}{l}\right)\right)\left(\frac{l}{n\pi}\right)^2\right]_0^{l/2} \\ +\left[(l-x)\left(-\cos\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi}\right) - (-1)\left(-\sin\left(\frac{n\pi x}{l}\right)\right)\left(\frac{l}{n\pi}\right)^2\right]_{l/2}^{l}\end{array}\right\}$$

$$= \frac{4b}{l^2}\left\{\begin{array}{l}\left[-\frac{l}{2}\frac{l}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)\frac{l^2}{n^2\pi^2} - 0 + 0\right] \\ +\left[0 - 0 + \frac{l}{2}\frac{l}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)\frac{l^2}{n^2\pi^2}\right]\end{array}\right\}$$

$$= \frac{4b}{l^2}\left[2\sin\left(\frac{n\pi}{2}\right)\frac{l^2}{n^2\pi^2}\right]$$

$$B_n = \frac{8b}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$(4) \Rightarrow y(x,t) = \sum_{n=1}^{\infty}\frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

3. **A string is stretched with fixed end points x = 0 and x = l is initially in a position given by $y(x,0) = y_0\sin^3\frac{\pi x}{l}$. It is released from rest from this position. Find the displacement y at any distance x from one end at anytime t.**

Solution:

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$

From the given problem, we get the following boundary conditions.

$(i)\, y(0,t) = 0\,\forall t > 0$

$(ii)\, y(l,t) = 0\,\forall t > 0$

$(iii)\, \frac{\partial y(x,0)}{\partial t} = 0$

$(iv)\, y(x,0) = k(lx - x^2) = f(x)$

The correct solution which satisfies our boundary conditions is given by

$y(x,t) = (A\cos px + B\sin px)(C\cos apt + D\sin apt) \rightarrow (1)$

Apply (i) in (1) we get,

$$y(0,t) = A(C\cos apt + D\sin apt)$$

$$\Rightarrow 0 = A(C\cos apt + D\sin apt)$$

$$\Rightarrow A = 0,\ C\cos apt + D\sin apt) \neq 0$$

Putting A= 0 in (1) we get

$$y(x,t) = (B\sin px)(C\cos apt + D\sin apt) \rightarrow (2)$$

Applying (ii) in (2) we get

$$y(l,t) = (B\sin pl)(C\cos apt + D\sin apt)$$

$$0 = (B\sin pl)(C\cos apt + D\sin apt)$$

$$\textit{If } B = 0 \textit{ then we get trivial solution}$$

$$B \neq 0,\ (C\cos apt + D\sin apt) \neq 0$$

$$\therefore \sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$\Rightarrow p = \frac{n\pi}{l}$$

$$(2) \Rightarrow y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \rightarrow (3)$$

Before applying condition (iii) differentiate (3) partially with respect to t

$$\frac{\partial y(x,t)}{\partial t} = B\sin\frac{n\pi x}{l}\left(-C\sin\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \rightarrow (I)$$

Apply (iii) in (I) we get

$$\frac{\partial y(x,0)}{\partial t} = B\sin\frac{n\pi x}{l}\left(D\frac{n\pi a}{l}\right)$$

$$\Rightarrow 0 = B\sin\frac{n\pi x}{l}\left(D\frac{n\pi a}{l}\right)$$

$$\textit{Here } B \neq 0,\ \sin\frac{n\pi x}{l} \neq 0,\ \frac{n\pi a}{l} \neq 0$$

$$\therefore D = 0$$

$$(3) \Rightarrow y(x,t) = BC\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$y(x,t) = B_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \rightarrow (II),\ \textit{where } B_n = BC$$

Since the partial differential equation is linear, any linear combinationof solutions of the form (4) with n = 1, 2, 3, … is also a solution of the equation.

The solution (II) can be written as

$$y(x,t)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \rightarrow (4)$$

Apply (iv) in (4)

$$y(x,0)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$f(x)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$y_0 \sin^3\frac{\pi x}{l}=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$\frac{y_0}{4}\left[3\sin\frac{\pi x}{l}-\sin\frac{3\pi x}{l}\right]=B_1\sin\frac{\pi x}{l}+B_2\sin\frac{2\pi x}{l}+B_3\sin\frac{3\pi x}{l}+...$$

By Equating like coefficients

$$B_1=\frac{3y_0}{4}, B_2=0, B_3=-\frac{y_0}{4}, B_4=B_5=...=0$$

Substitute these values in (4) we get

$$y(x,t)=\frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi at}{l}-\frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l}$$

Problems on vibrating on strings with non – zero initial velocity

4. A tightly stretched string with fixed end points x = 0 and x = *l* is initially at rest in its equilibrium position. If it is set vibrating by giving each point a velocity kx(*l*-x). Find the displacement of the string at any time.

Solution:

The wave equation is $\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$

From the given problem, we get the following boundary conditions.

$(i)\, y(0,t)=0 \forall t>0$

$(ii)\, y(l,t)=0 \forall t>0$

$(iii)\, y(x,0)=0$

$(iv)\, \frac{\partial y(x,0)}{\partial t}=kx(l-x)=f(x)$

The correct solution which satisfies our boundary conditions is given by

$$y(x,t)=(A\cos px+B\sin px)(C\cos apt+D\sin apt) \rightarrow (1)$$

Apply (i) in (1) we get,

$$y(0,t)=A(C\cos apt+D\sin apt)$$

$$\Rightarrow 0=A(C\cos apt+D\sin apt)$$

$$\Rightarrow A=0,\ C\cos apt+D\sin apt)\neq 0$$

Putting A= 0 in (1) we get

$$y(x,t) = (B\sin px)(C\cos apt + D\sin apt) \rightarrow (2)$$

Applying (ii) in (2) we get

$$y(l,t) = (B\sin pl)(C\cos apt + D\sin apt)$$

$$0 = (B\sin pl)(C\cos apt + D\sin apt)$$

$$If\ B = 0\, then\ we\ get\, trivial\ solution$$

$$B \neq 0,\ (C\cos apt + D\sin apt) \neq 0$$

$$\therefore \sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$\Rightarrow p = \frac{n\pi}{l}$$

$$(2) \Rightarrow y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \rightarrow (3)$$

Apply (iii) in (3)

$$y(x,0) = B\sin\frac{n\pi x}{l}C$$

$$0 = BC\sin\frac{n\pi x}{l}$$

$$\Rightarrow C = 0$$

$$(3) \Rightarrow y(x,0) = BD\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$y(x,0) = B_n\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

Since the partial differential equation is lin ear, any linear combition of solutions of the form (4) with n = 1, 2, 3, … is also a solution of the equation.The general solution can be written as

$$y(x,t) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \rightarrow (4)$$

Differentiate Partially (4) with respect to t we get,

$$\frac{\partial y(x,t)}{\partial t} = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}.\frac{n\pi a}{l} \rightarrow (5)$$

$$Apply\,(iv)\,in\,(5)\,we\ get$$

$$\frac{\partial y(x,0)}{\partial t} = \sum_{n=1}^{\infty} B_n\frac{n\pi a}{l}\sin\frac{n\pi x}{l}$$

$$f(x) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l} \rightarrow (6),\ where\ \ b_n = B_n\frac{n\pi a}{l}$$

Equation (6) Represents Half rane sine series.

$$\therefore b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\int_0^l k(lx - x^2)\sin\frac{n\pi x}{l}$$

$$= \frac{2k}{l}\left[(lx - x^2)\frac{l}{n\pi}\left(-\cos\frac{n\pi x}{l}\right) + (l - 2x)\sin\frac{n\pi x}{l}.\frac{l^2}{n^2\pi^2} - 2\cos\frac{n\pi x}{l}\frac{l^3}{n^3\pi^3}\right]_0^l$$

$$= \frac{2k}{l}\left[-2(-1)^n\frac{l^3}{n^3\pi^3} + \frac{2l^3}{n^3\pi^3}\right] = \frac{4kl^2}{n^3\pi^3}\left[1 - (-1)^n\right] = \begin{cases} \frac{8kl^2}{n^3\pi^3}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$$B_n = \frac{l}{n\pi a}b_n$$

$$\Rightarrow B_n = \frac{l}{n\pi a}\begin{cases} \frac{8kl^2}{n^3\pi^3}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$$\Rightarrow B_n = \begin{cases} \frac{8kl^3}{an^4\pi^4}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$$(4) \Rightarrow y(x,t) = \sum_{n=1,3,5,..}^{\infty} \frac{8kl^4}{an^4\pi^4}\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

5. Find the displacement of a tightly stretched string of a length 7cms vibrating between fixed end points if initial displacement is $10\sin\left(\frac{3\pi x}{7}\right)$ and initial velocity is $15\sin\left(\frac{9\pi x}{l}\right)$

Solution :

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$

From the given problem, we get the following boundary conditions.

$(i)\, y(0,t) = 0 \,\forall t > 0$

$(ii)\, y(l,t) = 0 \,\forall t > 0$

$(iii)\, y(x,0) = 10\sin\frac{3\pi x}{7}$

$(iv)\, \frac{\partial y(x,0)}{\partial t} = 15\sin\frac{9\pi x}{7}$

The correct solution which satisfies our boundary conditions is given by

$$y(x,t) = (A\cos px + B\sin px)(C\cos apt + D\sin apt) \to (1)$$

Apply (i) in (1) we get,

$$y(0,t) = A(C\cos apt + D\sin apt)$$

$$\Rightarrow 0 = A(C\cos apt + D\sin apt)$$

$$\Rightarrow A = 0,\ C\cos apt + D\sin apt) \neq 0$$

Putting A= 0 in (1) we get

$$y(x,t) = (B\sin px)(C\cos apt + D\sin apt) \to (2)$$

Applying (ii) in (2) we get

$$y(l,t) = (B\sin pl)(C\cos apt + D\sin apt)$$

$$0 = (B\sin pl)(C\cos apt + D\sin apt)$$

$$\textit{If } B = 0 \textit{ then we get trivial solution}$$

$$B \neq 0,\ (C\cos apt + D\sin apt) \neq 0$$

$$\therefore \sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$\Rightarrow p = \frac{n\pi}{l}$$

$$(2) \Rightarrow y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \to (3)$$

$$y(x,t) = \sin\frac{n\pi x}{l}\left(BC\cos\frac{n\pi at}{l} + BD\sin\frac{n\pi at}{l}\right)$$

$$y(x,t) = \sin\frac{n\pi x}{l}\left(B_n\cos\frac{n\pi at}{l} + C_n\sin\frac{n\pi at}{l}\right)$$

The general solution is

$$y(x,t) = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{l}\left(B_n\cos\frac{n\pi at}{l} + C_n\sin\frac{n\pi at}{l}\right) \to (4)$$

Apply (iii) in (4) we get

$$y(x,0) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l} = 10\sin\frac{3\pi x}{7}$$

$$B_1\sin\frac{\pi x}{l} + B_2\sin\frac{2\pi x}{l} + B_3\sin\frac{3\pi x}{7} + \ldots = 10\sin\left(\frac{3\pi x}{7}\right)$$

Equating like coefficients, we get $B_3 = 10 \to (I)$

From (4) we get

$$\frac{\partial y(x,t)}{\partial t}=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{l}\left(-B_n\cos\frac{n\pi at}{l}\frac{n\pi a}{l}+C_n\sin\frac{n\pi at}{l}.\frac{n\pi a}{l}\right)\rightarrow(5)$$

Apply (iv) in (5) we get

$$\frac{\partial y(x,0)}{\partial t}=\sum_{n=1}^{\infty}C_n\frac{n\pi a}{l}\sin\frac{n\pi x}{l}=15\sin\frac{9\pi x}{7}$$

$$\Rightarrow C_9\frac{9\pi a}{l}=15$$

$$C_9=\frac{15l}{9\pi a}\rightarrow(II)$$

The remaining C_n's are zero

Substitute (I) & (II) in (4)

$$\Rightarrow y(x,t)=10\cos\frac{3\pi at}{l}\sin\frac{3\pi x}{l}+\frac{15l}{9\pi a}\sin\frac{9\pi at}{l}\sin\frac{9\pi x}{l}$$

$$l=7\Rightarrow y(x,t)=10\cos\frac{3\pi at}{7}\sin\frac{3\pi x}{7}+\frac{105}{9\pi a}\sin\frac{9\pi at}{7}\sin\frac{9\pi x}{7}$$

ONE DIMENSIONAL HEAT FLOW EQUATION

Problems with zero boundary values.

6. **A uniform bar of lenth l through which heat flow is insulated at its sides. The ends are kept at zero temperature. If the initial temperature at the interior points of the bar is given by $k\left(lx-x^2\right), 0<x<l$. Find the temperature distribution in the bar after time t.**

   Solution:

   The heat equation is $\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}$

   The heat equation satisfying the boundary conditions are

   $(i)u(0,t)=0$

   $(ii)u(l,t)=0$

   $(iii)u(x,0)=f(x)=k(lx-x^2)$

   The correct solution is

$u(x,t)=Ae^{-\alpha^2p^2t}\left(B\cos px+C\sin px\right)\rightarrow(1)$

$Apply\ (i)\, in\, (1)\, we\ get$

$u(0,t)=ABe^{-\alpha^2p^2t}$

$0=ABe^{-\alpha^2p^2t}$

$B=0\ \&\ e^{-\alpha^2p^2t}\neq 0, A\neq 0$

$(1)\Rightarrow u(x,t)=AC\sin pxe^{-\alpha^2p^2t}\rightarrow(2)$

Applying (ii) in (2) we get

$u(l,t)=AC\sin px.e^{-\alpha^2p^2t}$

$0=AC\sin px.e^{-\alpha^2p^2t}$

$\Rightarrow \sin pl=0$

$pl=n\pi$

$p=\dfrac{n\pi}{l}$

$(2)\Rightarrow u(x,t)=AC\sin\dfrac{n\pi x}{l}e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)}$

$u(x,t)\ =B_n\sin\dfrac{n\pi x}{l}e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)}$

The general solution is

$u(x,t)=\sum_{n=1}^{\infty}B_n\sin\dfrac{n\pi x}{l}e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)}\rightarrow(3)$

Apply (iii) in (3) we get,

$u(x,0)=\sum_{n=1}^{\infty}B_n\sin\dfrac{n\pi x}{l}$

$f(x)=\sum_{n=1}^{\infty}B_n\sin\dfrac{n\pi x}{l}\rightarrow(4)$

(4) Represents half range sine series

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\int_0^l k(lx-x^2)\sin\frac{n\pi x}{l}$$

$$= \frac{2k}{l}\left[\left(lx-x^2\right)\frac{l}{n\pi}\left(-\cos\frac{n\pi x}{l}\right)+\left(l-2x\right)\sin\frac{n\pi x}{l}.\frac{l^2}{n^2\pi^2}-2\cos\frac{n\pi x}{l}\frac{l^3}{n^3\pi^3}\right]_0^l$$

$$= \frac{2k}{l}\left[-2(-1)^n\frac{l^3}{n^3\pi^3}+\frac{2l^3}{n^3\pi^3}\right]=\frac{4kl^2}{n^3\pi^3}\left[1-(-1)^n\right]=\begin{cases}\frac{8kl^2}{n^3\pi^3}, & n\ is\ odd\\ 0, & n\ is\ even\end{cases}$$

$$(3)\Rightarrow u(x,t)=\sum_{n=1,3,5,..}^{\infty}\frac{8kl^2}{n^3\pi^3}\sin\frac{n\pi x}{l}e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)}$$

7. **A uniform bar of lenth l through which heat flow is insulated at its sides. The ends are kept at zero temperature. If the initial temperature at the interior points of the bar is given by $k\sin^3\frac{\pi x}{l},\ 0<x<l$. Find the temperature distribution in the bar after time t.**

Solution:

The heat equation is $\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}$

The heat equation satisfying the boundary conditions are

$(i) u(0,t)=0$

$(ii) u(l,t)=0$

$(iii) u(x,0)=f(x)=k\sin^3\frac{\pi x}{l}$

The correct solution is

$u(x,t)=Ae^{-\alpha^2p^2t}\left(B\cos px+C\sin px\right)\rightarrow(1)$

Apply (i) *in* (1) *we get*

$u(0,t)=ABe^{-\alpha^2p^2t}$

$0=ABe^{-\alpha^2p^2t}$

$B=0\ \&\ e^{-\alpha^2p^2t}\neq 0, A\neq 0$

$(1)\Rightarrow u(x,t)=AC\sin pxe^{-\alpha^2p^2t}\rightarrow(2)$

Applying (ii) in (2) we get

$$u(l,t) = AC\sin px.e^{-\alpha^2 p^2 t}$$

$$0 = AC\sin px.e^{-\alpha^2 p^2 t}$$

$$\Rightarrow \sin pl = 0$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

$$(2) \Rightarrow u(x,t) = AC\sin\frac{n\pi x}{l}e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)}$$

$$u(x,t) \;= B_n \sin\frac{n\pi x}{l}e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)}$$

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)} \to (3)$$

Apply (iii) in (3) we get,

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \to (4)$$

$$k\sin^3\frac{\pi x}{l} = B_1\sin\frac{\pi x}{l} + B_2\sin\frac{2\pi x}{l} + B_3\sin\frac{3\pi x}{l} + ...$$

$$\frac{k}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right] = B_1\sin\frac{\pi x}{l} + B_2\sin\frac{2\pi x}{l} + B_3\sin\frac{3\pi x}{l} + ...$$

Equating the like coefficients we get

$$B_1 = \frac{3k}{4}, B_2 = 0, B_3 = \frac{k}{4}, \text{The remaining } \mathrm{B}_n\text{'}s \text{ are zero}$$

$$(3) \Rightarrow u(x,t) = \frac{3k}{4}\sin\frac{\pi x}{l}e^{-\left(\frac{\pi^2\alpha^2}{l^2}\right)t} - \frac{k}{4}\sin\frac{3\pi x}{l}e^{-\left(\frac{9\pi^2\alpha^2}{l^2}\right)t}$$

8. A rod of 30 cm has its ends A and B are kept at $20^0 c$ and $40^0 c$ respectively until steady state conditions prevails. The temperature at A is then suddenly raised to $90^0 c$ and the same time that B is lowered to $30^0 c$. Find the temperature distribution in the rod at time t. also show that the temperature at the midpoint of the rod remains unaltered for all time, regardless of the material of the rod.

Solution:

The heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \rightarrow (1)$$

when the steady state conditions prevails $\frac{\partial u}{\partial t} = 0$

$$(1) \Rightarrow \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$\Rightarrow \frac{d^2 u}{dx^2} = 0$$

$$\Rightarrow u(x) = ax + b \rightarrow (2)$$

Now the boundary conditions are

$(i)\, u(0) = 20$

$(ii)\, u(10) = 40$

Applying (i) in (2) we get

b = 20

$(2) \Rightarrow u(x) = ax \;+\; 20 \rightarrow (3)$

Applying (ii) in (3) we get

$a = 2$

$(3) \Rightarrow u(x) = 2x + 20$

Hence the steady state, the temperature function is given by $u(x) = 2x + 20$

Now the temperature at A is raised to $50^0 c$ and the temperature at B is lowered to $10^o c$. That is , the steady state is changed to unsteady state. For this unsteady state the temperature distribution is given by

$u(x) = 2x + 20$

Now the new boundary conditions are

$(i)\, u(0,t) = 50 \,\forall t > 0$

$(ii)\, u(10,t) = 10 \,\forall t > 0$

$(iii) u(x,0) = 2x + 20$

The correct solution is

$$u(x,t) = (A\cos px + B\sin px)e^{-\alpha^2 p^2 t} \rightarrow (4)$$

Apply (i) and (ii) in (4) we get

$$u(x,t) = Ae^{-\alpha^2 p^2 t} = 50$$

$$u(l,t) = (A\cos pl + B\sin pl)e^{-\alpha^2 p^2 t} = 10$$

It is not possible to find the constants A & B. since we have infinite number of values for A & B. Therefore in this case we split the solution u(x,t) into two parts

$u(x,t) = u_s(x) + u_t(x,t) \to (5)$

Where $u_s(x)$ is a solution of the equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ and is a function of x alone and satisfying the conditions $u_s(0) = 50 \,\&\, u_s(10) = 10 \,\&\, u_t(x,t)$ is a transient solution satisfying (5) which decreases at t increases.

To find $u_s(x)$:

$u_s(x) = a_1 x + b_1 \to (6)$

Applying the condition $\mathrm{u}_s(0) = b_1 = 50$

$(6) \Rightarrow a_1 x + 50 \to (7)$

Applying the condition $\mathrm{u}_s(10) = 10a_1 + 50 = 10$

$\Rightarrow a_1 = -4$

$(7) \Rightarrow u_s(x) = -4x + 50$

To find $u_t(x,t)$:

$u(x,t) = u_s(x) + u_t(x,t)$

$\Rightarrow u_t(x,t) = u(x,t) - u_s(x) \to (9)$

Now we have to find the boundary conditions for $u_t(x,t)$

Putting x = 0 in (9) we get

$u_t(0,t) = u(0,t) - u_s(0)$

$\quad = 50 - 50$

$u_t(0,t) = 0$

$Putting\ x = 10\ in(9)\ we\ get$

$u_t(10,t) = u(10,t) - u_s(0)$

$\quad = 10 - 10$

$u_t(10,t) = 0$

Putting t = 0 in (9) we get

$u_t(x,0) = u(x,0) - u_s(x)$

$\quad = 2x + 20 + 4x - 50$

$u_t(x,0) = 6x - 30$

Now the function $u_t(x,t)$ we have the following boundary conditions

$(i)\, u_t(0,t) = 0$

$(ii)\, u_t(10,t) = 0$

$(iii)\, u_t(x,0) = 6x - 30$

Applying the first two conditions we get the general solution as

$$u_t(x,t)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} e^{-\left(\frac{n^2\pi^2\alpha^2}{100}t\right)} \to (10)$$

Applying the (iii) in equation (10) we get

$$u_t(x,0)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} = 6x-30$$

$$b_n=\frac{2}{10}\int_0^{10}(6x-30)\sin\frac{n\pi x}{10}dx$$

$$=\frac{6}{5}\int_0^{10}(x-5)\sin\frac{n\pi x}{10}dx$$

$$=\frac{6}{5}\left[-(x-5)\cos\frac{n\pi x}{10}.\frac{10}{n\pi}+\sin\frac{n\pi x}{10}.\frac{100}{n^2\pi^2}\right]_0^{10}$$

$$=\frac{6}{5}\left[\frac{-50}{n\pi}(-1)^n-\frac{50}{n\pi}\right]$$

$$=-\frac{60}{n\pi}\left[1+(-1)^n\right]$$

$$=\begin{cases} 0 & :\text{n is odd} \\ -\dfrac{120}{n\pi} & :\text{n is even}\end{cases}$$

$$(10)\Rightarrow u_t(x,t)=\sum_{n=2,4,6,..}^{\infty}\frac{-120}{n\pi}\sin\frac{n\pi x}{10}e^{-\left(\frac{n^2\pi^2\alpha^2}{100}t\right)} \to (11)$$

Substitute (8) and (11) in (5) we get

$$\mathrm{u}(x,t)=-4x+50+\sum_{n=2,4,6,..}^{\infty}\frac{-120}{n\pi}\sin\frac{n\pi x}{10}e^{-\left(\frac{n^2\pi^2\alpha^2}{100}t\right)}$$

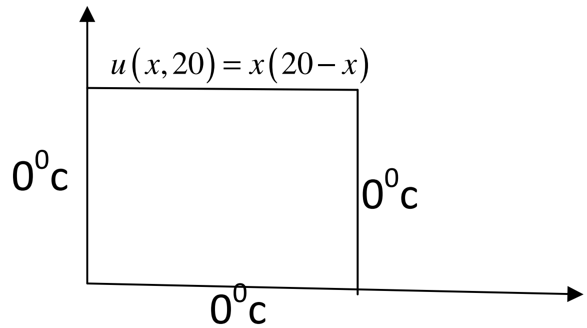
TWO DIMENSIONAL HEAT FLOW EQUATION

9. A square plate is bounded by the lines $x=0, y=0$, $x=20$ & $y=20$ .Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x,20)=x(20-x)$ when

0 < x < 20 while the other three edges are kept at 0^0 C. Find the steady state temperature in the plate.

Solution:

Let us take the sides of the plate be $l = 20$.Let $u(x, y)$ satisfies the Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \rightarrow (1)$.



The boundary conditions are

(i) $u(0, y) = 0, 0 < x < l$

(ii) $u(l, y) = 0, 0 < x < l$

(iii) $u(x, 0) = 0, 0 < x < l$

(iv) $u(x, l) = x(l - x), 0 < x < l$

The correct solution should be

$$u(x, y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \rightarrow (2)$$

Applying (i) in (2), we get

$$u(0, y) = A(Ce^{py} + De^{-py}) = 0$$

$$\Rightarrow A = 0$$

Applying (ii) in (2), we get

$$u(x, y) = B\sin px\left(Ce^{py} + De^{-py}\right) \rightarrow (2)$$

$Apply\,(ii)\,in\,(2)\,we\ get,$

$$u(l, y) = B\sin pl\left(Ce^{py} + De^{-py}\right)$$

$$0 = B\sin pl\left(Ce^{py} + De^{-py}\right), \quad Here\ \left(Ce^{py} + De^{-py}\right) \neq 0, B \neq 0$$

$$\Rightarrow \sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

$$\therefore u(x, y) = B\sin\frac{n\pi x}{l}\left(Ce^{\frac{n\pi y}{l}} + De^{-\frac{n\pi y}{l}}\right) \rightarrow (3)$$

Apply (iii) in (3) we get

$$u(x,0) = B\sin\frac{n\pi x}{l}(C+D)$$

$$0 = B\sin\frac{n\pi x}{l}(C+D)$$

$$C+D=0.\ \sin ce \sin\frac{n\pi x}{l} \neq 0, B \neq 0$$

$$\Rightarrow D = -C$$

$$(3) \Rightarrow u(x,y) = B\sin\frac{n\pi x}{l}C\left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right)$$

$$u(x,y) = 2BC\sin\frac{n\pi x}{l}.\frac{1}{2}\left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right)$$

$$Consider\ B_n = 2BC,\ and\ \frac{1}{2}\left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right) = \sinh\frac{n\pi y}{l}$$

$$\Rightarrow u(x,y) = B_n\sin\frac{n\pi x}{l}\sinh\frac{n\pi y}{l}$$

Most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}\sinh\frac{n\pi y}{l} \to (4)$$

Aplly (iv) *in* (4) *we get*

$$u(x,l) = \sum_{n=1}^{\infty} B_n\sinh n\pi.\sin\frac{n\pi x}{l}$$

$$f(x) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l} \to (5)\quad where\ b_n = B_n\sinh n\pi$$

(5) represents Half range Fourier Sine series

$$\therefore b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}$$

$$= \frac{2}{l}\left[\left(lx - x^2\right)\frac{l}{n\pi}\left(-\cos\frac{n\pi x}{l}\right) + (l-2x)\sin\frac{n\pi x}{l}.\frac{l^2}{n^2\pi^2} - 2\cos\frac{n\pi x}{l}\frac{l^3}{n^3\pi^3}\right]_0^l$$

$$= \frac{2}{l}\left[-2(-1)^n\frac{l^3}{n^3\pi^3} + \frac{2l^3}{n^3\pi^3}\right] = \frac{4l^2}{n^3\pi^3}\left[1-(-1)^n\right] = \begin{cases} \frac{8l^2}{n^3\pi^3}, & n\ is\ odd \\ 0, & n\ is\ even \end{cases}$$

$$\Rightarrow B_n = \begin{cases} \frac{8l^2}{n^3\pi^3}\cos echn\pi : n\, is\, odd \\ 0 \qquad\qquad : n\, is\, even \end{cases}$$

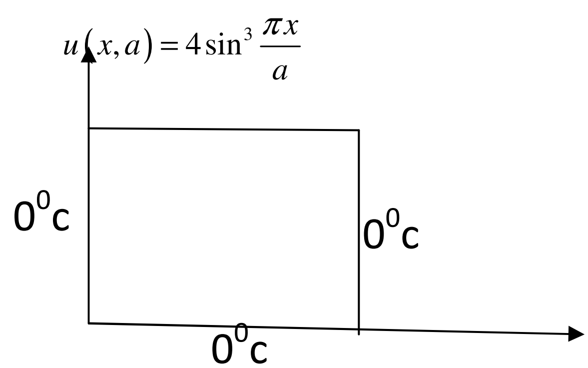
$$\therefore u(x,y) = \sum_{n=1,3,5,..}^{\infty} \frac{8l^2}{n^3\pi^3}\cos echn\pi \sin\frac{n\pi x}{l}\sinh\frac{n\pi y}{l}$$

$$l = 20 \Rightarrow u(x,y) = \frac{3200}{\pi^3}\sum_{n=1,3,5,..}^{\infty}\frac{1}{n^3}\cos echn\pi\sin\frac{n\pi x}{20}\sinh\frac{n\pi y}{20}$$

10. A square plate is bounded by the lines $x=0, y=0$, $x=a$ & $y=a$ .Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x,a)=4\sin^3\left(\frac{\pi x}{a}\right)$ when 0 < x < a while the other three edges are kept at 0^0 C. Find the steady state temperature in the plate.

Solution:

Let us take the sides of the plate be $l=a$ .Let $u(x,y)$ satisfies the Laplace's equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.



The boundary conditions are

(i) $u(0,y)=0, 0<y<a$

(ii) $u(a,y)=0, 0<y<a$

(iii) $u(x,0)=0, 0<x<a$

(iv) $u(x,a)=4\sin^3\frac{\pi x}{a}, 0<x<a$

The correct solution should be

$$u(x,y)=(A\cos px+B\sin px)(Ce^{py}+De^{-py}) \rightarrow (1)$$

Applying (i) in (1), we get

$$u(0,y)=A(Ce^{py}+De^{-py})=0$$

$$\Rightarrow A=0$$

Applying (ii) in (2), we get

$u(x,y) = B\sin px\left(Ce^{py} + De^{-py}\right) \rightarrow (2)$

$Apply\,(ii)\,in\,(2)\,we\ get,$

$u(a,y) = B\sin pl\left(Ce^{py} + De^{-py}\right)$

$0 = B\sin pl\left(Ce^{py} + De^{-py}\right), \quad Here\ \left(Ce^{py} + De^{-py}\right) \neq 0, B \neq 0$

$\Rightarrow \sin pa = 0$

$\sin pa = \sin n\pi$

$pa = n\pi$

$$p = \frac{n\pi}{a}$$

$$\therefore u(x,y) = B\sin\frac{n\pi x}{a}\left(Ce^{\frac{n\pi y}{a}} + De^{-\frac{n\pi y}{a}}\right) \rightarrow (3)$$

Apply (iii) in (3) we get

$$u(x,0) = B\sin\frac{n\pi x}{a}(C+D)$$

$$0 = B\sin\frac{n\pi x}{a}(C+D)$$

$$C + D = 0.\ \sin ce \sin\frac{n\pi x}{a} \neq 0, B \neq 0$$

$\Rightarrow D = -C$

$$(3) \Rightarrow u(x,y) = B\sin\frac{n\pi x}{a}C\left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right)$$

$$u(x,y) = 2BC\sin\frac{n\pi x}{a}.\frac{1}{2}\left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right)$$

$$Consider\ \ B_n = 2BC,\ and\ \frac{1}{2}\left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right) = \sinh\frac{n\pi y}{a}$$

$$\Rightarrow u(x,y) = B_n\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

$Most\ general\ solution\,is$

$$u(x,y) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a} \rightarrow (4)$$

$Aplly\,(iv)\,in\,(4)\,we\ get$

$$u(x,a) = \sum_{n=1}^{\infty} B_n\sinh n\pi.\sin\frac{n\pi x}{a}$$

$Apply\,(iv)\,in\,(4)\,we\;get$

$$u(x,a) = B_1 \sinh \pi \sin \frac{n\pi x}{a} + B_2 \sinh 2\pi \sin \frac{2\pi x}{a} + B_3 \sinh 3\pi \sin \frac{3\pi x}{a} + \ldots$$

$$4\sin^3 \frac{\pi x}{a} = B_1 \sinh \pi \sin \frac{n\pi x}{a} + B_2 \sinh 2\pi \sin \frac{2\pi x}{a} + B_3 \sinh 3\pi \sin \frac{3\pi x}{a} + \ldots$$

$$4.\frac{1}{4}\left[3\sin \frac{\pi x}{a} - \sin \frac{3\pi x}{a}\right] = B_1 \sinh \pi \sin \frac{n\pi x}{a} + B_2 \sinh 2\pi \sin \frac{2\pi x}{a} + B_3 \sinh 3\pi \sin \frac{3\pi x}{a} + \ldots$$

$$\Rightarrow B_1 = 3\cos ech\pi, B_2 = 0, B_3 = -\cos ech3\pi, B_4 = B_5 = \ldots = 0$$

$$(4) \Rightarrow u(x,y) = 3\cos ech\pi \sin \frac{\pi x}{a} \sinh \frac{\pi y}{a} - \cos ech3\pi \sin \frac{3\pi x}{a} \sinh \frac{3\pi y}{a}$$

MA 6351 TRANSFORMS & PARTIAL DIFFERENTIAL EQUATIONS
UNIT IV – FOURIER TRANSFORM
PART – A

1. State Fourier integral theorem.

Sol. If f(x) is piecewise continuous derivative and absolutely integrable in (– ∞, ∞) then

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{is(x-t)}\,dt\,ds \quad (or)\quad f(x)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos[s(x-t)]\,dt\,ds$$

2. Define Fourier transform pair.

Sol. Fourier transform of f(x) is

$$F(s)=F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx$$

Its Inverse Fourier transform is

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\,ds = F^{-1}[F(s)]$$

3. Define Fourier cosine transform pair.

Sol. Fourier cosine transform of f(x) is

$$F_c(s)=F_c[f(x)]=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\cos sx\,dx$$

Its Inverse Fourier cosine transform is

$$f(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} F_c[f(x)]\cos sx\,ds$$

4. Define Fourier sine transform pair.

Sol. Fourier sine transform of f(x) is

$$F_s(s)=F_s[f(x)]=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\sin sx\,dx$$

Its Inverse Fourier sine transform is

$$f(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} F_s[f(x)]\sin sx\,ds$$

5. State Parseval's identity for Fourier transform.

Sol. If F(s) is the Fourier transform of f(x) then

$$\int_{-\infty}^{\infty}|F(s)|^2\,ds=\int_{-\infty}^{\infty}|f(x)|^2\,dx$$

6. State Parseval's identity for Fourier sine and cosine transform.

Sol. If F_s(s) and F_c(s) are the Fourier sine and Fourier cosine transform of f(x) respectively then

$$\int_{0}^{\infty}[F_s(s)]^2 ds=\int_{0}^{\infty}[f(x)]^2\,dx \quad and \quad \int_{0}^{\infty}[F_c(s)]^2 ds=\int_{0}^{\infty}[f(x)]^2\,dx$$

7. Define the convolution of two functions for Fourier transform.

Sol. The convolution of two functions f(x) and g(x) is defined by

$$(f*g)(x)=f(x)*g(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,g(x-t)\,dt$$

8. State convolution theorem

Sol. If F[f(x)] = F(s) and F[g(x)] = G(s) then $F[f(x)*g(x)]=F(s).G(s)$

9. Solve the integral equation $\int_0^\infty f(x)\cos\lambda x\,dx = e^{-\lambda}$

Sol. Given $\int_0^\infty f(x)\cos\lambda x\,dx = e^{-\lambda}$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\lambda x\,dx = \sqrt{\frac{2}{\pi}}\;e^{-\lambda}$$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\;e^{-\lambda}$$

$$f(x) = F_c^{-1}\left[\sqrt{\frac{2}{\pi}}\;e^{-\lambda}\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}e^{-\lambda}\cos\lambda x\,d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty e^{-\lambda}\cos\lambda x\,d\lambda$$

$$= \frac{2}{\pi}\left[\frac{e^{-\lambda}}{1+x^2}(-\cos\lambda x + \lambda\sin\lambda x)\right]_0^\infty$$

$$= \frac{2}{\pi}\left[\{0\} - \left\{\frac{1}{1+x^2}(-1+0)\right\}\right]$$

$$(i.e.)\, f(x) = \frac{2}{\pi}\,\frac{1}{1+x^2}$$

10. Find f(x) if its sine transform is e^{-as}

Sol. The inverse Fourier sine transform is given by

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s[f(x)]\sin sx\,ds$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-as}\sin sx\,ds$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-as}}{a^2+x^2}(-a\sin sx - x\cos sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[\{0\} - \left\{\frac{1}{a^2+x^2}(0-x)\right\}\right]$$

$$= \sqrt{\frac{2}{\pi}}\;\frac{x}{x^2+a^2}$$

11. State the Fourier transform of the derivatives of a function.

Sol. $F[f'(x)] = (-is)F(s)$

$$F[f''(x)] = (-is)^2 F(s)$$

$$F[f'''(x)] = (-is)^3 F(s)$$

$$------------$$

Ingeneral, $F[f^{(n)}(x)] = (-is)^n F(s)$

12. Find f(x) if its cosine transform is $f_c(p) = \begin{cases} \dfrac{1}{\sqrt{2\pi}}\left(a - \dfrac{s}{2}\right), & s < 2a \\ 0, & s \geq 2a \end{cases}$

Sol. The inverse Fourier cosine transform is given by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c[f(x)] \cos sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^{2a} \frac{1}{\sqrt{2\pi}} \left(a - \frac{s}{2} \right) \cos sx ds + \int_{2a}^{\infty} 0 ds \right]$$

$$= \frac{1}{\pi} \left[\left(a - \frac{s}{2} \right) \left(\frac{\sin sx}{x} \right) - \left(-\frac{1}{2} \right) \left(\frac{-\cos sx}{x^2} \right) \right]_0^{2a}$$

$$= \frac{1}{\pi} \left[\left\{ 0 - \frac{\cos 2ax}{2x^2} \right\} - \left\{ 0 - \frac{1}{2x^2} \right\} \right]$$

$$= \frac{1}{\pi x^2} \frac{1 - \cos 2ax}{2}$$

$$= \frac{\sin^2 ax}{\pi x^2}$$

13. Find the sine transform of $\dfrac{1}{x}$

Sol. $F_s[f(x)] = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^{\infty} f(x) \sin sx \, dx$

$$F_s\left[\frac{1}{x}\right] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{x} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{s}{t} \sin t \, \frac{dt}{s}$$

Put $sx = t$
$s\,dx = dt$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin t}{t} \, dt$$

$$\boxed{\int_0^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\pi}{2}$$

$$= \sqrt{\frac{\pi}{2}}$$

14. Prove that F[af(x) + bg(x)] = aF(s) + bG(s) [Linearity property on Fourier transform]

Sol. We have $F[f(x)] = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{isx} \, dx = F(s)$

$$F[a\,f(x) + b\,g(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [a\,f(x) + b\,g(x)]\, e^{isx} \, dx$$

$$= a \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} \, dx \; + \; b \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\, e^{isx} \, dx$$

$$= a\,F(s) + b\,G(s)$$

15.Prove (i)$F_c[af(x) + bg(x)] = aF_c(s) + bG_c(s)$[Linear property on Fourier cosine transform]

(ii)$F_s[af(x) + bg(x)] = aF_s(s) + bG_s(s)$[Linear property on Fourier sine transform]

Sol. (*i*)
$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx = F_c(s)$$

$$F_c[a\,f(x)+b\,g(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty [a\,f(x)+b\,g(x)]\cos sx\,dx$$

$$= a\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx + b\sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\cos sx\,dx$$

$$= a\,F_c(s) + b\,G_c(s)$$

(*ii*)
$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx = F_s(s)$$

$$F_s[a\,f(x)+b\,g(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty [a\,f(x)+b\,g(x)]\sin sx\,dx$$

$$= a\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx + b\sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\sin sx\,dx$$

$$= a\,F_s(s) + b\,G_s(s)$$

16. Prove that $F[f(x-a)] = e^{ias}F(s)$ **[Time shifting property]**

Sol. We have $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx = F(s)$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)\,e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{is(t+a)}\,dt$$

Put $x - a = t$
$dx = dt$

$$= e^{ias}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{ist}\,dt$$

$$= e^{ias}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$$

$$= e^{ias}F(s)$$

17. Prove that $F[e^{iax}f(x)] = F(s+a)$ **[Frequency shifting property]**

Sol. We have $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx = F(s)$

$$F[e^{iax}f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iax}f(x)\,e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{i(s+a)x}\,dx$$

$$= F(s+a)$$

18. Prove that $(i)\ F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right),\ a > 0$ **[Change of scale property]**

$(ii)\ F_s[f(ax)] = \frac{1}{a} F_s\left(\frac{s}{a}\right)$

$(iii)\ F_c[f(ax)] = \frac{1}{a} F_c\left(\frac{s}{a}\right)$

Sol. (i) We have $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = F(s)$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is\frac{t}{a}}\, \frac{dt}{a}$$

Put a x = t
a dx = dt

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\frac{s}{a}t}\, dt$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right)$$

(ii) We have $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx\, dx = F_s(s)$

$$F_s[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(ax) \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin\left(\frac{st}{a}\right) \frac{dt}{a}$$

Put a x = t
a dx = dt

$$= \frac{1}{a} \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin\left(\frac{s}{a}\right) t\, dt$$

$$= \frac{1}{a} F_s\left(\frac{s}{a}\right)$$

(iii) We have $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx\, dx = F_c(s)$

$$F_c[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(ax) \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos\left(\frac{st}{a}\right) \frac{dt}{a}$$

Put a x = t
a dx = dt

$$= \frac{1}{a} \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos\left(\frac{s}{a}\right) t\, dt$$

$$= \frac{1}{a} F_c\left(\frac{s}{a}\right)$$

19. If $\overline{f}(\lambda)$ is the Fourier transform of $f(x)$, find the Fourier transform of $f(x - a)$ and $f(ax)$.

Sol. $F[f(x-a)] = e^{ia\lambda}\, \overline{f}(\lambda)$ [see the solution in problem 16 & 18(i)]

and $F[f(ax)] = \frac{1}{a} \overline{f}\left(\frac{\lambda}{a}\right)$

20. Prove that [Modulation property]

$(i)\ F[f(x)\cos ax]=\frac{1}{2}[F(s+a)+F(s-a)]$ $(ii)\ F_s[f(x)\cos ax]=\frac{1}{2}[F_s(s+a)+F_s(s-a)]$

$(iii)\ F_s[f(x)\sin ax]=\frac{1}{2}[F_c(s-a)-F_c(s+a)]$ $(iv)\ F_c[f(x)\cos ax]=\frac{1}{2}[F_c(s+a)+F_c(s-a)]$

$(v)\ F_c[f(x)\sin ax]=\frac{1}{2}[F_s(a+s)+F_s(a-s)]$

Sol. (i) We have $F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = F(s)$

$$F[f(x)\cos ax]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos ax\, e^{isx}\, dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left(\frac{e^{iax}+e^{-iax}}{2}\right) e^{isx}\, dx$$

$$=\frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x}\, dx+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i(s-a)x}\, dx\right]$$

$$=\frac{1}{2}[F(s+a)+F(s-a)]$$

(ii) We have $F_s[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\, dx = F_s(s)$

$$F_s[f(x)\cos ax]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos ax\sin sx\, dx$$

2sinAcosB = sin(A + B) + sin(A – B)

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\,\frac{1}{2}[\sin(s+a)x+\sin(s-a)x]\, dx$$

$$=\frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(s+a)x\, dx+\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(s-a)x\, dx\right]$$

$$=\frac{1}{2}[F_s(s+a)+F_s(s-a)]$$

(iii) We have $F_s[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\, dx = F_s(s)$

$$F_s[f(x)\sin ax]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin ax\sin sx\, dx$$

2sinAsinB = cos(A – B) – cos(A + B)

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\,\frac{1}{2}[\cos(s-a)x-\cos(s+a)x]\, dx$$

$$=\frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(s-a)x\, dx-\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(s+a)x\, dx\right]$$

$$=\frac{1}{2}[F_c(s-a)-F_c(s+a)]$$

(iv) We have $F_c[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\, dx = F_c(s)$

$$F_c[f(x)\cos ax]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos ax\cos sx\, dx$$

2cosAcosB = cos(A + B) + cos(A – B)

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ \frac{1}{2}[\cos(s+a)x + \cos(s-a)x]\ dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ \cos(s+a)x\ dx + \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ \cos(s-a)x\ dx\right]$$

$$= \frac{1}{2}\ [F_c(s+a) + F_c(s-a)]$$

(v) We have $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ \cos sx\ dx = F_c(s)$

$$F_c[f(x)\sin a\,x] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin a\,x\ \cos sx\ dx$$

2sinAcosB = sin(A + B) + sin(A – B)

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ \frac{1}{2}[\sin(a+s)x + \sin(a-s)x]\ dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ \sin(a+s)x\ dx + \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ \sin(a-s)x\ dx\right]$$

$$= \frac{1}{2}\ [F_s(a+s) + F_s(a-s)]$$

21. Prove that (i) $F[f(-x)] = F(-s)$ $\quad (ii)$ $F[\overline{f(-x)}] = \overline{F(s)}$ $\quad (iii)$ $F[\overline{f(x)}] = \overline{F(-s)}$

Sol. (i) We have $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ e^{isx}\ dx = F(s)$

$$F[f(-x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-x)\ e^{isx}\ dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t)\ e^{-ist}\ (-dt)$$

Put $-x$ = t
– dx = dt

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\ e^{-ist}\ dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\ e^{i(-s)t}\ dt$$

$$= F(-s)$$

(ii) We have $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\ e^{isx}\ dx$

$$\overline{F(s)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)}\ e^{-isx}\ dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} \overline{f(-t)}\ e^{ist}\ (-dt)$$

Put $-x$ = t
– dx = dt

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-t)}\ e^{ist}\ dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-x)}\ e^{isx}\ dx$$

$$= F[\overline{f(-x)}]$$

(iii) We have $F(s) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$F(-s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

$$\overline{F(-s)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)}\, e^{isx}\, dx$$

$$= F[\overline{f(x)}]$$

22. Prove that (i) $\displaystyle\int_0^{\infty} F_c(s) G_c(s)\, ds = \int_0^{\infty} f(x)\, g(x)\, dx$

(ii) $\displaystyle\int_0^{\infty} F_s(s) G_s(s)\, ds = \int_0^{\infty} f(x)\, g(x)\, dx$

Sol. (i) $\displaystyle\int_0^{\infty} F_c(s) G_c(s)\, ds = \int_0^{\infty} F_c(s) \left\{ \sqrt{\frac{2}{\pi}} \int_0^{\infty} g(x) \cos sx\, dx \right\} ds$

$$= \int_0^{\infty} g(x) \left\{ \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(s) \cos sx\, ds \right\} dx$$

$$= \int_0^{\infty} g(x)\, f(x)\, dx$$

$$= \int_0^{\infty} f(x)\, g(x)\, dx$$

(ii) $\displaystyle\int_0^{\infty} F_s(s) G_s(s)\, ds = \int_0^{\infty} F_s(s) \left\{ \sqrt{\frac{2}{\pi}} \int_0^{\infty} g(x) \sin sx\, dx \right\} ds$

$$= \int_0^{\infty} g(x) \left\{ \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(s) \sin sx\, ds \right\} dx$$

$$= \int_0^{\infty} g(x)\, f(x)\, dx$$

$$= \int_0^{\infty} f(x)\, g(x)\, dx$$

23. Give an example for self-reciprocal under Fourier transform.

Sol. $e^{-\frac{x^2}{2}}$ is self-reciprocal under Fourier transform.

24. Give an example for self-reciprocal under Fourier cosine transform.

Sol. $e^{-\frac{x^2}{2}}$ is self-reciprocal under Fourier cosine transform.

25. Give an example for self-reciprocal under both Fourier sine and cosine transform.

Sol. $\dfrac{1}{\sqrt{x}}$ is self-reciprocal under both Fourier sine and cosine transform.

PART – B

1. Find the Fourier transform of $f(x)=\begin{cases} a^2-x^2, & |x|<a \\ 0, & |x|\ge a \end{cases}$

Hence deduce that $(i)\ \int_0^\infty \frac{\sin s - s\cos s}{s^3}\,ds=\frac{\pi}{4}$ $\quad (ii)\ \int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos\frac{s}{2}\,ds=\frac{3\pi}{16}$

$(iii)\ \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds=\frac{\pi}{15}$

Sol. $$F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} 0.e^{isx}\,dx+\int_{-a}^{a}(a^2-x^2)\,e^{isx}\,dx+\int_a^\infty 0.e^{isx}\,dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2-x^2)(\cos sx+i\sin sx)\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2-x^2)\cos sx\,dx+i\,\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2-x^2)\sin sx\,dx$$

$$=\frac{2}{\sqrt{2\pi}}\int_0^a (a^2-x^2)\cos sx\,dx+0$$

$$=\sqrt{\frac{2}{\pi}}\left[(a^2-x^2)\left(\frac{\sin sx}{s}\right)-(-2x)\left(\frac{-\cos sx}{s^2}\right)+(-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^a$$

$$=\sqrt{\frac{2}{\pi}}\left[\left\{0-\frac{2a\cos as}{s^2}+\frac{2\sin as}{s^3}\right\}-\{0-0+0\}\right]$$

$(i.e.)\ F[f(x)]=2\sqrt{\frac{2}{\pi}}\left[\frac{\sin as-as\cos as}{s^3}\right]$

When a = 1, we have $F[f(x)]=2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s-s\cos s}{s^3}\right]$

Using ***inverse Fourier transform***, we have

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F[f(x)]\,e^{-isx}\,ds$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s-s\cos s}{s^3}\right)(\cos sx-i\sin sx)\,ds$$

$$=\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s-s\cos s}{s^3}\right)\cos sx\,ds-i\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s-s\cos s}{s^3}\right)\sin sx\,ds$$

$$=\frac{4}{\pi}\int_0^\infty\left(\frac{\sin s-s\cos s}{s^3}\right)\cos sx\,ds-0$$

$$\int_0^\infty\left(\frac{\sin s-s\cos s}{s^3}\right)\cos sx\,ds=\frac{\pi}{4}f(x)\ -------\ (1)$$

Put $x=0$ *in equation* (1) *we get*

$$\int_0^\infty\left(\frac{\sin s-s\cos s}{s^3}\right)ds=\frac{\pi}{4}f(0)$$

$f(x)=a^2-x^2$
$f(x)=1-x^2$
$f(0)=1-0=1$

$$=\frac{\pi}{4}(1)=\frac{\pi}{4}\quad \textit{This proves } (i)$$

$$Put \;\; x=\frac{1}{2} \;\; in \;\; equation \;\; (1) \;\; we \;\; get$$

$$\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)\cos\frac{s}{2}\,ds = \frac{\pi}{4} f\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{4}\left(1-\frac{1}{4}\right) = \frac{\pi}{4}\left(\frac{3}{4}\right) = \frac{3\pi}{16} \qquad This \;\; proves \;\; (ii)$$

Using **Parseval's identity,** we have

$$\int_{-\infty}^{\infty} |F(s)|^2\,ds = \int_{-\infty}^{\infty} |f(x)|^2\,dx$$

$$\int_{-\infty}^{\infty}\left(2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]\right)^2 ds = \int_{-\infty}^{-1} 0.dx + \int_{-1}^{1}(1-x^2)^2\,dx + \int_1^\infty 0.dx$$

$$\frac{8}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \int_{-1}^{1}(1-x^2)^2\,dx$$

$$\frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\int_0^1 (1-x^2)^2\,dx$$

$$= 2\int_0^1 (1+x^4-2x^2)\,dx$$

$$= 2\left[x+\frac{x^5}{5}-\frac{2x^3}{3}\right]_0^1$$

$$= 2\left[\left\{1+\frac{1}{5}-\frac{2}{3}\right\}-\{0+0-0\}\right]$$

$$= 2\left[\frac{8}{15}\right]$$

$$\frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{16}{15}$$

$$\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{15} \qquad This \;\; proves \;\; (iii)$$

2. Find the Fourier sine and cosine transform of e^{-ax}

Sol. $F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\sin sx - s\cos sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[\{0\}-\left\{\frac{1}{a^2+s^2}(0-s)\right\}\right]$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{s}{s^2+a^2}$$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\{0\} - \left\{ \frac{1}{a^2+s^2}(-a+0) \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$$

3. Find the Fourier transform of $f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0 \quad , & |x| \ge 1 \end{cases}$

Hence deduce that $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$

Sol. $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} \, dx$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-1} 0.e^{isx} \, dx + \int_{-1}^{1} (1-|x|)\, e^{isx} \, dx + \int_{1}^{\infty} 0.e^{isx} \, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)(\cos sx + i\sin sx) \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)\cos sx \, dx + i\,\frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)\sin sx \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1-|x|)\cos sx \, dx + 0$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 (1-x)\cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{-\cos sx}{s^2}\right) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\left\{0 - \frac{\cos s}{s^2}\right\} - \left\{0 - \frac{1}{s^2}\right\} \right]$$

$(i.e.)$ $F[f(x)] = \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s^2}\right]$

Using **Parseval's identity,** we have

$$\int_{-\infty}^{\infty} |F(s)|^2 \, ds = \int_{-\infty}^{\infty} |f(x)|^2 \, dx$$

$$\int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s^2}\right] \right)^2 ds = \int_{-\infty}^{-1} 0.dx + \int_{-1}^{1} (1-|x|)^2 \, dx + \int_{1}^{\infty} 0.dx$$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{1-\cos s}{s^2}\right)^2 ds = \int_{-1}^{1} (1-|x|)^2 \, dx$$

$$\frac{4}{\pi}\int_0^\infty \left(\frac{1-\cos s}{s^2}\right)^2 ds = 2\int_0^1 (1-x)^2\, dx$$

$$\frac{4}{\pi}\int_0^\infty \left(\frac{1-\cos 2t}{4t^2}\right)^2 2dt = 2\left[\frac{(1-x)^3}{-3}\right]_0^1$$

Put s = 2t
ds = 2dt

$$\frac{8}{16\pi}\int_0^\infty \left(\frac{1-\cos 2t}{t^2}\right)^2 dt = 2\left[\{0\}-\left\{-\frac{1}{3}\right\}\right]$$

$$\frac{1}{2\pi}\int_0^\infty \left(\frac{2\sin^2 t}{t^2}\right)^2 dt = \frac{2}{3}$$

$$\frac{4}{2\pi}\int_0^\infty \left(\frac{\sin^2 t}{t^2}\right)^2 dt = \frac{2}{3}$$

$$(i.e.)\ \int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$$

4. Find the Fourier transform of $f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

Hence deduce that $(i)\ \int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$ $\qquad (ii)\ \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$

Sol. $$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-1} 0.e^{isx}\, dx + \int_{-1}^{1} (1)\, e^{isx}\, dx + \int_{1}^{\infty} 0.e^{isx}\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} (\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} \cos sx\, dx + i\,\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} \sin sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{1} \cos sx\, dx + 0$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin sx}{s}\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin s}{s} - 0\right]$$

$$(i.e.)\ F[f(x)] = \sqrt{\frac{2}{\pi}}\,\frac{\sin s}{s}$$

Using ***inverse Fourier transform***, we have

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\left(\frac{\sin s}{s}\right)(\cos sx - i\sin sx)\, ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin s}{s}\right)\cos sx\, ds - i\,\frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin s}{s}\right)\sin sx\, ds$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin s}{s}\right)\cos sx\,ds - 0$$

$$\int_0^\infty \left(\frac{\sin s}{s}\right)\cos sx\,ds = \frac{\pi}{2} f(x)$$

Put $x = 0$ we get

$$\int_0^\infty \frac{\sin s}{s}\,ds = \frac{\pi}{2} f(0)$$

$$= \frac{\pi}{2}(1)$$

$f(x) = 1$
$f(0) = 1$

$$(i.e.)\ \int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$$

Using **Parseval's identity,** we have

$$\int_{-\infty}^\infty |F(s)|^2\,ds = \int_{-\infty}^\infty |f(x)|^2\,dx$$

$$\int_{-\infty}^\infty \left(\sqrt{\frac{2}{\pi}}\,\frac{\sin s}{s}\right)^2 ds = \int_{-\infty}^{-1} 0.dx + \int_{-1}^{1}(1)^2\,dx + \int_1^\infty 0.dx$$

$$\frac{2}{\pi}\int_{-\infty}^\infty \left(\frac{\sin s}{s}\right)^2 ds = \int_{-1}^{1} dx$$

$$\frac{4}{\pi}\int_0^\infty \left(\frac{\sin s}{s}\right)^2 ds = [x]_{-1}^{1}$$

$$= 1-(-1)$$

$$\frac{4}{\pi}\int_0^\infty \left(\frac{\sin s}{s}\right)^2 ds = 2$$

$$(i.e.)\ \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

5. Find the Fourier cosine transform of e^{-4x}. Hence deduce that $\int_0^\infty \frac{\cos 2x}{x^2+16}\,dx = \frac{\pi}{8}e^{-8}$ and $\int_0^\infty \frac{x\sin 2x}{x^2+16}\,dx = \frac{\pi}{2}e^{-8}$

Sol. $F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$

$$F_c[e^{-4x}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-4x}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-4x}}{16+s^2}(-4\cos sx + s\sin sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[\{0\} - \left\{\frac{1}{16+s^2}(-4+0)\right\}\right]$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{4}{s^2+16}$$

Using ***inverse Fourier cosine transform***, we have

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c[f(x)] \cos sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{4}{s^2+16} \right) \cos sx \, ds$$

$$f(x) = \frac{8}{\pi} \int_0^\infty \frac{\cos sx}{s^2+16} \, ds$$

$$\int_0^\infty \frac{\cos sx}{s^2+16} \, ds = \frac{\pi}{8} f(x)$$

$$\int_0^\infty \frac{\cos sx}{s^2+16} \, ds = \frac{\pi}{8} e^{-4x} \quad ---------- (1)$$

Put $x = 2$, *we get*

$$\int_0^\infty \frac{\cos 2s}{s^2+16} \, ds = \frac{\pi}{8} e^{-8}$$

$$\int_0^\infty \frac{\cos 2x}{x^2+16} \, dx = \frac{\pi}{8} e^{-8}$$

Differentiate (1) w.r.t. x, we get

$$\frac{d}{dx} \int_0^\infty \frac{\cos sx}{s^2+16} \, ds = \frac{\pi}{8} \frac{d}{dx} (e^{-4x})$$

$$\int_0^\infty \frac{\partial}{\partial x} \left(\frac{\cos sx}{s^2+16} \right) ds = \frac{\pi}{8} \frac{d}{dx} (e^{-4x})$$

$$\int_0^\infty \left(\frac{-\sin sx . s}{s^2+16} \right) ds = \frac{\pi}{8} (e^{-4x})(-4)$$

$$\int_0^\infty \frac{s \sin sx}{s^2+16} \, ds = \frac{\pi}{2} e^{-4x}$$

Put $x = 2$, *we get*

$$\int_0^\infty \frac{s \sin 2s}{s^2+16} \, ds = \frac{\pi}{2} e^{-8}$$

$$\int_0^\infty \frac{x \sin 2x}{x^2+16} \, dx = \frac{\pi}{2} e^{-8}$$

6. Find the Fourier sine and cosine transform of e^{-x} and hence find the Fourier sine transform of $\frac{x}{1+x^2}$ and Fourier cosine transform of $\frac{1}{1+x^2}$

Sol. $$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$F_c[e^{-x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+s^2} (-\cos sx + s \sin sx) \right]_0^\infty$$

$$=\sqrt{\frac{2}{\pi}}\left[\{0\}-\left\{\frac{1}{1+s^2}(-1+0)\right\}\right]$$

$$=\sqrt{\frac{2}{\pi}}\;\frac{1}{s^2+1}$$

$$F_s[e^{-x}]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{1+s^2}(-\sin sx-s\cos sx)\right]_0^\infty$$

$$=\sqrt{\frac{2}{\pi}}\left[\{0\}-\left\{\frac{1}{1+s^2}(0-s)\right\}\right]$$

$$=\sqrt{\frac{2}{\pi}}\;\frac{s}{s^2+1}$$

Now, $F_c\left[\frac{1}{1+x^2}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{1+x^2}\cos sx\,dx \quad ------ (1)$

Using ***inverse Fourier cosine transform***, we have

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty F_c[f(x)]\,\cos sx\,ds$$

$$e^{-x}=\sqrt{\frac{2}{\pi}}\int_0^\infty\sqrt{\frac{2}{\pi}}\left(\frac{1}{s^2+1}\right)\cos sx\;ds$$

$$e^{-x}=\frac{2}{\pi}\int_0^\infty \frac{\cos sx}{s^2+1}\,ds$$

$$\int_0^\infty \frac{\cos sx}{s^2+1}\,ds=\frac{\pi}{2}e^{-x}$$

$$\int_0^\infty \frac{\cos sx}{x^2+1}\,dx=\frac{\pi}{2}e^{-s}$$

Put $x = s$
and $s = x$

Equation (1) becomes

$$F_c\left[\frac{1}{1+x^2}\right]=\sqrt{\frac{2}{\pi}}\;\frac{\pi}{2}\;e^{-s}$$

$$=\sqrt{\frac{\pi}{2}}\;e^{-s}$$

$$F_s\left[\frac{x}{1+x^2}\right]=-\frac{d}{ds}F_c\left[\frac{1}{1+x^2}\right]$$

$$=-\frac{d}{ds}\left[\sqrt{\frac{\pi}{2}}\;e^{-s}\right]$$

$$=-\sqrt{\frac{\pi}{2}}\;e^{-s}(-1)$$

$$=\sqrt{\frac{\pi}{2}}\;e^{-s}$$

7. Find the Fourier transform of $f(x)=\begin{cases} a-|x|\,, & |x|<a \\ 0\,, & |x|\geq a \end{cases}$

Hence deduce that $(i)\ \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2} \qquad (ii)\ \int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$

Sol.

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-a} 0.e^{isx}\, dx + \int_{-a}^{a} (a-|x|)\, e^{isx}\, dx + \int_{a}^{\infty} 0.e^{isx}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a-|x|)(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a-|x|)\cos sx\, dx + i\frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a-|x|)\sin sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{a} (a-|x|)\cos sx\, dx + 0$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a (a-x)\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[(a-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{-\cos sx}{s^2}\right) \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[\left\{0 - \frac{\cos sa}{s^2}\right\} - \left\{0 - \frac{1}{s^2}\right\} \right]$$

$$(i.e.)\ F[f(x)] = \sqrt{\frac{2}{\pi}}\ \frac{1-\cos as}{s^2}$$

Using ***inverse Fourier transform***, we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{1-\cos as}{s^2}\right) (\cos sx - i \sin sx)\, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{1-\cos as}{s^2}\right) \cos sx\, ds - i\frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{1-\cos as}{s^2}\right) \sin sx\, ds$$

$$= \frac{2}{\pi} \int_0^\infty \left(\frac{1-\cos as}{s^2}\right) \cos sx\, ds - 0$$

$$\int_0^\infty \left(\frac{1-\cos as}{s^2}\right) \cos sx\, ds = \frac{\pi}{2} f(x)$$

Put $x=0$ *we get*

$$\int_0^\infty \left(\frac{1-\cos as}{s^2}\right) ds = \frac{\pi}{2} f(0)$$

$$\int_0^\infty \left(\frac{1-\cos 2t}{\frac{4t^2}{a^2}} \right) \frac{2\,dt}{a} = \frac{\pi}{2}(a)$$

Put as = 2t
ads = 2dt

$f(x)$ = a – | x |
$f(0)$ = a – 0 = a

$$2a\int_0^\infty \left(\frac{2\sin^2 t}{4t^2} \right) dt = \frac{\pi a}{2}$$

$$\int_0^\infty \left(\frac{\sin t}{t} \right)^2 dt = \frac{\pi}{2}$$

This proves (i)

Using **Parseval's identity,** we have

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

$$\int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} \left[\frac{1-\cos as}{s^2} \right] \right)^2 ds = \int_{-\infty}^{-a} 0.dx + \int_{-a}^{a} (a-|x|)^2\, dx + \int_{a}^{\infty} 0.dx$$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{1-\cos as}{s^2} \right)^2 ds = \int_{-a}^{a} (a-|x|)^2\, dx$$

$$\frac{4}{\pi} \int_0^\infty \left(\frac{1-\cos as}{s^2} \right)^2 ds = 2\int_0^a (a-x)^2\, dx$$

$$\frac{4}{\pi} \int_0^\infty \left(\frac{1-\cos 2t}{4t^2/a^2} \right)^2 \frac{2dt}{a} = 2\left[\frac{(a-x)^3}{-3} \right]_0^a$$

Put as = 2t
ads = 2dt

$$\frac{8a^3}{16\pi} \int_0^\infty \left(\frac{1-\cos 2t}{t^2} \right)^2 dt = 2\left[\{0\} - \left\{ -\frac{a^3}{3} \right\} \right]$$

$$\frac{a^3}{2\pi} \int_0^\infty \left(\frac{2\sin^2 t}{t^2} \right)^2 dt = \frac{2a^3}{3}$$

$$\frac{4}{2\pi} \int_0^\infty \left(\frac{\sin^2 t}{t^2} \right)^2 dt = \frac{2}{3}$$

$$(i.e.) \int_0^\infty \left(\frac{\sin t}{t} \right)^4 dt = \frac{\pi}{3}$$

8. Find the Fourier transform of $e^{-\frac{x^2}{2}}$

Sol. $$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F\left[e^{-\frac{x^2}{2}} \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[x^2 - 2isx\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[(x-is)^2 - i^2 s^2\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[(x-is)^2\right]} e^{-\frac{s^2}{2}} \, dx$$

$$= \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-is)^2} \, dx$$

$$= \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x-is}{\sqrt{2}}\right)^2} \, dx$$

$$\boxed{\begin{aligned} Put \quad \frac{x-is}{\sqrt{2}} &= t \\ \frac{dx}{\sqrt{2}} &= dt \end{aligned}}$$

$$= \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \sqrt{2} \, dt$$

$$= \frac{e^{-\frac{s^2}{2}}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \, dt$$

$$\boxed{\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}}$$

$$= \frac{e^{-\frac{s^2}{2}}}{\sqrt{\pi}} \sqrt{\pi}$$

(i.e.) $F[e^{-\frac{x^2}{2}}] = e^{-\frac{s^2}{2}}$

9. Find the Fourier cosine transform of $e^{-a^2x^2}$ and hence find $F_s[x e^{-a^2x^2}]$

Sol. $$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx$$

$$F_c\left[e^{-a^2x^2}\right] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-a^2x^2} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{2} \int_{-\infty}^{\infty} e^{-a^2x^2} \cos sx \, dx$$

$$= \frac{1}{\sqrt{2\pi}} R.P. \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} R.P. \int_{-\infty}^{\infty} e^{-\left[a^2x^2 - isx\right]} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} R.P. \int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{is}{2a}\right)^2 - \frac{i^2 s^2}{4a^2}\right]} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} R.P. \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} e^{-\frac{s^2}{4a^2}} \, dx$$

$$= \frac{e^{-\frac{s^2}{4a^2}}}{\sqrt{2\pi}} R.P. \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} \, dx$$

$$= \frac{e^{-\frac{s^2}{4a^2}}}{\sqrt{2\pi}} R.P. \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a}$$

$$\boxed{Put \quad ax - \frac{is}{2a} = t \qquad a\,dx = dt}$$

$$= \frac{e^{-\frac{s^2}{4a^2}}}{a\sqrt{2\pi}} R.P. \sqrt{\pi}$$

$$(i.e.)\ F_c[e^{-a^2x^2}] = \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}}$$

$$F_s[x e^{-a^2x^2}] = -\frac{d}{ds} F_c[e^{-a^2x^2}]$$

$$= -\frac{d}{ds}\left[\frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}}\right]$$

$$= -\frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}} \left(\frac{-2s}{4a^2}\right)$$

$$= \frac{s}{2\sqrt{2}\,a^3} e^{-\frac{s^2}{4a^2}}$$

10. Find the Fourier transform of $f(x) = e^{-a|x|}$, $a > 0$. Hence deduce that $(i)\ \int_0^{\infty} \frac{\cos xt}{a^2+t^2} dt = \frac{\pi}{2a} e^{-a|x|}$ $\ (ii)\ \int_0^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{2}$ $\ (iii)\ \int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$ and also prove that $(iv)\ F\left[x e^{-a|x|}\right] = i\sqrt{\frac{2}{\pi}} \frac{2as}{(s^2+a^2)^2}$

Sol.

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} (\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} \cos sx\, dx + i\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} \sin sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-ax} \cos sx\, dx + 0$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx)\right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}} \left[\{0\} - \left\{\frac{1}{a^2+s^2}(-a+0)\right\}\right]$$

$$(i.e.)\ F[f(x)] = \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$$

Using ***inverse Fourier transform***, we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\, ds$$

$$e^{-a|x|} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2+a^2}\right) (\cos sx - i\sin sx)\, ds$$

$$= \frac{a}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{s^2+a^2}\right) \cos sx\, ds - i\frac{a}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{s^2+a^2}\right) \sin sx\, ds$$

$$= \frac{2a}{\pi} \int_0^{\infty} \frac{\cos sx}{s^2+a^2}\, ds - 0$$

$$\int_0^{\infty} \frac{\cos sx}{s^2+a^2}\, ds = \frac{\pi}{2a} e^{-a|x|}$$

$$(i.e.)\ \int_0^{\infty} \frac{\cos xt}{t^2+a^2}\, dt = \frac{\pi}{2a} e^{-a|x|} \quad \textit{This proves } (i)$$

Put $x = 0$ *and* $a = 1$, *we get*

$$\int_0^{\infty} \frac{1}{t^2+a^2}\, dt = \frac{\pi}{2}$$

$$(i.e.)\ \int_0^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{2} \quad \textit{This proves } (ii)$$

Using **Parseval's identity,** we have

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

$$\int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}}\, \frac{a}{s^2+a^2}\right)^2 ds = \int_{-\infty}^{-\infty} \left[e^{-a|x|}\right]^2\, dx$$

$$\frac{2a^2}{\pi} \int_{-\infty}^{\infty} \frac{1}{(s^2+a^2)^2}\, ds = \int_{-\infty}^{-\infty} \left[e^{-a|x|}\right]^2\, dx$$

$$\frac{4a^2}{\pi} \int_0^{\infty} \frac{ds}{(s^2+a^2)^2} = 2\int_0^{\infty} e^{-2ax}\, dx$$

$$\frac{2a^2}{\pi} \int_0^{\infty} \frac{ds}{(s^2+a^2)^2} = \left[\frac{e^{-2ax}}{-2a}\right]_0^{\infty}$$

$$= \frac{1}{-2a}[0-1]$$

$$\frac{2a^2}{\pi} \int_0^{\infty} \frac{ds}{(s^2+a^2)^2} = \frac{1}{2a}$$

$$\int_0^{\infty} \frac{ds}{(s^2+a^2)^2} = \frac{\pi}{4a^3}$$

put $a = 1$, *we get*

$$\int_0^{\infty} \frac{ds}{(s^2+1)^2} = \frac{\pi}{4}$$

$$(i.e.)\ \int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{4} \quad \textit{This proves } (iii)$$

By the property, $F[x\,f(x)] = (-i)\dfrac{d}{ds}F[f(x)]$

$$F[xe^{-a|x|}] = (-i)\frac{d}{ds}F[e^{-a|x|}]$$

$$= (-i)\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\,\frac{a}{s^2+a^2}\right]$$

$$= (-i)\sqrt{\frac{2}{\pi}}\left[\frac{-a}{(s^2+a^2)^2}\,(2s)\right]$$

$$= i\,\sqrt{\frac{2}{\pi}}\,\frac{2as}{(s^2+a^2)^2} \qquad \textit{This proves (iv)}$$

11. Find the Fourier sine and cosine transform of x^{n-1}, $0< n<1$, $x>0$ and hence prove that $\frac{1}{\sqrt{x}}$ is self reciprocal under both Fourier sine and cosine transforms.

Sol. Consider $F_c[f(x)] - i\,F_s[f(x)] = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^\infty f(x)\cos sx\,dx - i\,\sqrt{\dfrac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$

$$F_c[f(x)] - i\,F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)(\cos sx - i\sin sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,e^{-isx}\,dx$$

$$F_c[x^{n-1}] - i\,F_s[x^{n-1}] = \sqrt{\frac{2}{\pi}}\int_0^\infty x^{n-1}\,e^{-isx}\,dx$$

$$\boxed{\int_0^\infty x^{n-1}\,e^{-ax}\,dx = \frac{\Gamma(n)}{a^n}}$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n)}{(is)^n}$$

$$= (-i)^n\,\sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n)}{s^n}$$

$$= \left(\cos\frac{n\pi}{2} - i\,\sin\frac{n\pi}{2}\right)\sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n)}{s^n}$$

$$\boxed{\begin{aligned} -i &= \cos\frac{\pi}{2} - i\,\sin\frac{\pi}{2} \\ (-i)^n &= \left(\cos\frac{\pi}{2} - i\,\sin\frac{\pi}{2}\right)^n \\ &= \cos\frac{n\pi}{2} - i\,\sin\frac{n\pi}{2} \end{aligned}}$$

Equating R.P and I.P, we get

$$F_c[x^{n-1}] = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n)}{s^n}\cos\frac{n\pi}{2} \quad \text{--------} \quad (1)$$

$$F_s[x^{n-1}] = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n)}{s^n}\sin\frac{n\pi}{2} \quad \text{--------} \quad (2)$$

Put $n = \frac{1}{2}$ *in equation* (1), *we have*

$$F_c[x^{\frac{1}{2}-1}] = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(1/2)}{s^{1/2}}\cos\frac{\pi}{4}$$

$$F_c\left[\frac{1}{\sqrt{x}}\right] = \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{\pi}}{\sqrt{s}}\,\frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{s}}$$

$$Put \ \ n=\frac{1}{2} \ in \ \ equation \ (1), \ we \ \ have$$

$$F_s[x^{\frac{1}{2}-1}]=\sqrt{\frac{2}{\pi}}\ \frac{\Gamma(1/2)}{s^{1/2}}\sin\frac{\pi}{4}$$

$$F_s\left[\frac{1}{\sqrt{x}}\right]=\sqrt{\frac{2}{\pi}}\ \frac{\sqrt{\pi}}{\sqrt{s}}\frac{1}{\sqrt{2}}$$

$$=\frac{1}{\sqrt{s}}$$

Hence $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier sine and cosine transforms.

Now, $F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\ dx$

$$F\left[\frac{1}{\sqrt{|x|}}\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{|x|}}(\cos sx+i\sin sx)\ dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{|x|}}\cos sx\ dx\ +\ i\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{|x|}}\sin sx\ dx$$

$$=\frac{2}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{1}{\sqrt{x}}\ \cos sx\ dx\ +\ 0$$

$$=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{1}{\sqrt{x}}\ \cos sx\ dx$$

$$=F_c\left[\frac{1}{\sqrt{x}}\right]$$

$$=\frac{1}{\sqrt{s}}$$

12. Find the Fourier sine transform of $\frac{e^{-ax}}{x}$

Sol. $F_s[f(x)]=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\ \sin sx\, dx$

$$F_s\left[\frac{e^{-ax}}{x}\right]=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{e^{-ax}}{x}\ \sin sx\, dx$$

Diff. w.r.t. 's' on both sides we get

$$\frac{d}{ds}F_s\left[\frac{e^{-ax}}{x}\right]=\frac{d}{ds}\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{e^{-ax}}{x}\ \sin sx\, dx$$

$$=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{\partial}{\partial s}\left(\frac{e^{-ax}}{x}\ \sin sx\right)dx$$

$$=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{e^{-ax}}{x}\ \cos sx\, .x\ dx$$

$$=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} e^{-ax}\ \cos sx\ dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx+s\sin sx)\right]_0^{\infty}$$

$$=\sqrt{\frac{2}{\pi}}\left[\{0\}-\left\{\frac{1}{a^2+s^2}(-a+0)\right\}\right]$$

$$\frac{d}{ds}F_s\left[\frac{e^{-ax}}{x}\right]=\sqrt{\frac{2}{\pi}}\;\frac{a}{s^2+a^2}$$

Integrating w.r.t. 's' we get

$$F_s\left[\frac{e^{-ax}}{x}\right]=\sqrt{\frac{2}{\pi}}\int\frac{a}{s^2+a^2}ds$$

$$=a\sqrt{\frac{2}{\pi}}\left[\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)\right]$$

$$\boxed{\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)}$$

$$=\sqrt{\frac{2}{\pi}}\;\tan^{-1}\left(\frac{s}{a}\right)$$

13. Find the Fourier cosine transform of $\dfrac{e^{-ax}}{x}$

Sol. $$F_c[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$F_c\left[\frac{e^{-ax}}{x}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{e^{-ax}}{x}\cos sx\,dx$$

Diff. w.r.t. 's' on both sides we get

$$\frac{d}{ds}F_c\left[\frac{e^{-ax}}{x}\right]=\frac{d}{ds}\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{e^{-ax}}{x}\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{\partial}{\partial s}\left(\frac{e^{-ax}}{x}\cos sx\right)dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{e^{-ax}}{x}(-\sin sx.x)\;dx$$

$$=-\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\;dx$$

$$=-\sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\sin sx-s\cos sx)\right]_0^\infty$$

$$=-\sqrt{\frac{2}{\pi}}\left[\{0\}-\left\{\frac{1}{a^2+s^2}(0-s)\right\}\right]$$

$$\frac{d}{ds}F_c\left[\frac{e^{-ax}}{x}\right]=-\sqrt{\frac{2}{\pi}}\;\frac{s}{s^2+a^2}$$

Integrating w.r.t. 's' we get

$$F_c\left[\frac{e^{-ax}}{x}\right]=-\sqrt{\frac{2}{\pi}}\int\frac{s}{s^2+a^2}ds$$

$$\boxed{\int\frac{x\,dx}{x^2+a^2}=\frac{1}{2}\log(x^2+a^2)}$$

$$=-\sqrt{\frac{2}{\pi}}\;\frac{1}{2}\;\log(s^2+a^2)$$

$$=-\frac{1}{\sqrt{2\pi}}\;\log(s^2+a^2)$$

14. Find the Fourier sine and cosine transform of $x e^{-ax}$

Sol.

$$F_s[x e^{-ax}] = -\frac{d}{ds} F_c[e^{-ax}]$$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2 + s^2} (-a \cos sx + s \sin sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\{0\} - \left\{ \frac{1}{a^2 + s^2} (-a + 0) \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{s^2 + a^2}$$

$$F_s[x e^{-ax}] = -\frac{d}{ds} \left[\sqrt{\frac{2}{\pi}} \frac{a}{s^2 + a^2} \right]$$

$$= -\sqrt{\frac{2}{\pi}} \left[\frac{-a}{(s^2 + a^2)^2} (2s) \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{2as}{(s^2 + a^2)^2}$$

$$F_c[x e^{-ax}] = \frac{d}{ds} F_s[e^{-ax}]$$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2 + s^2} (-a \sin sx - s \cos sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\{0\} - \left\{ \frac{1}{a^2 + s^2} (0 - s) \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{s}{s^2 + a^2}$$

$$F_c[x e^{-ax}] = \frac{d}{ds} \left[\sqrt{\frac{2}{\pi}} \frac{s}{s^2 + a^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{(s^2 + a^2)(1) - s(2s)}{(s^2 + a^2)^2}$$

$$= \sqrt{\frac{2}{\pi}} \frac{a^2 - s^2}{(s^2 + a^2)^2}$$

15. Verify Parseval's theorem of Fourier transform for the function $f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x > 0 \end{cases}$

Sol. $F(s) = F[f(x)] = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} 0.e^{isx}\, dx + \int_{0}^{\infty} e^{-x}.e^{isx}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(1-is)x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(1-is)x}}{-(1-is)} \right]_0^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 - \frac{1}{-(1-is)} \right]$$

$$(i.e.)\ F(s) = \frac{1}{\sqrt{2\pi}} \frac{1}{1-is}$$

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} F(s)\overline{F(s)}\, ds$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{1-is}\ \frac{1}{\sqrt{2\pi}} \frac{1}{1+is} ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+s^2} ds$$

$$= \frac{2}{2\pi} \int_{0}^{\infty} \frac{ds}{1+s^2}$$

$$= \frac{1}{\pi} \left[\frac{1}{1} \tan^{-1}\left(\frac{s}{1} \right) \right]_0^{\infty}$$

$$= \frac{1}{\pi} \left[\frac{\pi}{2} - 0 \right]$$

$$= \frac{1}{2}$$

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{0} 0.\, dx + \int_{0}^{\infty} (e^{-x})^2\, dx$$

$$= \int_{0}^{\infty} e^{-2x}\, dx$$

$$= \left[\frac{e^{-2x}}{-2} \right]_0^{\infty}$$

$$= \left[0 - \frac{1}{-2} \right]$$

$$= \frac{1}{2}$$

$$\therefore \int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

Hence Parseval's theorem is verified.

16. Use transform methods to evaluate $i)\ \int_0^\infty \frac{dx}{(x^2+1)(x^2+4)}$ $\quad ii)\ \int_0^\infty \frac{x^2 dx}{(x^2+9)(x^2+25)}$

Sol. (i) Let $f(x)=e^{-x}$ and $g(x)=e^{-2x}$

Then $F_c(s)=\sqrt{\frac{2}{\pi}}\frac{1}{s^2+1}$ and $G_c(s)=\sqrt{\frac{2}{\pi}}\frac{2}{s^2+4}$

We have $\int_0^\infty F_c(s)G_c(s)\,ds=\int_0^\infty f(x)\,g(x)\,dx$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{1}{s^2+1}\sqrt{\frac{2}{\pi}}\frac{2}{s^2+4}ds=\int_0^\infty e^{-x}e^{-2x}\,dx$$

$$\frac{4}{\pi}\int_0^\infty \frac{ds}{(s^2+1)(s^2+4)}=\int_0^\infty e^{-3x}\,dx$$

$$=\left[\frac{e^{-3x}}{-3}\right]_0^\infty$$

$$=\left[0-\frac{1}{-3}\right]$$

$$\frac{4}{\pi}\int_0^\infty \frac{ds}{(s^2+1)(s^2+4)}=\frac{1}{3}$$

$$(i.e.)\ \int_0^\infty \frac{dx}{(x^2+1)(x^2+4)}=\frac{\pi}{12}$$

(ii) Let $f(x)=e^{-3x}$ and $g(x)=e^{-5x}$

Then $F_s(s)=\sqrt{\frac{2}{\pi}}\frac{s}{s^2+9}$ and $G_s(s)=\sqrt{\frac{2}{\pi}}\frac{s}{s^2+25}$

We have $\int_0^\infty F_s(s)G_s(s)\,ds=\int_0^\infty f(x)\,g(x)\,dx$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{s}{s^2+9}\sqrt{\frac{2}{\pi}}\frac{s}{s^2+25}ds=\int_0^\infty e^{-3x}e^{-5x}\,dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{s^2 ds}{(s^2+9)(s^2+25)}=\int_0^\infty e^{-8x}\,dx$$

$$=\left[\frac{e^{-8x}}{-8}\right]_0^\infty$$

$$=\left[0-\frac{1}{-8}\right]$$

$$\frac{2}{\pi}\int_0^\infty \frac{s^2 ds}{(s^2+9)(s^2+25)}=\frac{1}{8}$$

$$(i.e.)\ \int_0^\infty \frac{x^2 dx}{(x^2+9)(x^2+25)}=\frac{\pi}{16}$$

17. Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ **using transforms.**

Sol. Let $f(x)=e^{-ax}$ and $g(x)=e^{-bx}$

Then $F_c(s)=\sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}$ and $G_c(s)=\sqrt{\frac{2}{\pi}}\frac{b}{s^2+b^2}$

We have $\int_0^\infty F_c(s)G_c(s)\,ds = \int_0^\infty f(x)\,g(x)\,dx$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\,\frac{a}{s^2+a^2}\,\sqrt{\frac{2}{\pi}}\,\frac{b}{s^2+b^2}\,ds = \int_0^\infty e^{-ax}\,e^{-bx}\,dx$$

$$\frac{2ab}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)} = \int_0^\infty e^{-(a+b)x}\,dx$$

$$= \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= \left[0 - \frac{1}{-(a+b)}\right]$$

$$\frac{2ab}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)} = \frac{1}{a+b}$$

$$(i.e.)\ \int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

18. Using Parseval's identity, calculate $i)\ \int_0^\infty \frac{dx}{(x^2+a^2)^2}$ $\quad ii)\ \int_0^\infty \frac{x^2dx}{(x^2+4)^2}$

Sol. (i) Let $f(x) = e^{-ax}$ then $F_c(s) = \sqrt{\frac{2}{\pi}}\,\frac{a}{s^2+a^2}$

Using Parseval's identity for Fourier cosine transform, we have

$$\int_0^\infty [F_c(s)]^2\,ds = \int_0^\infty [f(x)]^2\,dx$$

$$\int_0^\infty \left(\sqrt{\frac{2}{\pi}}\,\frac{a}{s^2+a^2}\right)^2 ds = \int_0^\infty (e^{-ax})^2\,dx$$

$$\frac{2a^2}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)^2} = \int_0^\infty e^{-2ax}\,dx$$

$$= \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$= \left[0 - \frac{1}{-2a}\right]$$

$$\frac{2a^2}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)^2} = \frac{1}{2a}$$

$$(i.e.)\ \int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$$

(ii) Let $f(x) = e^{-2x}$ then $F_s(s) = \sqrt{\frac{2}{\pi}}\,\frac{s}{s^2+a^2}$

Using Parseval's identity for Fourier sine transform, we have

$$\int_0^\infty [F_s(s)]^2\,ds = \int_0^\infty [f(x)]^2\,dx$$

$$\int_0^\infty \left(\sqrt{\frac{2}{\pi}}\frac{s}{s^2+4}\right)^2 ds = \int_0^\infty (e^{-2x})^2\,dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{s^2 ds}{(s^2+4)^2} = \int_0^\infty e^{-4x}\,dx$$

$$= \left[\frac{e^{-4x}}{-4}\right]_0^\infty$$

$$= \left[0 - \frac{1}{-4}\right]$$

$$\frac{2}{\pi}\int_0^\infty \frac{s^2 ds}{(s^2+4)^2} = \frac{1}{4}$$

$$(i.e.)\ \int_0^\infty \frac{x^2 dx}{(x^2+4)^2} = \frac{\pi}{8}$$

19. State and prove convolution theorem for Fourier transform.

Statement: If F[f(x)] = F(s) and F[g(x)] = G(s) then $F[f(x)*g(x)] = F(s).G(s)$

Proof.

$$F[f(x)*g(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[f(x)*g(x)]\,e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,g(x-t)\,dt\right]e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-t)\,e^{isx}\,dx\right]dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-t)\,e^{isx}\,e^{ist}\,e^{-ist}\,dx\right]dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-t)\,e^{is(x-t)}\,d(x-t)\right]e^{ist}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,G(s)\,e^{ist}\,dt$$

$$= G(s)\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{ist}\,dt$$

$$= G(s)\,F(s)$$

(i.e.) $F[f(x)*g(x)] = F(s).G(s)$

20. State and prove Parseval's identity for Fourier transform.

Statement: If F(s) is the Fourier transform of f(x) then

$$\int_{-\infty}^{\infty} |F(s)|^2\,ds = \int_{-\infty}^{\infty} |f(x)|^2\,dx$$

Proof. By convolution theorem for Fourier transform, we have

$$F[f(x)*g(x)] = F(s).G(s)$$

$$\therefore F^{-1}[F(s)G(s)] = f(x)*g(x)$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)G(s)\,e^{-isx}\,ds = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,g(x-t)\,dt$$

$$\Rightarrow \int_{-\infty}^{\infty} F(s)G(s)\,e^{-isx}\,ds = \int_{-\infty}^{\infty} f(t)\,g(x-t)\,dt$$

Putting $x = 0$, we get

$$\int_{-\infty}^{\infty} F(s)G(s)\,ds = \int_{-\infty}^{\infty} f(t)\,g(-t)\,dt \quad \text{-------} \; (1)$$

$$Let \;\; g(-t) = \overline{f(t)} \quad \text{-------} \; (2)$$

$$(i.e.)\; g(t) = \overline{f(-t)}$$

$$G(s) = F[g(x)] = F[g(t)]$$

$$= F[\overline{f(-t)}]$$

$$= F[\overline{f(-x)}]$$

$$= \overline{F(s)} \quad (by \;\; property)$$

$$(i.e.)\; G(s) = \overline{F(s)} \quad \text{----------} \; (3)$$

Substituting (2) and (3) in equation (1) we have

$$\int_{-\infty}^{\infty} F(s)\overline{F(s)}\,ds = \int_{-\infty}^{\infty} f(t)\overline{f(t)}\,dt$$

(i.e.) $$\int_{-\infty}^{\infty} |F(s)|^2\,ds = \int_{-\infty}^{\infty} |f(x)|^2\,dx$$

21. Find the Fourier sine transform of $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$

Sol. $$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\,\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x\,\sin sx\,dx + \int_1^2 (2-x)\,\sin sx\,dx + \int_2^{\infty} 0.\,\sin sx\,dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[x\left(\frac{-\cos sx}{s}\right) - (1)\left(\frac{-\sin sx}{s^2}\right) \right]_0^1 + \sqrt{\frac{2}{\pi}} \left[(2-x)\left(\frac{-\cos sx}{s}\right) - (-1)\left(\frac{-\sin sx}{s^2}\right) \right]_1^2$$

$$= \sqrt{\frac{2}{\pi}} \left[\left\{ \frac{-\cos s}{s} + \frac{\sin s}{s^2} \right\} - \{0+0\} \right] + \sqrt{\frac{2}{\pi}} \left[\left\{ 0 - \frac{\sin 2s}{s^2} \right\} - \left\{ \frac{-\cos s}{s} - \frac{\sin s}{s^2} \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2\sin s}{s^2} - \frac{\sin 2s}{s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2\sin s - 2\sin s \cos s}{s^2} \right]$$

$$= 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s\,(1-\cos s)}{s^2} \right]$$

22. Find the Fourier sine and cosine transform of $f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x > a \end{cases}$

Sol. $$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\,\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^a \sin x\,\sin sx\,dx + \int_a^{\infty} 0.\,\sin sx\,dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \frac{1}{2}[\cos(s-1)x - \cos(s+1)x]\,dx$$

2sinAsinB = cos(A – B) – cos(A + B)

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s-1)x}{s-1} - \frac{\sin(s+1)x}{s+1} \right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left\{ \frac{\sin(s-1)a}{s-1} - \frac{\sin(s+1)a}{s+1} \right\} - \{0-0\} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s-1)a}{s-1} - \frac{\sin(s+1)a}{s+1} \right]$$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^a \sin x \cos sx \, dx + \int_a^\infty 0. \cos sx \, dx \right]$$

2cosAsinB = sin(A + B) – sin(A – B)

$$= \sqrt{\frac{2}{\pi}} \int_0^a \frac{1}{2} [\sin(s+1)x - \sin(s-1)x] \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{-\cos(s+1)x}{s+1} + \frac{\cos(s-1)x}{s-1} \right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left\{ \frac{-\cos(s+1)a}{s+1} + \frac{\cos(s-1)a}{s-1} \right\} - \left\{ \frac{-1}{s+1} + \frac{1}{s-1} \right\} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left\{ \frac{(s-1)[-\cos sa \cos a + \sin sa \sin a] + (s+1)[\cos sa \cos a + \sin sa \sin a]}{(s+1)(s-1)} \right\} - \left\{ \frac{-(s-1)+(s+1)}{(s+1)(s-1)} \right\} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2s \sin sa \sin a + 2\cos sa \cos a}{s^2-1} - \frac{2}{s^2-1} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s \sin sa \sin a + \cos sa \cos a - 1}{s^2-1} \right]$$

UNIT – V

Z – TRANSFORMS

PART – A

1. Find Z$\left[\dfrac{1}{n+1}\right]$.

Solution : Z$\left[f(n)\right]=\sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$Z\left(\frac{1}{n+1}\right) = \sum_{n=0}^{\infty}\frac{1}{n+1}z^{-n}$$

$$= 1+\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}+\frac{1}{4}z^{-3}+\ldots$$

$$= 1+\frac{\left(\frac{1}{z}\right)}{2}+\frac{\left(\frac{1}{z^2}\right)}{3}+\frac{\left(\frac{1}{z^3}\right)}{4}+\ldots\ldots$$

$$= z\left[\frac{1}{z}+\frac{\left(\frac{1}{z}\right)^2}{2}+\frac{\left(\frac{1}{z}\right)^3}{3}+\ldots\ \ldots\right]$$

$$= z\left[-\log\left(1-\frac{1}{z}\right)\right]$$

$$= -z\log\left(1-\frac{1}{z}\right)$$

$$= -z\log\left(\frac{z-1}{z}\right)$$

$$= z\log\left(\frac{z-1}{z}\right)^{-1}$$

$$= z\log\left(\frac{z}{z-1}\right)$$

2. Find $Z\left[e^{an}\right]$.

Solution : $Z\left[(a)^n\right]=\dfrac{z}{z-a}$

$$Z\left[e^{an}\right]=Z\left[\left(e^{a}\right)^n\right]$$

$$= \frac{z}{z-e^{a}}$$

3. Find $Z[\cos n\theta]\ and\ Z[\sin n\theta]$.

Solution : $Z\left[(a)^n\right]=\dfrac{z}{z-a}$

Put a = $e^{i\theta}$

$$Z\left[\left(e^{i\theta}\right)^n\right]= \frac{z}{z-e^{i\theta}}$$

$$Z\left[e^{in\theta}\right]=\frac{z}{z-(\cos\theta+i\sin\theta)}$$

$$Z[\cos n\theta+i\sin n\theta] = \frac{z}{(z-\cos\theta)-i\sin\theta}$$

$$= \frac{z}{(z-\cos\theta)-i\sin\theta} \times \frac{(z-\cos\theta)+i\sin\theta}{(z-\cos\theta)+i\sin\theta}$$

$$= \frac{z(z-\cos\theta)+i\,z\sin\theta}{(z-\cos\theta)^2+\sin^2\theta}$$

$$= \frac{z(z-\cos\theta)+i\,z\sin\theta}{z^2+\cos^2\theta-2z\cos\theta+\sin^2\theta}$$

$$= \frac{z(z-\cos\theta)+i\,z\sin\theta}{z^2-2z\cos\theta+1}$$

$$Z[\cos n\theta]+i\,Z[\sin n\theta]=\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}+\frac{i\,z\sin\theta}{z^2-2z\cos\theta+1}$$

Equating the Real and Imaginary parts,

$$Z[\cos n\theta]= \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$$

$$Z[\sin n\theta] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

4. State the initial and final value theorem of Z-transform

Solution : Initial Value Theorem : if $Z[f(n)]=F(Z)$, $then \lim\limits_{z\to\infty}F(Z)=f(0)$

Final Value Theorem : If $Z[f(t)]=F(z)$, $then\ \lim\limits_{t\to\infty} f(t)=\lim\limits_{z\leftarrow 1}(z-1)F(Z)$

5. Find the equation generated by $y_n = a + b3^n$.

Solution : $y_n = a + b3^n$

$Y_{n+1}= a + b.\ 3^{n+1} = a + 3b\ 3^n$

$Y_{n+2}= a + b.\ 3^{n+2} = a + 9b\ 3^n$

Eliminating 'a' and 'b' ,

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 1 & 3 \\ y_{n+2} & 1 & 9 \end{vmatrix} = 0$$

$$y_n[9-3]-1[9y_{n+1}-3y_{n+2}]+1[y_{n+1}-y_{n+2}]=0$$

$$6\,y_n-9y_{n+1}+3y_{n+2}+y_{n+1}-y_{n+2}=0$$

$$6y_n-8y_{n+1}+2\,y_{n+2}=0$$

$$y_{n=2} - 4y_{n+1} + 3y_n = 0$$

PART – B

1. Find $Z\left[2^n \sinh 3n\right]$

Solution:

$$Z\left[(a)^n\right] = \frac{z}{z-a}$$

$$\therefore \quad Z\left[2^n \sinh 3n\right] = Z\left[2^n\left(\frac{e^{3n} - e^{-3n}}{2}\right)\right]$$

$$= \frac{1}{2} Z\left[2^n\left(e^{3n} - e^{-3n}\right)\right]$$

$$= \frac{1}{2}\left[Z\left(2^n e^{3n}\right) - Z\left(2^n e^{3n}\right)\right]$$

$$= \frac{1}{2}\left\{Z\left[\left(2e^3\right)^n\right] - Z\left[\left(2e^{-3}\right)^n\right]\right\}$$

$$= \frac{1}{2}\left[\frac{z}{z-2e^3} - \frac{z}{z-2e^{-3}}\right]$$

$$= \frac{z}{2}\left[\frac{\left(z-2e^{-3}\right) - \left(z-2e^3\right)}{\left(z-2e^3\right)\left(z-2e^{-3}\right)}\right]$$

$$= \frac{z}{2}\left[\frac{2e^3 - 2e^{-3}}{z^2 - 2ze^{-3} - 2ze^3 + 4}\right]$$

$$= \frac{z}{2}\left[\frac{2\left(e^3 - e^{-3}\right)}{z^2 - 2z\left(e^3 + e^{-3}\right) + 4}\right]$$

$$= \frac{z}{2}\left[\frac{2.2\sinh 3}{z^2 - 2z2\cosh 3 + 4}\right]$$

$$= \frac{z}{2}\left[\frac{4\sinh 3}{z^2 - 4z\cosh 3 + 4}\right]$$

$$= \frac{2z\sinh 3}{z^2 - 4z\cosh 3 + 4}$$

2. Find $Z\left[\frac{2n+3}{(n+1)(n+2)}\right]$

Solution:

$$\frac{2n+3}{(n+1)(n+2)} = \frac{A}{n+1}+\frac{B}{n+2}$$

$$2n+3 = A(n+2)+B(n+1)$$

Put n = -1 put n = -2

-2+3 = A -4+3 = - B

A = 1 B = 1

$$\frac{2n+3}{(n+1)(n+2)} = \frac{1}{n+1}+\frac{1}{n+2}$$

$$Z\left[\frac{2n+3}{(n+1)(n+2)}\right] = Z\left[\frac{1}{n+1}+\frac{1}{n+2}\right]$$

$$= Z\left[\frac{1}{n+1}\right]+Z\left[\frac{1}{n+2}\right]$$

We know,

$$Z\left[\frac{1}{n+1}\right]= z\log\left(\frac{z}{z-1}\right)$$

$$Z\left[\frac{1}{n+2}\right]= z^2\log\left(\frac{z}{z-1}\right)-z$$

$$\therefore \quad Z\left[\frac{2n+3}{(n+1)(n+2)}\right] = z\log\left(\frac{z}{z-1}\right) - z^2\log\left(\frac{z}{z-1}\right)-z$$

3. Find $Z\left(e^{-at}\sin\omega t\right)$

Solution:

$$Z\left[e^{-at}\ f(t)\right]=\{F(z)\}_{z\to ze^{aT}}$$

Here, f(t) = sin ωt

$$Z\left[e^{-at}\sin\omega t\right] = \left[Z(\sin\omega t)\right]_{z\to ze^{aT}}$$

$$=\left\{\frac{z\sin\omega T}{z^2-2z\cos\omega T+1}\right\}_{z\to ze^{aT}}$$

$$=\frac{ze^{aT}\sin\omega T}{z^2e^{2aT}-2ze^{aT}\cos\omega T+1}$$

4. Find Z[n³]

Solution :

$$Z[n^3] = Z\left[n\times n^2\right]$$

$$= -z\frac{d}{dz}\left[Z\left(n^2\right)\right]$$

$$= -z\frac{d}{dz}\left[\frac{z^2+z}{(z-1)^3}\right]$$

$$= -z\left[\frac{(z-1)^3(2z+1)-\left(z^2+z\right)3(z-1)^2}{(z-1)^6}\right]$$

$$= -z(z-1)^2\left[\frac{(z-1)(2z+1)-3(z^2+z)}{(z-1)^6}\right]$$

$$= -z\left[\frac{(z-1)(2z+1)-3(z^2+z)}{(z-1)^4}\right]$$

$$= -z\left[\frac{2z^2+z-2z-1-3z^2-3z}{(z-1)^4}\right]$$

$$= -\ z\left[\frac{-z^2-4z-1}{(z-1)^4}\right]$$

$$= \frac{z(z^2+4z+1)}{(z-1)^4}$$

$$= \frac{z^3+4z^2+z}{(z-1)^4}$$

5. Find $Z\left[(n-1)a^{n-1}\right]$

Solution :

$$Z\left[(n-1)a^{n-1}\right] = Z\left[(n-1)a^n a^{-1}\right]$$

$$= Z\left[na^n a^{-1}\right] - Z\left[a^n a^{-1}\right]$$

$$= Z\left[na^{-1}\right]_{z\to\frac{z}{a}} - Z\left[a^{n-1}\right]$$

$$= a^{-1}\left[Z(n)\right]_{z\to\frac{z}{a}} - \frac{1}{z-a}$$

$$= \frac{1}{a}\left[\frac{z}{(z-1)^2}\right]_{z\to\frac{z}{a}} - \frac{1}{z-a}$$

$$= \frac{1}{a}\frac{\frac{z}{a}}{\left(\frac{z}{a}-1\right)^2} - \frac{1}{z-a}$$

$$= \frac{\frac{z}{a^2}}{\left(\frac{z-a}{a}\right)^3} - \frac{1}{z-a}$$

$$= \frac{z}{a^2}\times\frac{a^2}{(z-a)^2} - \frac{1}{z-a}$$

$$= \frac{z}{(z-a)^2} - \frac{1}{z-a}$$

$$= \frac{z-z+a}{(z-a)^2}$$

$$= \frac{a}{(z-a)^2}$$

6. Find the Z-transform of $Z[\cos(t+T)]$

Solution:

$$Z[f(t+T)] = z\,F(z) - z\,f(0)$$

Here, f(t) =cost , f(0) = cos 0 = 1

$$Z[\cos(t+T)] = z\,Z[\cos t] - z\,f(0)$$

$$= z\frac{z(z-\cos T)}{z^2-2z\cos T+1} - z$$

$$= \frac{z^2(z-\cos T)}{z^2-2z\cos T+1} - z$$

$$= \frac{z^3 - z^2\cos T - z(z^2-2z\cos T+1)}{z^2-2z\cos T+1}$$

$$= \frac{z^3 - z^2\cos T - z^3 + 2z^2\cos T - z}{z^2-2z\cos T+1}$$

$$= \frac{z^2\cos T - z}{z^2-2z\cos T+1}$$

7. If $F(z) = \dfrac{z(z-\cos aT)}{z^2-2z\cos aT+1}$ find f(0) and $\lim\limits_{t\to\infty} f(t)$

Solution:

By Initial value Theorem,

$$f(0) = \lim_{z\to\infty} F(z)$$

$$= \lim_{z\to\infty} \frac{z(z-\cos aT)}{z^2-2z\cos aT+1}$$

$$= \lim_{z\to\infty} \frac{z^2 - z\cos aT}{z^2-2z\cos aT+1}$$

$$= \lim_{z\to\infty} \frac{\frac{d}{dz}[z^2 - z\cos aT]}{\frac{d}{dz}[z^2-2z\cos aT+1]} \qquad [L-Hospital\,rule]$$

$$= \lim_{z\to\infty} \frac{2z-\cos aT}{2z-2\cos aT} \qquad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{z\to\infty} \frac{\frac{d}{dz}[2z-\cos aT]}{\frac{d}{dz}[2z-2\cos aT]} \qquad [L-Hospital\,rule]$$

$$= \lim_{z\to\infty} \frac{2}{2}$$

$= 1$

8. Find $Z^{-1}\left[\dfrac{z^3+3z}{(z-1)^2(z^2+1)}\right]$

Solution :

Let $F(z)=\left[\dfrac{z^3+3z}{(z-1)^2(z^2+1)}\right]$

$\therefore \quad \dfrac{F(z)}{z}=\left[\dfrac{z^2+3}{(z-1)^2(z^2+1)}\right]$

$$\frac{z^2+3}{(z-1)^2(z^2+1)} = \frac{A}{z-1}+\frac{B}{(z-1)^2}+\frac{Cz+D}{z^2+1}$$

$$z^2+3=A(z-1)(z^2+1)+B(z^2+1)+(Cz+D)(z-1)^2$$

Put z = 1 => 4 = 2B => $\boxed{B=2}$

Z=0 =>3 = -A+B+D

Substituting B=2 , we get ,

1 = -A+D (1)

Equating coefficient of z^3 ,

0=A + C (2)

Put z=2 => 7 = 5A+5B+2C+D (3)

Substituting B=2 in (3)

7= 5A+10+2C+D

-3 = 5A+2C+D (4)

From (2) , A = -C

-3 = 5(A) + 2(-A)+D

-3 = 5A-2A+D

-3 = 3A+D (5)

(1) – (5) => 4 = - 4A

$\boxed{A=-1}$

Substituting A = -1 and B= 2 in (1)

1+2+D = 3

$\boxed{D=0}$

Substituting A = -1 , B=2 , C=1 and D=0 in (A)

$$\frac{F(z)}{z} = \frac{-1}{z-1} + \frac{2}{(z-1)^2} + \frac{z}{z^2+1}$$

$$F(z) = \frac{-z}{z-1} + \frac{2z}{(z-1)^2} + \frac{z^2}{z^2+1}$$

$$Z^{-1}\left[\frac{z^3+3z}{(z-1)^2(z^2+1)}\right] = -(1)^n + 2n + \cos\frac{n\pi}{2}$$

9. Find $Z^{-1}\left[\frac{z(z+1}{(z-1)^3}\right]$

Solution :

Let $F(z) = \left[\frac{z(z+1}{(z-1)^3}\right]$, $Z^{-1}[F(z)] = f(n)$

$$z^{n-1}F(z) = \frac{z^n(z+1)}{(z-1)^3}$$

The poles are z=1 (pole of order 3)

$$\operatorname{Res}\{z^{n-1}F(z)\}_{z=1} = \lim_{z\to1}\frac{1}{2}\frac{d^2}{dz^2}\left[(z-1)^3\frac{z^n(z+1)}{(z-1)^3}\right]$$

$$= \lim_{z\to1}\frac{1}{2}\frac{d^2}{dz^2}\left[z^n(z+1)\right]$$

$$= \lim_{z\to1}\frac{1}{2}\frac{d}{dz}\left[z^n + (z+1)nz^{n-1}\right]$$

$$= \lim_{z\to1}\frac{1}{2}\left[nz^{n-1} + n\{(z+1)(n-1)z^{n-2} + z^{n-1}\}\right]$$

$$= \frac{1}{2}\left[n(1)^{n-1} + n\{2(n-1)(1)^{n-2} + (1)^{n-1}\}\right]$$

$$= \frac{1}{2}\left[n(1)^{n-1} + 2n(n-1)(1)^{n-2} + n(1)^{n-1}\right]$$

$$= \frac{1}{2}\left[2n(1)^{n-1} + 2n(n-1)(1)^{n-2}\right]$$

$$= n(1)^{n-1} + n(n-1)(1)^{n-2}$$

$$Z^{-1}\left[\frac{z(z+1}{(z-1)^3}\right] = n(1)^{n-1} + n(n-1)(1)^{n-2}, \; n \ge 0$$

10. Solve $y_{n+2} + 4y_{n+1} + 3y_n = 2^n$ with $y_0 = 0$ *and* $y_1 = 1$ using Z – Transform.

Solution:

$$y_{n+2} + 4y_{n+1} + 3y_n = 2^n$$

Taking Z – Transform,

$$Z[y_{n+2}] + 4Z[y_{n+1}] + 3Z[y_n] = Z[2^n]$$

$$z^2\bar{y} - z^2y_0 - zy_1 + 4[z\bar{y} - zy_0] + 3\bar{y} = \frac{z}{z-2}$$

Given $y_0 = 0, y_1 = 1$

$$z^2\bar{y} - z + 4z\bar{y} + 3\bar{y} = \frac{z}{z-2}$$

$$\bar{y}\left(z^2 + 4z + 3\right) = \frac{z}{z-2} + z$$

$$\bar{y}\left[(z+1)(z+3)\right] = \frac{z}{z-2} + z$$

$$\bar{y} = \frac{z}{(z-2)(z+1)(z+3)} + \frac{z}{(z+1)(z+3)} \qquad \text{(A)}$$

Now,

$$\frac{z}{(z-2)(z+1)(z+3)} = \frac{A}{z-2} + \frac{B}{z+1} + \frac{C}{z+3}$$

$$z = A(z+1)(z+3) + B(z-2)(z+3) + C(z-2)(z+1)$$

Put z = -1	put z = -3	put z=2
-1 = 6B	-3 = 10C	2 = 15A
B = 1/6	C = - 3/10	A = 2/15

$$\frac{z}{(z-2)(z+1)(z+3)} = \frac{2}{15}\frac{1}{z-2} + \frac{1}{6}\frac{1}{z+1} - \frac{3}{10}\frac{1}{z+3} \qquad \text{(B)}$$

Also,

$$\frac{z}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}$$

$$z = A(z+3) + B(z+1)$$

Put z = -3	put z = -1
-3 = -2B	-1 = 2A
B = 3/2	A = - ½

$$\frac{z}{(z+1)(z+3)} = -\frac{1}{2}\frac{1}{z+1} + \frac{3}{2}\frac{1}{z+3}$$

$$\bar{y} = \frac{2}{15}\frac{1}{z-2} + \frac{1}{6}\frac{1}{z+1} - \frac{3}{10}\frac{1}{z+3} - \frac{1}{2}\frac{1}{z+1} + \frac{3}{2}\frac{1}{z+3}$$

$$Z[y_n] = \frac{2}{15}\frac{1}{z-2} + \frac{1}{6}\frac{1}{z+1} - \frac{3}{10}\frac{1}{z+3} - \frac{1}{2}\frac{1}{z+1} + \frac{3}{2}\frac{1}{z+3}$$

$$y_n = \frac{2}{15}Z^{-1}\left[\frac{1}{z-2}\right] + \frac{1}{6}Z^{-1}\left[\frac{1}{z+1}\right] - \frac{3}{10}Z^{-1}\left[\frac{1}{z+3}\right] - \frac{1}{2}Z^{-1}\left[\frac{1}{z+1}\right] + \frac{3}{2}Z^{-1}\left[\frac{1}{z+3}\right]$$

$$= \frac{2}{15}2^{n-1} + \frac{1}{6}(-1)^{n-1} - \frac{3}{10}(-3)^{n-1} - \frac{1}{2}(-1)^{n-1} + \frac{3}{2}(-3)^{n-1}$$

$$= \frac{1}{15}2^n + \left(\frac{1}{6} - \frac{1}{2}\right)(-1)^{n-1} - (-3)^n\left[\frac{1}{10} - \frac{1}{2}\right]$$

$$= \frac{2^n}{15} + \left(\frac{1-3}{6}\right)(-1)^{n-1} - (-3)^n\left(\frac{1-5}{10}\right)$$

$$= \frac{2^n}{15} + \frac{1}{3}(-1)^n + \frac{2}{5}(-3)^n, n \geq 0$$