UNIT-I GROUPS AND RINGS

Groups: Definition and Properties-Homomorphism-Isomorphism-Cyclic groups-Cosets Lagrange's theorem.
Rings: Definition and examples-sub rings-Integral domain-Field-Integer modulo n-Ring homomorphism.

UNIT I
GROUPS AND RINGS
PART-A

1. State any two properties of a group.

Closure property: a*b $\in$ G, for all a,b $\in$ G

Associative property: (a*b)*c=a*(b*c), for all a,b,c $\in$ G

2. Define Homomorphism of groups.

Let (G,*) and (G,o) be two groups and f be a function from G into G1. Then f is called a ***homomorphism*** of G into G1 if for all a,b ∈ G,

f(a*b) = f(a) o f(b).

3. Give an example of Homomorphism of groups.

Consider the group (Z,+). Define f:Z →Z by f(n) = 3n for all n∈ Z

Here the function f is from the group (Z,+) to (Z,+)

Let n,m ∈ Z then we get n+m ∈ Z and we have f(n+m) = 3(n+m) = 3n + 3m = f(n) + f(m)

Hence the function f is a homomorphism.

4. Define Isomorphism.

Let (G,*) and (G',o) be two groups and f : G → G' be a homomorphism of groups then f is called a isomorphism if f is a bijective(one-to-one and onto) function.

5. Give any two Example of Isomorphism.

Example:1

Consider the function f: Z → Z by f(x) = x, Now we have to show that f is a homomorphism.

Take any two elements x, y belongs to Z ,Then x + y belongs to Z, Hence f(x+y) = x + y = f(x) + f(y)

Hence f is homomorphism.

Since the function f(x) = x is bijective. f is an isomorphism.

Example :2

Consider the function f: Z → Z by f(x) = x. Take any two elements x,y belongs to Z ,Then x + y belongs to Z, Hence f(x+y) = x + y = f(x) + f(y) Hence f is homomorphism.

Since the function f(x) = x is bijective. f is an isomorphism.

6. Show that ($Z_{5,}, +_5$) is a cyclic group.

$+_5$	[0]	[1]	[2]	[3]	[4]
[0]	0	1	2	3	4
[1]	1	2	3	4	0
[2]	2	3	4	0	1
[3]	3	4	0	1	2
[4]	4	0	1	2	3

$1^1 = 1$

$1^2 = 1 +_5 1 = 2$

$1^3 = 1 +_5 1^2 = 1 +_5 2 = 3$

$1^4 = 1 +_5 1^3 = 1 +_5 3 = 4$

$1^5 = 1 +_5 1^4 = 1 +_5 4 = 0$

Hence ($Z_{5,}, +_5$) is a cyclic group and 1 is a generator.

7. Prove that the group $H = (Z_4, +)$ is cyclic.

Here the operation is addition, so we have multiplies instead of powers. We find that both [1] and [3] generate *H*. For the case of [3], we have

1.[3]=[3], 2.[3]=[2], 3.[3]=[1], and 4.[3]=[0].

Hence *H*=<[3]>=<[1]>.Hence $H = (Z_4, +)$ is cyclic

8. Prove that $U_9 = \{1,2,4,5,7,8\}$ is cyclic group.

Here we find that $2^1=2$, $2^2=4$, $2^3=8$, $2^4=7$, $2^5=5$, $2^6=1$,

So U_9 is a cyclic group of order 6 and $U_9 = \langle 2 \rangle$ and also true that $U_9 = \langle 5 \rangle$

because $5^1=5$, $5^2=7$, $5^3=8$, $5^4=4$, $5^5=2$, $5^6=1$.

9. Define Left coset and Right coset of the group.

If H is a subgroup of G,then for each $a \in G$,the set $aH = \{ah / h \in H\}$is called a l eft coset of H in G and $Ha = \{ha / h \in H\}$is a right coset of H in G.

10. Consider the group Z_4= {[0],[1],[2],[3]} of integers modulo 4. Let H={[0],[2]} be a subgroup of Z_4 under $+_4$. Find the left cosets of H.

[0] + [H] = {[0],[2]} = H

[1] + [H] = {[1],[3]}

[2] + [H] = {[2] ,[4]} = {[2],[0]} = {[0],[2]} = H

[3] + [H] = {[3] , [5]} = {[3],[1]} = {[1],[3]} = [1] + H

∴ [0] + H = [2] + H = H and [1] + H = [3] + H are the two distinct left cosets of H in Z_4

11. State Lagrange's theorem for finite groups. Is the converse true?

If G is a finite group and H is a sub group of G, then the order of H is a divisor of order of G. The converse of Lagrange's theorem is false.

12. Define ring and give an example of a ring with zero-divisors.

An algebraic system (R,+,.) is called a ring if the binary operation + and . satisfies the following conditions.

(i) (a+b)+c=a+(b+c) a,b,c $\in$ R

(ii) There exists an element 0 ϵ R called zero element such that a+0 = 0+a =a for all a $\in$ R

(iii) For all aϵR,a+(-a) =(-a)+a = 0,-a is the negative of a.

(iv) a+b =b+a for all a,b ϵR

(v) (a.b).c =a.(b.c) for all a,b,c $\in$ R

The operation * is distributive over + i.e.,for any a,b,c ϵ R, a.(b+c) = a.b +a.c , (b+c).a = b.a +c.a In otherwords, if R is an abelian group under addition with the properties (iv) and (v) then R is a ring.

Example:The ring ($Z_{10}, +_{10}, X_{10}$) is not an integral domain.Since $5X_{10}2$,yet $5 \neq 0, 2 \neq 0$ in Z_{10}.

13. Define unit and multiplicative inverse of a Ring.

Let R be a ring with unity u. If a $\in$ R and there exists b $\in$ R such that ab=ba=u, then b is called a multiplicative inverse of a and a is called a unit of R.

14. Define integral domain and give an example.

Let R be a commutative ring with unity. Then R is called an integral domain if R has no proper divisors of zero.

Example: (Z,+,●) is an integral domain and Q,R,C are integral domain under addition and multiplication

15. Define Field and give an example.

A commutative ring (R,+,●) with identity is called a field if every non-zero element has a multiplicative inverse. Thus (R,+,●) is a field if

(i) (R,+) is abelian group and

(ii) (R-{0},●) is also abelian group.

Example: (R,+,●) is a field.

16. Give an example of a ring which is not a field.

(Z,+,●) is a ring but not a field, if every non-zero element need not a multiplicative inverse.

17. Define Integer modulo n.

Let $n \in Z^+$, n>1. For a,b $\in$ Z, we say that " a is congruent to b modulo n", and we write $a \equiv b (\bmod n)$, if $n | (a - b)$, or equivalently, $a = b + kn$ for some $k \in Z$.

18. Determine the values of the integer n>1 for the given congruence 401 $\equiv$ 323(mod n) is true.

401-323=78=2.3.13 there are five possible divisors (n>1),namely 2,3,6,26,39.

19. Determine the values of the integer n>1 for the given congruence 57 $\equiv$ 1(mod n) is true.

57-1=56=2^3.7. So there are six divisors, namely 2,4,8,14,28,56

20. Determine the values of the integer n>1 for the given congruence 68 $\equiv$ 37(mod n)is true.

68-37=31, prime, consequently n=31.

21. Determine the values of the integer n>1 for the given congruence 49 $\equiv$ 1(mod n)is true.

49-1=48=2^4.3. So there are nine possible values for n>1, namely 2,4,8,16,3,6,12,24,48.

UNIT-II-FINITE FIELDS AND POLYNOMIALS

Polynomial rings-Irreducible polynomial over finite fields-Factorization of polynomials over finite fields

1. Define polynomial.

Given a ring (R,+,.), an expression of the form $f(x)=a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x^1 + a_0 x^0,\ where\ \ a_i \in R \quad for\ \ 0 \le i \le n,$ is called a polynomial in the indeterminate x with coefficients from R.

2. Define Field.

A field is a nonempty set F of elements with two operations '+' (called addition) and '·' (called multiplication) satisfying the following axioms. For all a, b, c ∈ F:

(i) F is closed under + and · ; i.e., a + b and a · b are in F.
(ii) Commutative laws: a + b = b + a, a · b = b · a.
(iii) Associative laws: (a + b) + c = a + (b + c), a · (b · c) = (a · b) · c.
(iv) Distributive law: a · (b + c) = a · b + a · c.

3. What is meant by a finite field?

A field containing only finitely many elements is called a finite field, A finite field is simply a field Whose underlying set is finite. Eg: F_2, whose element 0 and 1.

4. What is meant by polynomial ring?

If R is a ring , then under the operations of addition and multiplication + and .,(R[x],+,.) is a ring ,called the polynomial ring, or ring of polynomials over R.

5. Define root of the polynomial.

Let R be a ring with unity u and let f(x) $\in R(x)$,with degree f(x)$\ge$1. If r and f(r)=z, then r is called a root of the polynomial f(x)

6. When do you you say that f(x) is a divisor of g(x)?

Let F be a field. For f(x), g(x) $\in F(x)$, where f(x) is not a zero a polynomial , we all f(x) a divisor of g(x) if there exists h(x) $\in F(x)$ with f(x)h(x)=g(x). In this situation we also say that f(x) divides g(x) and that g(x) is a multiple of f(x)

7. Find the roots of f(x)=x2-2Q x .

f(x)= x^2-2=$\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)$

Since $\sqrt{2}$ and -$\sqrt{2}$ are irrational numbers , f(x) has no roots.

8.Find all roots of f(x)=x2+4x if f(x) z x

Z_{12}={0,1,2,3,4,5,6,7,8,9,10,11}
f(0)=0+0=0 $\therefore$ 0 is a root of f)x)
f(1)1+4=5
f(2)=4+8=12=0
So 2 is a root.
f(3)=21, f(4)32
f(5)=45,f(6)=60=0
So 6 is a root
f(7)=77, f(8)=96=0
So 8 is aroot.
f(9)=81+36=117. f(10)100+40=140
f(11)=121+44=165
Thus x=0,2,6,8 are the roots of f(x)

9. State division algorithm

Let f(x),g(x) $\in F(x)$ with f(x) not the zero polynomial. There exists unique polynomials q(x), r(x) $\in F(x)$such that g(x)=q(x)f(x)+r(x),where r(x)=0 or degree r(x)< degree f(x).

10. State the remainder theorem.

The remainder theorem:
For f(x) $\in F(x)$and a$\in F$, the remainder in the division of f(x) by x-a is f(a).

11. Determine all polynomials of degree 2 in z [x].

The polynomials are

(i) x^2
(ii) x^2+x
(iii) x^2+1
(iv) x^2+x+1

12. State the factor theorem.

If f(x) and a f(x) $\in F$, then x-a ia a factor of f(x) if and only if a is a root of f(x).

13. Determine polynomial h(x) of degree 5 and polynomial k(x)of degree 2 such that degree of h(x)k(x) is 3.

Choose h(x)=$4x^5+x$ of degree 5 and k(x)$3x^2$ of degree 2. Then h(x)k(x)= $(4x^5+x)(3x^2)=12x^7+3x^3=0+3x^3$ which is of degree 3.

14. Define reducible and irreducible polynomials .

Let f(x) , with F a field and degree f(x) ≥ 2. We call f(x) reducible over F if there exists g(x),h(x) ,where f(x)=g(x)h(x) and each of g(x),h(x) has degree ≥ 1. If f(x) is not reducible it is called irreducible or prime.

15. Give example for reducible and irreducible polynomials .

The polynomial f(x) $=x^4+2x^2+1$ is reducible . Since $x^4+2x^2+1=(x^2+1)^2$

The polynomial x^2+1 is irreducible in Q[x] and R[x]but in C[x] it is reducible.

16. Verify the polynomial x2+x+1 over Z ,Z irreducible or not.

The polynomial $x^2+x+1=(x+2)(x+2)$ is irreducible over Z_3

The polynomial $x^2+x+1=(x+5)(x+3)$ is irreducible overZ_7.

17. What is meant by monic polynomial?

A polynomial f(x) is called monic if its leading coefficients is 1, the unity of F.

Example: x^2+2x+1

18. When do you say that 2 polynomials are relatively prime?

If f(x),g(x) and their gcd is 1, then f(x) and g(x)are calle d relatively prime.

19. What is the characteristic of R?

Let (R,+,.) be a ring. If there is least positive integer n such that nr=z(the zero of R) for all r $\in$ R, the we say that R has characteristic n and write characteristic n. When no such integer exists , R is said to be characteristic 0.

20. Find the characteristic of the following rings a) (Z ,+,.) b)(Z ,+,.) and Z [x]

The ring (Z_3,+,.) has characteristic 3.

The ring (Z_4,+,.) has characteristic 4

Z_3[x] has characteristic 3.

21. Give an example of a polynomial f(x) R x where f(x) has degree 8, is reducible but has no real roots.

Choose f(x)=$(x^2+9)^4$ is of degree 8, is reducible but has no real roots.

22. Write f(x)= 2x 1 5x 5x 3 4x 3 z x as the product of unit and three monic polynomials.

$$f(x)=\left(2x^2+1\right)\left(5x^3-5x+3\right)\left(4x-3\right)$$
$$=2\left(x^2+4\right)5\left(x^3-x+2\right)4\left(x-6\right)$$
$$=40\left(x^2+4\right)\left(x^3-x+2\right)4\left(x-6\right)$$
$$=5\left(x^2+4\right)\left(x^3-x+2\right)4\left(x-6\right)$$

Here each polynomial is monic.

23. If f(x) and g(x) are relatively prime and F x where F is any field , show that there is no element a $\in F$ such that f(a)=0 and g(a)=0

Suppose there exists a $\in F$ such that f(a)=0 and g(a)=0. Then (x-a) would be a factor of both f(x) and g(x). So (x-a) would divide the gcd of both f(x) and g(x).But this is a contradiction since f(x) and g(x) are relatively prime.

UNIT-III DIVISIBILITY THEORY AND CANONICAL DECOMPOSITION

Division algorthim-Base b-representations-Number patterns-Prime and Composite Numbers-GCD-Euclidean algorithm-Fundamental theorem of arithmetic-LCM

UNIT-III
DIVISIBILITY THEORY AND CANONICAL DECOMPOSITIONS
PART-A

1. **Write about divisible.**
   An integer b is divisible by an integer a, not zero, if there is an integer x such that b = ax, and we write a/b. m In case b is not divisible by a, we write a\b.
2. **Define division algorithm**.
   Given any integers a and b, with a > 0, there exist unique integers q and r such that b =qa + r, 0 < r < a. If a\b, then r satisfies the stronger inequalities z < r < a.
3. **Define greatest common divisor of b.**

The integer a is a common divisor of b and c in case a/b and a/c. Since there is only a finite number of divisors of any nonzero integer, there is only a finite number of common divisors of b and c, except in the case b=c=0, If at least one of b and c is not 0, the greatest among their common divisors is called the greatest common divisor of b and c and is denoted by (b, c).

4. **Define Euclidean algorithm.**

Given integers b and c > 0, we make a repeated application of the division algorithm, to obtain a series of equations

$b = cq_1 + r_1, \quad 0<r_1<c$
$c = r_1q_2 + r_2, \quad 0<r_2<r_1$
$r_1= r_1q_3 + r_3, \quad 0<r_2<r_1$
……….. …..
$r_{j-2} = r_{j-1}q_j + r_j, \quad 0<r_2<r_1$
$r_{j-1} = r_jq_{j+1}$

The greatest common divisor (b, c) of b and c is r_j, the last nonzero remainder in the division process. Values of x_0 and y_0 In (b, c) = bx_0+cy_0 can be obtained by writing each r_i as a linear combination of b and c.

5. **Solve by Euclidean algorithm for b=288 and c=158.**

288=158.2-28
158=28.6-10
28=10.3-2
10=2.5

6. **Define least common multiple.**

The integers $a_1,a_2,….a_n$. all different from zero, have a common multiple b if a_i/b for i=1,2,….n. The least of the positive common multiples is called the least common multiple [le, and it is denoted by $[a_1,a_2,….a_n]$.

7. **Define prime number.**

An integer p>1 is called a prime number, or a prime , in case there is no divisor d of satisfying 1<d<p.

8. **Define Composite number with example**.

If an integer a>1 is not a prime, it is called a composite number. Eg: 4,6,8,9….

9. **State the binomial theorem.**

For any integer n≥1 and any real numbers x and y $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

10. **Define arithmetical function with example**.

A function f(n) defined for all natural numbers n is called an arithmetical function. Eg:x^2+x-3

11. **Prove that if n is an even number, then 3^n+1 is divisible by 2; if n is an odd number, k then 3^n+1 is divisible by 2^2; if n is any number, whether even or odd, then 3^n+1 is not divisible by 2^m with m≥3.**

Since the square of an odd number minus 1 is a multiple of 8, when n=2m we have $3n=3^{2m}=(3^m)^2=8a+1$, and therefore$3^n+1=2(4a+1)$. When n=2m+1, we have $3^n+1=3^{2m}+1=3(8a+1)+1=4(6a+1)$. Since 4a+1 and 6a+1 are odd , the statement is true.

12. **Show that if $1<a_1<a_2….<a_{n-1}<a_n$, then there exist i and j with i<j, such that a_i/a_j.**

Let $a_i=2^{n_i}b_i,n_i≥0$), b_i is odd. Since among 1,2,….,2n, there are only n distinct odd numbers $b_1,….,b_{n+1}$ are not all distinct, in other words, among them there are some equal odd numbers, Let $b_i=b_j$. Then ai/aj.

13. **Define square number with example.**

If an integer a is a square of some other integer, then a is called a square number.Eg:4,9,16…

14. **Find the greatest common divisor of 525 and 231.**

From 525=2.231+63
231=3.63+42
63=1.42+21
42=2.21
Therefore g.c.d.(525.231)=21

UNIT-IV-DIOPHANTINE EQUATIONS AND CONGRUENCES

Linear Diaphantine equations-Congruence's-Linear congruence's-Congruence's applications-Divisibility tests-Modular exponentiation -Chinese remainder theorem-2x2 linear system.

UNIT IV
DIOPHANTINE QUATIONS AND CONGRUENCES
PART A

1. **Define linear Diophantine equation.**

   Any linear equation in two variables having integral coefficients can be put in the form $ax + by = c$ where a, b, c are given integers.

2. **State about the solution of linear Diophantine equation.**

   Consider the equation $ax + by = c$ ----(1), in which x and y are integers. If $a=b=c=0$, then every pair (x, y) of integers is a solution of (1), whereas if $a = b = 0$ and $c \neq 0$, then (1) has no solution. Now suppose that at least one of a and b is nonzero, and let g = gcd (a, b). If g/c then (1)has no solution.

3. **Write the solution of ax + by =c.**

If the pair (x_1, y_1) is one integral solution, then all others are of the form x = x_1 + kb/g , y = y_1=ka/g where k is an integer and g=gcd (a, b)

4. **Define unimodular with example.**

A square matrix U with integral elements is called unimodular if det(U)=±1.Eg: Identity matrix

5. **Define Pythagorean triangle.**

We wish to solve the equation $x^2+y^2=z^2$ in positive integers. The two most familiar solutions are 3,4,5 and 5,12,13. We refer to such a triple of positive integers as a Pythagorean triple or a Pythagorean triangle, since in geometric terms x and y are the legs of a right triangle with hypotenuse z.

6. **Write the legs of the Pythagorean triangles.**

The legs of the Pythagorean triangles.

$X=r^2-s^2$

$Y=2rs$

$Z=r^2+s^2$

7. **Define congruent and not congruent.**

If an d integer m, not zero, divides the difference a-b, we say that a is congruent to b modulo m and write $a\equiv b \pmod m$. If a-b is not divisible by m, we say that a is not congruent to b modulo m, and in this case we write $a \neq b \pmod m$.

8. **Define residue.**

If $x\equiv y \pmod m$ then y is called a residue of x modulo m.

9. **Define complete residue**

A set $x_1, x_2, \ldots, x_m$ is called a complete residue system modulo m if for every integer y there is one and only one x_j such that $y\equiv x_j \pmod m$.

10. **State Chinese Remainder Theorem.**

Let $m_1, m_2, \ldots, m_r$ denote r positive integers that are relatively prime in pairs, and let $a_1, a_2, \ldots, a_r$ denote any r integers. Then the congruences

$$x\equiv a_1 \pmod{m_1}$$
$$x\equiv a_2 \pmod{m_2}$$
$$\ldots\ldots$$
$$\ldots\ldots$$
$$x\equiv a_r \pmod{m_r}$$

have common solutions. If x_0 is one such solution, then an integer x satisfies the congruences the above equations iff x is of the form $x=x_0+km$ for some integer k. Here $m=m_1m_2\ldots m_r$.

11. **Define n-th power residue modulo p.**

If (a, p)=1 and $x_n \equiv a \pmod p$ has a solution, then a is called an n-th power residue modulo p.

12. **Define Euler's criterion.**

If p is an odd prime and (q, p)=1, then $x^2\equiv a \pmod p$ has two solutions or no solution according as $a^{(p-1)/2}\equiv$ or $\equiv -1 \pmod p$.

UNIT-V-CLASSICAL THEOREMS AND MULTIPLICATIVE FUNCTIONS

Wilson's theorem-Fermat's little theorem-Euler's theorem-Euler's phi-functions-Tau and Sigma functions

UNIT V
CLASSICAL THEOREMS AND MULTIPLICATIVE FUNCTIONS
PART A

1. **State Wilson's theorem**
   The Wilson's theorem states that, if p is a prime, then $(p-1)! \equiv -1 (\text{mod } p)$
2. **State Fermat's theorem.**
   Let p denote a prime. If p/a then $a^{p-1} \equiv 1(\text{mod } p)$. For every integer a, $a^p \equiv a(\text{mod } p)$.
3. **State Euler's generalization of Fermat's theorem.**
   If (a, m)=1, then $a^{\phi(m)} \equiv 1(\text{mod } m)$.
4. **State Fermat's little theorem**
   If p is a prime and $a \not\equiv 0(\text{mod } p)$, then $a^{p-1} \equiv 1 (\text{mod } p)$
5. **Explain the Exponent of an integer modulo n.**
   Let n be a natural number >1 and a an integer prime to n. if the infinite sequence $a, a^2, a^3, \ldots \equiv 1 (\text{mod } n)$. Suppose that a^δ is the first number in the sequence $\equiv 1 (\text{mod } n)$. then a is said to belong to the Exponent of an integer modulo n
6. **Define improper divisor of n**
   Every integer n is a divisor of itself. It is called the improper divisor of n . All other divisors of n are called proper divisors .
7. **Define Eulers Phi function**
   $\phi(n)$ is the number of non-negative integers less than n that are relatively prime to n. In other words, if $n>1$ then $\phi(n)$ is the number of elements in Un, and $\phi(1)=1$.
8. **If p is a prime, the only elements of Up which are their own inverses are [1] and $[p-1]=[-1]$.**
   Note that $[n]$ is its own inverse if and only if $[n2]=[n]2=[1]$ if and only if $n2\equiv1(\text{mod} p)$ if and only if $p|(n2-1)=(n-1)(n+1)$. This is true if and only if $p|(n-1)$ or $p|(n+1)$. In the first case, $n\equiv1(\text{mod} p)$, i.e., $[n]=[1]$. In the second case, $n\equiv-1\equiv p-1(\text{mod} p)$, i.e., $[n]=[p-1]$.
9. **Find the remainder of 97! When divided by 101.**

First we will apply Wilson's theorem to note that $100! \equiv -1 \pmod{101}$. When we decompose the factorial, we get that: $(100)(99)(98)(97!) \equiv -1 \pmod{101}$. Now we note that $100 \equiv -1 \pmod{101}$, $99 \equiv -2 \pmod{101}$, and $98 \equiv -3 \pmod{101}$.

Hence: $(-1)(-2)(-3)(97!) \equiv -1 \pmod{101} (-6)(97!) \equiv -1 \pmod{101} (6)(97!) \equiv 1 \pmod{101}$. Now we want to find a modular inverse of 6 (mod 101). Using the division algorithm, we get that: $101=6(16)+56=5(1)+11=6+5(-1)1=6+[101+6(-16)](-1)1=101(-1)+6(17)$

Hence, 17 can be used as an inverse for 6 (mod 101). It thus follows that: $(17)(6)(97!) \equiv (17)1 \pmod{101} 97! \equiv 17 \pmod{101}$ Hence, 97! has a remainder of 17 when divided by 101.

10. For prime p≥5, determine the remainder when (p−4)! is divided by p.

By *Wilson*'s theorem, $(p-1)! \equiv -1 \pmod{p}$. Therefore

$-1 \equiv (p-1)(p-2)(p-3) \cdot (p-4)! \equiv -6 \cdot (p-4)! \pmod{p}$.

If $p=6k+1$, multiplying both sides of the congruence by k gives $(p-4)! \equiv -k = -(p-1)/6 \pmod{p}$.
If $p=6k-1$, multiplying both sides of the congruence by k gives $(p-4)! \equiv k = (p+1)/6 \pmod{p}$.

11. Find the remainder of 53! when divided by 61.

We know that by Wilson's theorem $60! \equiv -1 \pmod{61}$. Decomposing 60!, we get that: $(60)(59)(58)(57)(56)(55)(54)(53)(52)51! \equiv -1 \pmod{61} (-1)(-2)(-3)(-4)(-5)(-6)(-7)(-8)(-9) 51! \equiv -1 \pmod{61} (-362880)51! \equiv -1 \pmod{61} (362880)51! \equiv 1 \pmod{61} (52)51! \equiv 1 \pmod{61}$ We will now use the division algorithm to find a modular inverse of 52 (mod 61): $61=52(1)+952=9(5)+79=7(1)+27=2(3)+11=7+2(-3)1=7+[9+7(-1)](-3)1=9(-3)+7(4)1=9(-3)+[52+9(-5)](4)1=52(4)+9(-23)1=52(4)+[61+52(-1)](-23)1=61(-23)+52(27)$ Hence 27 can be used as an inverse (mod 61). We thus get that: $(27)(52)51! \equiv (27)1 \pmod{61} 51! \equiv 27 \pmod{61}$
Hence the remainder of 51! when divided by 61 is 2.

12. What is the remainder of 149! when divided by 139?

From Wilson's theorem we know that $138! \equiv -1 \pmod{139}$. We are now going to multiply both sides of the congruence until we get up to 149!:
$149! \equiv (149)(148)(147)(146)(145)(144)(143)(142)(141)(140)(139)(-1) \pmod{139} 149! \equiv (10)(9)(8)(7)(6)(5)(4)(3)(2)(1)(0)(-1) \pmod{139} 149! \equiv 0 \pmod{139}$. Hence the remainder of 149! when divided by 139 is 0.

13. Define congruence in one variable

A congruence of the form $ax \equiv b \pmod{m}$ where x is an unknown integer is called a linear congruence in one variable.

14. Let p be a prime. A positive integer m is its own inverse modulo p iff p divides m + 1 or p divides m − 1.

Suppose that m is its own inverse. Thus $m.m \equiv 1 \pmod{p}$. Hence $p \mid m^2 - 1$.then $p \mid (m-1)$ or $p \mid (m+1)$.