PREPARED BY

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MSAJCE

EC3354 SIGNALSANDSYSTEMS

OBJECTIVES:
- To understand the basic properties of signal &systems
- To know the methods of characterization of LTI systems in timedomain
- To analyze continuous time signals and system in the Fourier and Laplacedomain
- To analyze discrete time signals and system in the Fourier and Z transformdomain

UNITI CLASSIFICATION OF SIGNALSANDSYSTEMS 12
Standard signals- Step, Ramp, Pulse, Impulse, Real and complex exponentials and Sinusoids_ Classification of signals – Continuous time (CT) and Discrete Time (DT) signals, Periodic & Aperiodic signals, Deterministic & Random signals, Energy & Power signals - Classification of systems- CT systems and DT systems- – Linear & Nonlinear, Time-variant & Time-invariant, Causal & Non-causal, Stable & Unstable.

UNITII ANALYSIS OF CONTINUOUSTIMESIGNALS 12
Fourier series for periodic signals - Fourier Transform – properties- Laplace Transforms and properties

UNITIII LINEAR TIME INVARIANT CONTINUOUSTIMESYSTEMS 12
Impulse response - convolution integrals- Differential Equation- Fourier and Laplace transforms in Analysis of CT systems - Systems connected in series / parallel.

UNITIV ANALYSIS OF DISCRETETIMESIGNALS 12
Baseband signal Sampling – Fourier Transform of discrete time signals (DTFT) – Properties of DTFT- Z Transform &Properties

UNITV LINEAR TIME INVARIANT-DISCRETETIMESYSTEMS 12
Impulse response – Difference equations-Convolution sum- Discrete Fourier Transform and Z Transform Analysis of Recursive & Non-Recursive systems-DT systems connected in series and parallel.

TOTAL: 60 PERIODS

OUTCOMES:
At the end of the course, the student should be able to:
- To be able to determine if a given system islinear/causal/stable
- Capable of determining the frequency components present in a deterministicsignal
- Capable of characterizing LTI systems in the time domain and frequencydomain
- To be able to compute the output of an LTI system in the time and frequencydomains

TEXT BOOK:
1. Allan V.Oppenheim, S.Wilsky and S.H.Nawab, "Signals and Systems", Pearson, 2015.(Unit 1- V)

REFERENCES
1. B. P. Lathi, "Principles of Linear Systems and Signals", Second Edition, Oxford,2009.
2. R.E.Zeimer, W.H.Tranter and R.D.Fannin, "Signals & Systems - Continuous and Discrete", Pearson,2007.
3. John Alan Stuller, "An Introduction to Signals and Systems", Thomson,2007.

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UNIT – I
CLASSIFICATION OF SIGNALS AND SYSTEMS

1.1 INTRODUCTION:

A signal, as stated before is a function of one or more independent variables. A signal is a quantitative description of a physical phenomenon, event or process. More precisely, a signal is a function, usually of one variable in time. However, in general, signals can be functions of more than one variable, e.g., image signals. Signals are functions of one or more variables.

Systems respond to an input signal by producing an output signal .
Examples of signals include:

1. A voltage signal: voltage across two points varying as a function oftime.
2. A force pattern: force varying as a function of 2-dimensionalspace.
3. A photograph: color and intensity as a function of 2-dimensionalspace.
4. A video signal: color and intensity as a function of 2-dimensional space andtime.

A continuous-time signal is a quantity of interest that depends on an independent variable, where we usually think of the independent variable as time. Two examples are the voltage at a particular node in an electrical circuit and the room temperature at a particular spot, both as functions oftime.

A discrete-time signal is a sequence of values of interest, where the integer index can be thought of as a time index, and the values in the sequence represent some physical quantity of interest.

A signal was defined as a mapping from a set of the independent variable (domain) to the set of the dependent variable (co-domain). A system is also a mapping, but across signals, or across mappings. That is, the domain set and the co-domain set for a system are both sets of signals, and corresponding to each signal in the domain set, there exists a unique signal in the co-domain set.

System description

The system description specifies the transformation of the input signal to the output signal. In certain cases, a system has a closed form description. E.g. the continuous-time system with description y (t) = x(t) + x(t-1); where x(t) is the input signal and y(t) is the output signal.

1.2 Continuous-time and discrete-timesystems

- Physically, a system is an interconnection of components, devices, etc., such as a computer or an aircraft or a powerplant.
- Conceptually, a system can be viewed as a black box which takes in an input signal x(t) (or x[n]) and as a result generates an output signal y(t) (or(y[n]).
- A system is continuous-time (discrete-time) when its I/O signals are continuous-time (discrete-time).

1.3 ElementarySignals:

The elementary signals are used for analysis of systems. Such signals are,

- Step
- Impulse
- Ramp
- Exponential
- Sinusoidal

1.3.1 Unit stepsignal:

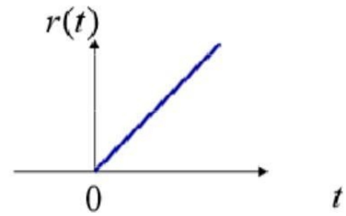
- Unit Step Sequence: The unit step signal has amplitude of 1 for positive value and amplitude of 0 for negative value of independentvariable.
- It have two different parameter such as CT unit step signal u(t) and DT unit step signal u(n).
- The mathematical representation of CT unit step signal u(t) is givenby,

1.3.2 RampSignal:

- The amplitude of every sample is linearly increased with the positive value of independentvariable.
- Mathematical representation of CT unit ramp signal is givenby,

The Ramp function

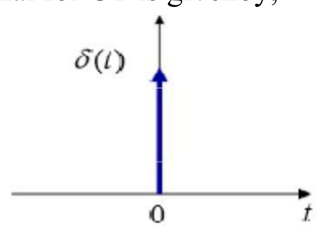
$$r(t) = t \cdot u(t)$$



1.3.3 Unit impulsefunction:

- Amplitude of unit impulse approaches 1 as the width approaches zero and it has zero value at all othervalues.
- The mathematical representation of unit impulse signal for CT is givenby,

$$\delta(0) = \infty$$
$$\delta(t) = 0, t \neq 0$$
$$\int_{-\infty}^{t} \delta(t)dt = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$



- It is used to determine the impulse response ofsystem.

1.3.4 Sinusoidalsignal:

- A continuous time sinusoidal signal is givenby,

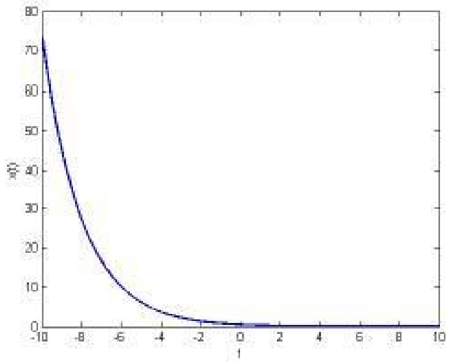
$$x(t) = A\cos(\Omega_0 t + \alpha)$$

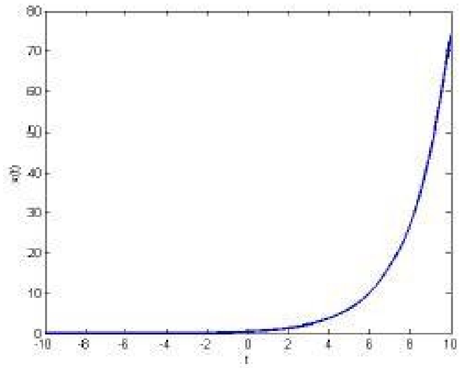
Where, A– amplitude - phase angle inradians

1.3.5 Exponentialsignal:

- It is exponentially growing or decayingsignal.
- Mathematical representation for CT exponential signalis,

$$x(t) = Ce^{at}, \quad where\ C, a \in \mathbb{C}$$





1.4 Classification of CT and DTsignals:

- **Periodic and non-periodicSignals**

A periodic function is one which has been repeating an exact pattern for an infinite period of time and will continue to repeat that exact pattern for aninfinite time. That is, a periodic function x(t) is one for which

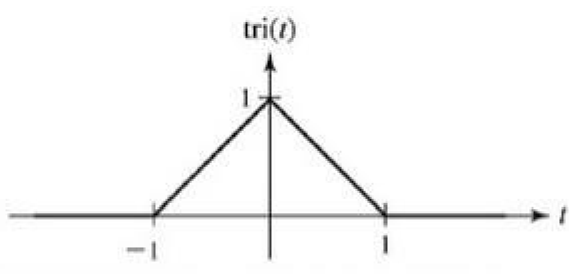
$$x(t) = x(t+nT)$$

for any integer value of n, where T >0 is the period of the function and $-\infty < t < \infty$. The signal repeats itself every T sec. Of course, it also repeats every 2 T,3T and nT. Therefore, 2T, 3T and nT are all periods of the function because the function repeats over any of those intervals. The minimum positive interval over which a function repeats itself is called the fundamental period T0.T0 is the smallest value that satisfies the condition x (t) = x (t+T0). The fundamental frequency f 0 of a periodic function is the reciprocal of the fundamental period f 0=1/T0. It is measured in Hertz and is the number of cycles (periods) per second. The fundamental angular frequency ω0 measured in radians per second is ω0=2πT0= 2πf0. A signal that does not satisfy the condition in (2.1) is said to be a periodic or non-periodic.

- **Deterministic and RandomSignals**

Deterministic Signals are signals who are completely defined for any instant of time, there is no uncertainty with respect to their value at any point of time. They can also be described mathematically, at least approximately. Let a function be defined as

$$\mathrm{tri}(t) = \begin{cases} 1-|t|, & -1 < t < 1 \\ 0, & otherwise \end{cases}$$



A random signal is one whose values cannot be predicted exactly and cannot be described by any exact mathematical function, they can be approximately described.

- **Energy and PowerSignals:**

Consider v(t) to be the voltage across a resistor R producing a current i(t). The instantaneous power p(t) per ohm is defined as, Total energy E and average power P on a per-ohm basis are

$$p(t) = \frac{v(t)i(t)}{R} = i^2(t)$$

For an arbitrary continuous-time signal x (t), the normalized energy content E of x(t) is defined as,

$$E = \int_{-\infty}^{\infty} i^2(t)\,dt \quad \text{joules}$$

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} i^2(t)\,dt \quad \text{watts}$$

The normalized average power P of x (t) is defined as,

$$E = \int_{-\infty}^{\infty} |x(t)|^2\,dt$$

Similarly, for a discrete-time signal x[n], the normalized energy content ***E*** of x[n] is defined as,

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

The normalized average power P of x[n] is defined as,

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

1.5 CT Systems and DTSystems:

A system is defined as a physical device which contains set of elements or functional blocks and that generates a response or output signal for a given input.

1.6 Classification ofsystem:

The systems are classified as,

- Static & dynamicsystem
- Time invariant and variantsystem
- Linear and non linearsystem
- Causal and non causalsystem
- Stable and unstablesystem

1.6.1 Static and dynamic system:

- Static system is said to be a memorylesssystem.
- The output does not depend the past or futureinput.
- It only depends the present input for anoutput.

Eg, y(n) = x(n)

- Dynamic system is said to be as system withmemory.
- Its output depend the past values of input for anoutput.

Eg. Y(n) = x(n) + x(n - 1)

- This static and dynamic systems are otherwise called as memoryless and system with memory.

1.6.2 Systems with and withoutmemory:

- A system is called memory less if the output at any time t (or n) depends only on the input at time t (or n); in other words, independent of the input at times before of after t (or n). Examples of memory lesssystems:

$$y(t) = Rx(t) \qquad \text{or} \qquad y[n] = \left(2x[n] - x^2[n]\right)^2$$

Examples of systems with memory:

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau \qquad \text{or} \qquad y[n] = x[n-1].$$

1.6.3 Time invariant and time variantsystem:

- If the time shifts in the input signals results in corresponding time shift in the output, then the system is called as time invariant.
- The input and output characteristics do not change withtime.
- For a continuous timesystem,

f[x(t1 – t2)] = y(t1 – t2)

- For a discrete time system,

F[x(n - k)] = y(n - k)

- If the above relation does not satisfy, then the system is said to be a time variantsystem.
- A system is called time-invariant if the way it responds to inputs does not change over time:

$$x(t) \to y(t) \qquad \Rightarrow \qquad x(t-t_0) \to y(t-t_0), \qquad \text{for any } t_0$$

$$x[n] \to y[n] \qquad \Rightarrow \qquad x[n-n_0] \to y[n-n_0], \qquad \text{for any } n_0.$$

Examples of time-invariant systems:

- The RC circuit considered earlier provided the values of R or C areconstant.

$$y[n] = x[n-1].$$

Examples of time-varying systems:

- The RC circuit considered earlier if the values of R or C change overtime.

$$y(t) = x(2t) \text{ since}$$

$$x(t) \to x(2t) \qquad \text{but} \qquad x(t-t_0) \to x(2t-t_0).$$

- Most physical systems are slowly time-varying due to aging, etc. Hence, they can be considered time-invariant for certain time periods in which its behavior does not changesignificantly.

1.6.4 Linear and non linearsystem:

- A system is said to be linear if it satisfies the superpositionprinciple.
- Superposition principle states that the response to a weighted sum of input signal be equal to the weighted sum of the output corresponding to each of the individualinput signal
- The continuous system is linearif,

$$F[a1x1(t) + a2x2(t)] = a1y1(t) + a2y2(t)$$

- The discrete system is linearif,

$$F[a1x1(n) + a2x2(n)] = a1y1(n) + a2y2(n)$$

- Otherwise the system is non linear.
- A system is called linear if its I/O behavior satisfies the additivity and homogeneity properties:

$$\left.\begin{array}{l} x_1(t) \rightarrow y_1(t) \\ x_2(t) \rightarrow y_2(t) \end{array}\right\} \quad \Rightarrow \quad \begin{array}{c} (x_1(t) + x_2(t)) \rightarrow (y_1(t) + y_2(t)) \\ (ax_1(t)) \rightarrow (ay_1(t)) \end{array}$$

for any complex constant a.

- Equivalently, a system is called linear if its I/O behavior satisfies the superposition property:

$$\left.\begin{array}{l} x_1(t) \rightarrow y_1(t) \\ x_2(t) \rightarrow y_2(t) \end{array}\right\} \quad \Rightarrow \quad (ax_1(t) + bx_2(t)) \rightarrow (ay_1(t) + by_2(t))$$

where any complex constants a and b.

1.6.5 Causal and non causalsystem:

- A causal system is one whose output depends upon the present and past inputvalues.
- If the system depends the future input values, the system is said to be non causal.

Eg. for causalsystem.

$$Y(t) = x(t) + x(t - 1)$$
$$Y(n) = x(n) + x(n - 3)$$

Eg. For non causal system,

$$Y(t) = x(t+3) + x2(t)$$
$$Y(n) = x(2n)$$

- A system is called causal or non-anticipative if the output at any time t (or n) depends only on the input at times t or before t (or n or before n); in other words, independent of the input at times after t (or n). All memory less systems are causal. Physical systems where the time is the independent variable arecausal.
- Non-causal systems may arise in applications where the independent variable is not the time such as in the image processingapplications.

Examples of causal systems:

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau \qquad \text{or} \qquad y[n] = x[n-1].$$

Examples of non-causal systems:

$$y(t) = x(-t) \qquad \text{or} \qquad y[n] = \frac{1}{3}\left(x[n-1] + x[n] + x[n+1]\right)$$

1.6.6 Stable and unstablesystem:

- When every bounded input produces bounded output then the system is called as stable system or bounded input bounded output (BIBOstable).
- Otherwise the system is unstable.
- A system is called stable if it produces bounded outputs for all boundedinputs.
- Stability in a physical system generally results from the presence of mechanisms that dissipate energy, such as the resistors in a circuit, friction in a mechanical system,etc.

Sample Problems:

<u>Problems:</u>

Determine whether the following system is linear or not.

$$\frac{dy}{dt} + 3ty(t) = t^2x(t)$$

For an input $x_1(t)$ and for the corresponding output $y_1(t)$ the differential equation can be written as,

$$\frac{dy_1(t)}{dt} + 3ty_1(t) = t^2x_1(t) \qquad \text{--------(1)}$$

For an input $x_2(t)$ and for the corresponding output $y_2(t)$ the differential equation can be written as,

$$\frac{dy_2(t)}{dt} + 3ty_2(t) = t^2x_2(t) \qquad \text{--------(2)}$$

(1) $\times a$ + (2) $\times b$ *and rearrranging,*

$$\frac{d[ay_1(t) + by_2(t)]}{dt} + 3t[ay_1(t) + by_2(t)] = t^2[ax_1(t) + bx_2(t)]$$

$$\frac{d}{dt}\underbrace{[ay_1(t) + by_2(t)]}_{\text{weighted sum of output}} + 3t\,\underbrace{[ay_1(t) + by_2(t)]}_{\text{weighted sum of output}} = t^2\underbrace{[ax_1(t) + bx_2(t)]}_{\text{weighted sum of input}}$$

From the above equation, we can note that the weighted sum of inputs to the system produces an output which is also equal to weighted sum of outputs corresponding to each of the individual inputs.
Therefore, the system is linear.

Determine whether the system $y(n) = 2x(n) + \frac{1}{x(n-1)}$ **is linear or not**

$y(n) = T[x(n)] = 2x(n) + \frac{1}{x(n-1)}$

For an input $x_1(n)$,

$y_1(n) = T[x_1(n)] = 2x_1(n) + \frac{1}{x_1(n-1)}$ ----------------(1)

For an input $x_2(n)$,

$y_2(n) = T[x_2(n)] = 2x_2(n) + \frac{1}{x_2(n-1)}$ ------------------(2)

Weighted sum of outputs is given by

$ay_1(n) + by_2(n) = 2ax_1(n) + \frac{a}{x_1(n-1)} + 2bx_2(n) + \frac{b}{x_2(n-1)}$ -------------(3)

Output due to weighted sum of inputs is

$y_3(n) = T[ax_1(n) + bx_2(n)] = 2[ax_1(n) + bx_2(n)] + \frac{1}{[ax_1(n-1) + bx_2(n-1)]}$ --------(4)

$y_3(n) \neq ay_1(n) + by_2(n)$

Therefore, the system is non-linear.

Determine whether the following systems are time-invariant or not.

(1) $y(t) = tx(t)$

$y(t) = T[x(t)] = tx(t)$

The output due to delayed input is

$y(t,T) = T[x(t-T)] = tx(t-T)$

If the output is delayed by T, we get

$y(t-T) = (t-T)x(t-T)$

$y(t,T) \neq y(t-T)$

Therefore, the system is time-variant

(2) $y(n) = x(2n)$

$y(n) = T[x(n)] = x(2n)$

If the input is delayed by k units of time then the output is ,

$y(n,k) = T[x(n-k)] = x(2n-k)$

Outputdelayedbykunitsoftimeis,

$y(n-k) = x[2(n-k)] = x[2n-2k]$

$y(n,k) \neq y(n-k)$

Therefore, the system is time –variant.

Problem

Find whether the signal $x(t) = 2\cos(10t+1) - \sin(4t-1)$ **is periodic or not.**

Time period of $2\cos(10t+1)$ is $T_1 = \frac{2\pi}{10} = \frac{\pi}{5}$ sec

Time period of $\sin(4t-1)$ is $T_2 = \frac{2\pi}{4} = \frac{\pi}{2}$ sec

The ratio of two periods is $\frac{T_1}{T_2} = \frac{\pi/5}{\pi/2} = \frac{2}{5}$

The ratio of two periods is a rational number.
Therefore, the sum of two signals are periodic and the period is given by

$$T = 2T_2 = 5T_1 = 2\times\frac{\pi}{2} = 5\times\frac{\pi}{5} = \pi \quad \text{sec}$$

---

ii) Find thesummation $\sum_{n=-\infty}^{\infty} e^{2n}\delta(n-2)$

$$\delta(n-2) = 1 \quad for \;\; n = 2$$
$$= 0 \quad for \;\; n \neq 2$$

$$\sum_{n=-\infty}^{\infty} e^{2n}\delta(n-2) = e^{2n}\delta(n-2)\Big|_{n=2} = e^4$$

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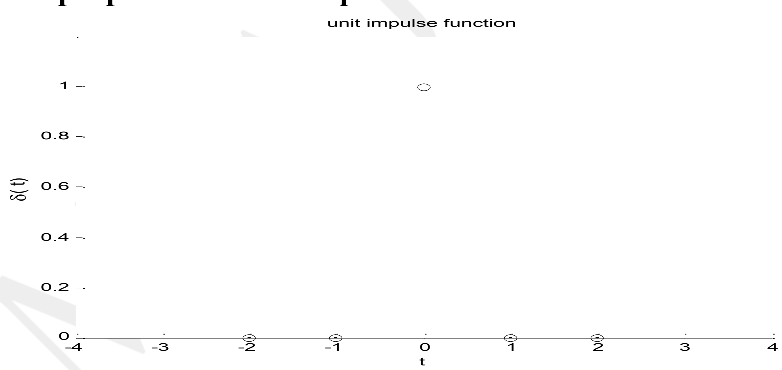
iii) Explain the properties of unit impulsefunction.



The impulse function has zero amplitude everywhere except at t = 0.

$$\delta(t) = 1 \qquad t = 0$$
$$= 0 \qquad t \neq 0$$

At t=0, the amplitude is infinity such that the area under the curve is equal to one.

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

---

iv) Find the fundamental period T of the continuous timesignal $x(t) = 20\cos\left(10\pi t + \frac{\pi}{6}\right)$

$$x(t) = 20\cos\left(10\pi t + \frac{\pi}{6}\right)$$

$$\Omega_0 = 10\pi$$

$$T = \frac{2\pi}{\Omega_0} = \frac{2\pi}{10\pi} = \frac{1}{5}\text{sec}$$

Derive the relationship between unit step and delta function.

$$\delta(t) \xrightarrow{\text{integrate}} u(t)$$

$$u(t) \xrightarrow{\text{differentiate}} \delta(t)$$

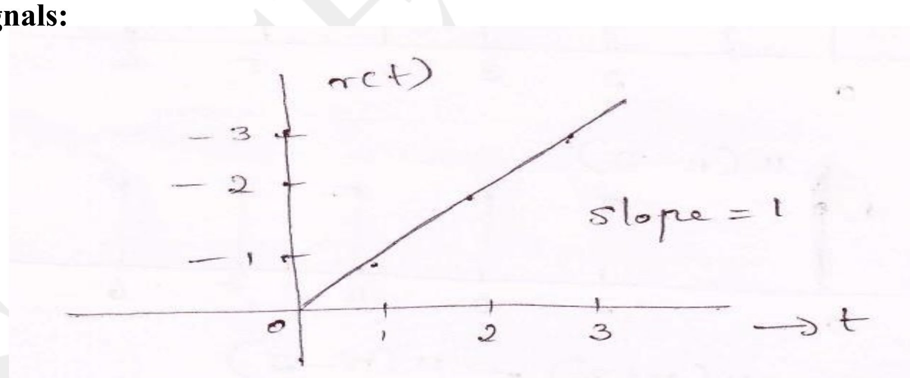
$$\delta(t) = \text{unit impulse function}$$

$$u(t) = \text{unit step function}$$

$$u(t) = 1 \qquad t \geq 0$$
$$= 0 \qquad t < 0$$

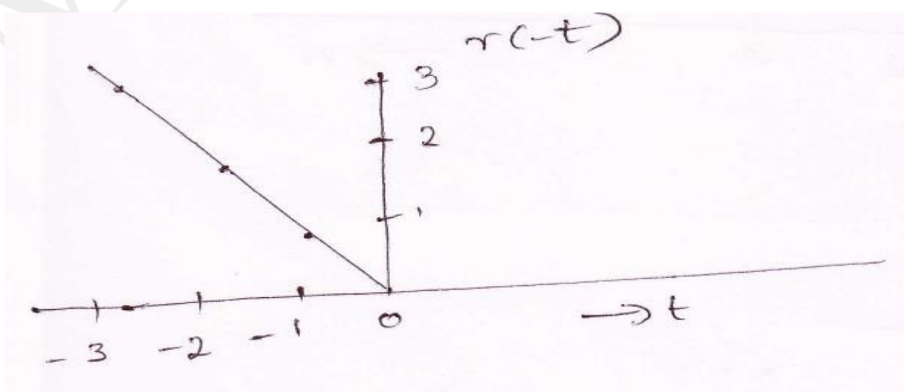
$$\frac{du(t)}{dt} = 1 \qquad t = 0$$
$$= 0 \qquad t \neq 0 \qquad \Rightarrow \delta(t)$$

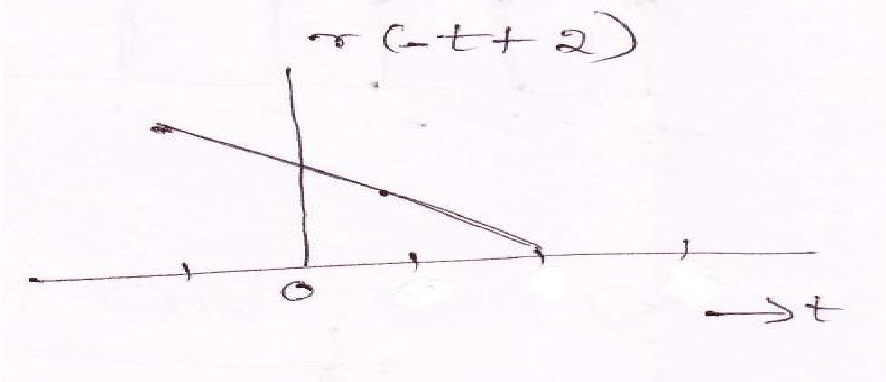
Sketch the following signals:



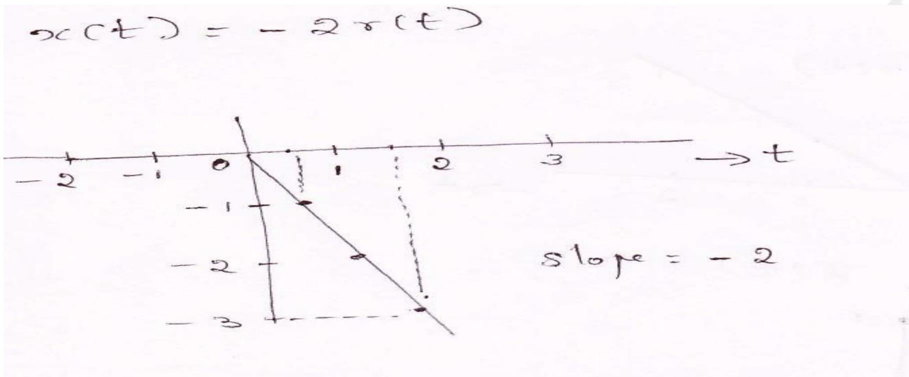
(1) $x(t) = r(t)$

(2) $x(t) = r(-t+2)$





(3) $x(t) = -2r(t)$ where r(t) is a ramp signal.



.Determine whether the following systems are time invariant or not.

- **i) Y(t) =tx(t)**
- **ii) Y(n) =x(2n)**

Solution:

i) Y(t) =tx(t)

Y(t) = T[x(t)] = tx(t)

The output due to delayed input is,

Y(t,T) = T[x(t - T)] = tx(t - t)

If the output is delayed by T, we get

Y(t -T) = (t - T) x(t - T)

The system does not satisfy the condition, y(t,T) = y(t – T).

Then the system is time invariant.

ii) Y(n) = x(2n)

Y(n) =x(2n)

Y(n) = T[x(n)] = x(2n)

If the input is delayed by K units of time then the output is,

Y(n,k) = T[x(n-k)] = x(2n-k)

The output delayed by k units of time is,

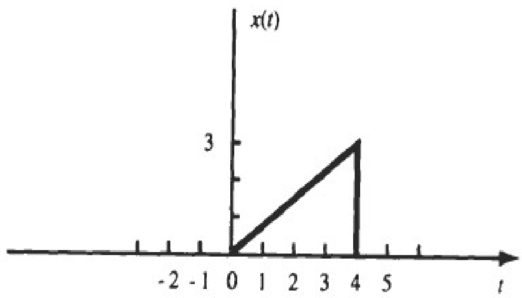
Y(n-k) = x[2(n-k)]

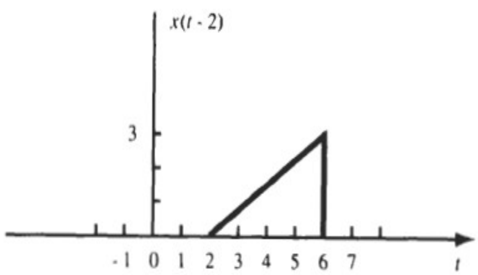
Therefore, y(n,k) is not equal to y(n-k). Then the system is time variant.

1.

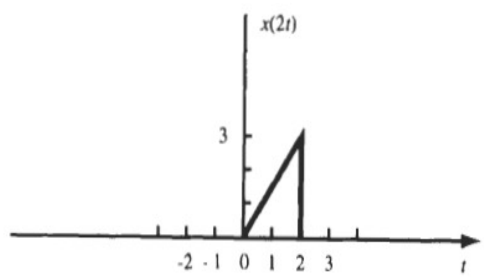
A continuous-time signal $x(t)$ is shown in Fig. Sketch and label each of the following signals.

(*a*) $x(t-2)$; (*b*) $x(2t)$; (*c*) $x(t/2)$; (*d*) $x(-t)$

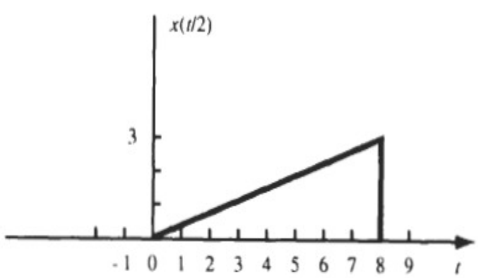




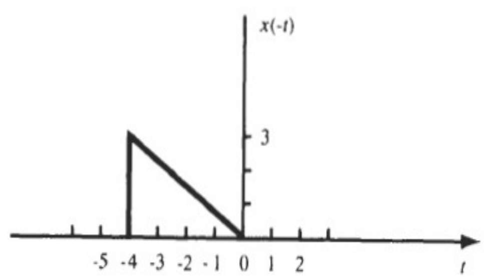
(*a*)



(*b*)



(*c*)

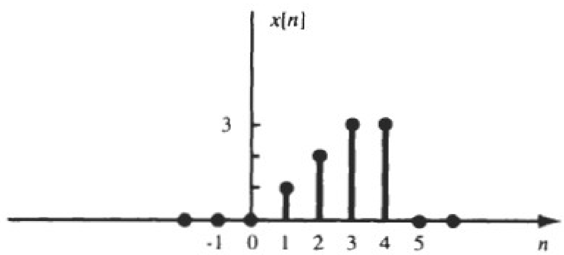


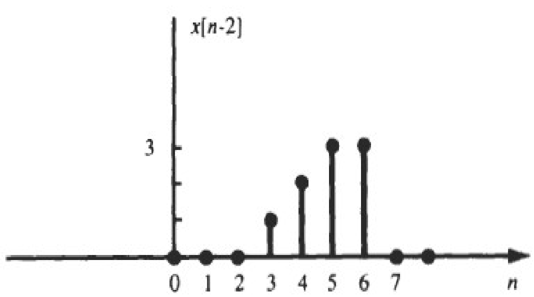
(*d*)

2.

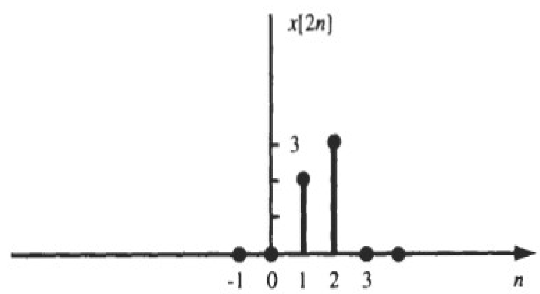
A discrete-time signal $x[n]$ is shown in Fig. Sketch and label each of the following signals.

(a) $x[n-2]$; (b) $x[2n]$; (c) $x[-n]$; (d) $x[-n+2]$

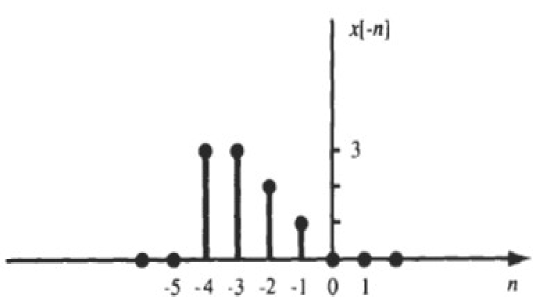




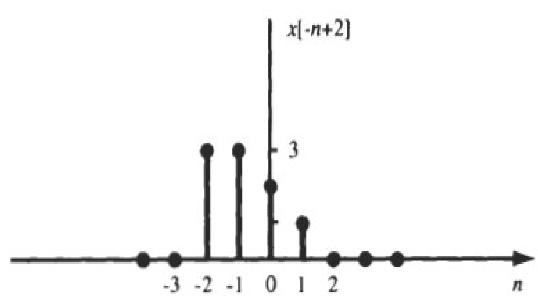
(a)



(b)



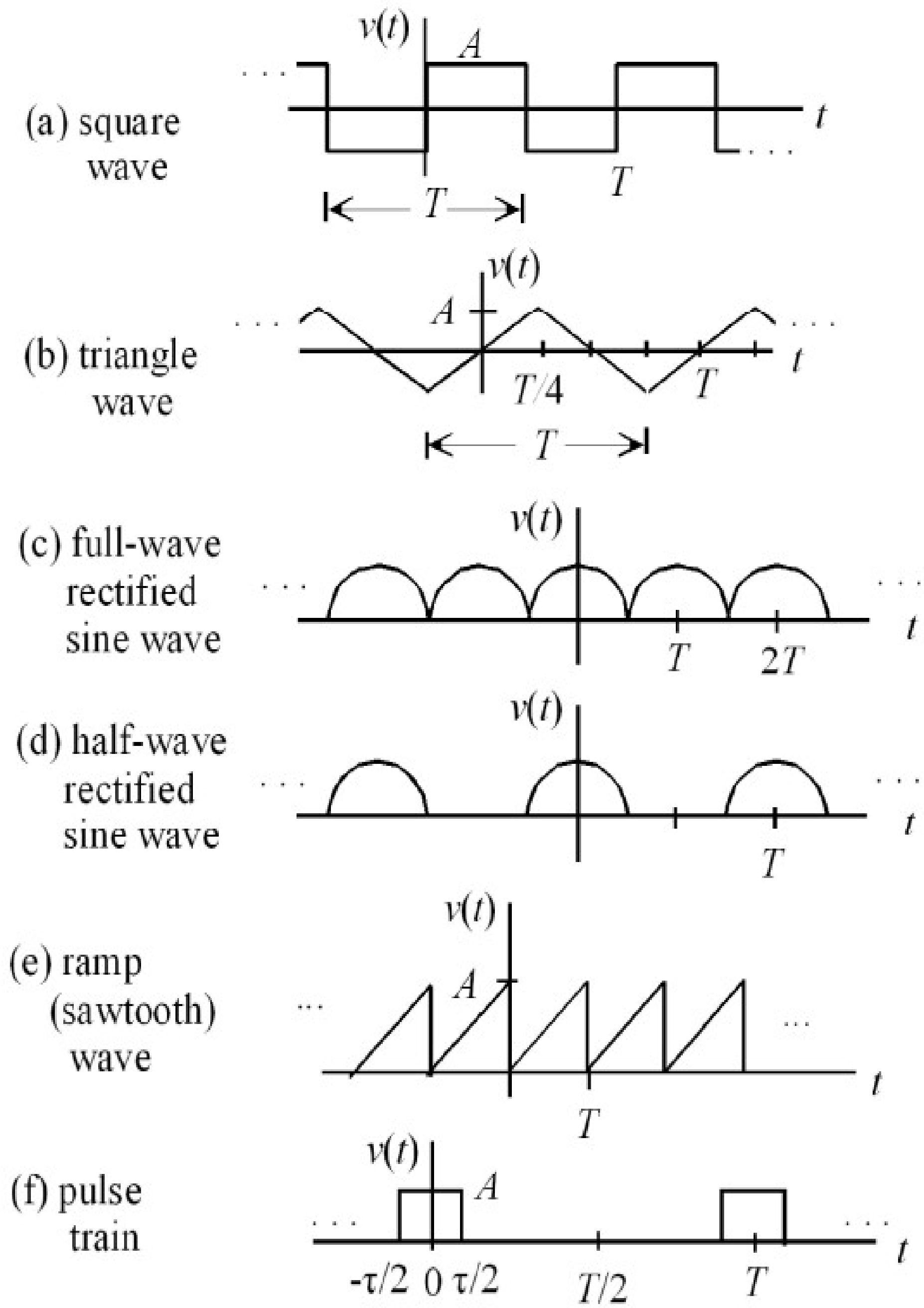
(c)



(d)

Periodic Signals

Consider the following signals. All of them are periodic:



A general method for finding the period of a sum of periodic signals:

Let $x(t) = x_1(t) + x_2(t) + \cdots + x_N(t)$ be the sum of periodic signals $x_1(t), x_2(t), \ldots, x_N(t)$.

1. Find the fundamental period of each of the individual signals, for example, denote T_i to be the period of the i-th signal.
2. Calculate the ratio $R_i = T_1/T_i$ for $i = 2, \ldots, N$. If any of these ratios is not a rational number, the composite signal $x(t)$ is not periodic.
3. Simplify the ratios, and let k_0 be the least common multiple of the denominators of the R_i.
4. The fundamental period of $x(t)$ is $T_0 = k_0 T_1$.

Determine whether or not each of the following signals is periodic. If a signal is periodic, determine its fundamental period.

(a) $x(t) = \cos\left(2t + \dfrac{\pi}{4}\right)$

(b) $x(t) = \cos^2 t$

(c) $x(t) = (\cos 2\pi t)u(t)$

(d) $x(t) = e^{j\pi t}$

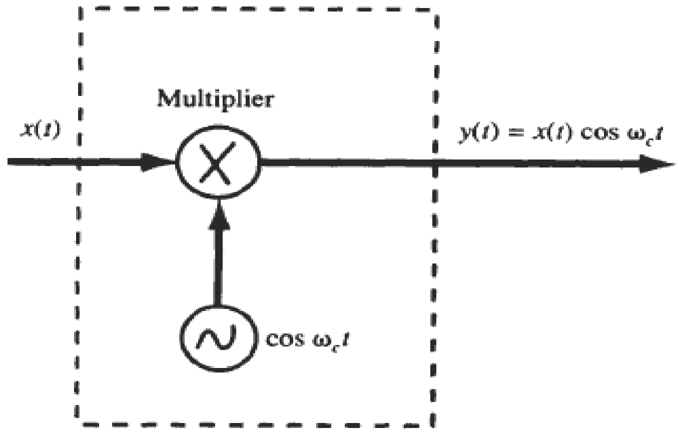
(e) $x[n] = e^{j[(n/4)-\pi]}$

(f) $x[n] = \cos\left(\dfrac{\pi n}{8}\right)$

(g) $x[n] = \cos\left(\dfrac{n}{2}\right)\cos\left(\dfrac{\pi n}{4}\right)$

(h) $x[n] = \cos\left(\dfrac{\pi n}{4}\right) + \sin\left(\dfrac{\pi n}{8}\right) - 2\cos\left(\dfrac{\pi n}{2}\right)$

Ans.
(a) Periodic, period $= \pi$ (b) Periodic, period $= \pi$
(c) Nonperiodic (d) Periodic, period = 2
(e) Nonperiodic (f) Periodic, period = 16
(g) Nonperiodic (h) Periodic, period = 16



Consider the system shown in Fig. Determine whether it is (*a*) memoryless, (*b*) causal, (*c*) linear, (*d*) time-invariant, or (*e*) stable.

(*a*) From Fig. we have

$$y(t) = \mathbf{T}\{x(t)\} = x(t)\cos\omega_c t$$

Since the value of the output $y(t)$ depends on only the present values of the input $x(t)$, the system is memoryless.

(*b*) Since the output $y(t)$ does not depend on the future values of the input $x(t)$, the system is causal.

(*c*) Let $x(t) = \alpha_1 x(t) + \alpha_2 x(t)$. Then

$$\begin{aligned} y(t) = \mathbf{T}\{x(t)\} &= [\alpha_1 x_1(t) + \alpha_2 x_2(t)]\cos\omega_c t \\ &= \alpha_1 x_1(t)\cos\omega_c t + \alpha_2 x_2(t)\cos\omega_c t \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned}$$

Thus, the superposition property is satisfied and the system is linear.

(*d*) Let $y_1(t)$ be the output produced by the shifted input $x_1(t) = x(t - t_0)$. Then

$$y_1(t) = \mathbf{T}\{x(t - t_0)\} = x(t - t_0)\cos\omega_c t$$

But

$$y(t - t_0) = x(t - t_0)\cos\omega_c(t - t_0) \neq y_1(t)$$

Hence, the system is not time-invariant.

(*e*) Since $|\cos\omega_c t| \le 1$, we have

$$|y(t)| = |x(t)\cos\omega_c t| \le |x(t)|$$

Thus, if the input $x(t)$ is bounded, then the output $y(t)$ is also bounded and the system is BIBO stable.

UNIT – II
ANALYSIS OF CONTINUOUS TIME SIGNALS

2.1 Fourier Series representation of CT periodicsignals

Any periodic function of time x(t) can be represented by an infinite series called the Fourier Series.

(i) Trigonometric FourierSeries

$$x(t)=a_0+\sum_{n=1}^{\infty}a_n\cos n\Omega_0 t+\sum_{n=1}^{\infty}b_n\sin n\Omega_o t$$

a_0 , a_n , b_n *are Fourier Coefficients*

$\Omega_0 = 2\pi f$ *where* f *=fundamental frequency* $\& T=\frac{1}{f}$

$$a_0=\frac{1}{T}\int_0^T x(t)dt$$

$$a_n=\frac{2}{T}\int_0^T x(t)\cos(n\Omega_0 t)dt$$

$$b_n=\frac{2}{T}\int_0^T x(t)\sin(n\Omega_0 t)dt$$

(ii) CosineSeries

$$x(t)=a_0+\sum_{n=1}^{\infty}a_n\cos n\Omega_0 t+\sum_{n=1}^{\infty}b_n\sin n\Omega_o t$$

a_0 , a_n , b_n *are FourierCoefficients*

$$x(t)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos n\Omega_0 t+b_n\sin n\Omega_0 t\right)$$

$$x(t)=A_0+\sum_{n=1}^{\infty}A_n\cos\left(n\Omega_0 t+\theta_n\right)$$

where $A_0=a_0$

$$A_n=\sqrt{a_n^2+b_n^2}$$

$$\theta_n=-\tan^{-1}\left(\frac{b_n}{a_n}\right)$$

note:*Trigonometric identity*

$$A\sin C+B\cos C=\sqrt{A^2+B^2}\cos\left(C-\tan^{-1}\left(\frac{A}{B}\right)\right)$$

(iii) Exponential FourierSeries

$$x(t)=\sum_{n=-\infty}^{\infty}C_n e^{jn\Omega_0 t}$$

$$C_n = \frac{1}{T} \int_T x(t) e^{-jn\Omega_0 t} \, dt$$

2.2 Properties of CT FourierSeries

Using exponential Fourier series, a periodic signal x(t) with period T can be expressed as

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{jn\Omega_0 t}$$

$$C_n = \frac{1}{T} \int_T x(t) e^{-jn\Omega_0 t} \, dt$$

1. Linearity

If $x_1(t)$ *and* $x_2(t)$ are two periodic signals with period T with Fourier series coefficients C_{n1} *and* C_{n2} then Fourier series coefficient of linear combination of $x_1(t)$ *and* $x_2(t)$ is given by

$FS[Ax_1(t)+Bx_2(t)]=AC_{n1}+BC_{n2}$

Proof:

Fourier series coefficient of $[Ax_1(t)+Bx_2(t)]$ is $=\frac{1}{T}\int_T [Ax_1(t)+Bx_2(t)]e^{-jn\Omega_0 t}dt$

$$=\frac{1}{T}\int_T Ax_1(t)e^{-jn\Omega_0 t}dt+\frac{1}{T}\int_T Bx_2(t)e^{-jn\Omega_0 t}dt=AC_{n1}+BC_{n2}$$

2. TimeShifting

If the Fourier series coefficient of $x(t)$ is C_n then the Fourier series coefficient of the time shifted signal $x(t-t_0)$ is $e^{-jn\Omega_0 t_0}C_n$

Proof: Fourier series coefficient of $x(t-t_0)$ is $=\frac{1}{T}\int_T x(t-t_0)e^{-jn\Omega_0 t}dt$

$$let\ t-t_0=m \Rightarrow t=m+t_0 \quad \&dt=dm$$

$$=\frac{1}{T}\int_T x(m)e^{-jn\Omega_0(m+t_0)}dm$$

$$=e^{-jn\Omega_0 t_0}\frac{1}{T}\int_T x(m)e^{-jn\Omega_0 m}dm=e^{-jn\Omega_0 t_0}C_n$$

3. TimeReversal

If the Fourier series coefficient of $x(t)$ is C_n then the Fourier series coefficient of the time reversed signal $x(-t)$ is $-C_{-n}$

Proof: Fourier series coefficient of $x(-t)$ is $=\frac{1}{T}\int_T x(-t)e^{-jn\Omega_0 t}dt$

$$let\ -t=m \Rightarrow t=-m \quad \&dt=-dm$$

$$=-\frac{1}{T}\int_T x(m)e^{-jn\Omega_0(-m)}dm=-C_{-n}$$

4. Timescaling

If the Fourier series coefficient of $x(t)$ is C_n then the Fourier series coefficient of the time scaled signal $x(\beta t)$ is $\frac{1}{\beta}C_{n/\beta}$

Proof: Fourier series coefficient of $x(\beta t)$ is $=\frac{1}{T}\int_T x(\beta t)e^{-jn\Omega_0 t}dt$

$$let\ \beta t=m \Rightarrow t=\frac{m}{\beta} \quad \&dt=\frac{dm}{\beta}$$

$$=\frac{1}{\beta}\times\frac{1}{T}\int_T x(m)e^{-jn\Omega_0\left(\frac{m}{\beta}\right)}dm$$

$$=\frac{1}{\beta}\times\frac{1}{T}\int_T x(m)e^{-jm\Omega_0\left(\frac{n}{\beta}\right)}dm=\frac{1}{\beta}C_{n/\beta}$$

5. Conjugation

If the Fourier series coefficient of $x(t)$ is C_n then the Fourier series coefficient of the complex conjugate of the signal $x^*(t)$ is C^*_{-n}

Proof:

$$x(t)=\sum_{n=-\infty}^{\infty}C_n e^{jn\Omega_0 t}$$ taking complex conjugate on both sides,

$$x^*(t)=\sum_{n=-\infty}^{\infty}C_n^* e^{-jn\Omega_0 t}$$

let $l=-n$

$$x^*(t)=\sum_{l=-\infty}^{\infty}C^*_{-l}e^{jl\Omega_0 t}$$ changing the variable

$$x^*(t)=\sum_{n=-\infty}^{\infty}C^*_{-n}e^{jn\Omega_0 t}\quad\Rightarrow FS\left[x^*(t)\right]=C^*_{-n}$$

6. Differentiation

If the Fourier series coefficient of $x(t)$ is C_n then the Fourier series coefficient of the signal $\frac{dx(t)}{dt}$ is $jn\Omega_0 C_n$

Proof:

$$x(t)=\sum_{n=-\infty}^{\infty}C_n e^{jn\Omega_0 t}$$ differentiatingw.r.t t on both sides,

$$\frac{dx(t)}{dt}=\sum_{n=-\infty}^{\infty}C_n e^{jn\Omega_0 t}jn\Omega_0$$

$$\frac{dx(t)}{dt}=\sum_{n=-\infty}^{\infty}jn\Omega_0 C_n e^{jn\Omega_0 t}\quad\Rightarrow FS\left[\frac{dx(t)}{dt}\right]=jn\Omega_0 C_n$$

7. Integration

If the Fourier series coefficient of $x(t)$ is C_n then the Fourier series coefficient ofthe signal $\int_{-\infty}^{t}x(t)dt$ is $\frac{C_n}{jn\Omega_0}$

Proof:

$$x(t)=\sum_{n=-\infty}^{\infty}C_n e^{jn\Omega_0 t}$$ integratingw.r.t t on both sides,

$$\int_{-\infty}^{t}x(t)dt=\sum_{n=-\infty}^{\infty}C_n e^{jn\Omega_0 t}\frac{1}{jn\Omega_0}$$

$$\int_{-\infty}^{t} x(t)dt = \sum_{n=-\infty}^{\infty} \frac{C_n}{jn\Omega_0} e^{jn\Omega_0 t} \qquad \Rightarrow FS\left[\int_{-\infty}^{t} x(t)dt\right] = \frac{C_n}{jn\Omega_0}$$

8. Multiplication

If $x(t)$ *and* $y(t)$ are two periodic signals with period T with Fourier series coefficients C_n *and* D_n then Fourier series coefficient of product of $x(t)$ *and* $y(t)$ isgiven by $\sum_{l=-\infty}^{\infty} C_l D_{(n-l)}$

Proof:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\Omega_0 t} \qquad \Rightarrow \qquad x(t) = \sum_{l=-\infty}^{\infty} C_l e^{jl\Omega_0 t}$$

Fourier series coefficient of $x(t)\,y(t)$ is = $\frac{1}{T}\int_T x(t)y(t)e^{-jn\Omega_0 t}dt$

$$= \frac{1}{T}\int_T y(t)e^{-jn\Omega_0 t}\sum_{l=-\infty}^{\infty} C_l e^{jl\Omega_0 t} dt$$

$$= \sum_{l=-\infty}^{\infty} C_l \frac{1}{T}\int_T y(t)e^{-j(n-l)\Omega_0 t}dt$$

$$= \sum_{l=-\infty}^{\infty} C_l D_{(n-l)}$$

9. Parseval's Relation for periodic signal

If the Fourier series coefficient of $x(t)$ is C_n then, $\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |C_n|^2$

$\frac{1}{T}\int_T |x(t)|^2 dt$ = *average power in the signal*

Proof:

$$\frac{1}{T}\int_T |x(t)|^2 dt = \frac{1}{T}\int_T [x(t)x^*(t)]dt = \frac{1}{T}\int_T x(t)\left[\sum_{n=-\infty}^{\infty} C_n e^{jn\Omega_0 t}\right]^* dt$$

$$= \frac{1}{T}\int_T x(t)\sum_{n=-\infty}^{\infty} C_n^* e^{-jn\Omega_0 t} dt = \sum_{n=-\infty}^{\infty} C_n^* C_n = \sum_{n=-\infty}^{\infty} |C_n|^2$$

2.3 laplacetransform

The Unilateral Laplace Transform

The Unilateral Laplace Transform is applied to the signals that are causal. The Unilateral Laplace Transform of a signal *x(t)* is defined by

$$L[x(t)] = X(s) = \int_0^\infty x(t)e^{-st}dt$$

Properties of Unilateral Laplace transform

1 ***Linearity***

$L[x_1(t)] = X_1(s)$

$L[x_2(t)] = X_2(s)$

$L[Ax_1(t)+Bx_2(t)] = AX_1(s)+BX_2(s)$

Proof:

$$L[Ax_1(t)+Bx_2(t)] = \int_0^\infty [Ax_1(t)+Bx_2(t)]e^{-st}\,dt$$

$$= A\int_0^\infty x_1(t)e^{-st}dt + B\int_0^\infty x_2(t)e^{-st}dt$$

$$= AX_1(s)+BX_2(s)$$

2 ***Shifting in timedomain***

$L[x(t)] = X(s)$

$L[x(t-t_0)] = e^{-st_0}X(s)$

Proof:

$$L[x(t-t_0)] = \int_0^\infty [x(t-t_0)]e^{-st}dt$$

$let \quad t-t_0 = p \Rightarrow t = t_0+p \quad and \quad dt = dp$

$$= \int_0^\infty x(p)e^{-s(t_0+p)}dp$$

$$= e^{-st_0}\int_0^\infty x(p)e^{-sp}dp = e^{-st_0}X(s)$$

3 ***Shifting in frequencydomain***

$L[x(t)] = X(s)$

$L[e^{-at}x(t)] = X(s+a)$

Proof:

$$L[e^{-at}x(t)] = \int_0^\infty e^{-at}x(t)e^{-st}dt = \int_0^\infty x(t)e^{-(s+a)t}dt = X(s+a)$$

4 ***Differentiation in time***

$L[x(t)] = X(s)$

$$L\left[\frac{dx(t)}{dt}\right]=sX(s)-x(0^-)$$

Proof:

$$L\left[\frac{dx(t)}{dt}\right]=\int_0^\infty \frac{dx(t)}{dt}e^{-st}\,dt$$

$$let\ u=e^{-st} \quad and \quad dv=dx(t)$$

$$du=-se^{-st}dt \quad and \quad v=x(t) \quad \Rightarrow\int u\,dv=uv-\int v\,du$$

$$L\left[\frac{dx(t)}{dt}\right]=\left[e^{-st}x(t)\right]_0^\infty-\int_0^\infty x(t)\left(-se^{-st}\right)dt$$

$$=-x(0^-)+s\int_0^\infty x(t)e^{-st}dt=sX(s)-x(0^-$$

Note:

$$L\left[\frac{d^2x(t)}{dt^2}\right]=s[sX(s)-x(0^-)]-\frac{dx(0^-)}{dt}$$

$$=s^2X(s)-sx(0^-)-\frac{dx(0^-)}{dt}$$

5. Integration in time

$$L[x(t)]=X(s)$$

$$L\left[\int_{-\infty}^{t}x(\tau)d\tau\right]=\frac{X(s)}{s}+\frac{\int_{-\infty}^{0^-}x(\tau)d\tau}{s}$$

proof:

$$\int_{-\infty}^{t}x(\tau)d\tau=\int_{-\infty}^{0^-}x(\tau)d\tau+\int_{0}^{t}x(\tau)d\tau$$

$$L\left[\int_{-\infty}^{0^-}x(\tau)d\tau\right]=\frac{\int_{-\infty}^{0^-}x(\tau)d\tau}{s}$$

$$L\left[\int_0^t x(\tau)d\tau\right]=\int_0^\infty\int_0^t x(\tau)d\tau e^{-st}\,dt$$

$$let\ u=\int_0^t x(\tau)d\tau\Rightarrow du=x(t)dt$$

$$dv=e^{-st}dt \quad \Rightarrow v=\frac{-e^{-st}}{s}$$

$$\int_0^\infty\int_0^t x(\tau)d\tau e^{-st}dt=\left[\int_0^t x(\tau)d\tau\,\frac{(-e^{-st})}{s}\right]_0^\infty-\int_0^\infty\left(\frac{-e^{-st}}{s}\right)x(t)dt$$

$$=0+\frac{X(s)}{s}$$

$$L\left[\int_{-\infty}^{t} x(\tau)d\tau\right] = \frac{X(s)}{s} + \frac{\int_{-\infty}^{0^-} x(\tau)d\tau}{s}$$

6. Scaling in Time

$$L\left[x(t)\right] = X(s)$$
$$L\left[x(at)\right] = \frac{1}{a} X\left(\frac{s}{a}\right)$$

Proof:

$$L\left[x(at)\right] = \int_{0}^{\infty}\left[x(at)\right]e^{-st}\, dt$$

$$let \quad at = p \Rightarrow t = \frac{p}{a} \quad and \quad dt = \frac{dp}{a}$$

$$L\left[x(at)\right] = \int_{0}^{\infty} x(p)e^{-\left(\frac{s}{a}\right)p}\frac{dp}{a} = \frac{1}{a} X\left(\frac{s}{a}\right)$$

7. Differentiation in s-domain (frequencydifferentiation)

$$L\left[x(t)\right] = X(s)$$
$$L\left[-tx(t)\right] = \frac{dX(s)}{ds}$$

Proof:

$$X(s) = \int_{0}^{\infty} x(t)e^{-st}dt$$

$$\frac{dX(s)}{ds} = \int_{0}^{\infty} x(t)\frac{d\left(e^{-st}\right)}{ds}dt = \int_{0}^{\infty}\left(-tx(t)e^{-st}\right) dt$$

$$\Rightarrow L\left[-tx(t)\right] = \frac{dX(s)}{ds}$$

Note:

$$L\left[(-t)^{n} x(t)\right] = \frac{d^{n} X(s)}{ds^{n}}$$

8. Integration in s-domain (frequencyintegration)

$$L\left[x(t)\right] = X(s)$$
$$L\left[\frac{x(t)}{t}\right] = \int_{s}^{\infty} X(s)ds$$

Proof:

$$X(s) = \int_0^{\infty} x(t)e^{-st}\,dt$$

$$\int_s^{\infty} X(s)ds = \int_0^{\infty} x(t)\{\int_s^{\infty} e^{-st}\,ds\}dt = \int_0^{\infty} x(t)\frac{1}{t}e^{-st}dt$$

$$\Rightarrow L\left[\frac{x(t)}{t}\right] = \int_s^{\infty} X(s)ds$$

9. initial and Final valuetheorem

$$L[x(t)] = X(s)$$

Initial Value theorem:

$$\lim_{t\to 0} x(t) = \lim_{s\to\infty} sX(s)$$

Proof: $L\left[\frac{dx(t)}{dt}\right] = sX(s) - x(0^-)$

$$\int_0^{\infty} \frac{dx(t)}{dt} e^{-st}\,dt = sX(s) - x(0^-)$$

$$\lim_{s\to\infty} \int_0^{\infty} \frac{dx(t)}{dt} e^{-st}dt = 0 = \lim_{s\to\infty} sX(s) - x(0^-)$$

$$\lim_{s\to\infty} sX(s) = x(0)$$

Final Value theorem:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Proof: $\int_0^{\infty} \frac{dx(t)}{dt} e^{-st}\,dt = sX(s) - x(0^-)$

$$\lim_{s\to 0} \int_0^{\infty} \frac{dx(t)}{dt} e^{-st}dt = \lim_{s\to 0} [sX(s) - x(0^-)]$$

$$\int_0^{\infty} \frac{dx(t)}{dt}dt = \lim_{s\to 0} [sX(s) - x(0^-)]$$

$$\lim_{t\to\infty} x(t) - x(0) = \lim_{s\to 0} sX(s) - x(0)$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

10. Convolution in Time

$$L[x_1(t)] = X_1(s)$$

$$L[x_2(t)] = X_2(s)$$

$$L[x_1(t)*x_2(t)] = X_1(s)X_2(s)$$

By definition , $x_1(t)*x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau$

Proof:

$$L[x_1(t)*x_2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau e^{-st}dt$$

$$let\ \ t-\tau = p \quad \Rightarrow t = \tau + p\ \&dt = dp$$

$$L[x_1(t)*x_2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)x_2(p)d\tau e^{-s(\tau+p)}dp$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-s\tau}d\tau \int_{-\infty}^{\infty} x_2(p)e^{-sp}dp = X_1(s)X_2(s)$$

11. Convolution in frequency(multiplication)

$$L[x(t)] = X(s)$$

$$L[g(t)] = G(s)$$

$$L[x(t)g(t)] = \frac{1}{2\pi j}[X(s)*G(s)]$$

Note:

$$(i)\quad X(s)*G(s) = \int_{-\infty}^{\infty} X(u)G(s-u)du \quad \Rightarrow \textit{by definition}$$

$$(ii)\quad L^{-1}[X(s)] = x(t) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} X(s)e^{st}ds \Rightarrow \textit{by definition}$$

<u>Proof:</u>

Inverse Laplace Transform of $\frac{1}{2\pi j}[X(s)*G(s)]$ is

$$= \frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{1}{2\pi j}\int_{-\infty}^{\infty} X(u)G(s-u)du e^{st}ds$$

let $s - u = p \Rightarrow s = u + p$ *and* $ds = dp$

$$= \frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{1}{2\pi j}\int_{-\infty}^{\infty} X(u)G(p)du e^{(u+p)t}dp$$

$$= \frac{1}{2\pi j}\int_{-\infty}^{\infty} X(u)e^{ut}du \frac{1}{2\pi j}\int_{-\infty}^{\infty} G(p)e^{pt}dp = x(t)g(t)$$

<u>2.4 Continuous Time Fourier Transform</u>

$$X(j\Omega) = F[x(t)] \quad \textit{and} \quad x(t) = F^{-1}[X(j\Omega)]$$

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \qquad \textit{for all } \Omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega \qquad \textit{for all } t$$

Properties of CT Fourier Transform
1.Linearity

$$F[x_1(t)] = X_1(j\Omega)$$

$$F[x_2(t)] = X_2(j\Omega)$$

$$F[Ax_1(t) + Bx_2(t)] = AX_1(j\Omega) + BX_2(j\Omega)$$

<u>Proof:</u>

$$F\left[Ax_1(t)+Bx_2(t)\right]=\int_{-\infty}^{\infty}\left[Ax_1(t)+Bx_2(t)\right]e^{-j\Omega t}dt$$

$$=A\int_{-\infty}^{\infty}x_1(t)e^{-j\Omega t}dt+B\int_{-\infty}^{\infty}x_2(t)e^{-j\Omega t}dt$$

$$=AX_1(j\Omega)+BX_2(j\Omega)$$

2. Shifting in timedomain

$F\left[x(t)\right]=X(j\Omega)$

$F\left[x(t-t_0)\right]=e^{-j\Omega t_0}X(j\Omega)$

Proof:

$$F\left[x(t-t_0)\right]=\int_{-\infty}^{\infty}\left[x(t-t_0)\right]e^{-j\Omega t}dt$$

$let \quad t-t_0=p \Rightarrow t=t_0+p \quad and \quad dt=dp$

$$=\int_{-\infty}^{\infty}x(p)e^{-j\Omega(t_0+p)}dp$$

$$=e^{-j\Omega t_0}\int_{-\infty}^{\infty}x(p)e^{-j\Omega p}dp=e^{-j\Omega t_0}X(j\Omega)$$

3. Shifting in frequencydomain

$F\left[x(t)\right]=X(j\Omega)$

$F\left[e^{-j\Omega_0 t}x(t)\right]=X(j\Omega+j\Omega_0)$

Proof:

$$F\left[e^{-j\Omega_0 t}x(t)\right]=\int_{-\infty}^{\infty}e^{-j\Omega_0 t}x(t)e^{-j\Omega t}dt=\int_{-\infty}^{\infty}x(t)e^{-(j\Omega_0+j\Omega)t}dt=X(j\Omega+j\Omega_0)$$

4. Scaling in Time

$F\left[x(t)\right]=X(j\Omega)$

$F\left[x(at)\right]=\frac{1}{a}X\left(\frac{j\Omega}{a}\right)$

Proof:

$$F\left[x(at)\right]=\int_{-\infty}^{\infty}\left[x(at)\right]e^{-j\Omega t}dt$$

$let \quad at=p \Rightarrow t=\frac{p}{a} \quad and \quad dt=\frac{dp}{a}$

$$F\left[x(at)\right]=\int_{-\infty}^{\infty}x(p)e^{-\left(\frac{j\Omega}{a}\right)p}\frac{dp}{a}=\frac{1}{a}X\left(\frac{j\Omega}{a}\right)$$

5. Time reversal

$F\left[x(t)\right]=X(j\Omega)$

$F\left[x(-t)\right]=X(-j\Omega)$

Proof:

$$F[x(-t)]=\int_{-\infty}^{\infty}x(-t)e^{-j\Omega t}dt=\int_{-\infty}^{\infty}x(t)e^{j\Omega t}dt=\int_{-\infty}^{\infty}x(t)e^{-(-j\Omega)t}=X(-j\Omega)$$

6. Differentiation in time

$$F[x(t)]=X(j\Omega)$$

$$F\left[\frac{dx(t)}{dt}\right]=j\Omega X(j\Omega)$$

Proof:

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\Omega)e^{j\Omega t}d\Omega \qquad \textit{for all } t$$

$$\frac{dx(t)}{dt}=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\Omega)\frac{d}{dt}[e^{j\Omega t}]d\Omega$$

$$\frac{dx(t)}{dt}=\frac{1}{2\pi}\int_{-\infty}^{\infty}j\Omega X(j\Omega)e^{j\Omega t}d\Omega dt$$

$$\Rightarrow F\left[\frac{dx(t)}{dt}\right]=j\Omega X(j\Omega)$$

note:

$$F\left[\frac{d^n x(t)}{dt^n}\right]=(j\Omega)^n X(j\Omega)$$

7. Differentiation in frequency

$$F[x(t)]=X(j\Omega)$$

$$F[tx(t)]=j\frac{dX(j\Omega)}{d\Omega}$$

Proof:

$$X(j\Omega)=\int_{-\infty}^{\infty}x(t)\,e^{-j\Omega t}dt$$

$$\frac{dX(j\Omega)}{d\Omega}=\int_{-\infty}^{\infty}x(t)\frac{d(e^{-j\Omega t})}{d\Omega}dt$$

$$=\int_{-\infty}^{\infty}-jt\,x(t)e^{-j\Omega t}\,dt$$

$$F[tx(t)]=j\frac{dX(j\Omega)}{d\Omega}$$

8. Duality

$$F[x(t)]=X(j\Omega)$$

$$F[X(t)]=2\pi x(-j\Omega)$$

Proof:

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\Omega)e^{j\Omega t}d\Omega \qquad \textit{for all } t$$

$$2\pi x(t) = \int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(j\Omega) e^{-j\Omega t} d\Omega \qquad = F[X(j\Omega)]$$

changing t to $j\Omega$

$$2\pi x(-j\Omega) = \int_{-\infty}^{\infty} X(t) e^{-j\Omega t} d\Omega \qquad = F[X(t)]$$

9. Convolution in Time

$F[x_1(t)] = X_1(j\Omega)$

$F[x_2(t)] = X_2(j\Omega)$

$F[x_1(t) * x_2(t)] = X_1(j\Omega) X_2(j\Omega)$

By definition , $x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau$

Proof:

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau e^{-j\Omega t}\, dt$$

let $t - \tau = p \Rightarrow t = \tau + p \;\&\; dt = dp$

$$F[x_1(t) * x_2(\mathrm{t})] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau) x_2(p) d\tau e^{-j\Omega(\tau+p)}\, dp$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-j\Omega\tau} d\tau \int_{-\infty}^{\infty} x_2(p) e^{-j\Omega p} dp = X_1(j\Omega) X_2(j\Omega)$$

10. Convolution in frequency(multiplication)

$F[x(t)] = X(j\Omega)$

$F[g(t)] = G(j\Omega)$

$F[x(t)g(t)] = \frac{1}{2\pi}[X(j\Omega) * G(j\Omega)]$

Note:

(*i*) $X(j\Omega) * G(j\Omega) = \int_{-\infty}^{\infty} X(u) G(j\Omega - u) du \quad \Rightarrow$ *by definition*

(*ii*) $F^{-1}[X(j\Omega)] = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t}\, d\Omega \Rightarrow$ *by definition*

Proof:

Inverse Fourier Transform of $\frac{1}{2\pi}[X(j\Omega) * G(j\Omega)]$ is

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} X(u) G(j\Omega - u) du\, e^{j\Omega t} d\Omega$$

let $s - u = p \Rightarrow s = u + p$ *and* $ds = dp$

$$= \frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{1}{2\pi j}\int_{-\infty}^{\infty} X(u)G(p)du\,e^{(u+p)t}dp \qquad = \frac{1}{2\pi j}\int_{-\infty}^{\infty} X(u)e^{ut}\,du\,\frac{1}{2\pi j}\int_{-\infty}^{\infty} G(p)e^{pt}\,dp = x(t)g(t)$$

<u>Parseval's Theorem</u>

$$F\left[x(t)\right] = X(j\Omega)$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\Omega)|^2 d\Omega$$

<u>Proof:</u>

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)x^*(t)dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega \qquad \Rightarrow x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\Omega)\,e^{-j\Omega t}d\Omega$$

$$\int_{-\infty}^{\infty} x(t)x^*(t)dt = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\Omega)e^{-j\Omega t}d\Omega\right]dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\left[\int_{-\infty}^{\infty} x(t)e^{j\Omega t}dt\right]d\Omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\left[\int_{-\infty}^{\infty} x(t)e^{j\Omega t}dt\right]d\Omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\Omega)\,X(j\Omega)d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\Omega)|^2 d\Omega$$

Continuous Time Fourier Transform:

Viewed periodic functions in terms of frequency components ((Fourier series) as well as ordinary functions of time

Viewed LTI systems in terms of what they do to frequency components (frequency response)

Viewed LTI systems in terms of what they do to time-domain signals (convolution with impulse response)

- View aperiodic functions in terms of frequency components via Fouriertransform

The Fourier expansion coefficient $X[k]$ (in a_k OWN) of a periodic signal is $x_T(t) = x_T(t+T)$ is

$$X[k] = \frac{1}{T}\int_T x_T(t)e^{-jk\omega_0 t}dt \qquad (k = 0, \pm 1, \pm 2, \cdots)$$

and the Fourier expansion of the signal is:

$$x_T(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}$$

which can also be written as:

$$x_T(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty}(TX[k])e^{jk\omega_0 t} = \frac{\omega_0}{2\pi}\sum_{k=-\infty}^{\infty} X(k\omega_0)e^{jk\omega_0 t} \qquad (a)$$

where $X(k\omega_0)$ is defined as

$$X(k\omega_0) \triangleq T\,X[k] = \int_T x_T(t)e^{-jk\omega_0 t}dt \qquad (b)$$

When theperiodof $x_T(t)$ approaches infinity $T \to \infty$, the periodic signal $x_T(t)$ becomes a non-periodic signal $x(t)$ and the following willresult:

Interval between two neighboring frequency components becomes zero:

$$T \to \infty \Longrightarrow \omega_0 = 2\pi/T \to 0$$

Discrete frequency becomes continuous frequency:

$$k\omega_0\big|_{\omega_0 \to 0} \Longrightarrow \omega$$

Summation of the Fourier expansion in equation (a) becomes an integral:

$$x(t) \triangleq \lim_{T\to\infty} x_T(t) = \lim_{\omega_0\to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\omega_0) e^{jk\omega_0 t}\omega_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega$$

the second equal sign is due to the general fact:

$$\lim_{\Delta x\to 0} \sum_{k=-\infty}^{\infty} f(k\Delta x)\Delta x = \int_{-\infty}^{\infty} f(x)dx$$

Time integral over in equation (b) becomes over the entire time axis:

$$X(\omega) \triangleq \lim_{T\to\infty} X(k\omega_0) = \lim_{T\to\infty} \int_T x_T(t)e^{-jk\omega_0 t}dt = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

In summary, when the signalisnon-periodic $x(t) = \lim_{T\to\infty} x_T(t)$, the Fourier expansion becomes Fourier transform. The forward transform (analysis)is:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \quad \text{or} \quad X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}dt$$

and the inverse transform (synthesis) is:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = \int_{-\infty}^{\infty} X(f)e^{j2\pi f t}df$$

Notethat $X(\omega)$ isdenoted by $X(j\omega)$ in OWN.

Comparing Fourier coefficient of a periodic signal $x_T(t)$ with with Fourier spectrum of a non-periodicsignal $x(t)$

$$X[k] = \frac{1}{T}\int_T x_T(t)e^{-jk\omega_0 t}dt, \qquad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

we see that thedimension of $X(\omega)$ is different fromthatof $X[k]$:

$$[X(\omega)] = [X[k]][t] = \frac{[X[k]]}{[\omega]}$$

If $|X[k]|^2$ represents the *energy* contained in the kth frequency component of a periodic signal $x_T(t)$, then $|X(\omega)|^2$ represents the energy *density* of a non-periodic signal

$x(t)$ distributed along the frequency axis. We can only speak of the energy contained ina particular frequencyband $\omega_1 < \omega < \omega_2$:

2.5 InverseTransforms

If we have the full sequence of Fourier coefficients for a periodic signal, we can reconstruct it by multiplying the complex sinusoids of frequency ω0k by the weights Xk and summing:

$$x(n) = \sum_{k=0}^{p-1} X_k e^{ik\omega_0 n} \qquad x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$$

We can perform a similar reconstruction for aperiodic signals

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{i\omega n}\, d\omega \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}\, d\omega$$

These are called the **inverse transforms**.

2.6 PROBLEMS:

Example0:

Consider the unit impulse function:

$$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

Example 1:

If the spectrum of a signal $x(t)$ is a delta function in frequency domain $X(j\omega) = 2\pi\ \delta(\omega)$, the signal can be found to be:

$$x(t) = \mathcal{F}^{-1}[X(j\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\ \delta(\omega) e^{j\omega t} d\omega = e^0 = 1$$

i.e.,

$$\mathcal{F}[x(t)] = \int_{-\infty}^{\infty} e^{-j\omega t} dt = 2\pi\ \delta(\omega)$$

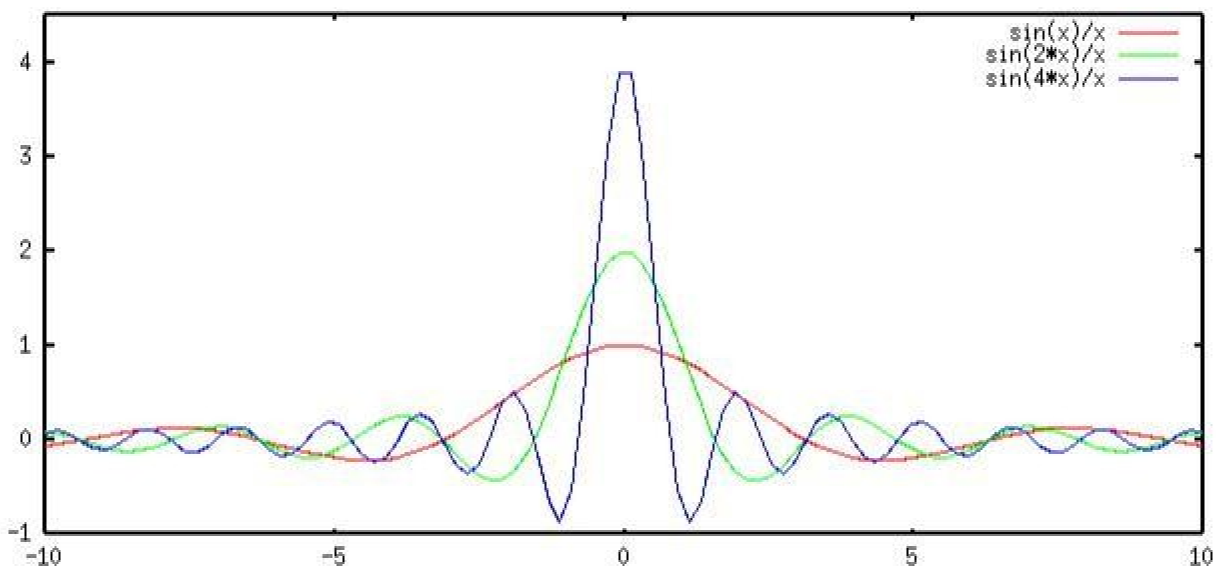
Example 2:

$$x(t) = \begin{cases} 1 & |t| < a \\ 0 & \text{else} \end{cases}$$

The spectrum is

$$X(j\omega) = \int_{-a}^{a} e^{-j\omega t} dt = \frac{1}{-j\omega} e^{-j\omega t}|_{-a}^{a} = \frac{2}{\omega} sin(a\omega)$$

This is the sinc function withaparameter a, as shown in thefigure.



Note that the height of the main peak is $2a$ and it gets taller andnarroweras a gets larger. Also note

$$\int_{-\infty}^{\infty} X(j\omega) d\omega = 2 \int_{-\infty}^{\infty} \frac{sin(a\omega)}{\omega} d\omega = 2\pi$$

When approachesinfinity, forall , and the spectrumbecomes

Recall that the Fourier coefficient of is
which represents the energy contained in thesignal at $k = 0$(DC component atzero frequency), andthespectrum $X(j\omega) = X[k]/\omega$ is the energy density or distribution which is infinity at zerofrequency.

The integral in the above transform is an important formula to be used frequently later:

$$\int_{-\infty}^{\infty} e^{-j\omega t} dt = 2\pi\delta(\omega) \quad \text{or} \quad \int_{-\infty}^{\infty} e^{-j2\pi f t} dt = \delta(f)$$

which can also be written as

$$\int_{-\infty}^{\infty} e^{-j\omega t} dt = \int_{-\infty}^{\infty} [cos(\omega t) + j\ sin(-j\omega t)] dt = \int_{-\infty}^{\infty} cos(\omega t) dt = \delta(f)$$

Switching and in the equation above, we alsohave $x(t) = sin(\omega_0 t) = \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}]$
representing a superposition of an infinite number of cosine functions of all frequencies, which cancel each other any where along the time axisexcept at $t = 0$ where they add up to infinity, an impulse.

Example 3:

$$x(t) = cos(\omega_0 t) = \frac{1}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}]$$

The spectrum of the cosine function is

The spectrum of the sine function

can be similarly obtained to be

$$X(j\omega) = \frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] = -j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

Again, these spectra represent the energy density distribution of the sinusoids, while the corresponding Fourier coefficients

$$X[k] = \mathcal{F}[cos(\omega_0 t)] = \frac{1}{2}[\delta[k-1] + \delta[k+1]]$$

and

$$X[k] = \mathcal{F}[sin(\omega_0 t)] = \frac{1}{2j}[\delta[k-1] - \delta[k+1]]$$

represent the energy containedat frequency $\omega = \omega_0$.

Continuous Time Fourier Transform-PROBLEMS

$$X(j\Omega) = F[x(t)] \qquad and \qquad x(t) = F^{-1}[X(j\Omega)]$$

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \qquad for\ all\ \Omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega \qquad for\ all\ t$$

1. Find the inverse Fourier transformof $(i)\ \delta(\Omega)$ $(ii)\ \delta(\Omega-\Omega_0)$

(i) $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\Omega)\, e^{j\Omega t} d\Omega \qquad = \frac{1}{2\pi}$$

Since, $$\delta(\Omega)=1 \ \ for\ \ \Omega=0$$
$$=0 \ \ for\ \ \Omega\neq 0$$

$$F^{-1}[\delta(\Omega)] = \frac{1}{2\pi}$$

$$F^{-1}[2\pi\delta(\Omega)] = 1$$

$$F[1] = 2\pi\delta(\Omega)$$

(ii) $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\Omega-\Omega_0) e^{j\Omega t} d\Omega \qquad = \frac{e^{j\Omega_0 t}}{2\pi}$$

Since, $$\delta(\Omega-\Omega_0)=1 \ \ for\ \ \Omega=\Omega_0$$
$$=0\ for\ \ \Omega\neq\Omega_0$$

$$F^{-1}[\delta(\Omega-\Omega_0)] = \frac{e^{j\Omega_0 t}}{2\pi}$$

$$F^{-1}[2\pi\delta(\Omega-\Omega_0)] = e^{j\Omega_0 t}$$

$$F\left[e^{j\Omega_0 t}\right] = 2\pi\delta(\Omega-\Omega_0)$$

<u>Fourier transform of standard signals:</u>

a) $x(t) = \cos(\Omega_0 t)$

$$\cos(\Omega_0 t) = \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}$$

$$X(j\Omega) = \left[\frac{F\left[e^{j\Omega_0 t}\right] + F\left[e^{-j\Omega_0 t}\right]}{2}\right] = \frac{1}{2}\left[2\pi\delta(\Omega-\Omega_0) + 2\pi\delta(\Omega+\Omega_0)\right]$$

$$= [\pi\delta(\Omega-\Omega_0) + \pi\delta(\Omega+\Omega_0)] = \pi[\delta(\Omega-\Omega_0) + \delta(\Omega+\Omega_0)]$$

b) $x(t) = \sin(\Omega_0 t)$

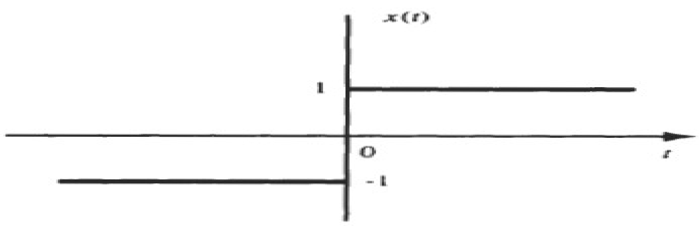
$$\sin\left(\Omega_0 t\right) = \frac{e^{j\Omega_0 t} - e^{-j\Omega_0 t}}{2j}$$

$$X\left(j\Omega\right) = \left[\frac{F\left[e^{j\Omega_0 t}\right] - F\left[e^{-j\Omega_0 t}\right]}{2j}\right] = \frac{1}{2j}\left[2\pi\delta(\Omega-\Omega_0) - 2\pi\delta(\Omega+\Omega_0)\right]$$

$$= -j\left[\pi\delta(\Omega-\Omega_0) - \pi\delta(\Omega+\Omega_0)\right] = -j\pi\left[\delta(\Omega-\Omega_0) - \delta(\Omega+\Omega_0)\right]$$

c) $x(t) = \mathrm{sgn}\left(t\right)$

The given function $x(t) = \mathrm{sgn}\left(t\right)$ **is known as signum function and is defined as,**



$$\mathrm{sgn}(t) = \begin{cases} 1 & if \quad t > 0 \\ 0 & if \quad t = 0 \\ -1 & if \quad t < 0 \end{cases}$$

The function is not absolutely integrable. The Fourier transform of $x(t) = \mathrm{sgn}\left(t\right)$ **is obtained by considering thefunction** $\underset{a\to 0}{lt}\, e^{-a|t|}\,\mathrm{sgn}\left(t\right)$

$$X\left(j\Omega\right) = \underset{a\to 0}{lt}\left[\int_{-\infty}^{\infty} e^{-a|t|}\,\mathrm{sgn}(t)\,e^{-j\Omega t}\,dt\right]$$

$$= \underset{a\to 0}{lt}\left[-\int_{-\infty}^{0} e^{at}\, e^{-j\Omega t}\,dt + \int_{0}^{\infty} e^{-at}\, e^{-j\Omega t}\,dt\right]$$

$$= \underset{a\to 0}{lt}\left[\frac{-1}{a - j\Omega} + \frac{1}{a + j\Omega}\right] = \underset{a\to 0}{lt}\left[\frac{-2j\Omega}{a^2 + \Omega^2}\right]$$

$$= \frac{-2j}{\Omega} = \frac{2}{j\Omega}$$

d) $x(t) = u\left(t\right)$

$$u(t) = \begin{cases} 1 & for \quad t \geq 0 \\ 0 & for \quad t < 0 \end{cases}$$

This signal can be expressedas $u(t)=\frac{1}{2}+\frac{1}{2}\mathrm{sgn}(t)$

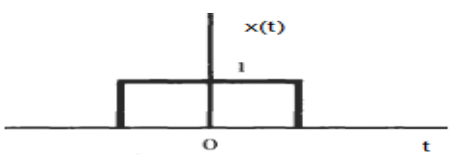
$$F[1]=2\pi\delta(\Omega)$$

$$F[\mathrm{sgn}(t)]=\frac{2}{j\Omega}$$

$$F[u(t)]=\frac{1}{2}[2\pi\delta(\Omega)]+\frac{1}{2}\left[\frac{2}{j\Omega}\right]$$

$$F[u(t)]=[\pi\delta(\Omega)]+\left[\frac{1}{j\Omega}\right]$$

e) Rectangular pulse $\Pi\left(\frac{t}{\tau}\right)$



Rectangular pulse ofwidth τ **extending from** $-\frac{\tau}{2} to \frac{\tau}{2}$ **and amplitude1.**

$$x(t)=\Pi\left(\frac{t}{\tau}\right)$$

$$x(t)=1 \qquad -\frac{\tau}{2} to \frac{\tau}{2}$$

$$=0 otherwise$$

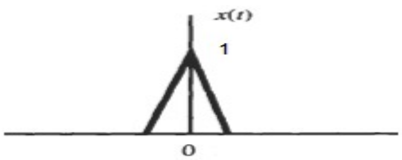
$$X(j\Omega)=\int_{-\infty}^{\infty}x(t)e^{-j\Omega t}dt=\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}e^{-j\Omega t}dt$$

$$=\left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}}=\frac{1}{-j\Omega}\left[e^{-j\Omega\frac{\tau}{2}}-e^{j\Omega\frac{\tau}{2}}\right]=\frac{2\sin\left(\frac{\Omega\tau}{2}\right)}{\Omega}$$

$$=\frac{2\tau\sin\left(\frac{\Omega\tau}{2}\right)}{\Omega\tau}=\frac{\tau\sin\left(\frac{\Omega\tau}{2}\right)}{\frac{\Omega\tau}{2}}=\tau\operatorname{sinc}\left(\frac{\Omega\tau}{2}\right)$$

Note: $\frac{\sin x}{x}=\sin c(x)$

f)Triangularpulse $\Delta\left(\frac{t}{\tau}\right)$



Triangular pulse of width τ **extending from** $-\frac{\tau}{2}$ *to* $\frac{\tau}{2}$ **and amplitude 1.**

$$x(t)=\Delta\left(\frac{t}{\tau}\right)$$

$$x(t)=1-\frac{2t}{\tau} \qquad 0<t<\frac{\tau}{2}$$

$$=1+\frac{2t}{\tau} \qquad -\frac{\tau}{2}<t<0$$

$$X(j\Omega)=\int_{-\infty}^{\infty}x(t)e^{-j\Omega t}dt$$

$$=\int_{-\frac{\tau}{2}}^{0}(1+\frac{2t}{\tau})e^{-j\Omega t}dt+\int_{0}^{\frac{\tau}{2}}(1+\frac{2t}{\tau})e^{-j\Omega t}dt$$

$$=\left(\frac{e^{-j\Omega t}}{-j\Omega}\right)_{-\frac{\tau}{2}}^{0}+\frac{2}{\tau}\left(\frac{t\,e^{-j\Omega t}}{-j\Omega}-\frac{e^{-j\Omega t}}{(j\Omega)^2}\right)_{-\frac{\tau}{2}}^{0}+\left(\frac{e^{-j\Omega t}}{-j\Omega}\right)_{0}^{\frac{\tau}{2}}-\frac{2}{\tau}\left(\frac{t\,e^{-j\Omega t}}{-j\Omega}-\frac{e^{-j\Omega t}}{(j\Omega)^2}\right)_{0}^{\frac{\tau}{2}}$$

$$=\left[\frac{1}{-j\Omega}+\frac{e^{j\Omega\frac{\tau}{2}}}{j\Omega}\right]+\frac{2}{\tau}\left[\frac{1}{(j\Omega)^2}+\frac{\tau}{2}\frac{e^{j\Omega\frac{\tau}{2}}}{(-j\Omega)}+\frac{e^{j\Omega\frac{\tau}{2}}}{(j\Omega)^2}\right]+e^{-j\Omega\frac{\tau}{2}}\left[\frac{1}{-j\Omega}+\frac{1}{j\Omega}\right]$$

$$-\frac{2}{\tau}\left[\frac{\tau}{2}\frac{e^{-j\Omega\frac{\tau}{2}}}{-j\Omega}-\frac{e^{-j\Omega\frac{\tau}{2}}}{(j\Omega)^2}+\frac{1}{(j\Omega)^2}\right]$$

$$=\frac{2}{\tau}\left[-\frac{2}{(j\Omega)^2}+\frac{e^{j\Omega\frac{\tau}{2}}}{(j\Omega)^2}+\frac{e^{-j\Omega\frac{\tau}{2}}}{(j\Omega)^2}\right]$$

$$=\frac{2}{\tau}\left[-\frac{2}{(j\Omega)^2}+\frac{e^{j\Omega\frac{\tau}{2}}}{(j\Omega)^2}+\frac{e^{-j\Omega\frac{\tau}{2}}}{(j\Omega)^2}\right]=\frac{2}{\tau}\left[\frac{e^{j\Omega\frac{\tau}{4}}-e^{-j\Omega\frac{\tau}{4}}}{j\Omega}\right]^2$$

$$==\frac{2}{\tau}\left[\frac{e^{j\Omega\frac{\tau}{4}}-e^{-j\Omega\frac{\tau}{4}}}{j\Omega}\right]^2=\frac{\tau^2}{16}\times\frac{8}{\tau}\left[\frac{\sin\left(\Omega\frac{\tau}{4}\right)}{\Omega\frac{\tau}{4}}\right]^2$$

$$=\frac{\tau}{2}\operatorname{sinc}^2\left(\Omega\frac{\tau}{4}\right)$$

2.5 LaplaceTransform

►Lapalcetransform isageneralization oftheFourier transform in the sensethat it allows –complex frequency‖ whereasFourieranalysis can on lyhandle –real frequency‖.LikeFourier transform,Lapalce transform allows us to analyzea–linearcircuit‖problem, no matterhow complicated the circuit is, in the frequency domain in stead of in he timedomain.

►Mathematically, it produces the benefit of converting a set of differential equations into a corresponding set of algebraic equations, which are much easier to solve. Physically, it produces more insight of the circuit and allows us to know the bandwidth, phase, and transfer characteristics important for circuit analysis and design.

►Most importantly, Laplace transform lifts the limit of Fourier analysis to allow us to find both the steady-state and –transient‖responses of a linear circuit. Using Fourier transform, one can only deal with he steady state behavior (i.e. circuit response under indefinite sinusoidal excitation).

►Using Laplace transform, one can find the response under any types of excitation (e.g. switching on and off at any given time(s), sinusoidal, impulse, square wave excitations, etc.

$$\boxed{\mathcal{L}[f(t)] = \mathbf{F}(s) = \int_0^{\infty} f(t)e^{-st}\,dt}$$

$f(t)$	Property	$\mathbf{F}(s)$
$f(t)$	Definition	$\int_0^{\infty} f(t)e^{-st}\,dt$
$f_1(t) + f_2(t)$	Linearity	$\mathbf{F}_1(s) + \mathbf{F}_2(s)$
$Kf(t)$	Linearity	$K\mathbf{F}(s)$
$\frac{df(t)}{dt}$	Differentiation	$s\mathbf{F}(s) - f(0)$
$\frac{d^2f(t)}{dt^2}$	Differentiation	$s^2\mathbf{F}(s) - sf(0) - \frac{df(0)}{dt}$
$\int_0^t f(t)\,dt$	Integration	$\frac{1}{s}\mathbf{F}(s)$
$tf(t)$	Complex differentiation	$-\frac{d\mathbf{F}(s)}{ds}$
$e^{-at}f(t)$	Complex translation	$\mathbf{F}(s + a)$
$f(t - a)\,u(t - a)$	Real translation	$e^{-as}\mathbf{F}(s)$

Application of Laplace Transform to Circuit Analysis

	Properties	Time Domain	Laplace Transform
1	Linearity	$a_1x_1(t)+a_2x_2(t)+\ldots+a_nx_n(t)$	$a_1X_1(s)+a_2X_2(s)+\ldots+a_nX_n(s)$
2	Frequency Shifting	$e^{-\alpha t}x(t)$	$F(s+\alpha)$
3	Time Delay	$x(t-a)u(t-a)$	$e^{-as}X(s)$
4	Time Scaling	$x(\alpha t)$	$\frac{1}{\alpha}X\left(\frac{s}{\alpha}\right)$
5	Time Differentiation	$\frac{d}{dt}x(t)$	$sX(s)-x(0^-)$
6	Time Integration	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{X(s)}{s}+\frac{1}{s}\int_{-\infty}^{0^-}x(\tau)d\tau$
7	Initial Value Theorem	$\lim_{t\to 0^+}x(t)$	$\lim_{s\to\infty}sX(s)=x(0^+)$
8	Final Value Theorem	$\lim_{t\to\infty}x(t)$	$\lim_{s\to 0}sX(s)=x(\infty)$
9	Time Convolution	$x(t)*y(t)$	$X(s)Y(s)$

1. *Linearity*

Assume $x(t)=a_1x_1(t)+a_2x_2(t)$ (a_1 and a_2 are time independent)

$X_1(s)=L[x_1(t)],\qquad X_2(s)=L[x_2(t)]$

then $X(s)=L[x(t)]=a_1X_1(s)+a_2X_2(s)$

Proof:

$$\begin{aligned}L[a_1x_1(t)+a_2x(t)] &= \int_0^\infty (a_1x_1(t)+a_2x(t))e^{-st}dt\\ &= \int_0^\infty a_1x_1(t)e^{-st}+a_2x(t)e^{-st}dt\\ &= a_1\int_0^\infty x_1(t)e^{-st}dt+a_2\int_0^\infty x(t)e^{-st}dt\\ &= a_1X_1(s)+a_2X_2(s)\end{aligned}$$

2. *Complex Frequency shift (s-shift) Theorem*

Assume $y(t)=x(t)e^{-\alpha t}$

$X(s)=L[x(t)]\qquad Y(s)=L[y(t)]$

Then $Y(s)=X(s+\alpha)$

3. *Time Delay Theorem*

Assume $L[x(t)]\equiv L[x(t)u(t)]=X(s)$

Then $L[x(t-t_0)u(t-t_0)]=e^{-st_0}X(s)\qquad (t_0>0)$

4. *Scaling*

Assume $X(s) = L[x(t)]$ then $L[x(at)] = \frac{1}{a}X(\frac{s}{a})$

Restriction: $a>0$

$x(at) \leftarrow a$ times fast (if $a>1$) or slow (if $a<1$) as $x(t)$

Proof:

$$
\begin{aligned}
L[x(at)] &= \int_0^\infty x(at)e^{-st}dt \\
&= \frac{1}{a}\int_0^\infty x(\tau)e^{-(s/a)\tau}d\tau, \quad \text{set } \tau = at \\
&= \frac{1}{a}X(\frac{s}{a})
\end{aligned}
$$

5. *Time Differentiation*

Assume $X(s) = L[x(t)]$

Then $L\left(\frac{dx(t)}{dt}\right) = sX(s) - x(0^-)$

Proof:

(1) Definition $L\left[\frac{dx(t)}{d(t)}\right] = \int_0^\infty \frac{dx(t)}{dt}e^{-st}dt = \int_0^\infty e^{-st}dx(t)$

(2) Integration by parts:

$$\int_a^b u(t)dv(t) = u(t)v(t)\Big|_{t=a}^{t=b} - \int_a^b v(t)du(t)$$

Make the following substitution:

$$x(t) \Rightarrow v(t)$$
$$e^{-st} \Rightarrow u(t)$$

$$
\begin{aligned}
L\left[\frac{d}{dt}x(t)\right] &= \int_0^\infty e^{-st}dx(t) \\
&= e^{-st}x(t)\Big|_{t=0}^{t=\infty} - \int_0^\infty x(t)de^{-st} \\
&= 0 - x(0^-) - (-s)\int_0^\infty x(t)e^{-st}dt \\
&= sX(s) - x(0^-)
\end{aligned}
$$

In the ROC of X(s), we must have $\lim_{t\to\infty}[e^{-st}x(t)] = 0$ otherwise the Laplace integral which is the area under $e^{-st}x(t)$ from 0 to t=∞, will become ∞.

ROC for Laplace Transform

Laplace transform of x(t) is $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$

The range of values of ‗s' for which the integral in the equation converges is referred to as the region of convergence(ROC).

2.8 Properties of ROC for LaplaceTransform

1. ROC of $X(s)$ consists of strips parallel to the $j\Omega$axis in the s-plane.
2. For rational Laplace Transform, the ROC does not contain anypoles.

   $X(s) = \dfrac{N(s)}{D(s)} \Rightarrow general\ form of\ rational\ LT$

   $Roots\ of\ \ D(s) = 0\ \ are\ \ poles$

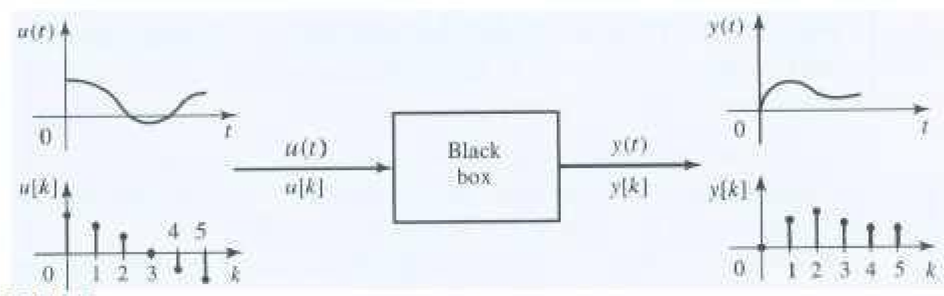
3. If *x(t)* is of finite duration and if there is at least one value of **s** for which the LT converges, then the ROC is the entires-plane.
4. If *x(t)* is right sided and if the line $\mathrm{Re}\{s\}=\sigma_0$is in the ROC , then all values of **s** forwhich $\mathrm{Re}\{s\}>\sigma_0$will also be in the ROC.
5. If *x(t)* is left sided and if the line $\mathrm{Re}\{s\}=\sigma_0$ is in the ROC , then all values of **s** for which $\mathrm{Re}\{s\}<\sigma_0$will also be in the ROC.
6. If *x(t)* is two sided and if the line $\mathrm{Re}\{s\}=\sigma_0$ is in the ROC , then the ROC will consist of a strip in the s-plane which includes the line $\mathrm{Re}\{s\}=\sigma_0$.

UNIT III

LINEAR TIME INVARIANT –CONTINUOUS TIME SYSTEMS

3.1 System:

A system is an operation that transforms input signal *x* into output signal *y*.



3.2 LTISystems

• Time Invariant

–X(t) y(t) &x(t-to) y(t-to)

• Linearity

– a1x1(t)+a2x2(t) a1y1(t)+a2y2(t)

– a1y1(t)+ a2y2(t)=T[a1x1(t)+a2x2(t)]

• Meet the description of many physicalsystems

• They can be modeledsystematically

– Non-LTI systems typically have no general mathema tical procedure to obtainsolution

Differential equation:

• This is a linear first order differential equatio n with constant coefficients (assuming a and bare constants)

$$\frac{d}{dt}y(t) - ay(t) = bx(t)$$

The general nth order linear DE with constant equations is

$$a_0 y(t) + a_1 \frac{d}{dt}y(t) + \ldots + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + a_n \frac{d^n}{dt^n}y(t) =$$

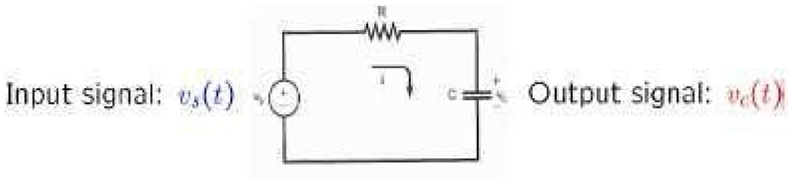
$$b_0 x(t) + b_1 \frac{d}{dt}x(t) + \ldots + b_{m-1}\frac{d^{m-1}}{dt^{m-1}}x(t) + b_m \frac{d^m}{dt^m}x(t)$$

which we can write as:

$$\sum_{k=0}^{n} a_k \frac{d^k}{dt^k}y(t) = \sum_{k=0}^{m} b_k \frac{d^k}{dt^k}x(t).$$

Linear constant-coefficient differential equations In RC circuit

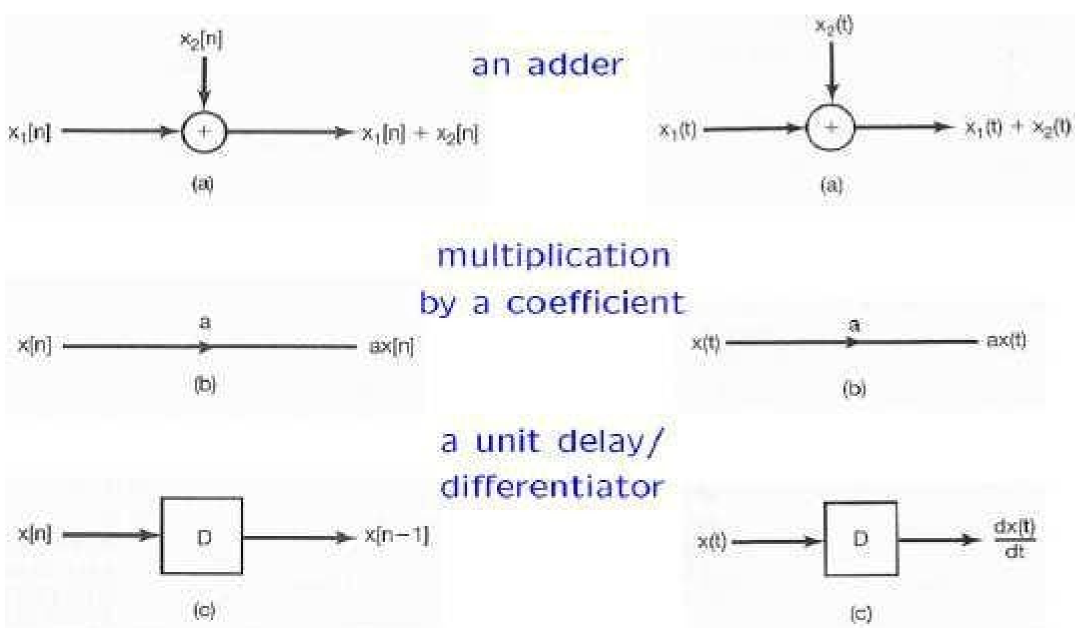
- To introduce some of the important ideas concerning systems specified by linearconstant-coefficient differential equations ,let us consider a first-order differentialequations:



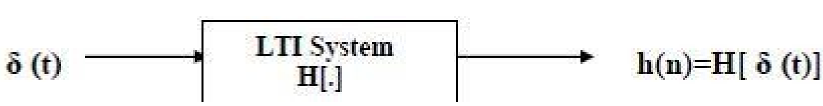
$$\Rightarrow \frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$

3.3 Block diagramrepresentations

Block diagram representations of first-order systems described by differential and difference equations



3.4 ImpulseResponse



This impulse response signal can be used to infer properties about the system's structure (LHS of difference equation or unforced solution). The system impulse response, $h(t)$ completely characterises a linear, time invariant system

3.5 Properties of System ImpulseResponse

Stable

A system is stable if the impulse response is absolutely summable

Causal

A system is causal if $h(t)$=0 when t<0

Finite/infinite impulse response

The system has a finite impulse response and hence no dynamics in $y(t)$ if there exists T>0, such that: $h(t)$=0 when t>T

3.6 ConvolutionIntegral

• An approach (available tool or operation) to desc ribe the input-output relationship for LTISystems

• In a LTIsystem
– d(t) h(t)
– Remember h(t) isT[d(t)]
– Unitimpulsefunction the impulseresponse
• It is possible to use h(t) to solve for any input -outputrelationship
• Any input can be expressed using the unit impulsefunction

3.6.1 Convolution Integral –Properties

- Commutative $x(t) * h(t) = h(t) * x(t)$
- Associative $[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$
- Distributive $x(t) * [h_1(t) + h_2(t)] = [x(t) * h_1(t)] + [x(t) * h_2(t)]$
- Thus, using commutative property:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k) \qquad 10$$

$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) - a_3\, y(n-3) - \ldots\ldots\ldots - a_{N-1}\, y(n-N+1) - a_N\, y(n-N) + b_1\, x(n-1) + b_2\, x(n-2) + b_3(n-3) - \ldots\ldots\ldots + b_{M-1}\, x(n-M+1) + b_M\, x(n-M)$

IIR FILTER STRUCTURE REALIZATION
DIRECT FORM I

A linear time invariant discrete-time systems characterized by the general linear constant coefficient difference equation

$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k) \qquad 10$$

$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) - a_3\, y(n-3) - \ldots\ldots\ldots - a_{N-1}\, y(n-N+1) - a_N\, y(n-N) + b_1\, x(n-1) + b_2\, x(n-2) + b_3(n-3) - \ldots\ldots\ldots + b_{M-1}\, x(n-M+1) + b_M\, x(n-M)$

Let
$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) - a_3\, y(n-3) - \ldots\ldots\ldots - a_{N-1}\, y(n-N+1) - a_N\, y(n-N) + w(n)$

$w(n) = b_1\, x(n-1) + b_2\, x(n-2) + b_3\, x(n-3) - \ldots\ldots\ldots + b_{M-1}\, x(n-M+1) + b_M\, x(n-M)$

The structure realization of above equations

we get

$$\left[x(t) * y(t)\right] * z(t) = \left[\int_{-\infty}^{\infty} x(\tau_{xy})\, y(t-\tau_{xy})\, d\tau_{xy}\right] * z(t)$$

D.1.3 Distributivity Property

Convolution is also distributive,

$$x(t) * \left[h_1(t) + h_2(t)\right] = x(t) * h_1(t) + x(t) * h_2(t).$$

$$x(t) * \left[h_1(t) + h_2(t)\right] = \int_{-\infty}^{\infty} x(t)\left[h_1(t-\tau) + h_2(t-\tau)\right] d\tau$$

$$x(t) * \left[h_1(t) + h_2(t)\right] = \int_{-\infty}^{\infty} x(t)\, h_1(t-\tau)\, d\tau + \int_{-\infty}^{\infty} x(t)\, h_2(t-\tau)\, d\tau$$

$$x(t) * \left[h_1(t) + h_2(t)\right] = x(t) * h_1(t) + x(t) * h_2(t)$$

3.7 BLOCK DIAGRAM REPRESENTATION-STRUCTUREREALIZATION

IIR Systems are represented in four different
ways Direct Form Structures Form I and Form II
Cascade Form Structure
Parallel Form Structure
Lattice and Lattice-Ladder structure.

DIRECT FORM STRUCTURE FOR IIR SYSTEMS

IIR systems can be described by a generalized equations as

$$y(n)=-\sum_{k=1}^{N} a_k\, y(n-k)+\sum_{k=0}^{M} b_k\, x(n-k)$$

Z transform is given as

$$H(z) = \sum_{K=0}^{M} b_k z^{-k} \Big/ 1+ \sum_{k=1}^{N} a_k z^{-k}$$

Here $H1(z) = \sum_{K=0}^{M} b_k z^{-k}$ And $H2(z) = 1+ \sum_{k=0}^{N} a_k z^{-k}$

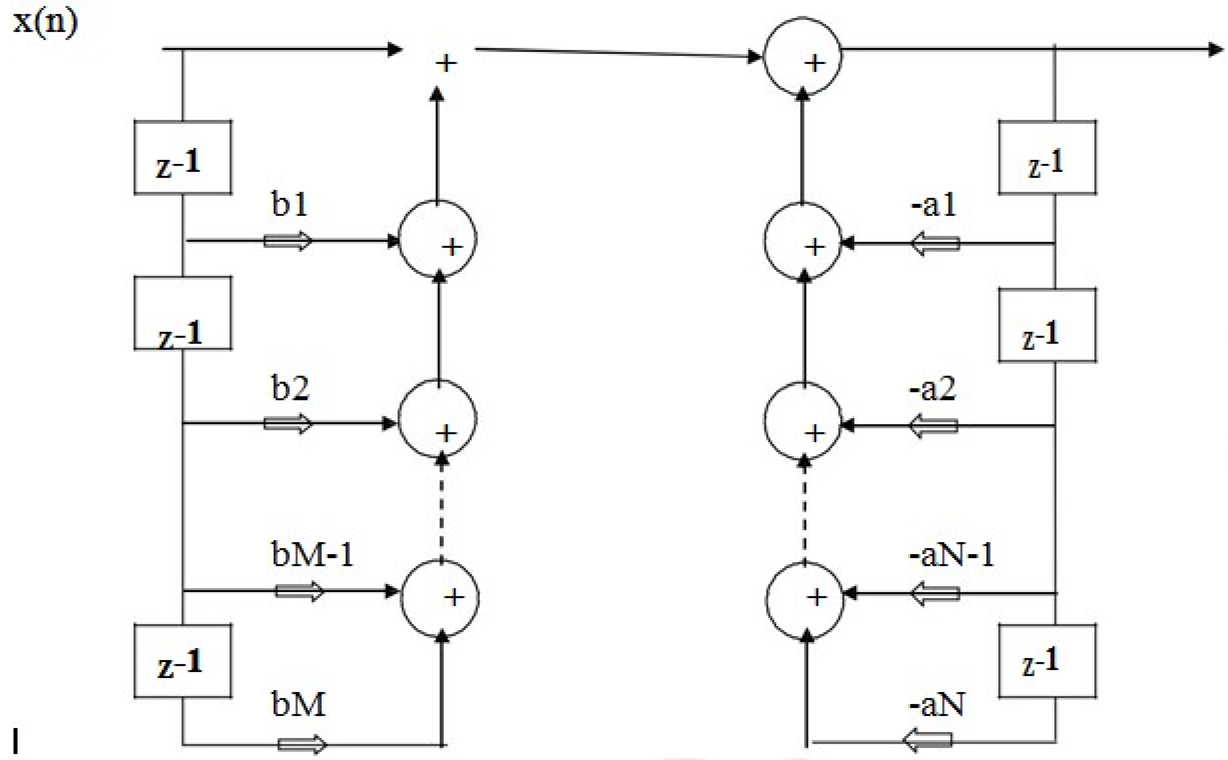
Overall IIR system can be realized as cascade of two function H1(z) and H2(z). Here H1(z) represents zeros of H(z) and H2(z) represents all poles of H(z).

1. Direct form I realization of H(z) can be obtained by cascading the realization of H1(z) which is all zero system first and then H2(z) which is all pole system.

· There are M+N-1 unit delay blocks. One unit delay block requires one memory location. Hence direct form structure requires M+N-1 memorylocations.

3. Direct Form I realization requires M+N+1 number of multiplications and M+N number of additions and M+N+1 number of memory locations.

DIRECT FORM -I



DIRECT FORM - II

1. Direct form realization of H(z) can be obtained by cascading the realization of H1(z) which is all pole system and H2(z) which is all zero system.

· Two delay elements of all pole and all zero system can be merged into single delayelement.

· Direct Form II structure has reduced memory requirement compared to Direct form I structure. Hence it is called canonicform.

· The direct form II requires same number of multiplications(M+N+1) andadditions (M+N) as that of direct formI.

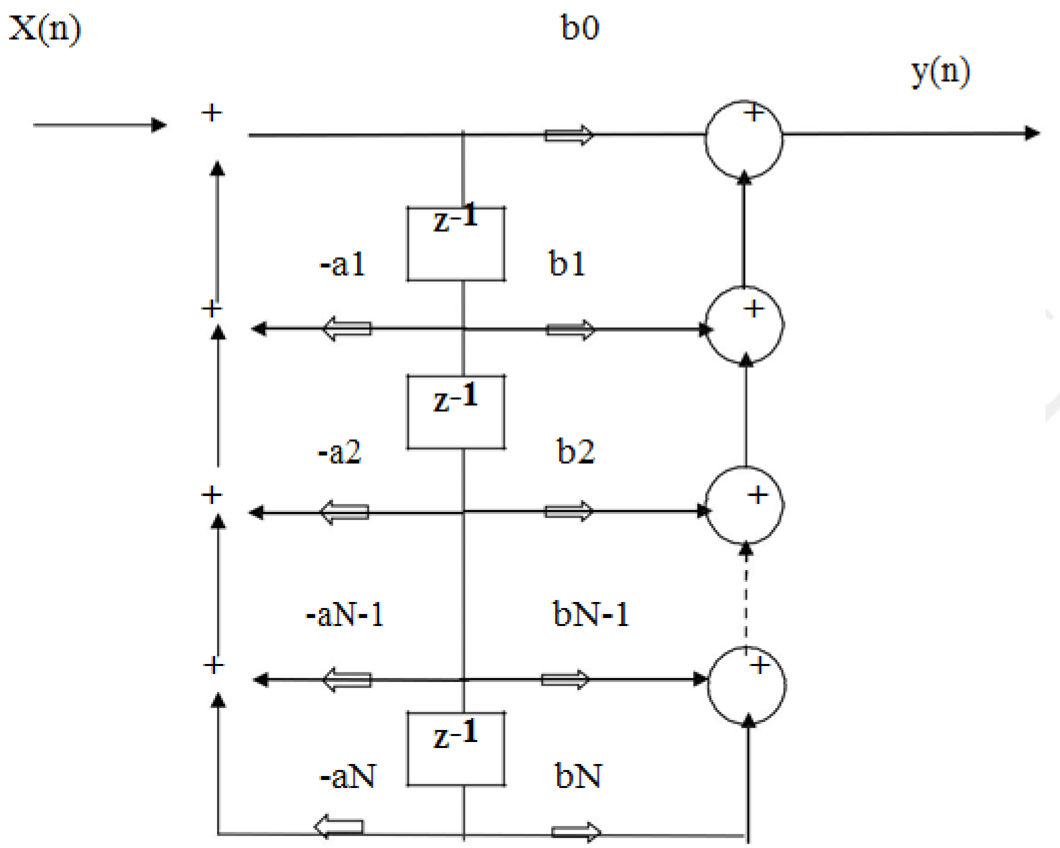


FIG - DIRECT FORM II REALIZATION OF IIR SYSTEM

CASCADE FORM STRUCTURE FOR IIR SYSTEMS

In cascade form, stages are cascaded (connected) in series. The output of one system is input to another. Thus total K number of stages are cascaded. The total system function 'H' is given by

$$H = H_1(z).H_2(z)\ldots\ldots\ldots H_k(z) \quad (1)$$

$$H = Y_1(z)/X_1(z).Y_2(z)/X_2(z).\ldots\ldots\ldots Y_k(z)/X_k(z) \quad (2)$$

$$H(z) = \prod_{k=1}^{k} H_k(z) \quad (3)$$

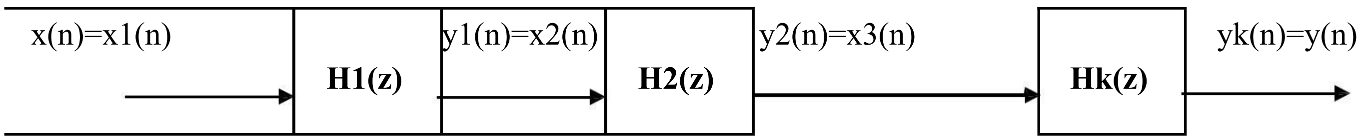


FIG - CASCADE FORM REALIZATION OF IIR SYSTEM

Each H1(z), H2(z)… etc is a second order section and it is realized by direct form 2.

PARALLEL FORM STRUCTURE FOR IIR SYSTEMS

System function for IIR systems is given as

$$H(z) = \sum_{K=0}^{M} b_k z^{-k} / 1 + \sum_{k=1}^{N} a_k z^{-k} \qquad (1)$$

$$= b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots\ldots + b_M z^{-M} / 1 + a_1 z^{-1} + a_2 z^{-2} + \ldots\ldots + a_N z^{-N} \qquad (2)$$

The above system function can be expanded in partial fraction as follows

$$H(z) = C + H_1(z) + H_2(z) \ldots\ldots\ldots\ldots\ldots\ldots + H_k(z) \qquad (3)$$

Where C is constant and Hk(z) is given as

$$Hk(z) = b_{k0} + b_{k1} z^{-1} / 1 + a_{k1} z^{-1} + a_{k2} z^{-2} \qquad (4)$$

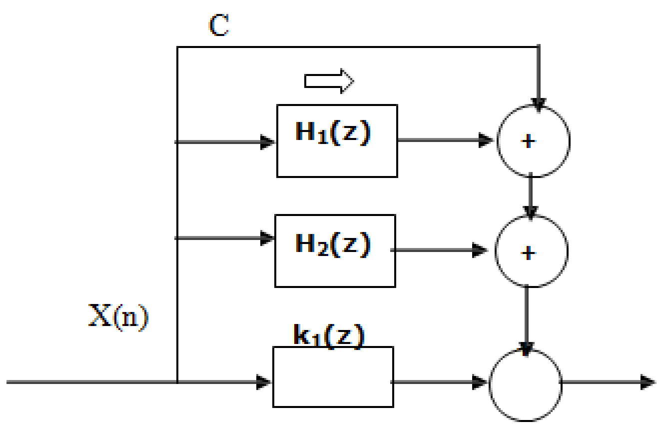
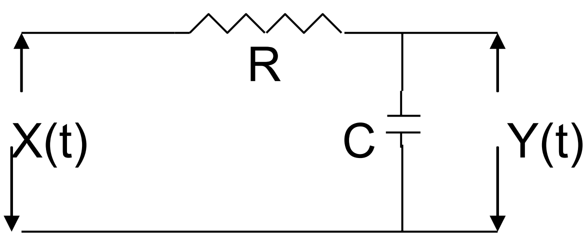


FIG - PARALLEL FORM REALIZATION OF IIR SYSTEM

3.8 PROBLEMS

1. Determine the output response of RC Low pass network shown in figure due toinput $x(t) = te^{-\frac{t}{RC}}$ by convolution.



For the circuit shown in figure,

$$x(t) = Ri(t) + \frac{1}{C}\int i(t)dt \qquad \text{------------------(1)}$$

$$y(t) = \frac{1}{C}\int i(t)dt \qquad \text{------------------(2)}$$

Taking LT of the equations and rearranging,

$$X(s) = RI(s) + \frac{I(s)}{sC} \qquad \text{------------------(1)}$$

$$Y(s) = \frac{I(s)}{sC} \qquad \text{------------------(2)}$$

$$X(s) = sRCY(s) + Y(s)$$

$$H(s) = \frac{Y(s)}{X(s)}$$

$$H(s) = \frac{1}{(sRC+1)} = \frac{\frac{1}{RC}}{(s+\frac{1}{RC})}$$

$$x(t) = te^{\frac{t}{RC}}$$

$$X(s) = \square \frac{1}{(s+\frac{1}{RC})^2}$$

$y(t) = x(t) * h(t)$

$$Y(s) = X(s)H(s) = \frac{\frac{1}{RC}}{(s+\frac{1}{RC})^3}$$

$$y(t) = L^{-1}(Y(s)) = \frac{1}{2RC} t^2 e^{-t/RC}$$

---

2. The Input and Output of a causal LTI system are related by the differentialequation $\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)$. Using Fourier Transform

(i) Find the Impulse response of thesystem

(ii) Find the response of the systemif $x(t) = e^{-3t}u(t)$

SOLUTION:

(i) Impulse response of thesystem:

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t) \quad \text{---------------(1)}$$

Applying Fourier Transform to the given Differential Eq.

$$j\Omega^2 Y(j\Omega) + j\Omega 6Y(j\Omega) + 8Y(j\Omega) = 2X(j\Omega)Y(j\Omega)$$

$$(j\Omega^2 + 6j\Omega + 8) = 2X(j\Omega)$$

$$Y(j\Omega) = \frac{2X(j\Omega)}{(j\Omega^2 + 6j\Omega + 8)} \quad \text{----------------------(2)}$$

Here $x(t) = \delta(t)$ and $X(j\Omega) = 1$

using partial fraction expansion

$$Y(j\Omega) = \frac{A}{j\Omega + 2} + \frac{B}{j\Omega + 4}$$

Set $j\Omega = -2$ then $A(j\Omega+2)+B(j\Omega+4)=2$, *gives* B = 1

Set $j\Omega = -4$ then $A(j\Omega+2)+B(j\Omega+4)=2$, *gives* A =–1

$$\therefore Y(j\Omega) = \frac{-1}{j\Omega + 2} + \frac{1}{j\Omega + 4}$$

By taking inverse FT , $y(t) = e^{-4t}u(t) - e^{-2t}u(t)$

(ii) If $x(t) = e^{-3t}u(t)$ then $X(j\Omega) = \frac{1}{j\Omega + 3}$

$$Y(j\Omega)(j\Omega^2 + j\Omega 6 + 8) = \frac{2}{j\Omega + 3}$$

$$Y(j\Omega) = \frac{2}{(j\Omega+3)(j\Omega^2+6j\Omega+8)}$$

using partial fraction expansion

$$Y(j\Omega) = \frac{A}{j\Omega + 2} + \frac{B}{j\Omega + 4} + \frac{C}{j\Omega + 3}$$

Solving for A,B & C, we get A=1 , B=1 and C=-2

Then by taking Inverse FT, weget

$y(t) = e^{-2t}u(t) + e^{-4t}u(t) - 2e^{-3t}u(t)$

---

3. Realize the system described by the differential equation

$$\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 7\frac{dy(t)}{dt} + 8y(t) = 5\frac{d^2x(t)}{dt^2} + 4\frac{dx(t)}{dt} + 7x(t)$$ in Direct FormII.

SOLUTION:

Apply Laplace transform for the given differential equation with assuming zero initial conditions.

$$s^3Y(S) + 4s^2Y(S) + 7sY(S) + 8Y(S) = 5s^2X(S) + 4sX(S) + 7X(S) \quad (1)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{5s^2 + 4s + 7}{s^3 + 4s^2 + 7s + 8}$$

$$\frac{5s^2 + 4s + 7}{s^3 + 4s^2 + 7s + 8} = \frac{Y(s)}{X(s)} \times \frac{W(s)}{W(s)}$$

$$let\ \frac{W(s)}{X(s)} = \frac{1}{s^3 + 4s^2 + 7s + 8} \quad \text{---------}(1)$$

$$let\ \frac{Y(s)}{W(s)} = 5s^2 + 4s + 7 \quad \text{-----------}(2)$$

from (1)

$$(s^3 + 4s^2 + 7s + 8)W(s) = X(s) \quad \text{-----}(3)$$

from (2)

$$Y(s) = (5s^2 + 4s + 7)W(s) \quad \text{---------}(4)$$

from (3)

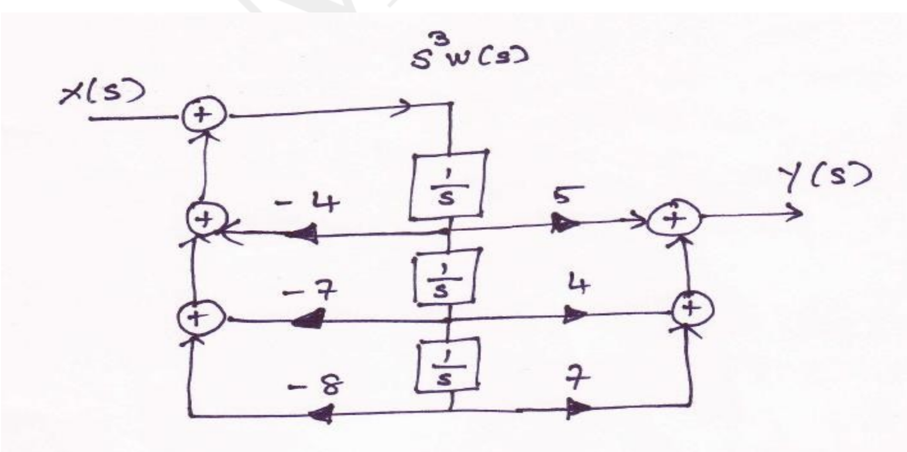
$$s^3W(s) + 4s^2W(s) + 7sW(s) + 8W(s) = X(s)$$

$$s^3W(s) = X(s) - 4s^2W(s) - 7sW(s) - 8W(s) \quad \text{-------}(5)$$

from (4)

$$Y(s) = 5s^2W(s) + 4sW(s) + 7W(s) \quad \text{---------}(6)$$

Equations (5) and (6) are used for implementing the system.



- -

4. Find the convolution of the followingsignals.

$x_1(t) = e^{-at}u(t)$; $x_2(t) = e^{-bt}u(t)$

SOLUTION:

$L[x_1(t) \times x_2(t)] = X_1(s)X_2(s)$

$x_1(t)=e^{-at}u(t)$; $X_1(s)=L[e^{-at}u(t)]= \frac{1}{s+a}$

$x_1(t)=e^{-bt}u(t)$; $X_2(s)=L[e^{-bt}u(t)]= \frac{1}{s+b}$

$Y(s) = X_1(s)X_2(s) = \frac{A}{s+a} + \frac{B}{s+b}$

$A= \frac{s+a}{(s+a)(s+b)} \mid s=-a$,then $A= \frac{1}{(b-a)}$

$A= \frac{s+b}{(s+a)(s+b)} \mid s=-b$,then $B= \frac{1}{(a-b)}$

$Y(s) = \frac{\frac{1}{(b-a)}}{s+a} - \frac{\frac{1}{(b-a)}}{s+b}$

$y(t) = \frac{1}{b-a}\left(e^{-at} - e^{-bt}\right)u(t)$

5.Find the convolution of the followingsignals:

$x(t) = e^{-2t}u(t)$

$h(t) = u(t+2).$

SOLUTION:

$L[x_1(t) \times x_2(t)] = X_1(s)X_2(s)$

$x_1(t)=e^{-2t}u(t)$ and $X_1(s)=L[e^{-2t}u(t)] = \frac{1}{s+2}$

$x_2(t)=L[u(t+2)]= \frac{e^{2S}}{s}$, [Using time shiftingproperty,$L[u(t)]= \frac{1}{s}$]

$L[x_1(t) \times x_2(t)]= \frac{e^{2S}}{s(s+2)}$

$Y(s) = \frac{e^{2S}}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$

$A = s \times \frac{1}{s(s+2)} \mid s=0$; $A = \frac{1}{2}$

$B=(s+2) \times \frac{1}{s(s+2)} \mid s=-2$; $B = -\frac{1}{2}$

$$Y_1(s) = \frac{1/2}{s} - \frac{1/2}{(s+2)}$$

$$Y(s) = e^{2S}\left(\frac{1}{2s} - \frac{1}{2(s+2)}\right)$$

$$Y(s) = \frac{e^{2S}}{2s} - \frac{e^{2S}}{2(s+2)}$$

$$y(t) = \frac{1}{2}u(t+2) - \frac{1}{2}e^{-2(t+2)}u(t+2)$$

6.i) Using Laplace transform , find the impulse response of an LTI system described by thedifferentialequation $\frac{d^2}{dt}y(t) - \frac{dy(t)}{dt} - 2y(t) = x(t)$

SOLUTION:

Apply Laplace transform for the given differential equation with assuming zero initial conditions .

$$s^2Y(S) - sY(S) - 2Y(S) = X(S)$$

$$Y(S)\left[s^2 - s - 2\right] = X(S)$$

$$H(S) = \frac{Y(S)}{X(S)} = \left[\frac{1}{s^2 - s - 2}\right]$$

$$= \frac{1}{\left[(s+1)(s-2)\right]}$$

$$= \frac{A}{(S-2)} + \frac{B}{(S+1)}$$

$$= \frac{1}{3}\frac{1}{(S-2)} - \frac{1}{3}\frac{1}{S+1}$$

$$h(t) = \frac{1}{3}e^{2t}u(t) - \frac{1}{3}e^{-t}u(t)$$

---

ii) Explain the properties of Convolutionintegral.

(1) Commutativeproperty

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

(2) Associative property

$$[x_1(t) * x_2(t)] * x_3(t) = x_1(t) * [x_2(t) * x_3(t)]$$

(3) Distributiveproperty

$$x_1(t) * [x_2(t) + x_3(t)] = [x_1(t) * x_2(t)] + [x_1(t) * x_3(t)]$$

(4) shift property

$$if\ x_1(t) * x_2(t) = z(t)\ then$$

$$x_1(t) * x_2(t-T) = z(t-T)$$

(5) Convolution with animpulse

$$x(t) * \delta(t) = x(t)$$

(6) Convolution with unitstep

$$x(t)*u(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

And $$x(t)*u(t-t_0) = \int_{-\infty}^{t-t_0} x(\tau)d\tau$$

(7) Width property

Let the duration of $x_1(t)$ be T_1

Let the durationof $x_2(t)$ be T_2

Then the duration of the signal obtained by convolving $x_1(t)$ and $x_2(t)$ is T_1+T_2

8.i) Solve the differential equation:

$$\frac{d^2}{dt}y(t)+4\frac{dy(t)}{dt}+5y(t)=5x(t) \text{ with } y(0^-)=1 \text{ and } \left.\frac{dy(t)}{dt}\right|_{0^-}=2 \text{ and } x(t)=u(t)$$

SOLUTION:

To solve the given differential equation we can use LT method.
Applying LT to the given differential Eq. With given initial conditions.

$$s^2Y(S)-sy(0^-)-\frac{dy(0^-)}{dt}+4\left(sY(s)-y(0^-)\right)+5Y(s)=5X(s)$$

$$s^2Y(S)-s-2+4(sY(s)-1)+5Y(s)=5X(s)$$

$$s^2Y(S)+4sY(s)+5Y(s)=\frac{5}{s}+s+6$$

$$(s^2+4s+5)Y(s)=\frac{5}{s}+s+6$$

$$Y(s)=\frac{5}{s(s^2+4s+5)}+\frac{s}{(s^2+4s+5)}+\frac{6}{(s^2+4s+5)} \quad \text{----------(1)}$$

Applying Partial fraction to the first term in Y(s)

$$\frac{5}{s(s^2+4s+5)}=\frac{A}{s}+\frac{Bs+C}{(s^2+4s+5)}$$

$$\frac{5}{s(s^2+4s+5)}=\frac{A(s^2+4s+5)+(Bs+C)s}{s(s^2+4s+5)}$$

$$\frac{5}{s(s^2+4s+5)}=\frac{A((s+2)^2+1)+(Bs+C)s}{s(s^2+4s+5)}$$

By solving this Eq. We can get the values of A, B and C

A=1, B=-1 and C=-4

By substituting the values in Eq. (1) and rearranging the terms we have

$$Y(s)=\frac{1}{s}-\frac{s}{((s+2)^2+1)}-\frac{4}{((s+2)^2+1)}+\frac{6}{((s+2)^2+1)}+\frac{s}{((s+2)^2+1)}$$

$$=\frac{1}{s}+\frac{2}{((s+2)^2+1)}$$

By taking inverse LT we can get y(t)

$$y(t)=u(t)+2e^{-2t}\sin t$$

---

ii) The system is described by the input output relation $\frac{d^2}{dt^2}y(t)+\frac{dy(t)}{dt}-3y(t)=\frac{dx(t)}{dt}+2x(t)$.
Find the system transfer function, frequency response and impulse response.

System Transfer function:

Applying Laplace Transform to the given differential equation with zero initial conditions,

$$s^2Y(S)+sY(S)-3Y(S)=sX(s)+2X(s)$$

$$Y(s)=\frac{(s+2)}{(s^2+s-3)}X(s)$$

$$\frac{Y(s)}{X(s)}=\frac{(s+2)}{(s^2+s-3)}=H(s)\Rightarrow systemtranferfunction$$

Frequence response of the System :

Applying FOURIER Transform to the given differential equation.

$$j\Omega^2Y(j\Omega)+j\Omega Y(j\Omega)-3Y(j\Omega)=j\Omega X(j\Omega)+2X(j\Omega)\,Y(j\Omega)(j\Omega^2+j\Omega-3)=(j\Omega+2)X(j\Omega)$$

$$Y(j\Omega)=\frac{(j\Omega+2)}{(j\Omega^2+j\Omega-3)}X(j\Omega)$$

$$H(j\Omega)=\frac{Y(j\Omega)}{X(j\Omega)}=\frac{(j\Omega+2)}{(j\Omega^2+j\Omega-3)}\Rightarrow Frequencyresponse$$

Impulse response of the system:

$$s^2Y(S)+sY(S)-3Y(S)=sX(s)+2X(s)$$

$$Y(s)=\frac{(s+2)}{(s^2+s-3)}X(s)$$

$$\frac{Y(s)}{X(s)}=\frac{(s+2)}{(s^2+s-3)}=H(s)$$

Impulse response of the systemh(t)

h(t) = inverse LT of H(s),

$$\frac{Y(s)}{X(s)}=\frac{(s+2)}{(s^2+s-3)}=H(s)$$

---

9. Draw the direct form I and II implementations of the system describedby

$$\frac{dy(t)}{dt}+5y(t)=3x(t)$$

Applying LT to the given differential Eq.

$$sY(s)+5Y(s)=3X(s)$$

$$(s+5)Y(s)=3X(s)$$

$$\frac{Y(s)}{X(s)}=\frac{3}{(s+5)}=H(s)$$

Direct form I

$$\frac{Y(s)}{X(s)} = \frac{3}{(s+5)} = \frac{\frac{3}{s}}{\left(1+\frac{5}{s}\right)}$$

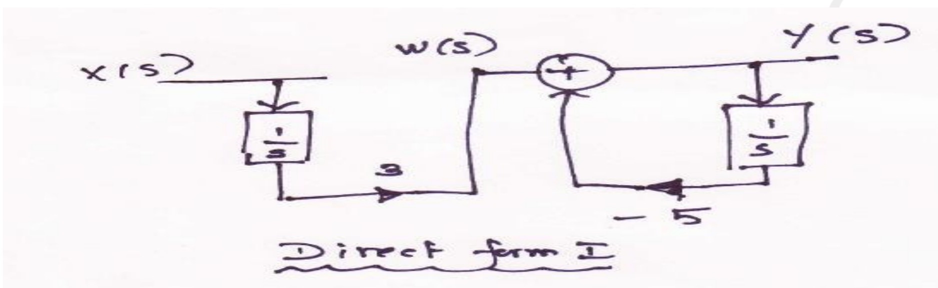
$$Y(s) + \frac{5}{s}Y(s) = \frac{3}{s}X(s) \quad \text{-----}(1)$$

$$let \ \frac{3}{s}X(s) = W(s) \quad \text{--------} \ (2)$$

from (1) *and* (2)

$$Y(s) = W(s) - \frac{5}{s}Y(s) \text{------}(3)$$

Using (2) and (3) the system is implemented asfollows:



Direct form II

$$H(s) = \frac{Y(s)}{X(s)} = \frac{3}{(s+5)} = \frac{Y(s)}{X(s)} \times \frac{W(s)}{W(s)}$$

$$let, \ \frac{W(s)}{X(s)} = \frac{1}{(s+5)} \text{-------}(1)$$

$$let, \ \frac{Y(s)}{W(s)} = 3 \quad \text{----------}(2)$$

from (1)

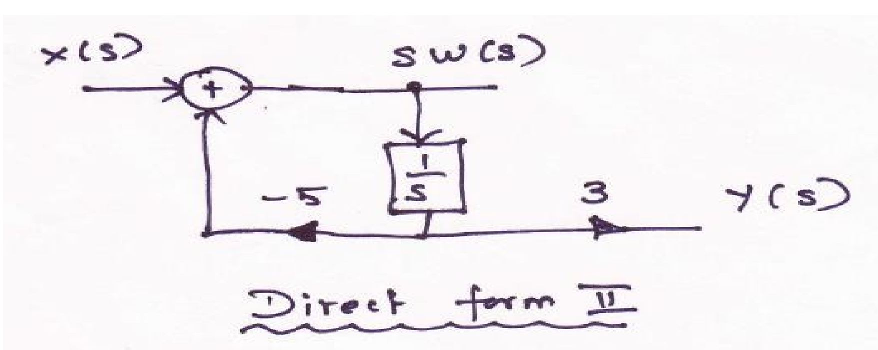
$$sW(s) + 5W(s) = X(s)$$

$$sW(s) = X(s) - 5W(s) \text{------}(3)$$

from (2)

$$Y(s) = 3W(s) \quad \text{---------}(4)$$

Using (3) and (4) the system is implemented as follows:



10. The input and the output of a causal CTI system are related by thedifferential equation $\frac{d^2y(t)}{dt^2} + 6\frac{d}{dt}y(t) + 8y(t) = 2x(t)$ .Find impulse response of the system.

SOLUTION:

Apply Laplace transform for the given differential equation with assuming zero initial conditions .

$s^2Y(S)+6sY(S)+8Y(S)=2X(S)$

$H(S)= \frac{Y(S)}{X(S)} = \frac{2}{[s^2+6s+8]}$

$x(t) = \delta(t)$ and $X(s) = 1$

$H(S)= \frac{2}{[s^2+6s+8]}X(s)$

$\therefore H(s) = \frac{A}{(s+2)} + \frac{B}{(s+4)} = \frac{2}{(s+2)(s+4)}$

$A=(s+2)\times \frac{2}{(s+4)(s+2)} \Big|_{s=-2} \quad \therefore A = 1$

$B=(s+4)\times \frac{2}{(s+4)(s+2)} \Big|_{s=-4} \quad \therefore B = -1$

$\therefore H(s) = \frac{1}{s+2} - \frac{1}{s+4}$

Taking inverse Laplace transform to get impulse response

$h(t) = e^{-2t}u(t) - e^{-4t}u(t)$

3.9 Convolution integralexamples

To find the output of the system with impulse response

$$h(t) = e^{-2t}, \quad t > 0$$

to the input

$$f(t) = \begin{cases} 0, & t < 0 \quad \text{(section 1)} \\ 1, & 0 \le t \le 1 \quad \text{(section 2)} \\ 0, & 1 < t \quad \text{(section 3)} \end{cases}$$

we will use the convolution integral

$$y(t) = \int_{-\infty}^{+\infty} f(\lambda)h(t-\lambda)d\lambda$$

Because the input function has three distinct regions t<0, 0<t<1 and 1<t, we will need to split up the integral into three parts.

Section 1: t<0

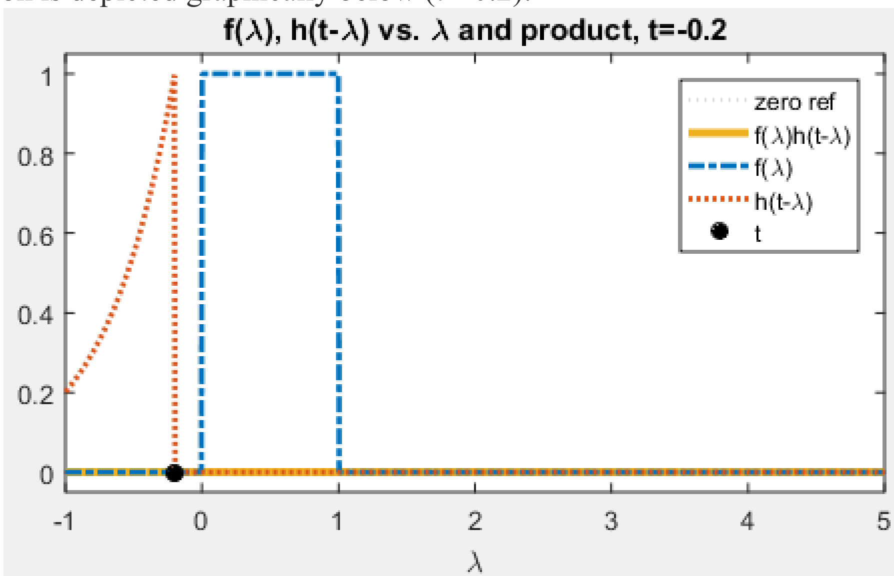
For t<0 the argument of the impulse function (t-λ) is always negative. Since h(t-λ)=0 for (t-λ)<0, the result of the integral is zero for t<0.

This situation is depicted graphically below (t=-0.2):

Section 1: t<0

For t<0 the argument of the impulse function (t-λ) is always negative. Since h(t-λ)=0 for (t-λ)<0, the result of the integral is zero for t<0.

This situation is depicted graphically below (t=-0.2):



The result for the first part of our solution is the integral of the yellow line (which is always zero),

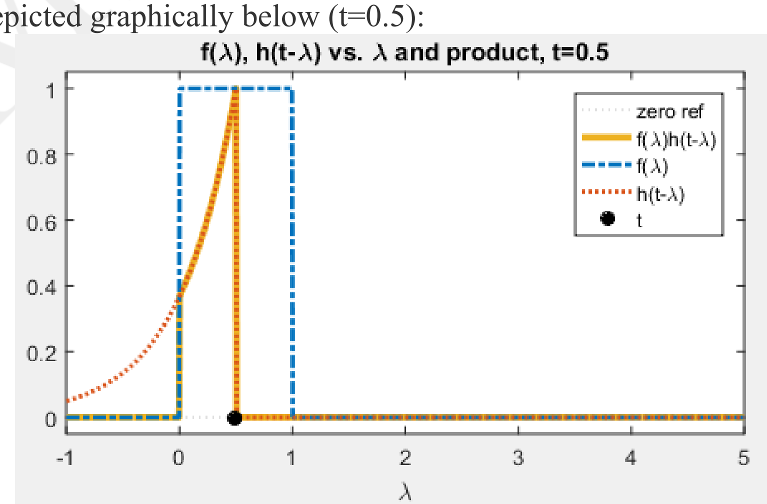
$$y(t) = 0, \quad t < 0$$

Section 2: 0<t<1

For 0<t<1 we need to evaluate the integral only from λ=0 to λ=t, since f(λ)=0 when λ<0, and h(t-λ)=0 when (t-λ)<0 (or, equivalently t<λ). So the integral becomes, in effect:

$$y(t) = \int_0^t f(\lambda)h(t-\lambda)d\lambda$$

This situation is depicted graphically below (t=0.5):



We can now evaluate the integral of the yellow line:

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} f(\lambda)\cdot h(t-\lambda)d\lambda \\ &= \int_0^t f(\lambda)\cdot h(t-\lambda)d\lambda = \int_0^t 1\cdot e^{-2(t-\lambda)}d\lambda \\ &= e^{-2t}\int_0^t e^{2\lambda}d\lambda = e^{-2t}\left(\frac{1}{2}e^{2t}\Big|_0^t\right) \\ &= e^{-2t}\left(\frac{1}{2}\left(e^{2t}-1\right)\right) \\ &= \frac{1}{2}\left(1-e^{-2t}\right) \end{aligned}$$

Thus, the result for the second part of the solution is

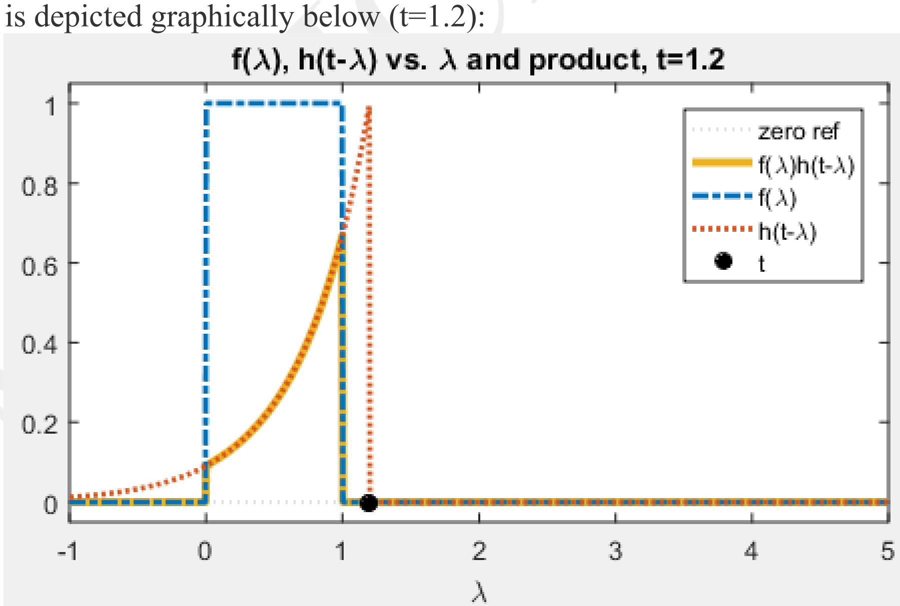
$$y(t) = \frac{1}{2}\left(1-e^{-2t}\right), \quad 0 \le t \le 1$$

Section 3: 1<t

For 1<t we need to evaluate the integral only from λ=0 to λ=1, since f(λ)=0 when λ<0 and when λ>1. So the integral becomes, in effect:

$$y(t) = \int_0^1 f(\lambda)\cdot h(t-\lambda)d\lambda$$

This situation is depicted graphically below (t=1.2):



We can now evaluate the integral:

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{+\infty} f(\lambda)\cdot h(t-\lambda)d\lambda \\
&= \int_{0}^{1} f(\lambda)\cdot h(t-\lambda)d\lambda = \int_{0}^{1} 1\cdot e^{-2(t-\lambda)}d\lambda \\
&= e^{-2t}\int_{0}^{1} e^{2\lambda}d\lambda = e^{-2t}\left(\frac{1}{2}e^{2t}\Big|_0^1\right) \\
&= e^{-2t}\left(\frac{1}{2}(e^{2}-1)\right) = e^{-2(t-1)}\left(\frac{1}{2}(1-e^{-2})\right)
\end{aligned}
$$

Thus, the result for the third part of the solution is:

$$y(t) = e^{-2(t-1)}\left(\frac{1}{2}(1-e^{-2})\right), \quad 1 < t$$

The Complete Answer

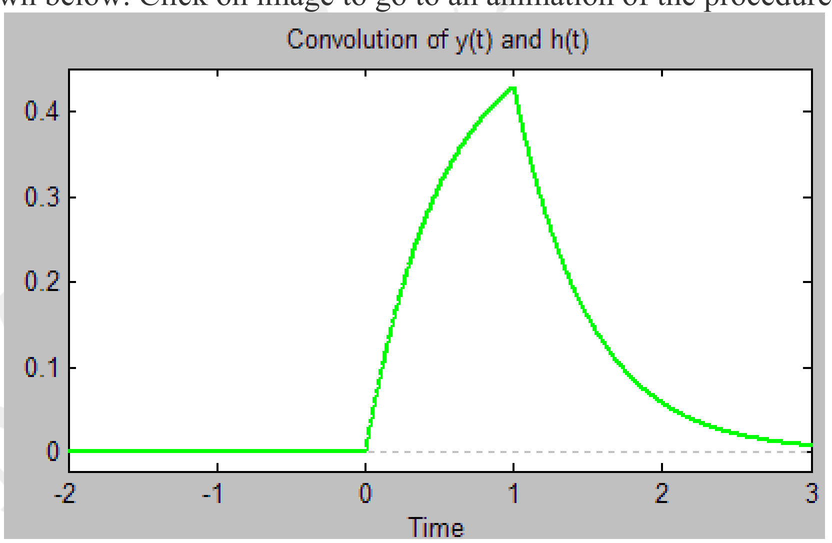
We can get the results for all time by combining the solutions from the three parts.

$$y(t) = 0, \quad t < 0$$

$$y(t) = \frac{1}{2}\left(1-e^{-2t}\right), \quad 0 \le t \le 1$$

$$y(t) = e^{-2(t-1)}\left(\frac{1}{2}(1-e^{-2})\right), \quad 1 < t$$

This result is shown below. Click on image to go to an animation of the procedure.

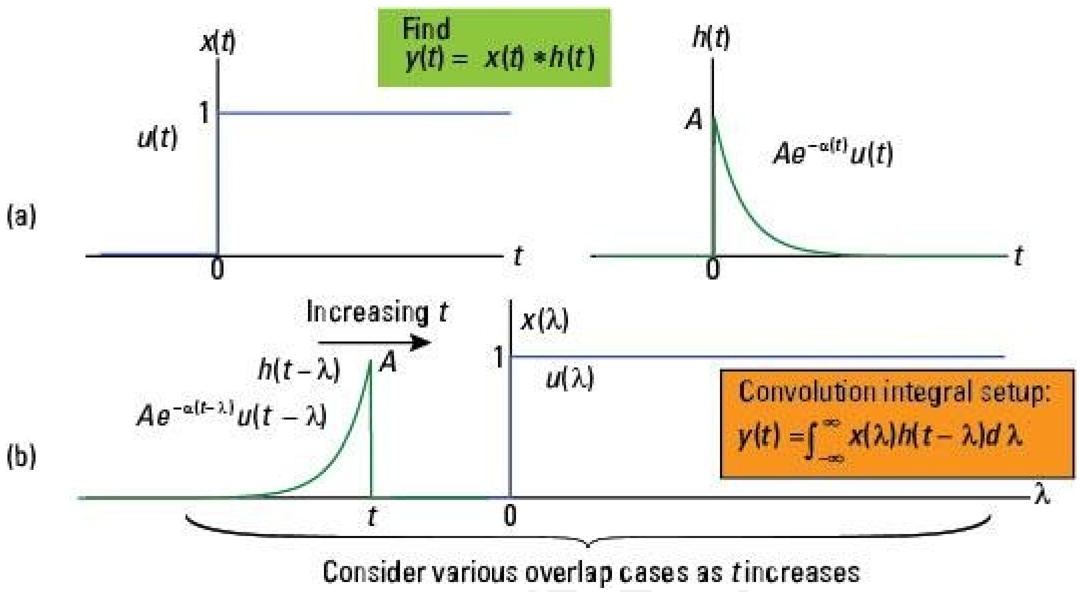


Continuous-time convolution

Here is a convolution integral example employing semi-infinite extent signals. Consider the convolution of $x(t) = u(t)$ (a unit step function) and

$$h(t) = e^{-\alpha t}u(t)$$

(a real exponential decay starting from $t = 0$). The figure provides a plot of the waveforms.



The output support interval is

$$[0+0, \infty+\infty) = [0, \infty)$$

You need two cases (steps) to form the analytical solution valid over the entire time axis.

- **Case 1:** Using Figure b, you can clearly see that for $t < 0$, it follows that$y(t)$ =0.
- **Case 2:** Again looking at Figure b, you see that for $t \geq 0$, some overlap always occurs betweenthe two signals of the integrand. The convolution integral outputis

$$y(t) = \int_0^t Ae^{-\alpha(t-\lambda)} \cdot 1\,\lambda = Ae^{-\alpha t} \cdot \frac{-e^{\alpha\lambda}}{\alpha}\Bigg|_0^t = \frac{A}{\alpha}\left[1 - e^{-\alpha t}\right]$$

Putting the two pieces together, the analytical solution for $y(t)$

$$y(t) = \frac{A}{\alpha}\left[1 - e^{-\alpha t}\right]u(t)$$

UNIT IV

ANALYSIS OF DISCRETE TIME SIGNALS

4.1 Samplingtheory

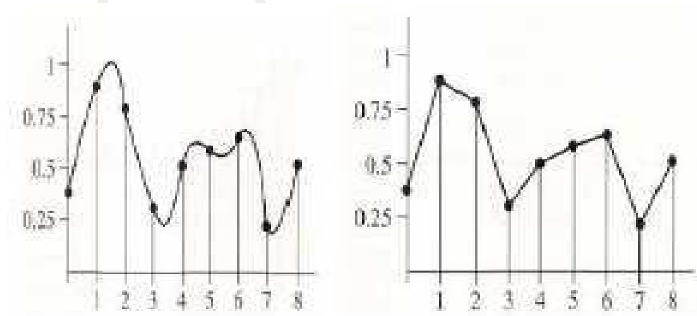
Let $x(t)$ be a continuous signal which is to be sampled, and that sampling is performed by measuring the value of the continuous signal every T seconds, which is called the sampling interval. Thus, the sampled signal $x[n]$ given by**:** $x[n] = x(nT)$, with $n = 0, 1, 2, 3, ...$

The sampling frequency or sampling rate fs is defined as the number of samples obtained in one second, or $fs = 1/T$. The sampling rate is measured in hertz or in samples per second.

The frequency equal to one-half of the sampling rate is therefore a bound on the highest frequency that can be unambiguously represented by the sampled signal. This frequency (half the sampling rate) is called the Nyquist frequency of the sampling system. Frequencies above the Nyquist frequency fN can be observed in the sampled signal, but their frequency is ambiguous. That is, a frequency component with frequency f cannot be distinguished from other components with frequencies $NfN + f$ and $NfN – f$ for nonzero integers N. This ambiguity is called aliasing. To handle this problem as gracefully as possible, most analog signals are filtered with an anti-aliasing filter (usually a low-pass filter with cutoff near the Nyquist frequency) before conversion to the sampled discrete representation.

► The theory of taking discrete sample values *(grid of color pixels)* from functions defined over continuous domains *(incident radiance defined over the film plane)* and then using those samples to reconstruct new functions that are similar to the original(reconstruction).

► **Sampler**: selects sample points on the imageplane
► **Filter**: blends multiple samplestogether



► **Samplingtheory**

Sampling Theorem: bandlimited signal can be reconstructed exactly if it is sampled at a rate atleast twice the maximum frequencycomponent in it."

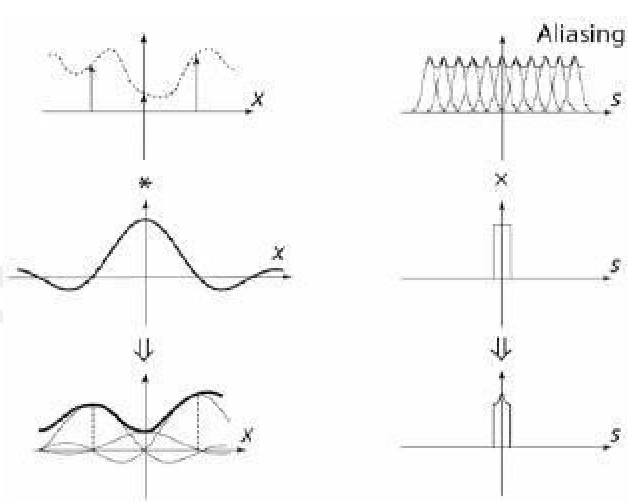
Consider a signal g(t) that is bandlimited.

The maximum frequency component of g(t) is fm. To recover the signal g(t) exactly from its samples it has to be sampled ata rate fs _ 2fm. The minimum required sampling rate fs = 2fm is called nyquist rate

A continuous time signal can be processed by processing its samples through a discrete time system. For reconstructing the continuous time signal from its discrete time samples without any error, the signal should be sampled at a sufficient rate that is determined by the sampling theorem.

4.2 Aliasing

Aliasing is a phenomenon where the high frequency components of the sampled signal interfere with each other because of inadequate sampling $\omega s < 2\omega m$. Aliasing



Aliasing leads to distortion in recovered signal. This is the reason why sampling frequency should be atleast twice the bandwidth of the signal.

.SAMPLING THEOREM

It is the process of converting continuous time signal into a discrete time signal by taking samples of the continuous time signal at discrete time instants.

X[n]= Xa(t) where t= nTs=n/Fs ….(1)

When sampling at a rate of fs samples/sec, if k is any positive or negative integer, we cannot distinguish between the samples values of fa Hz and a sine wave of (fa+ kfs) Hz. Thus (fa + kfs) wave is alias or image of fa wave.

Thus **Sampling Theorem** states that if the highest frequency in an analog signal is Fmax and the signal is sampled at the rate fs > 2Fmax then x(t) can be exactly recovered from its sample values. This sampling rate is called Nyquist rate of sampling. The imaging or aliasing starts after Fs/2 hence folding frequency is fs/2. If the frequency is less than or equal to 1/2 it will be represented properly.

Example:

Case1: X1(t) = cos 2∏(10) t Fs=40 Hz i.e t= n/Fs
x1[n]= cos 2∏(n/4)= cos(∏/2)n

Case2: X1(t) = cos 2∏(50) t Fs=40 Hz i.e t=
n/Fs x1[n]= cos 2∏(5n/4)= cos 2∏(1+¼)n
= cos
(∏/2)n

Thus the frequency 50 Hz, 90 Hz , 130 Hz … are alias of the frequency 10 Hz at the sampling rate of 40 samples/sec

.QUANTIZATION

The process of converting a discrete time continuous amplitude signal into a digital signal by expressing each sample value as a finite number of digits is called quantization. The error introducedin
representing the continuous values signal by a finite set of discrete value levels is called quantization error or quantization noise.

Example: $x[n] = 5(0.9)^n u(n)$ where 0 <n<∞ & fs= 1Hz

N	[n]	X_q [n] Rounding	X_q [n] Truncating	e_q [n]
0	5	5.0	5.0	0
1	4.5	4.5	4.5	0
2	4.05	4.0	4.0	-0.05
3	3.645	3.6	3.6	-0.045
4	3.2805	3.2	3.3	0.0195

Quantization Step/Resolution : The difference between the two quantization levels is called quantization step. It is given by $\Delta = X_{Max} - x_{Min} / L-1$ where L indicates Number of quantizationlevels.

.CODING/ENCODING

Each quantization level is assigned a unique binary code. In the encoding operation, the quantization
sample value is converted to the binary equivalent of that quantization level. If 16 quantization levels are present, 4 bits are required. Thus bits required in the coder is the smallest integer greater than or equal toLog_2L. i.e $b= Log_2L$

Thus Sampling frequency is calculated as fs=Bit rate / b.

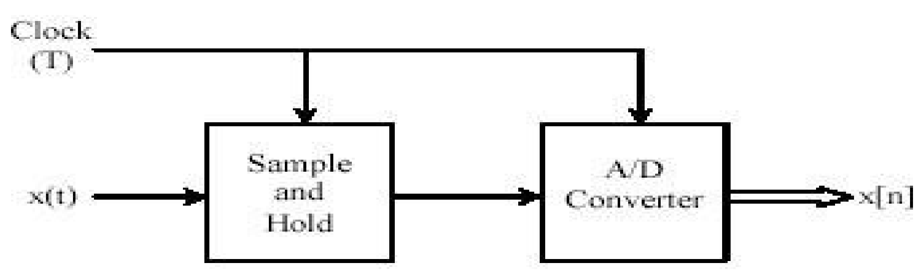
.ANTI-ALIASING FILTER

When processing the analog signal using DSP system, it is sampled at some rate depending upon the
bandwidth. For example if speech signal is to be processed the frequencies upon 3khz can be used. Hence the sampling rate of 6khz can be used. But the speech signal also contains some frequency components more than 3khz. Hence a sampling rate of 6khz will introduce aliasing. Hence signal should be band limited to avoidaliasing.

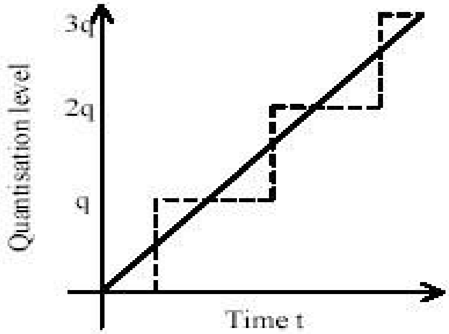
The signal can be band limited by passing it through a filter (LPF) which blocks or attenuates all the frequency components outside the specific bandwidth. Hence called as Anti aliasing filter or pre- filter. (BlockDiagram).

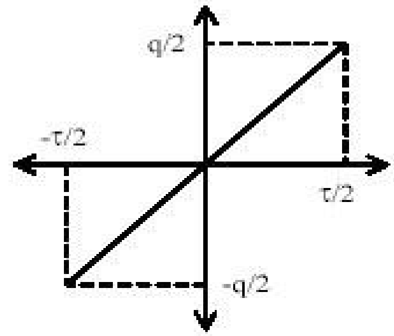
.SAMPLE-AND-HOLD CIRCUIT:

The sampling of an analogue continuous-time signal is normally implemented using a device called an
analogue-to-digitalconverter(A/D).Thecontinuous-timesignalisfirstpassedthroughadevicecalled a sample-and-hold (S/H) whose function is to measure the input signal value at the clock instant and hold
itfixedforatimeintervallongenoughfortheA/Doperationtocomplete.Analogue-to-digitalconversion ispotentiallyaslowoperation,andavariationoftheinputvoltageduringtheconversionmaydisruptthe operationoftheconverter.TheS/Hpreventssuchdisruptionbykeepingtheinputvoltageconstantduring the conversion. This is schematically illustrated byFigure.



After a continuous-time signal has been through the A/D converter, the quantized output may differ from the input value. Themaximum possible output value after the quantization process could be up to half the quantization level q above or q below the ideal output value. This deviation from the ideal output value is called the quantization error. In order to reduce this effect, we increases the number ofbits.





4.3 Sampling of Non-bandlimited Signal: Anti-aliasingFilter

Anti aliasing filter is a filter which is used before a signal sampler, to restrict the bandwidth of a signal to approximately satisfy the sampling theorem.

The potential defectors are all the frequency components beyond $f_S/2$ Hz.
We should have to eliminate these components from $x(t)$ before sampling $x(t)$.

As a result of this we lose only the components beyond the folding frequency $f_S/2$ Hz.
These frequency components cannot reappear to corrupt the components with frequencies below the folding frequency.

This suppression of higher frequencies can be accomplished by an ideal filter of bandwidth $f_S/2$ Hz. This filter is called the **anti-aliasing filter**.
The anti aliasing operation must be performed before the signal is sampled. The anti aliasing filter, being an ideal filter isunrealizable.
In practice, we use a steep cutoff filter, which leaves a sharply attenuated residual spectrum beyond the folding frequency $f_S/2$.

4.4 DISCRETE TIME FOURIERTRANSFORM

In mathematics, the **discrete-time Fourier transform (DTFT)** is one of the specific forms of Fourier analysis. As such, it transforms one function into another, which is called the *frequency domain* representation, or simply the "DTFT", of the original function (which is often a function in the time-domain). But the DTFT requires an input function that is *discrete*. Such inputs are often created by sampling a continuous function, like a person's voice.

Given a discrete set of real or complex numbers**:** $x[n],\ n \in \mathbb{Z}$ (integers),the **discrete-time Fourier transform** (or **DTFT**) of $x[n]$ is usually written**:**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-i\omega n}.$$

Often the $x[n]$ sequence represents the values (aka *samples*) of a continuous-time function, $x(t)$, at discrete moments in time: $t = nT$, where T is the sampling interval (in seconds), and $1/T = f_s$ is the sampling rate (samples per second). Then the DTFT provides an approximation of the continuous-time Fourier transform:

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-i2\pi f t}\, dt.$$

To understand this, consider the Poisson summation formula, which indicates that a periodic summation of function $X(f)$ can be constructed from the samples of function $x(t)$. The result is:

$$X_T(f) \stackrel{\text{def}}{=} \sum^{\infty} X(f - kf_s) \equiv T \sum^{\infty} x(nT)\, e^{-i2\pi f T n}. \quad \text{(Eq.2)}$$

The right-hand sides of Eq.2 and Eq.1 are identical with these associations:

$$x[n] = T \cdot x(nT)$$
$$\omega = 2\pi f T = 2\pi \left(\frac{f}{f_s}\right).$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-i\omega n}.$$

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$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-i2\pi f t}\, dt.$$

To understand this, consider the Poisson summation formula, which indicates that a periodic summation of function $X(f)$ can be constructed from the samples of function $x(t)$

4.5 Inversetransform

The following inverse transforms recover the discrete-time sequence**:**

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\cdot e^{i\omega n}\, d\omega$$
$$= T\int_{-\frac{1}{2T}}^{\frac{1}{2T}} X_T(f)\cdot e^{i2\pi f n T}\, df.$$

The integrals span one full period of the DTFT, which means that the x[n] samples are also the coefficients of a Fourier series expansion of the DTFT.

Infinite limits of integration change the transform into a continuous-time Fourier transform [inverse], which produces a sequence of Dirac impulses. That is**:**

$$\int_{-\infty}^{\infty} X_T(f)\cdot e^{i2\pi f t}\, df = \int_{-\infty}^{\infty} \left(T\sum_{n=-\infty}^{\infty} x(nT)\; e^{-i2\pi f T n}\right)\cdot e^{i2\pi f t}\, df$$
$$= \sum_{n=-\infty}^{\infty} T\cdot x(nT)\int_{-\infty}^{\infty} e^{-i2\pi f T n}\cdot e^{i2\pi f t}\, df$$
$$= \sum_{n=-\infty}^{\infty} x[n]\cdot \delta(t-nT).$$

4.6 Properties

- $*$ is the convolution between two signals
- $x[n]^*$ is the complex conjugate of the function *x[n]*
- $\rho_{xy}[n]$ represents the correlation between *x[n]* and *y[n]*.

x[n]^*\!

Property	Time domain $x[n]$	Frequency domain $X(\omega)$
Linearity	$ax[n] + by[n]$	$aX(e^{i\omega}) + bY(e^{i\omega})$
Shift in time	$x[n-k]$	$X(e^{i\omega})e^{-i\omega k}$
Shift in frequency (modulation)	$x[n]e^{ian}$	$X(e^{i(\omega-a)})$
Time reversal	$x[-n]$	$X(e^{-i\omega})$
Time conjugation	$x[n]^*$	$X(e^{-i\omega})^*$
Time reversal & conjugation	$x[-n]^*$	$X(e^{i\omega})^*$
Derivative in frequency	$\frac{n}{i}x[n]$	$\frac{dX(e^{i\omega})}{d\omega}$
Integral in frequency	$\frac{i}{n}x[n]$	$\int_{-\pi}^{\omega} X(e^{i\vartheta})d\vartheta$
Convolve in time	$x[n] * y[n]$	$X(e^{i\omega}) \cdot Y(e^{i\omega})$
Multiply in time	$x[n] \cdot y[n]$	$\frac{1}{2\pi}X(e^{i\omega}) * Y(e^{i\omega})$
Correlation	$\rho_{xy}[n] = x[-n]^* * y[n]$	$R_{xy}(\omega) = X(e^{i\omega})^* \cdot Y(e^{i\omega})$
Parseval's theorem	$E = \sum_{n=-\infty}^{\infty} x[n]y^*[n]$	$E = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{i\omega})Y^*(e^{i\omega})d\omega$

4.7 SYMMETRYPROPERTIES

The Fourier Transform can be decomposed into a real and imaginary part or into an even and odd part.

$$X(e^{i\omega}) = X_R(e^{i\omega}) + iX_I(e^{i\omega})$$

or

$$X(e^{i\omega}) = X_E(e^{i\omega}) + X_O(e^{i\omega})$$

Time Domain	Frequency Domain
$x[n]$	$X(e^{i\omega})$
$x^*[n]$	$X^*(e^{-i\omega})$
$x^*[-n]$	$X^*(e^{i\omega})$

4.8 *Z*-transforms

► **Definition:** The *Z* – transform of a discrete-time signal *x(n)* is defined as the powerseries:

$$X(z) = \sum_{k=-\infty}^{\infty} x(n) z^{-k} \qquad X(z) = Z[x(n)]$$

where *z* is a complex variable. The above given relations are sometimes called **the direct *Z* - transform** because they transform the time-domain signal *x(n)* into its complex-plane representation *X(z)*. Since *Z* – transform is an infinite power series, it exists only for those values of *z* for which this series converges.

The **region of convergence** of *X(z)* is the set of all values of *z* for which *X(z)* attains a finite value.

► For discrete-time systems, *z*-transforms play the same role of Laplace transforms do in continuous-timesystems

Bilateral forward Z transform

$$H[z] = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

Bilateral inverse Z transform

$$h[n] = \frac{1}{2\pi j} \oint_R H[z] z^{-n+1} dz$$

Z-transform Pairs

► $h[n] = d[n]$

Region of convergence: entire *z*-plane

$$H[z] = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = \sum_{n=0}^{0} \delta[n] z^{-n} = 1$$

► $h[n] = d[n\text{-}1]$

Region of convergence: entire *z*-plane

$$h[n\text{-}1] \Leftrightarrow z^{-1} H[z] \quad H[z] = \sum^{\infty} \delta[n-1] z^{-n} = \sum_{n=1}^{1} \delta[n-1] z^{-n} = z^{-1}$$

► **Inverse *z*-transform**

$$f[n] = \frac{1}{2\pi j} \oint_{c-j\infty}^{c+j\infty} F[z] z^{n-1} dz$$

ADVANTAGES OF Z TRANSFORM

1. The DFT can be determined by evaluating ztransform.
2. Z transform is widely used for analysis and synthesis of digitalfilter.
3. Z transform is used for linear filtering. z transform is also used for finding Linear convolution, cross-correlation and auto-correlations of sequences.
4. In z transform user can characterize LTI system(stable/unstable,causal/anti- causal) and its response to various signals by placements of pole and zeroplot.

ADVANTAGES OF ROC(REGION OF CONVERGENCE)

1. ROC is going to decide whether system is stable orunstable.
2. ROC decides the type of sequences causal oranti-causal.
3. ROC also decides finite or infinite durationsequences.

Z TRANSFORM PLOT

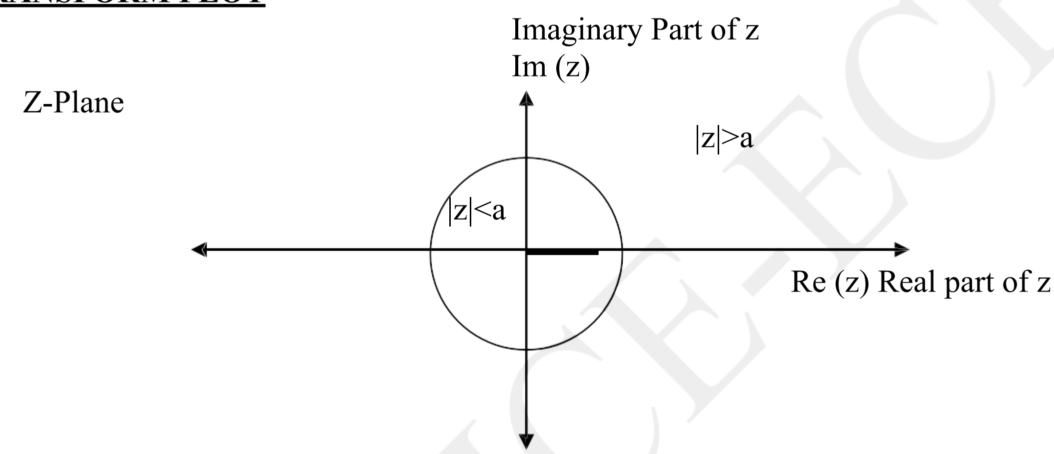


Fig show the plot of z transforms. The z transform has real and imaginary parts. Thus a plot of imaginarypartversusrealpartiscalledcomplexz-plane.Theradiusofcircleis1calledasunitcircle.

This complex z plane is used to show ROC, poles and zeros. Complex variable z is also expressed in polar form as $Z= re^{j\omega}$where r is radius of circle is given by |z| and ω is the frequency of the sequence in radians and given by$\angle$z.

S.No	Time Domain Sequence	Property	z Transform	ROC
1	$\delta(n)$ (Unit sample)		1	complete z plane
2	$\delta(n-k)$	Time shifting	z^{-k}	except z=0
3	$\delta(n+k)$	Time shifting	z^{k}	exceptz=∞
4	u(n) (Unit step)		$1/1- z^{-1} = z/z-1$	$\|z\| > 1$
5	u(-n)	Time reversal	$1/1- z$	$\|z\| < 1$
6	-u(-n-1)	Time reversal	$z/z- 1$	$\|z\| < 1$
7	n u(n) (Unit ramp)	Differentiation	$z^{-1} / (1- z^{-1})^2$	$\|z\| > 1$
8	a^n u(n)	Scaling	$1/1- (az^{-1})$	$\|z\| > \|a\|$
9	$-a^n$ u(-n-1)(Left side exponentialsequence)		$1/1- (az^{-1})$	$\|z\| < \|a\|$
10	n a^n u(n)	Differentiation	$a z^{-1} / (1- az^{-1})^2$	$\|z\| > \|a\|$
11	-n a^n u(-n-1)	Differentiation	$a z^{-1} / (1- az^{-1})^2$	$\|z\| < \|a\|$
12	a^n for 0 < n < N-1		$1- (a z^{-1})^N/ 1- az^{-1}$	$\|az^{-1}\| <\infty$ except z=0
13	1 for 0<n<N-1 or u(n) – u(n-N)	Linearity Shifting	$1- z^{-N}/ 1- z^{-1}$	$\|z\| > 1$
14	$\cos(\omega_0 n)$ u(n)		$\frac{1- z^{-1}\cos\omega_0}{1- 2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
15	$\sin(\omega_0 n)$ u(n)		$\frac{z^{-1}\sin\omega_0}{1- 2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
16	$a^n \cos(\omega_0 n)$ u(n)	Time scaling	$\frac{1- (z/a)^{-1}\cos\omega_0}{1- 2(z/a)^{-1}\cos\omega_0+(z/a)^{-2}}$	$\|z\| > \|a\|$
17	$a^n \sin(\omega_0 n)$ u(n)	Time scaling	$\frac{(z/a)^{-1}\sin\omega_0}{1- 2(z/a)^{-1}\cos\omega_0+(z/a)^{-2}}$	$\|z\| > \|a\|$

4.9-Properties Of Z transform

Z- Transform of a signal *x(n)* is defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Properties of z- transform

1. ***Linearity***

$x_1(n) \xleftrightarrow{z} X_1(z)$

$x_2(n) \xleftrightarrow{z} X_2(z)$

$$Ax_1(n)+Bx_2(n) \xleftrightarrow{z} AX_1(z)+BX_2(z)$$

Proof:

z- transform of $Ax_1(n)+Bx_2(n) = \sum_{n=-\infty}^{\infty}\left[Ax_1(n)+Bx_2(n)\right]z^{-n}$

$$=\sum_{n=-\infty}^{\infty}\left[Ax_1(n)\right]z^{-n}+\sum_{n=-\infty}^{\infty}\left[Bx_2(n)\right]z^{-n}=A\sum_{n=-\infty}^{\infty}\left[x_1(n)\right]z^{-n}+B\sum_{n=-\infty}^{\infty}\left[x_2(n)\right]z^{-n}$$

$$=AX_1(z)+BX_1(z)$$

2. _Shifting in timedomain_

$$x(n) \xleftrightarrow{z} X(z)$$

$$x(n-k) \xleftrightarrow{z} z^{-k}X(z)$$

Proof:

z- transformof $x(n-k)=\sum_{n=-\infty}^{\infty}\left[x(n-k)\right]z^{-n}$

$let\ n-k=m$

$n=m+k$

$$\sum_{n=-\infty}^{\infty}\left[x(n-k)\right]z^{-n}=\sum_{m=-\infty}^{\infty}\left[x(m)\right]z^{-(m+k)}=z^{-k}\sum_{m=-\infty}^{\infty}\left[x(m)\right]z^{-(m)}=z^{-k}X(z)$$

3.Time reversal

$$x(n) \xleftrightarrow{z} X(z)$$

$$x(-n) \xleftrightarrow{z} X(z^{-1})$$

Proof:

z- transform of $x(-n)=\sum_{n=-\infty}^{\infty}\left[x(-n)\right]z^{-n}$

$let\ -n=m$

$n=-m$

$$\sum_{n=-\infty}^{\infty}\left[x(-n)\right]z^{-n}=\sum_{m=-\infty}^{\infty}\left[x(m)\right]z^{(m)}=\sum_{m=-\infty}^{\infty}\left[x(m)\right](z^{-1})^{-m}=X(z^{-1})$$

4.scaling in z-domain
(multiplication by an exponential sequence)

$$x(n) \xleftrightarrow{z} X(z)$$

$$a^n x(n) \xleftrightarrow{z} X\left(\frac{z}{a}\right)$$

Proof:

z- transform of $a^n x(n)=\sum_{n=-\infty}^{\infty}\left[a^n x(n)\right]z^{-n}$

$$\sum_{n=-\infty}^{\infty}\left[x(n)\right](a^{-1}z)^{-n}=X(a^{-1}z)=X\left(\frac{z}{a}\right)$$

5. Differentiation in z-domain

$$x(n) \xleftrightarrow{z} X(z)$$
$$nx(n) \xleftrightarrow{z} -z\frac{dX(z)}{dz}$$

Proof:

z- transform of $x(n)$= $X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$

differentiating the z-transform, we have,

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x(n)(-n)z^{-n-1}$$
$$= -z^{-1}\sum_{n=-\infty}^{\infty} (nx(n))z^{-n}$$
$$-z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} (nx(n))z^{-n}$$

6.Convolution

$$x_1(n) \xleftrightarrow{z} X_1(z)$$
$$x_2(n) \xleftrightarrow{z} X_2(z)$$
$$x_1(n)*x_2(n) \xleftrightarrow{z} X_1(z)X_2(z)$$

Proof:
By definition,

$$x_1(n)*x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)$$

Z – transform of $x_1(n) * x_2(n)$= $\sum_{n=-\infty}^{\infty}[x_1(n) * x_2(n)]z^{-n}$

$$= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)z^{-n}$$

$let \quad n-k = p$

$\quad n = p + k$

$$= \sum_{p=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x_1(k)x_2(p)z^{-(p+k)}$$

$$= \sum_{p=-\infty}^{\infty} x_2(p)z^{-p} \sum_{k=-\infty}^{\infty} x_1(k)z^{-k} = X_2(z)\, X_1(z)$$

7.Correlation

$$x_1(n) \xleftrightarrow{z} X_1(z)$$
$$x_2(n) \xleftrightarrow{z} X_2(z)$$
$$\gamma_{x_1x_2} \xleftrightarrow{z} X_1(z)X_2(z^{-1})$$

Proof:
By definition,

Correlation between two signals $x_1(n) \& x_2(n)$ is given by

$\gamma_{x_1 x_2} = x_1(n) * x_2(-n)$

Z – transform of $x_1(n) * x_2(-n) = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(-n)] z^{-n}$

$= X_1(z)\, X_2(z^{-1})$

Using convolution and time reversal property of z- transform

Note:

Convolution property:

$x_1(n) * x_2(n) \xleftrightarrow{z} X_1(z) X_2(z)$

Time reversal property:

$x(-n) \xleftrightarrow{z} X(z^{-1})$

8.Conjugation

$x(n) \xleftrightarrow{z} X(z)$

$x^*(n) \xleftrightarrow{z} X^*(z^*)$

Proof:

z- transform of $x^*(n) = \sum_{n=-\infty}^{\infty} \left[x^*(n)\right] z^{-n}$

$= \sum_{n=-\infty}^{\infty} \left[x(n)\left(z^*\right)^{-n}\right]^* = \left[X(z^*)\right]^* = X^*(z^*)$

9.Initial value theorem

For causal signal $x(n)$

$x(0) = \lim_{z\to\infty} X(z)$

Where, $x(n) \xleftrightarrow{z} X(z)$

Proof:

For causal signal $x(n)$

$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \ldots\ldots$

$\lim_{z\to\infty} X(z) = x(0)$

10Final value theorem

For causal signal, $x(n)$

$x(n) \xleftrightarrow{z^+} X^+(z)$

If poles of $X^+(z)$ are within the unit circle in z-plane, then

$x(\infty) = \lim_{z\to 1}(z-1)\, X^+(z)$

Proof:

$$Z\{x(n+1)-x(n)\}=\lim_{k\to\infty}\sum_{n=0}^{k}[x(n+1)-x(n)]z^{-n}$$

$$zX^+(z)-zx(0)-X^+(z)=\lim_{k\to\infty}\sum_{n=0}^{k}[x(n+1)-x(n)]z^{-n}$$

$$(z-1)X^+(z)-zx(0)=\lim_{k\to\infty}\sum_{n=0}^{k}[x(n+1)-x(n)]z^{-n}$$

$$\lim_{z\to1}\{(z-1)X^+(z)-zx(0)\}=\lim_{z\to1}\lim_{k\to\infty}\sum_{n=0}^{k}[x(n+1)-x(n)]z^{-n}$$

$$=\lim_{z\to1}\lim_{k\to\infty}\{[x(1)-x(0)]+[x(2)-x(1)]+[x(3)-x(2)]+\cdots+[x(k+1)-x(k)]\}$$

$$=[x(\infty)-x(0)]$$

$$\lim_{z\to1}(z-1)X^+(z)-x(0)=[x(\infty)-x(0)]$$

$$x(\infty)=\lim_{z\to1}(z-1)X^+(z)$$

4.10-Properties of DISCRETE TIME FOURIER TRANSFORM (DTFT)

$$x(n)\xleftrightarrow{DTFT}X(e^{j\omega})$$

$$X(e^{j\omega})=\sum_{n=-\infty}^{\infty}x(n)e^{-j\omega n}\qquad\Rightarrow Analysis equation$$

$$x(n)=\frac{1}{2\pi}\int_{2\pi}X(e^{j\omega})e^{j\omega n}d\omega\qquad\Rightarrow synthesis\ equation$$

Properties of DTFT:

1.Linearity:

$$x_1(n)\xleftrightarrow{DTFT}X_1(e^{j\omega})$$

$$x_2(n)\xleftrightarrow{DTFT}X_2(e^{j\omega})$$

$$Ax_1(n)+Bx_2(n)\xleftrightarrow{DTFT}AX_1(e^{j\omega})+BX_2(e^{j\omega})$$

Where A , B are constants.

Proof:

$$\text{DTFT of } Ax_1(n)+Bx_2(n)=\sum_{n=-\infty}^{\infty}[Ax_1(n)+Bx_2(n)]e^{-j\omega n}$$

$$=\sum_{n=-\infty}^{\infty}[Ax_1(n)]e^{-j\omega n}+\sum_{n=-\infty}^{\infty}[Bx_2(n)]e^{-j\omega n}=A\sum_{n=-\infty}^{\infty}[x_1(n)]e^{-j\omega n}+B\sum_{n=-\infty}^{\infty}[x_2(n)]e^{-j\omega n}$$

$$=AX_1(e^{j\omega})+BX_2(e^{j\omega})$$

2.Periodicity

DTFT is periodic with period 2 π

$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$ where ‗k' is an integer.

$X(e^{j(\omega+2k\pi)}) = X(e^{j\omega})$

$$proof: \quad X(e^{j(\omega+2\pi k)}) = \sum_{n=-\infty}^{\infty} x(n) \; e^{-j(\omega+2\pi k)n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \; e^{-j2\pi kn} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = X(e^{j\omega})$$

$\sin ce \quad e^{-j2\pi kn} = 1$

$n = \text{int } eger$

3. Time shiftingproperty

$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$

$x(n-k) \xleftrightarrow{DTFT} e^{-j\omega k} X(e^{j\omega})$

$proof:$

$$DTFT \; of \quad x(n-k) is = \sum_{n=-\infty}^{\infty} x(n-k) e^{-j\omega n}$$

$n-k = p$

$n = k+p$

$$= \sum_{p=-\infty}^{\infty} x(p) e^{-j\omega(k+p)} = e^{-j\omega k} \sum_{p=-\infty}^{\infty} x(p) e^{-j\omega p} = e^{-j\omega k} X(e^{j\omega})$$

4. Time Reversalproperty

$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$

$x(-n) \xleftrightarrow{DTFT} X(e^{-j\omega})$

$proof:$

$$DTFT \; of \; x(-n) is = \sum_{n=-\infty}^{\infty} x(-n) e^{-j\omega n}$$

$let \quad -n = p$

$n = -p$

$$= \sum_{p=-\infty}^{\infty} x(p) e^{-j\omega(-p)} = \sum_{p=-\infty}^{\infty} x(p) (e^{-j\omega})^{-p} = X(e^{-j\omega})$$

5. Conjugation

$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$

$x^*(n) \xleftrightarrow{DTFT} X^*(e^{-j\omega})$

$proof:$

$$DTFT\ of\ \ x(n)\ is = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$DTFT\ of\ \ x^*(n)\ is = \sum_{n=-\infty}^{\infty} x^*(n)e^{-j\omega n}$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n)e^{j\omega n}\right]^*$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n)(e^{-j\omega})^{-n}\right]^* = X^*\left(e^{-j\omega}\right)$$

6. Frequency shifting

$$x(n) \xleftrightarrow{DTFT} X\left(e^{j\omega}\right)$$

$$e^{j\omega_0 n}x(n) \xleftrightarrow{DTFT} X\left(e^{j(\omega-\omega_0)}\right)$$

$proof:$

$$DTFT\ of\ x(n)\ is = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$DTFT\ of\ e^{j\omega_0 n}x(n)\ is = \sum_{n=-\infty}^{\infty} e^{j\omega_0 n}x(n)e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega-\omega_0)n} = X\left(e^{j(\omega-\omega_0)}\right)$$

7.Differentiation in frequency

$$x(n) \xleftrightarrow{DTFT} X\left(e^{j\omega}\right)$$

$$n\,x(n) \xleftrightarrow{DTFT} j\,\frac{dX\left(e^{j\omega}\right)}{d\omega}$$

$proof:$

$$DTFT\ of\ x(n)\ is = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = X\left(e^{j\omega}\right)$$

$$\frac{dX\left(e^{j\omega}\right)}{d\omega} = -j\sum_{n=-\infty}^{\infty} n x(n)e^{-j\omega n}$$

$$j\,\frac{dX\left(e^{j\omega}\right)}{d\omega} = \sum_{n=-\infty}^{\infty} n x(n)e^{-j\omega n}$$

$$\Rightarrow DTFT\ of\ \ nx(n) = j\,\frac{dX\left(e^{j\omega}\right)}{d\omega}$$

8. Convolution in time

$x_1(n) \xleftrightarrow{DTFT} X_1(e^{j\omega})$

$x_2(n) \xleftrightarrow{DTFT} X_2(e^{j\omega})$

$x_1(n) * x_2(n) \xleftrightarrow{DTFT} X_1(e^{j\omega}) X_2(e^{j\omega})$

Proof:

By definition,

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k)$$

DTFT of $x_1(n) * x_2(n) = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)] e^{-j\omega n}$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k) e^{-j\omega n}$$

$let \quad n-k = p \quad n = p+k$

$$= \sum_{p=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k) x_2(p) e^{-j\omega(p+k)}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) e^{-j\omega k} \sum_{p=-\infty}^{\infty} x_2(p) e^{-j\omega p} = X_1(e^{j\omega}) X_2(e^{j\omega})$$

9. Convolution in frequency (Multiplication)

$x_1(n) \xleftrightarrow{DTFT} X_1(e^{j\omega})$

$x_2(n) \xleftrightarrow{DTFT} X_2(e^{j\omega})$

$x_1(n) x_2(n) \xleftrightarrow{DTFT} \frac{1}{2\pi} [X_1(e^{j\omega}) * X_2(e^{j\omega})]$

Proof:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad \Rightarrow Analysis\ equation$$

$$x(n) = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \quad \Rightarrow synthesis\ equation$$

$$x_1(n) = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\omega}) e^{j\omega n} d\omega$$

$$x_2(n) = \frac{1}{2\pi} \int_{2\pi} X_2(e^{j\omega}) e^{j\omega n} d\omega$$

$let \quad x_2(n) = \frac{1}{2\pi} \int_{2\pi} X_2(e^{j\lambda}) e^{j\lambda n} d\lambda \qquad changing\ variable\ \omega\ to\ \lambda$

$$DTFT\ of\ \ x_1(n)x_2(n) = \sum_{n=-\infty}^{\infty}[x_1(n)x_2(n)]e^{-j\omega n}$$

$$=\sum_{n=-\infty}^{\infty} x_1(n)\frac{1}{2\pi}\int_{2\pi} X_2\left(e^{j\lambda}\right)e^{j\lambda n}\,d\lambda\, e^{-j\omega n}$$

$$=\frac{1}{2\pi}\int_{2\pi} X_2\left(e^{j\lambda}\right)\sum_{n=-\infty}^{\infty} x_1(n)e^{-j(\omega-\lambda)n}\,d\lambda$$

$$=\frac{1}{2\pi}\int_{2\pi} X_2\left(e^{j\lambda}\right)X_1(e^{j(\omega-\lambda)})d\lambda = \frac{1}{2\pi}\left[X_1\left(e^{j\omega}\right)*X_2\left(e^{j\omega}\right)\right]$$

10 .Parseval's relation for DT signals

$$x(n)\xleftrightarrow{DTFT}X(e^{j\omega})$$

$$\sum_{n=-\infty}^{\infty}|x(n)|^2 = \frac{1}{2\pi}\int_{2\pi}\left|X(e^{j\omega})\right|^2 d\omega$$

Proof:

$$X\left(e^{j\omega}\right)= \sum_{n=-\infty}^{\infty}x(n)\,e^{-j\omega n} \qquad \Rightarrow DTFT\ of\ \ x(n)$$

$$x(n)=\frac{1}{2\pi}\int_{2\pi} X\left(e^{j\omega}\right)e^{j\omega n}d\omega$$

$$x^*(n)=\frac{1}{2\pi}\int_{2\pi} X^*\left(e^{j\omega}\right)e^{-j\omega n}d\omega$$

$$\sum_{n=-\infty}^{\infty}|x(n)|^2= \sum_{n=-\infty}^{\infty}x(n)x^*(n)$$

$$=\sum_{n=-\infty}^{\infty} x(n)\frac{1}{2\pi}\int_{2\pi} X^*\left(e^{j\omega}\right)e^{-j\omega n}d\omega$$

$$=\frac{1}{2\pi}\int_{2\pi} X^*\left(e^{j\omega}\right)\left\{ \sum_{n=-\infty}^{\infty}x(n)e^{-j\omega n} \right\}d\omega$$

$$=\frac{1}{2\pi}\int_{2\pi}X^*\left(e^{j\omega}\right)X\left(e^{j\omega}\right)d\omega=\frac{1}{2\pi}\int_{2\pi}\left|X(e^{j\omega})\right|^2 d\omega$$

RELATIONSHIP BETWEEN FOURIER TRANSFORM AND Z TRANSFORM.

There is a close relationship between Z transform and Fourier transform. If we replace the complex variable z by $e^{-j\omega}$, then z transform is reduced to Fourier transform.

Z transform of sequence x(n) is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \qquad \textbf{(Definition ofz-Transform)}$$

Fourier transform of sequence x(n) is given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$ **(Definition of FourierTransform)**

Complex variable z is expressed in polar form as $Z= re^{j\omega}$ where $r= |z|$ and ω is $\angle z$. Thus we can be written as

$$X(z) = \sum_{n=-\infty}^{\infty} [\, x(n)\, r^{-n}]\, e^{-j\omega n}$$

$$X(z)\Big|_{z=e^{jw}} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

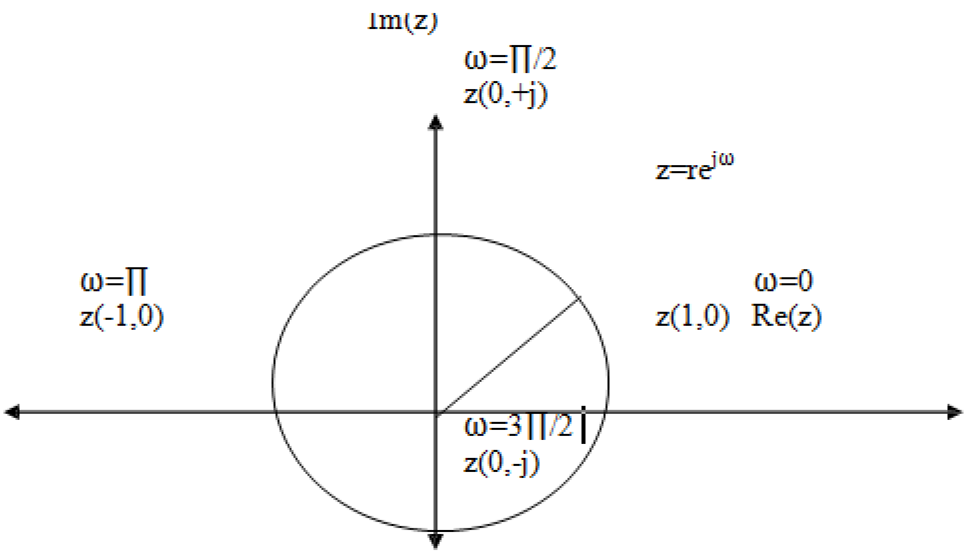
$$X(z)\Big|_{z=e^{jw}} = x(\omega) \qquad \text{at } |z| = \text{unitcircle.}$$

Thus, X(z) can be interpreted as Fourier Transform of signal sequence $(x(n)\, r^{-n})$. Here r^{-n} grows with n if r<1 and decays with n if r>1. X(z) converges for |r|= 1. hence Fourier transform may be viewed as Z transform of the sequence evaluated on unit circle. Thus The relationship between DFT and Z transform is given by

$$X(z)_{z=e^{j2\Pi kn}} = x(k)$$

The frequency $\omega=0$ is along the positive Re(z) axis and the frequency $\Pi/2$ is along the positive Im(z) axis. Frequency Π is along the negative Re(z) axis and $3\Pi/2$ is along the negative Im(z) axis.

Frequency scale on unit circle X(z)= X(ω) on unit circle



Z transform properties

Z-transform Properties

Properties of z - transform

1. Linearity

$$Z(x_1(nT)+x_2(nT)) = Z(x_1(nT)) + Z(x_2(nT))$$

2. Initial Value $x(0) = \lim_{z\to\infty} X(z)$

$$X(z) = x(0) + x(1)z^{-1} + \cdots$$

3. Final value $x(\infty) = \lim_{z\to 1}(1-z^{-1})X(z)$

$$x(\infty) = \lim_{s\to 0} sX(s)$$

$$\begin{array}{ccc} s \to 0 & \Leftrightarrow & z \to 1 \\ \dfrac{1}{s} & \Leftrightarrow & \dfrac{1}{1-z^{-1}} \\ s & \Leftrightarrow & 1-z^{-1} \\ sX(s) & \Leftrightarrow & (1-z^{-1})X(z) \\ \lim\limits_{s\to 0} sX(s) & \Leftrightarrow & \lim\limits_{z\to 1}(1-z^{-1})X(z) \end{array}$$

1. Periodicity:

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

2. Linearity:

$$ax_1[n] + bx_2[n] \longleftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

3. Time Shift:

$$x[n-n_0] \longleftrightarrow e^{-j\omega n_0}X(e^{j\omega})$$

4. Phase Shift:

$$e^{j\omega_0 n}x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})$$

5. Conjugacy:

$$x^*[n] \longleftrightarrow X^*(e^{-j\omega})$$

6. Time Reversal

$$x[-n] \longleftrightarrow X(e^{-j\omega})$$

7. Differentiation

$$nx[n] \longleftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$$

8. Parseval Equality

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

9. Convolution

$$y[n] = x[n] * h[n] \longleftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

10. Multiplication

$$y[n] = x_1[n]x_2[n] \longleftrightarrow Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\omega})X_2(e^{j(\omega-\theta)})d\theta$$

4.11 INVERSE Z TRANSFORM (IZT)

The signal can be converted from time domain into z domain with the help of z transform (ZT). Similar way the signal can be converted from z domain to time domain with the help of inverse z transform(IZT). The inverse z transform can be obtained by using two different methods.

1) Partial fraction expansion Method (PFE) / Application of residuetheorem
2) Power series expansion Method(PSE)

1. PARTIAL FRACTION EXPANSIONMETHOD

In this method X(z) is first expanded into sum of simple partial fraction.

$$X(z) = \frac{a_0 z^m + a_1 z^{m-1} + \ldots\ldots. + a_m}{b_0 z^n + b_1 zn^{n-1} + \ldots\ldots. + b_n} \qquad \text{for } m \le n$$

First find the roots of the denominator polynomial

$$X(z) = \frac{a_0 z^m + a_1 z^{m-1} + \ldots\ldots. + a_m}{(z - p_1)(z - p_2)\ldots\ldots(z - p_n)}$$

The above equation can be written in partial fraction expansion form and find the coefficient A_K and take IZT.

SOLVE USING PARTIAL FRACTION EXPANSION METHOD (PFE)

S.No	Function (ZT)	Time domain sequence	Comment
1	$\frac{1}{1-az^{-1}}$	$a^n u(n)$ for $\|z\| > a$	causal sequence
		$-a^n u(-n-1)$ for $\|z\| < a$	anti-causal sequence
2	$\frac{1}{1+z^{-1}}$	$(-1)^n u(n)$ for $\|z\| > 1$	causal sequence
		$-(-1)^n u(-n-1)$ for $\|z\| < a$	anti-causal sequence
3	$\frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}}$	$-2(3)^n u(-n-1) + (0.5)^n u(n)$ for $0.5<\|z\|<3$	stable system
		$2(3)^n u(n) + (0.5)^n u(n)$ for $\|z\|>3$	causal system
		$-2(3)^n u(-n-1) - (0.5)^n u(-n-1)$ for $\|z\|<0.5$	anti-causal system
4	$\frac{1}{1-1.5z^{-1}+0.5z^{-2}}$	$-2(1)^n u(-n-1) + (0.5)^n u(n)$ for $0.5<\|z\|<1$	stable system
		$2(1)^n u(n) + (0.5)^n u(n)$ for $\|z\|>1$	causal system
		$-2(1)^n u(-n-1) - (0.5)^n u(-n-1)$ for $\|z\|<0.5$	anti-causal system
5	$\frac{1+2z^{-1}+z^{-2}}{1-3/2\,z^{-1}+0.5z^{-2}}$	$2\delta(n)+8(1)^n u(n)-9(0.5)^n u(n)$ for $\|z\|>1$	causal system
6	$\frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}$	$(1/2-j3/2)(1/2+j1/2)^n u(n)+(1/2+j3/2)(1/2+j1/2)^n u(n)$	causal system
7	$\frac{1-(0.5)z^{-1}}{1-3/4\,z^{-1}+1/8\,z^{-2}}$	$4(-1/2)^n u(n) - 3(-1/4)^n u(n)$ for $\|z\|>1/2$	causal system
8	$\frac{1-1/2z^{-1}}{1-1/4z^{-2}}$	$(-1/2)^n u(n)$ for $\|z\|>1/2$	causal system

9	$\frac{z + 1}{3z^2 - 4z + 1}$	$\delta(n) + u(n) - 2(1/3)^n u(n)$ for $\|z\|>1$	causal system

10	$\frac{5z}{(z-1)(z-2)}$	$5(2^n-1)$ for $\lvert z\rvert>2$	causal system
11	$\frac{z^3}{(z-1)(z-1/2)^2}$	$4-(n+3)(1/2)^n$ for $\lvert z\rvert>1$	causal system

RESIDUE THEOREM METHOD

In this method, first find $G(z)= z^{n-1} X(Z)$ and find the residue of G(z) at various poles of X(z).

S. No	Function (ZT)	Time domain Sequence
1	$\frac{z}{z-a}$	For causal sequence $(a)^n u(n)$
2	$\frac{z}{(z-1)(z-2)}$	$(2^n -1) u(n)$
3	$\frac{z^2+z}{(z-1)^2}$	$(2n+1) u(n)$
4	$\frac{z^3}{(z-1)(z-0.5)^2}$	$4-(n+3)(0.5)^n u(n)$

3. POWER-SERIES EXPANSION METHOD

The z transform of a discrete time signal x(n) is given as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad (1)$$

Expanding the above terms we have

$$x(z) = \ldots..+x(-2)Z^2+ x(-1)Z+ x(0)+ x(1) Z^{-1} + x(2)Z^2+\ldots.. \qquad (2)$$

This is the expansion of z transform in power series form. Thus sequence x(n)is given as x(n) ={…..,x(-2),x(-1),x(0),x(1),x(2),…................}.

Power series can be obtained directly or by long division method.

SOLVE USING —POWER SERIES EXPANSION— METHOD

S.No	Function (ZT)	Time domain Sequence
1	$\frac{z}{z-a}$	For causal sequence $a^n u(n)$ For Anti-causal sequence $-a^n u(-n-1)$
2	$\frac{1}{1-1.5z^{-1}+0.5z^{-2}}$	$\{\underline{1},3/2,7/4,15,8,\ldots\ldots\}$ For $\|z\| > 1$ $\{\ldots 14,6,2,0,\underline{0}\}$ For $\|z\| < 0.5$
3	$\frac{z^2+z}{z^3-3z^2+3z-1}$	$\{0,1,4,9,\ldots\}$ For $\|z\| > 3$
4	$z^2(1-0.5z^{-1})(1+z^{-1})(1-z^{-1})$	$X(n) = \{1,-0.5,\underline{-1},0.5\}$
5	$\log(1+az^{-1})$	$(-1)^{n+1}a^n/n$ for $n\geq 1$ and $\|z\|>\|a\|$

Sample Problem:

1. **Obtain the z transformof,**

$$X(z) = \frac{1}{z^2(z-0.5)}$$

We expand $X(z)/z$ into simple fractions as

$$\frac{X(z)}{z} = \frac{1}{z^3(z-0.5)} = \frac{K_1}{z^3} + \frac{K_2}{z^2} + \frac{K_3}{z} + \frac{K_4}{z-0.5}$$

where

$$\begin{aligned}
K_1 &= z^3\frac{X(z)}{z}|_{z=0} = \frac{1}{z-0.5}|_{z=0} = -2 \\
K_2 &= \frac{1}{1!}\frac{d}{dz}z^3\frac{X(z)}{z}|_{z=0} = \frac{d}{dz}\frac{1}{z-0.5}|_{z=0} = \frac{-1}{(z-0.5)^2}|_{z=0} = -4 \\
K_3 &= \frac{1}{2!}\frac{d^2}{dz^2}z^3\frac{X(z)}{z}|_{z=0} = \frac{1}{2}\frac{d}{dz}\frac{-1}{(z-0.5)^2}|_{z=0} = \frac{1}{2}\frac{(-1)(-2)}{(z-0.5)^3}|_{z=0} = -8 \\
K_4 &= (z-0.5)\frac{X(z)}{z}|_{z=0.5} = \frac{1}{z^3}|_{z=0.5} = 8
\end{aligned}$$

Thus, $X(z)$ is expanded as

$$X(z) = -2z^{-2} - 4z^{-1} - 8 + \frac{8}{1-0.5z^{-1}}$$

---

2. **Find the inverse z transformof,**

$$X(z) = \frac{z^2+z+2}{(z-1)(z^2-z+1)}$$

by use of the partial-fraction expansion method.

With complex conjugate poles ($z_{2,3} = 0.5 \pm j0.866$ with $|z_{2,3}| = 1$) in the quadratic factor z^2-z+1, we expand $X(z)$ in simple partial fractions as

$$X(z) = \frac{4}{z-1} + \frac{-3z+2}{z^2-z+1} \text{ or } X(z) = \frac{4z^{-1}}{1-z^{-1}} + \frac{-3z^{-1}+2z^{-2}}{1-z^{-1}+z^{-2}}$$

Recalling that the z transform of damped cosine and sine functions are given by

$$Z[e^{-akT}\cos\omega kT] = \frac{1-e^{-aT}z^{-1}\cos\omega T}{1-2e^{-aT}z^{-1}\cos\omega T + e^{-2aT}z^{-2}}$$

$$Z[e^{-akT}\sin\omega kT] = \frac{e^{-aT}z^{-1}\sin\omega T}{1-2e^{-aT}z^{-1}\cos\omega T + e^{-2aT}z^{-2}},$$

we observe that the second expanded term in the expression of $X(z)$ above can be viewed as the z transform of a damped sinusoid. Actually, $X(z)$ can be rewritten as

$$\begin{aligned}
X(z) &= \frac{4z^{-1}}{1-z^{-1}} - 3\left(\frac{z^{-1}-0.5z^{-2}}{1-z^{-1}+z^{-2}}\right) + \frac{0.5z^{-2}}{1-z^{-1}+z^{-2}} \\
&= 4z^{-1}\frac{1}{1-z^{-1}} - 3z^{-1}\frac{1-0.5z^{-1}}{1-z^{-1}+z^{-2}} + z^{-1}\frac{0.5z^{-1}}{1-z^{-1}+z^{-2}}
\end{aligned}$$

UNIT V

LINEAR TIME INVARIANT DISCRETE TIME SYSTEMS

5.1 Introduction

A discrete-time system is anything that takes a discrete-time signal as input and generates a discrete-time signal as output.1 The concept of a system is very general. It may be used to model the response of an audio equalizer . In electrical engineering, continuous-time signals are usually processed by electrical circuits described by differentialequations.

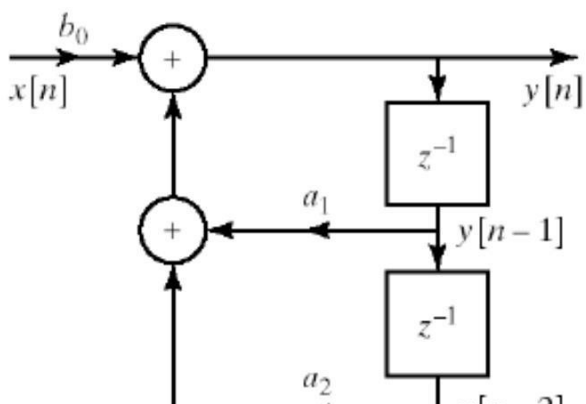
For example, any circuit of resistors, capacitors and inductors can be analyzed using mesh analysis to yield a system of differential equations. The voltages and currents in the circuit may then be computed by solving the equations. The processing of discrete-time signals is performed by discrete-time systems. Similar to the continuous-time case, we may represent a discrete-time system either by a set of difference equations or by a block diagram of its implementation.

For example, consider the following difference equation. y(n) = y(n-1)+x(n)+x(n-1)+x(n-2) This equation represents a discrete-time system. It operates on the input signal x(n)x(n) to produce the output signal y(n).

5.2 BLOCK DIAGRAMREPRESENTATION

Block diagram representation of

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n]$$

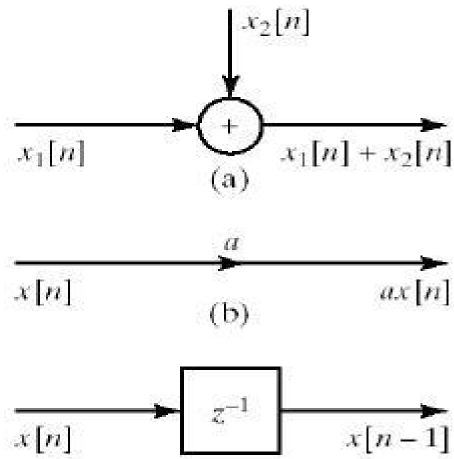


LTI systems with rational system function can be represented as constant-coefficient difference equation

• The implementation of difference equations requires delayed values ofthe
– input
– output

– intermediateresults

• The requirement of delayed elements implies need forstorage



• We also need meansof

– addition

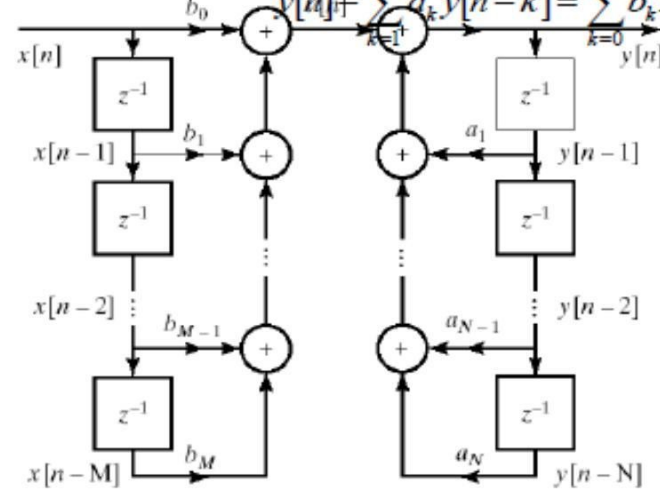
– multiplication

Direct Form I

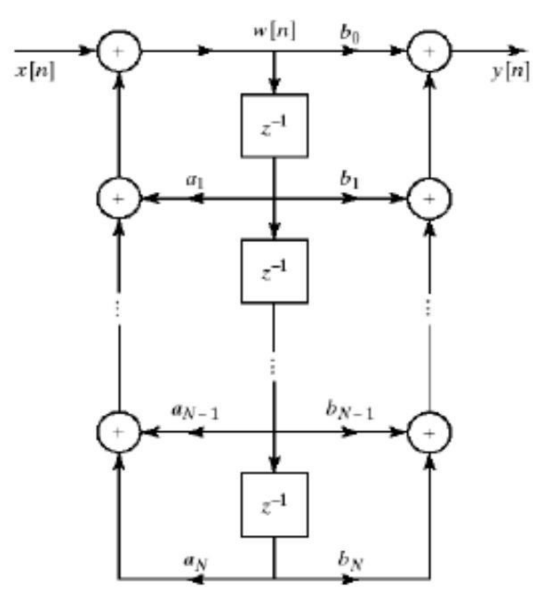
General form of difference equation

$$y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Alternative equivalent form



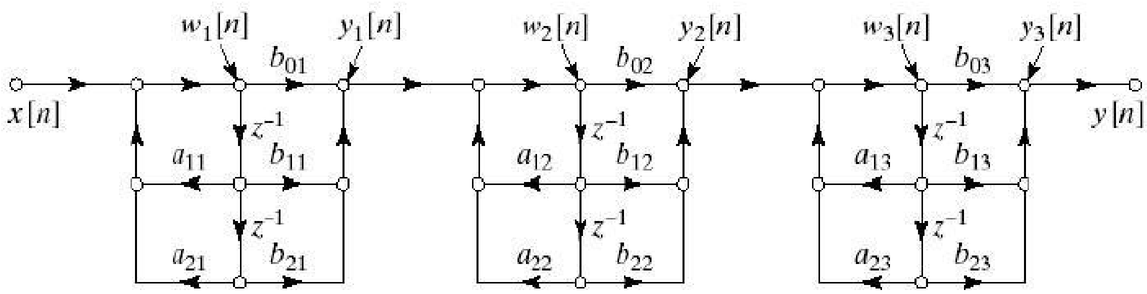
Direct Form II



- Cascadeform

General form for cascade implementation

$$H(z)=A\frac{\prod_{k=1}^{M_1}\left(1-f_k z^{-1}\right)\prod_{k=1}^{M_2}\left(1-g_k z^{-1}\right)\left(1-g_k^* z^{-1}\right)}{\prod_{k=1}^{N_1}\left(1-c_k z^{-1}\right)\prod_{k=1}^{N_2}\left(1-d_k z^{-1}\right)\left(1-d_k^* z^{-1}\right)}$$



Parallel form

► Represent system function using partial fractionexpansion

5.3 CONVOLUTIO NSUM

The convolution sum provides a concise, mathematical way to express the output of an LTI system based on an arbitrary discrete-time input signal and the system's response. The convolution sum is expressed as,

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

- Convolution is commutative

$$x[n] * h[n] = h[n] * x[n]$$

- Convolution is distributive

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

- Cascade connection:

$$y[n] = h_1[n] * [\ h_2[n] * x[n]\ ] = [\ h_1[n] * h_2[n]\ ] * x[n]$$

- Parallel connection

$$y[n] = h_1[n] * x[n] + h_2[n] * x[n]\ ] = [\ h_1[n] + h_2[n]\ ] * x[n]$$

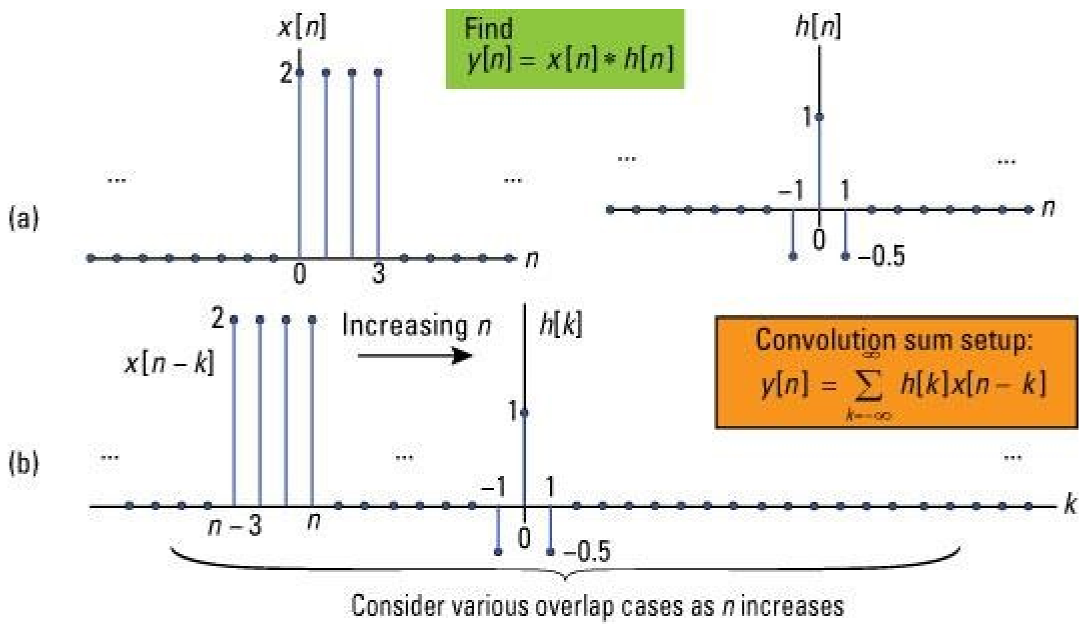
- LTI systems are stable iff

$$\sum_{k=-\infty}^{\infty} \left|h[k]\right| < \infty$$

For the case of discrete-time convolution, here are two convolution sum examples. The first employs finite extent sequences (signals) and the second employs semi-infinite extent signals. You encounter both types of sequences in problem solving, but finite extent sequences are the usual starting point when you're first working with the convolution sum.

Two finite length sequences

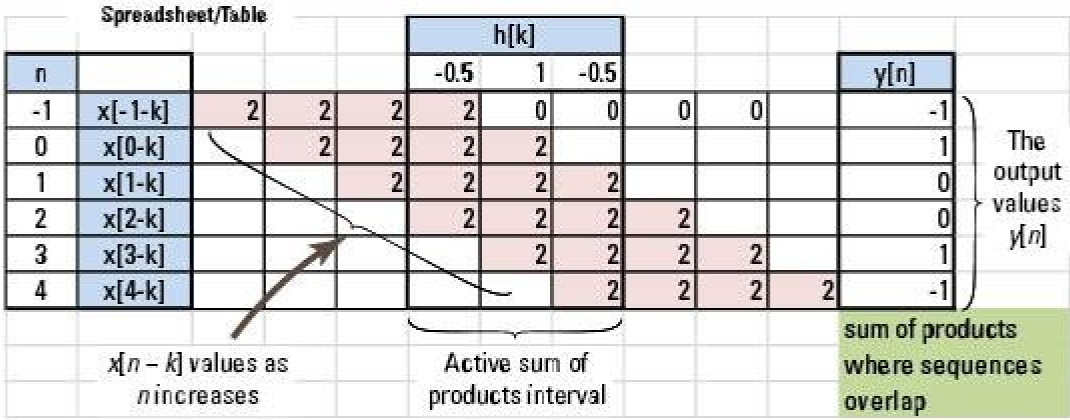
Consider the convolution sum of the two sequences $x[n]$ and $h[n]$, shown here, along with the convolution sum setup.



When convolving finite duration sequences, you can do the analytical solution almost by inspection or perhaps by using a table (even a spreadsheet) to organize the sequence values for each value of *n,* which produces a nonzero overlap between $h[k]$ and $x[n-k]$.

The support interval for the output follows the rule given for the continuous-time domain. The output $y[n]$ starts at the sum of the two input sequence starting points and ends at the sum of input sequence ending points. For the problem at hand this corresponds to $y[n]$ starting at $[0 + -1] = -1$ and ending at $[3 + 1] = 4$.

Looking at Figure b, you can see that as n increases from $n < -1$, first overlap occurs when $n = -1$. The last point of overlap occurs when $n - 3 = 1$ or $n = 4$. You can set up a spreadsheet table to evaluate the six sum-of-products related to the output support interval.

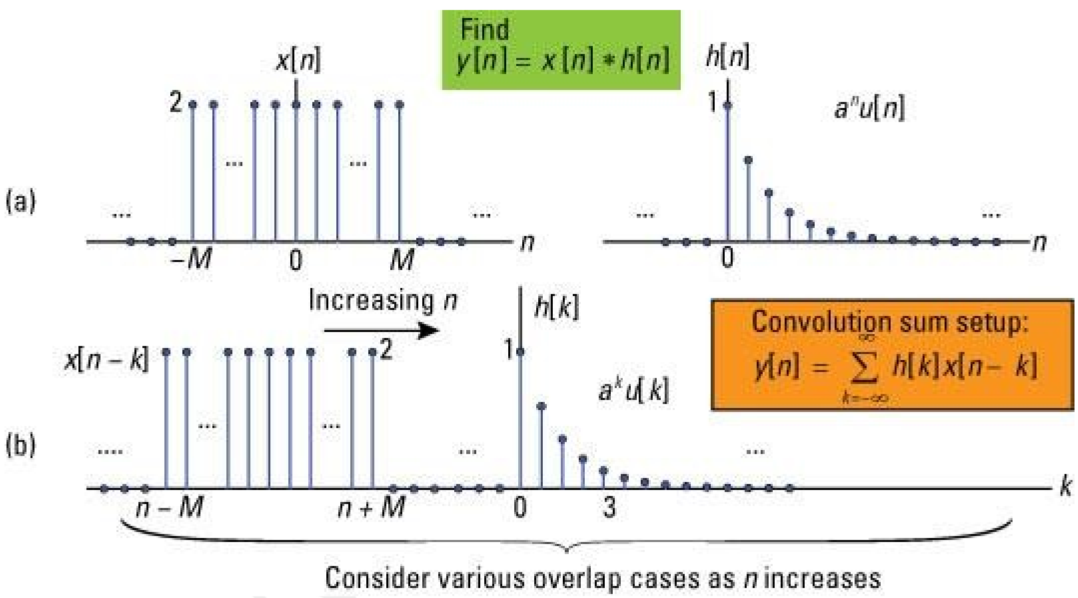


Spreadsheet/Table

n					h[k] -0.5	1	-0.5				y[n]
-1	x[-1-k]	2	2	2	2	0	0	0	0		-1
0	x[0-k]		2	2	2	2					1
1	x[1-k]			2	2	2	2				0
2	x[2-k]				2	2	2	2			0
3	x[3-k]					2	2	2	2		1
4	x[4-k]						2	2	2	2	-1

One finite and one semi-infinite sequence

As a second example of working with the convolution consider a finite duration pulse sequence of $2M + 1$ points convolved with the semi-infinite exponential sequence $a^n u[n]$ (a real exponential decay starting from $n = 0$). A plot of the waveforms is given here.



With the help of Figure b, you have three cases to consider in the evaluation of the convolution for all values of n. The support interval for the convolution is

$$[-M+0, M+\infty) = [-M, \infty)$$

Here are the steps for each case:

- **Case 1:** From Figure b, you see that for $n + M < 0$ or $n < -M$ no overlap occurs between thetwo sequences of the sum, so $y[n] = 0$.

- **Case 2:** Partial overlap between the two sequences occurs when $n + M \geq 0$ and $n - M \leq 0$ or $-M \leq n \leq M$. The sum limits start at $k = 0$ and end at $k = n + M$. Using the finite geometric seriessum formula, the convolution sum evaluatesto

$$y[n] = \sum_{k=0}^{n+M} 2 \cdot a^k = 2 \cdot \frac{1 - a^{n+M+1}}{1 - a}$$

- **Case 3:** Full overlap occurs when $n - M > 0$ or $n > M$. The sum limits under this case run from $k = n - M$ to $k = n + M$. Again, using the finite geometric series sum formula, the convolution sum evaluates to

$$y[n] = \sum_{k=n-M}^{n+M} 2 \cdot a^k = 2 \cdot \frac{a^{n-M} - a^{n+M+1}}{1 - a}$$

Putting the pieces together, the complete analytical solution for this problem is

$$y[n] = \begin{cases} 0, & n < -M \\ 2\dfrac{1 - a^{n+M+1}}{1 - a}, & -M \leq n \leq M \\ 2\dfrac{a^{n-M} - a^{n+M+1}}{1 - a}, & n > M \end{cases}$$

Graphically understanding convolution

Convolution can be seen as a graphical process:

1. Plot x[m]x[m] with dependent variablemm
2. Plot h[−m]h[−m] with dependent variable mm (hh reflected aroundm=0m=0).
3. Plot h[n−m]h[n−m] with dependent variable mm (nn can shift h[n−m]h[n−m] from −∞−∞ (all the way to the left) to ∞∞ (all the way to theright).
4. For each shift (i.e. nn), compute y[n]=∑∞m=−∞x[m]h[n−m] y[n]=∑m=−∞∞x[m]h[n−m](i.e., multiply x[m]h[n−m]x[m]h[n−m] and then sum the result).

1.

The input, $x[n]$, and output, $y[n]$, of a discrete-time LTI system are related by the convolution sum

$$y[n] \;=\; \sum_{k=-\infty}^{\infty} x[k]h[n-k] \;=\; \sum_{k=-\infty}^{\infty} h[k]x[n-k] \qquad (1)$$

where $h[n]$ is the impulse response of the system. Consider a system with impulse response and input shown in Fig. 1 and given by

$$h[n] \;=\; \alpha^n u[n] \qquad (2)$$

$$x[n] \;=\; u[n], \qquad (3)$$

where α is some constant such that $0 < \alpha < 1$. Using (1), the output of the system will be

$$y[n] \;=\; \sum_{k=-\infty}^{\infty} x[k]h[n-k] \;=\; \sum_{k=-\infty}^{\infty} u[k]\alpha^{n-k}u[n-k] \;=\; \alpha^n \sum_{k=-\infty}^{\infty} \alpha^{-k}u[k]u[n-k]. \qquad (4)$$

To help evaluate (4), the flipped and shifted version of the impulse response is shown in Fig. 2 for 2 cases. When $n < 0$, Fig. 2a, the curves do not overlap so $x[k]h[n-k] = 0$ for all values of k. Therefore,

$$y[n] \;=\; \alpha^n \sum_{k=-\infty}^{\infty} 0 \;=\; 0 \qquad\qquad n < 0 \qquad (5)$$

When $n > 0$, Fig. 2b, the curves overlap from $k = 0$ to $k = n$. Therefore, the product is only nonzero in this range, and (4) reduces to

$$y[n] \;=\; \alpha^n \sum_{\substack{k=-\infty \\ n}}^{\infty} \alpha^{-k}u[k]u[n-k]$$

When $n > 0$, Fig. 2b, the curves overlap from $k = 0$ to $k = n$. Therefore, the product is only nonzero in this range, and (4) reduces to

$$\begin{aligned} y[n] &= \alpha^n \sum_{k=-\infty}^{\infty} \alpha^{-k} u[k] u[n-k] \\ &= \alpha^n \sum_{k=0}^{n} \alpha^{-k} \\ &= \alpha^n \frac{1-(1/\alpha)^{n+1}}{1-(1/\alpha)} \\ y[n] &= \frac{1-\alpha^{n+1}}{1-\alpha} \qquad n \geq 0 \end{aligned} \tag{6}$$

Putting (5) and (6) together we have

$$y[n] = \begin{cases} \dfrac{1-\alpha^{n+1}}{1-\alpha} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

which can be combined as

$$y[n] = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right) u[n] \tag{7}$$

Sample plots of $x[n]$, $h[n]$, and $y[n]$ for $\alpha = 0.75$ are shown in Fig. 3.

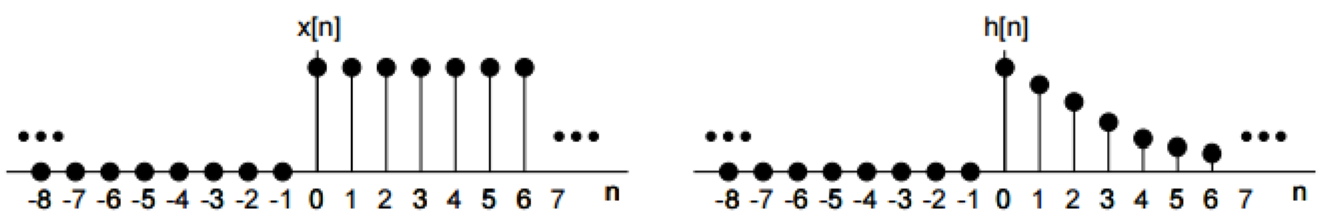


Figure 1: Input signal and system impulse response.

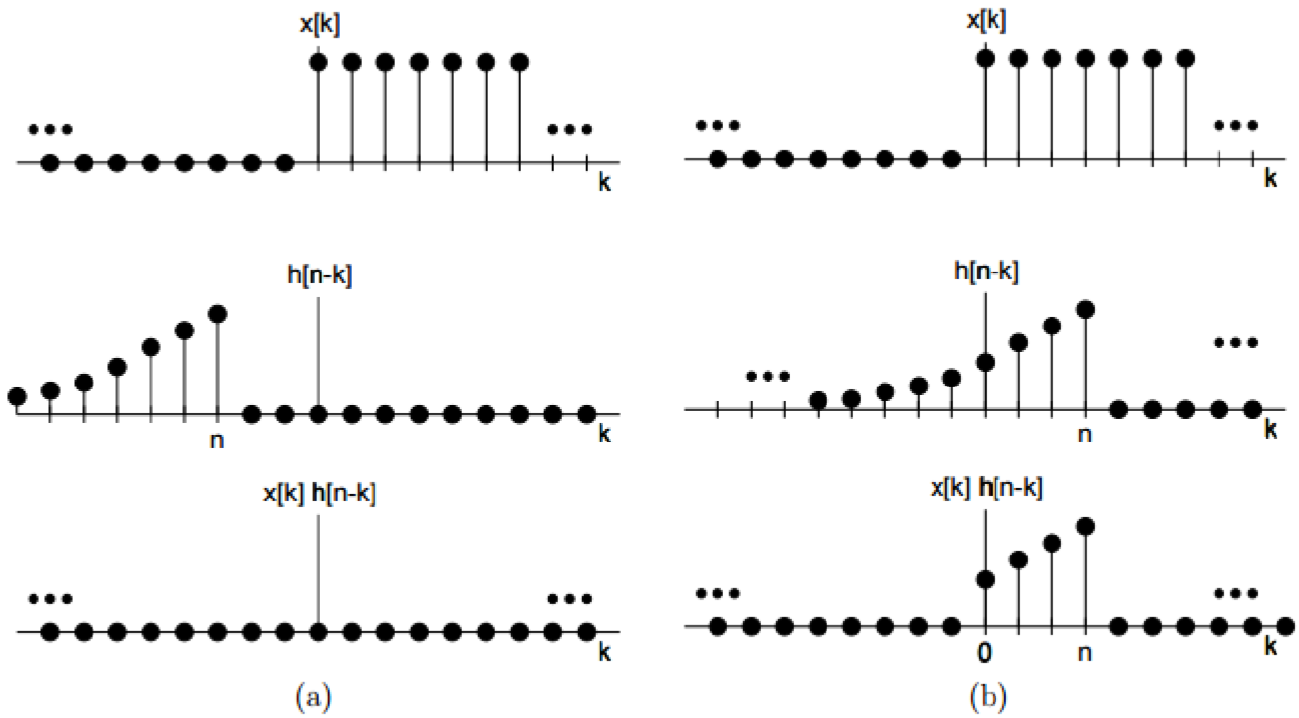


Figure 2: (a) Case 1: $n < 0$, curves do not overlap and (b) Case 2: $n > 0$, curves overlap between 0 and n.

LTI systems are causal if

$h[n] = 0 \quad n < 0$

5.4 LTI System analysis usingDTFT

LTI SYSTEMS ANALYSIS USING DTFT

- Consider $X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\angle X(e^{j\omega})}$ and $H(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{j\angle H(e^{j\omega})}$, then
  - magnitude

$$\left|Y(e^{j\omega})\right| = \left|X(e^{j\omega})\right|\left|H(e^{j\omega})\right|$$

  - phase

$$\angle Y(e^{j\omega}) = \angle X(e^{j\omega}) + \angle H(e^{j\omega})$$

Frequency response at $H(e^{j\omega}) = H(z)\big|_{|z|=1}$ is valid if ROC includes $|z| = 1$,

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

5.5 LTI SYSTEMS ANALYSIS USINGZ-TRANSFORM

- The z-transform of impulse response is called transfer or system function $H(z)$.

$$Y(z) = X(z)H(z).$$

- General form of LCCDE

System Function: Pole/zero Factorization

- Stability requirement can be verified.
- Choice of ROC determines causality.
- Location of zeros and poles determines the frequency response and phase

$$H(z) = \frac{b_0}{a_0}\frac{\prod_{k=1}^{M}\left(1 - c_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1 - d_k z^{-1}\right)}$$

POLE –ZERO PLOT

1. X(z) is a rational function, that is a ratio of two polynomials in z^{-1} orz.
   The roots of the denominator or the value of z for which X(z) becomes infinite, defines locations of the poles. The roots of the numerator or the value of z for which X(z) becomes zero, defines locations of the zeros.

2. ROC dos not contain any poles of X(z). This is because x(z) becomes infiniteat the locations of the poles. Only poles affect the causality and stability of the system.

3. **CASUALTY CRITERIA FOR LSISYSTEM**
   LSI system is causal if and only if the ROC the system function is exterior to the circle. i. e $|z| > r$. This is the condition for causality of the LSI system in terms of z transform. (The condition for LSI system to be causal is $h(n) = 0$ ….. $n<0$)

4. **STABILITY CRITERIA FOR LSISYSTEM**
   Bounded input x(n) produces bounded output y(n) in the LSI system only if

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

With this condition satisfied, the system will be stable. The above equation states that the LSI system is stable if its unit sample response is absolutely summable. This is necessary and sufficient condition for the stability of LSI system.

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n} \qquad \text{Z-Transform.……(1)}$$

Taking magnitude of both the sides

$$|H(z)| = \sum_{n=-\infty}^{\infty} h(n)z^{-n} \quad \text{.......................................} (2)$$

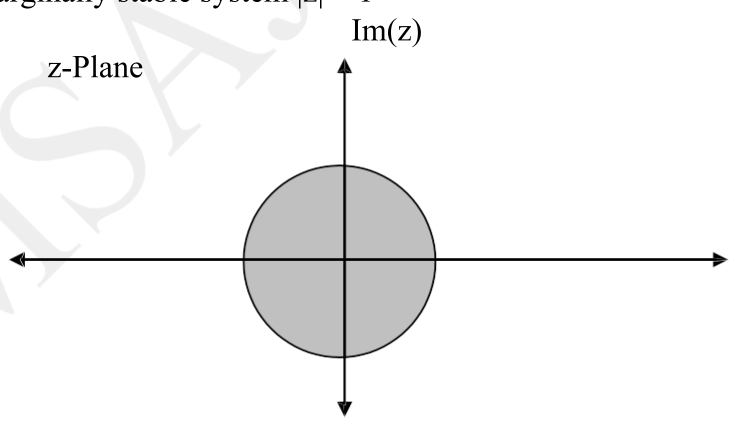
Magnitudes of overall sum is less than the sum of magnitudes of individual sums.

$$|H(z)| \le \sum_{n=-\infty}^{\infty} h(n)\, z^{-n}$$

$$|H(z)| \le \sum_{n=-\infty}^{\infty} |h(n)|\, |z^{-n}| \quad \text{....}(3)$$

5. If H(z) is evaluated on the unit circle $|z^{-n}|=|z|=1$.
Hence LSI system is stable if and only if the ROC the system function includes the unit circle. i.e $r < 1$. This is the condition for stability of the LSI system in terms of z transform. Thus

For stable system $|z| < 1$
For unstable system $|z| > 1$
Marginally stable system $|z| = 1$



Poles inside unit circle gives stable system. Poles outside unit circle gives unstable system. Poles on unit circle give marginally stable system.

6. A causal and stable system must have a system function that convergesfor $|z| > r < 1$.

STANDARD INVERSE TRANSFORMS

S. No	Function (ZT)	Causal Sequence $|z| > |a|$	Anti-causal sequence $|z| < |a|$
1	$\frac{z}{z-a}$	$(a)^n u(n)$	$-(a)^n u(-n-1)$
2	$\frac{z}{z-1}$	$u(n)$	$u(-n-1)$
3	$\frac{z^2}{(z-a)^2}$	$(n+1)a^n$	$-(n+1)a^n$
4	$\frac{z^k}{(z-a)^k}$	$1/(k-1)!\ (n+1)\ (n+2)\ldots\ldots a^n$	$-1/(k-1)!\ (n+1)\ (n+2)\ldots\ldots\ldots a^n$
5	1	$\delta(n)$	$\delta(n)$
6	Z^k	$\delta(n+k)$	$\delta(n+k)$
7	Z^{-k}	$\delta(n-k)$	$\delta(n-k)$

S.No	z Transform (Bilateral)	One sided z Transform (Unilateral)
1	z transform is an infinite power series because summation index varies from ∞ to -∞. Thus Z transform are given by $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$	One sided z transform summation index varies from 0 to ∞. Thus One sided z transform are given by $X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$
2	z transform is applicable for relaxed systems (having zero initial condition).	One sided z transform is applicable for those systems which are described by differential equations with non zero initial conditions.
3	z transform is also applicable for non-causal systems.	One sided z transform is applicable for causal systems only.
4	ROC of x(z) is exterior or interior to circle hence need to specify with z transform of signals.	ROC of x(z) is always exterior to circle hence need not to be specified.

5.6Sample Problems:

Unilateral z-transform

- Unilateral z-transform is used in the analysis of non-relaxed DTsystems.
- Non-relaxed DT systems are systems with non zero initialconditions.
- Unilateral Z- Transform of a signal *x(n)* is definedby

$$X(z)=\sum_{n=0}^{\infty}x(n)z^{-n}$$

time shifting property

$$x(n)\xleftrightarrow{z^+}X(z)$$

Case 1:

$$x(n-k)\xleftrightarrow{z^+}z^{-k}\left[\sum_{n=1}^{k}x(-n)z^{n}+X(z)\right]$$

$$X(z)=\sum_{n=0}^{\infty}x(n)z^{-n}$$

Proof:

Unilateralz-transformof $x(n-k)=\sum_{n=0}^{\infty}\left[x(n-k)\right]z^{-n}$

$$let\ n-k=m\ n$$
$$=m+k$$

$$\sum_{n=0}^{\infty}\left[x(n-k)\right]z^{-n}=\sum_{m=-k}^{\infty}\left[x(m)\right]z^{-(m+k)}=z^{-k}\sum_{m=-k}^{\infty}\left[x(m)\right]z^{-m}$$

$$=z^{-k}\left[\sum_{m=-k}^{-1}\left[x(m)\right]z^{-(m)}+\sum_{m=0}^{\infty}\left[x(m)\right]z^{-(m)}\right]$$

$$=z^{-k}\left[\sum_{m=1}^{k}\left[x(-m)\right]z^{(m)}+\sum_{m=0}^{\infty}\left[x(m)\right]z^{-(m)}\right]$$

$$=z^{-k}\left[\sum_{n=1}^{k}\left[x(-n)\right]z^{(n)}+\sum_{m=0}^{\infty}\left[x(m)\right]z^{-(m)}\right]$$

$$=z^{-k}\left[\sum_{n=1}^{k}x(-n)z^{n}+X(z)\right]$$

Note:

Unilateral z- transform of $y(n-3)$ is,

$$=z^{-3}\left[\sum_{n=1}^{3} y(-n)z^{n}+Y(z)\right]$$

$$=z^{-3}Y(z)+y(-1)z^{-2}+y(-2)z^{-1}+y(-3)$$

Case2:

$$x(n+k)\overset{z^{+}}{\longleftrightarrow} z^{k}\left[X(z)-\sum_{n=0}^{k-1}x(n)z^{-n}\right]$$

$$X(z)=\sum_{n=0}^{\infty}x(n)z^{-n}$$

Proof:

Unilateral z-transform of $x(n+k)=\sum_{n=0}^{\infty}\left[x(n+k)\right]z^{-n}$

$let\ n+k=m$

$n=m-k$

$$\sum_{n=0}^{\infty}\left[x(n+k)\right]z^{-n}=\sum_{m=k}^{\infty}\left[x(m)\right]z^{-(m-k)}=z^{k}\sum_{m=k}^{\infty}\left[x(m)\right]z^{-m}$$

$$=z^{k}\left[\sum_{m=0}^{\infty}\left[x(m)\right]z^{-(m)}-\sum_{m=0}^{k-1}\left[x(m)\right]z^{-(m)}\right]$$

$$=z^{k}\left[X(z)-\sum_{n=0}^{k-1}x(n)z^{-n}\right]$$

Discrete time systems has one more type ofclassification.

1. RecursiveSystems
2. Non-RecursiveSystems

S. No	Recursive Systems	Non-Recursive systems
1	In Recursive systems, the output depends upon past, present, future value of inputs as well as past output.	In Non-Recursive systems, the output depends only upon past, present or future values of inputs.
2	Recursive Systems has feedback from output to input.	No Feedback.
3	Examples y(n) = x(n) + y(n-2)	Y(n) = x(n) + x(n-1)

Example:

Find the step response of the system described by the difference equation

$y(n)+y(n-1)-2y(n-2)=x(n-1)+2x(n-2)$ **given the initialconditions**

$y(-1)=0.5$; $y(-2)=0.25$

Solution:

$y(n)+y(n-1)-2y(n-2)=x(n-1)+2x(n-2)$

Taking unilateral z- transform

$$Y(z)+\left[z^{-1}Y(z)+y(-1)\right]-2\left[z^{-2}Y(z)+z^{-1}y(-1)+y(-2)\right]=\left[z^{-1}X(z)+x(-1)\right]$$
$$+2\left[z^{-2}X(z)+z^{-1}x(-1)+x(-2)\right]$$

$$Y(z)+\left[z^{-1}Y(z)+0.5\right]-2\left[z^{-2}Y(z)+z^{-1}(0.5)+0.25\right]=\left[z^{-1}X(z)\right]+2\left[z^{-2}X(z)\right]$$

$\sin ce \quad x(n)=u(n) \quad \& x(-1)=0;\ x(-2)=0$

$$Y(z)-2z^{-2}Y(z)+z^{-1}Y(z)-z^{-1}=\left[z^{-1}X(z)\right]+2\left[z^{-2}X(z)\right]$$

$$Y(z)\left[1-2z^{-2}+z^{-1}\right]-z^{-1}=X(z)\left[z^{-1}+2z^{-2}\right]$$

$$Y(z)\frac{\left[z^2-2+z\right]}{z^2}=X(z)\frac{\left[z+2\right]}{z^2}+\frac{1}{z}$$

$\sin ce \quad x(n)=u(n) \quad \& X(z)=\dfrac{z}{z-1}$

$$Y(z)\frac{\left[z^2-2+z\right]}{z^2}=\frac{z}{(z-1)}\times\frac{\left[z+2\right]}{z^2}+\frac{1}{z}$$

$$Y(z)=\frac{z(z+2)}{(z+2)(z-1)(z-1)}+\frac{z}{(z+2)(z-1)}=\frac{z(z+2)+z(z-1)}{(z+2)(z-1)(z-1)}$$

$$=\frac{2z^2+z}{(z+2)(z-1)(z-1)}$$

$$\frac{Y(z)}{z}=\frac{2z+1}{(z+2)(z-1)(z-1)}=\frac{A}{(z+2)}+\frac{B}{(z-1)}+\frac{C}{(z-1)^2}$$

$$A=-\frac{1}{3};\ B=\frac{1}{3};\ C=1$$

$$Y(z)=\frac{-\frac{1}{3}z}{(z+2)}+\frac{\frac{1}{3}z}{(z-1)}+\frac{z}{(z-1)^2}$$

$y(n)=inverse\ z-transform\ of\ Y(z)$

$$y(n)=\frac{1}{3}u(n)-\frac{1}{3}(-2)^n u(n)+nu(n)$$

Example:

Find the difference equation description for the system with transfer function

$$H(Z)=\frac{5Z+2}{Z^2+3Z+2}$$

Ans:

$$H(Z)=\frac{5Z+2}{Z^2+3Z+2}=\frac{Y(Z)}{X(Z)}=\frac{5Z^{-1}+2Z^{-2}}{1+3Z^{-1}+2Z^{-2}}$$

$$(1+3Z^{-1}+2Z^{-2})Y(Z)=(5Z^{-1}+2Z^{-2})X(Z)$$

Taking inverse z transform,

$$y(n)+3y(n-1)+2y(n-2)=5x(n-1)+2x(n-2)$$

System analysis using DTFT

For LTI DT system

$$y(n) = x(n) * h(n)$$
$$y(n) = output$$
$$x(n) = input$$
$$h(n) = impulse response$$

In frequency domain,

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$
$$Y(e^{j\omega}) = DTFT\ of\ y(n)$$
$$X(e^{j\omega}) = DTFT\ of\ x(n)$$
$$H(e^{j\omega}) = DTFT\ of\ h(n)$$
$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \Rightarrow \text{frequency Response of the system}$$

Frequency response of the system

- Plot of magnitude of $H(e^{j\omega})$ Vs frequency ω is magnitude response of the system
- Plot of phase angle of $H(e^{j\omega})$ Vs frequency ω is phase response of the system

Example:

Determine and sketch the magnitude response of the system described by the difference equation $y(n) = \frac{1}{3}[x(n) + x(n-1) + x(n-2)]$

$$Y(e^{j\omega}) = \frac{1}{3}\left[X(e^{j\omega}) + e^{-j\omega}X(e^{j\omega}) + e^{-j2\omega}X(e^{j\omega})\right]$$
$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{3}\left[1 + e^{-j\omega} + e^{-j2\omega}\right] = \frac{e^{-j\omega}}{3}\left[1 + 2\cos(\omega)\right]$$
$$magnitude\ of\ H(e^{j\omega}) = \frac{1}{3} \times [1 + 2\cos(\omega)]$$

- Compute magnitude of $H(e^{j\omega})$ for values of frequency ω ranging from $-\pi$ to $+\pi$
- Plot of magnitude of $H(e^{j\omega})$ Vs frequency ω is magnitude response of the system

Example:

Consider a system consisting of the cascade of two LTI systems with frequency responses $H_1(e^{j\omega}) = \frac{2 - e^{-j\omega}}{1 + \frac{1}{2}e^{-j\omega}}$ **and** $H_2(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-2j\omega}}$**. Find the difference equation describing the overall system.**

Frequency response of the overall system is $H(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega})$

$$H(e^{j\omega}) = \frac{2 - e^{-j\omega}}{1 + \frac{1}{8}e^{-j3\omega}} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$Y(e^{j\omega}) + \frac{1}{8}e^{-j3\omega}Y(e^{j\omega}) = 2X(e^{j\omega}) - e^{-j\omega}X(e^{j\omega})$$

taking inverseDTFT ,

$$y(n) + \frac{1}{8}y(n-3) = 2x(n) - x(n-1)$$

Example:

Determine the frequency response of the system

described by the difference equation, $y(n) - \left(\frac{1}{6}\right)y(n-1) - \left(\frac{1}{6}\right)y(n-2) = x(n-1)$ **.**

taking DTFT , and rearranging,

$$\text{frequency response} = H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{e^{-j\omega}}{1 - \frac{1}{6}e^{-j\omega} - \frac{1}{6}e^{-j2\omega}}$$

1. Consider the system described by the differenceequation.

$$y[n] = x[n] + \frac{1}{3}x[n-1] + \frac{5}{4}y[n-1] - \frac{1}{2}y[n-2] + \frac{1}{16}y[n-3]$$

Here $N = 3$, $M = 1$. Order 3 homogeneous equation:

$$y[n] - \frac{5}{4}y[n-1] + \frac{1}{2}y[n-2] - \frac{1}{16}y[n-3] = 0 \qquad n \geq 2$$

The characteristic equation:

$$1 - \frac{5}{4}a^{-1} + \frac{1}{2}a^{-2} - \frac{1}{16}a^{-3} = 0$$

The roots of this third order polynomial is: $a_1 = a_2 = 1/2 \quad a_3 = 1/4$ and

$$y_h[n] = h[n] = A_1\left(\frac{1}{2}\right)^n + A_2 n\left(\frac{1}{2}\right)^n + A_3\left(\frac{1}{4}\right)^n, \quad n \geq 2$$

Let us assume $y[-1] = 0$ then (3.52) for this case becomes:

$$\begin{bmatrix} a_0 & 0 \\ a_1 & a_0 \end{bmatrix} \cdot \begin{bmatrix} y[0] \\ y[1] \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ -5/4 & 1 \end{bmatrix} \cdot \begin{bmatrix} y[0] \\ y[1] \end{bmatrix} = \begin{bmatrix} 1 \\ 1/3 \end{bmatrix} \Rightarrow y[0] = 1; y[1] = 19/12$$

with these we have the impulse response of this system:

$$h[n] = -\frac{4}{3}\left(\frac{1}{2}\right)^n + \frac{10}{3}n\left(\frac{1}{2}\right)^n + \frac{7}{3}\left(\frac{1}{4}\right)^n, \quad n \geq 0$$

2. Compute the impulse response of the system describedby,

$$y[n] - \frac{1}{2}y[n-1] = x[n].$$

Solution: if $x[n] = \delta[n]$, then $y[n] = h[n]$ is the impulse response.

$$y[n] = \frac{1}{2}y[n-1] + x[n]$$

$$\Rightarrow h[n] = \frac{1}{2}h[n-1] + \delta[n]$$

$$h[0] = \frac{1}{2}h[-1] + \delta[0]$$

If we assume condition of initial rest $h[-1] = 0$, then

$$h[0] = 1$$

$$h[1] = \frac{1}{2}h[0] + \delta[1] = \frac{1}{2} + 0 = \frac{1}{2}$$

$$h[2] = \frac{1}{2}h[1] + \delta[2] = (\frac{1}{2})^2$$

$$\vdots$$

$$h[n] = (\frac{1}{2})^n, \text{for } n \geq 0$$

$$h[n] = 0, \text{ for } n < 0$$

$$\Rightarrow h[n] = (\frac{1}{2})^n u[n]$$

- The response of the system is not limited to a finite time interval. This is called an <u>infinite impulse response</u> (IIR) system.

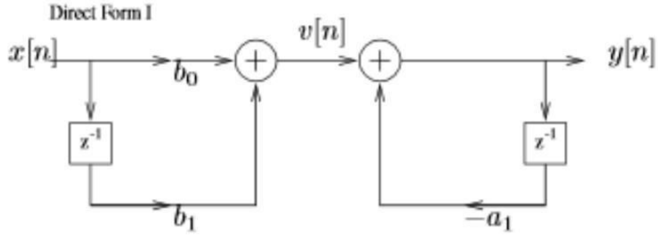
4. Obtain the structures realization of LTI system

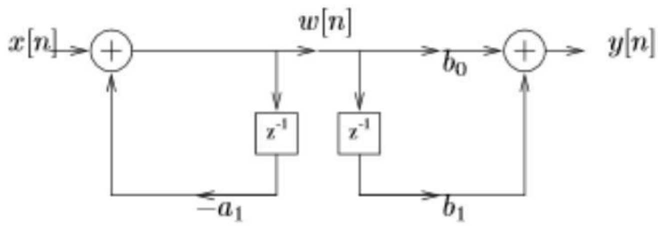
$$y[n] = -a_1\, y[n-1] + b_0\, x[n] + b_1\, x[n-1]$$

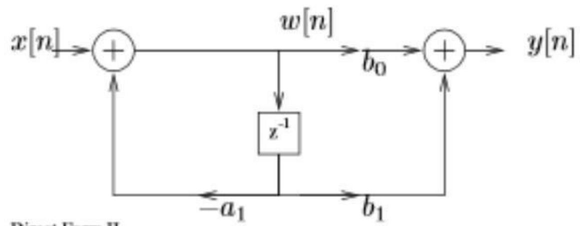
$$\begin{aligned} y[n] &= -a_1\, y[n-1] + v[n] \\ v[n] &= b_0\, x[n] + b_1\, x[n-1] \end{aligned}$$

$$\begin{aligned} w[n] &= -a_1\, w[n-1] + x[n] \\ y[n] &= b_0\, w[n] + b_1\, w[n-1] \end{aligned}$$

Direct Form I







Direct Form II

Generalizes to higher order systems described by difference equations.

$$y[n] = -\sum_{k=1}^{N} a_k\, y[n-k] + \sum_{k=0}^{M} b_k\, x[n-k]$$

$$\begin{aligned} v[n] &= \sum_{k=0}^{M} b_k\, x[n-k] \\ y[n] &= -\sum_{k=1}^{N} a_k\, y[n-k] + v[n] \end{aligned}$$

The first system $v[n] = \ldots$ is nonrecursive, where as the second system is recursive.

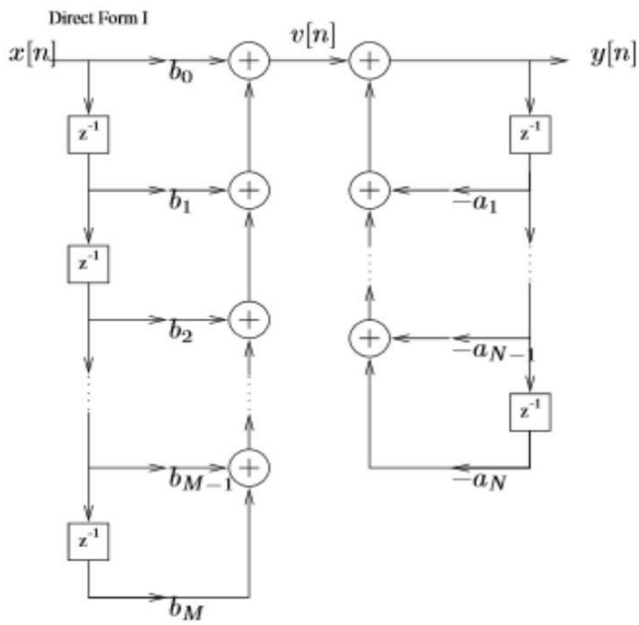
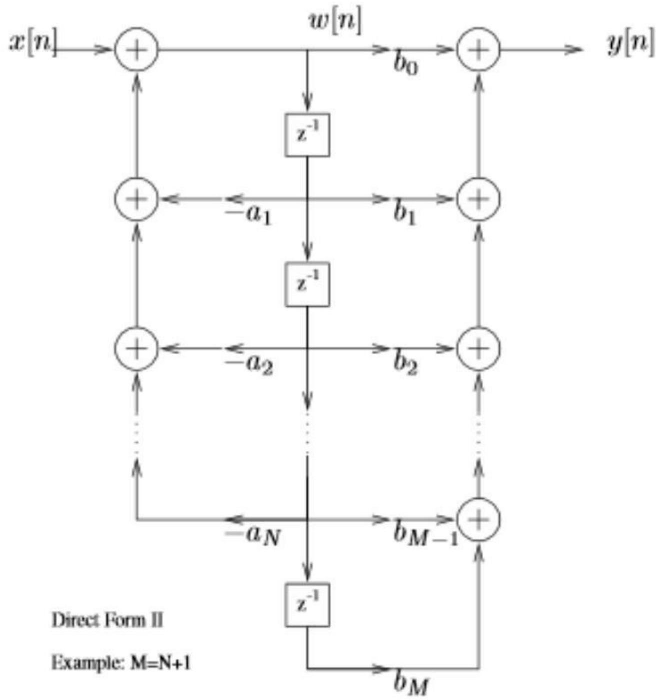


Figure 2.3: General Direct Form I.

$$w[n] = -\sum_{k=1}^{N} a_k \, w[n-k] + x[n]$$

$$y[n] = \sum_{k=0}^{M} b_k \, w[n-k]$$



Find the convolution of x(n)=[1,1,1,1,2,2,2,2] with h(n)=[3,3,0,0,0,0,3,3] by using matrix method.

Solution: By using matrix method, N=8

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} h(0) & h(7) & h(6) & h(5) & h(4) & h(3) & h(2) & h(1) \\ h(1) & h(0) & h(7) & h(6) & h(5) & h(4) & h(3) & h(2) \\ h(2) & h(1) & h(0) & h(7) & h(6) & h(5) & h(4) & h(3) \\ h(3) & h(2) & h(1) & h(0) & h(7) & h(6) & h(5) & h(4) \\ h(4) & h(3) & h(2) & h(1) & h(0) & h(7) & h(6) & h(5) \\ h(5) & h(4) & h(3) & h(2) & h(1) & h(0) & h(7) & h(6) \\ h(6) & h(5) & h(4) & h(3) & h(2) & h(1) & h(0) & h(7) \\ h(7) & h(6) & h(5) & h(4) & h(3) & h(2) & h(1) & h(0) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix}$$

Substituting the values, we get

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 & 0 & 0 & 0 & 0 & 3 \\ 3 & 3 & 3 & 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 & 3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 & 3 & 3 & 3 \\ 3 & 0 & 0 & 0 & 0 & 3 & 3 & 3 \\ 3 & 3 & 0 & 0 & 0 & 0 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} 3\times1+3\times1+3\times1+0\times1+0\times2+0\times2+0\times2+3\times2 \\ 3\times1+3\times1+3\times1+3\times1+0\times2+0\times2+0\times2+0\times2 \\ 0\times1+3\times1+3\times1+3\times1+3\times2+0\times2+0\times2+0\times2 \\ 0\times1+0\times1+3\times1+3\times1+3\times2+3\times2+0\times2+0\times2 \\ 0\times1+0\times1+0\times1+3\times1+3\times2+3\times2+3\times2+0\times2 \\ 0\times1+0\times1+0\times1+0\times1+3\times2+3\times2+3\times2+3\times2 \\ 3\times1+0\times1+0\times1+0\times1+0\times2+3\times2+3\times2+3\times2 \\ 3\times1+3\times1+0\times1+0\times1+0\times2+0\times2+3\times2+3\times2 \end{bmatrix} = \begin{bmatrix} 15 \\ 12 \\ 15 \\ 18 \\ 21 \\ 24 \\ 21 \\ 18 \end{bmatrix}$$

Therefore, the convoluted sum is $y(n)$ = [15, 12, 15, 18, 21, 24, 21, 18].

REALIZATION

A linear time invariant discrete-time systems characterized by the general linear constant coefficient difference equation

$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k) \qquad 1$$

Take Z-Transfomr,

$$Y(Z) = -\sum_{k=1}^{N} a_k\, Y(Z)\, Z^{-k} + \sum_{k=0}^{M} b_k\, X(Z)\, Z^{-k}$$

The transfer function

$$\frac{Y(Z)}{X(Z)} = H(Z) = \frac{\sum_{k=0}^{M} b_k Z^{-k}}{1+\sum_{k=1}^{N} a_k Z^{-k}} \qquad 2$$

There are various forms to implement above equation either in hardware or in software. For each set of equation, we can construct a block diagram consisting of an interconnection of delay elements, multipliers and adders. We referred to such a block diagram as a realization of the system.
The computational complexity refers to the number of operations (like multiplication, addition etc.,) required to compute an output value of the system

STRUCTURE FOR FIR FILTER

In general the FIR filter is described by the difference equation,

$$y(n) = \sum_{k=0}^{M-1} b_k\, x(n-k) \qquad 3$$

Take Z-Transfomr,

$$Y(Z) = \sum_{k=0}^{M-1} b_k\, X(Z)\, Z^{-k}$$

The transfer function

$$\frac{Y(Z)}{X(Z)} = H(Z) = \sum_{k=0}^{M-1} b_k\, X(Z)\, Z^{-k} \qquad 4$$

The unit sample response of the FIR system is identical to the coefficient b_k ie.,

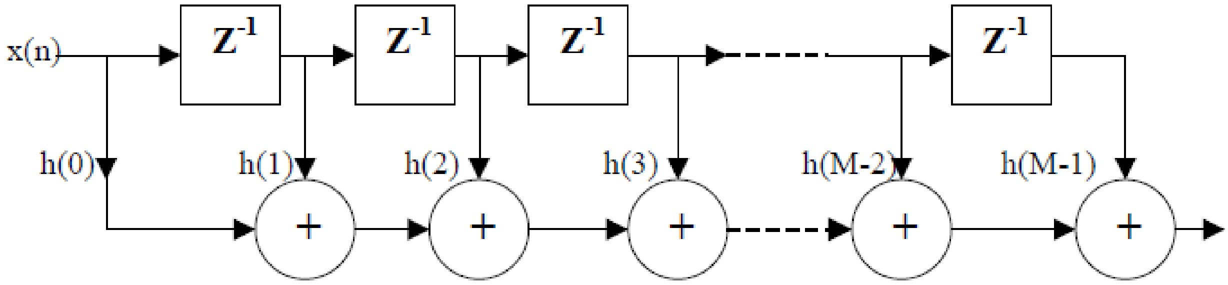
$$h(n) = \begin{cases} b_k, 0 \le n \le M-1 \\ 0, elsewhere \end{cases} \qquad 5$$

where, M = length of the FIR filter

Tapped Delay Line (TDL) or Transversal System (TS)

Direct form is one of the simplest structures. The direct realization can be obtained from the non-recursive difference equation given in equation

$$y(n) = \sum_{k=0}^{M-1} b_k\, x(n-k) \qquad 6$$



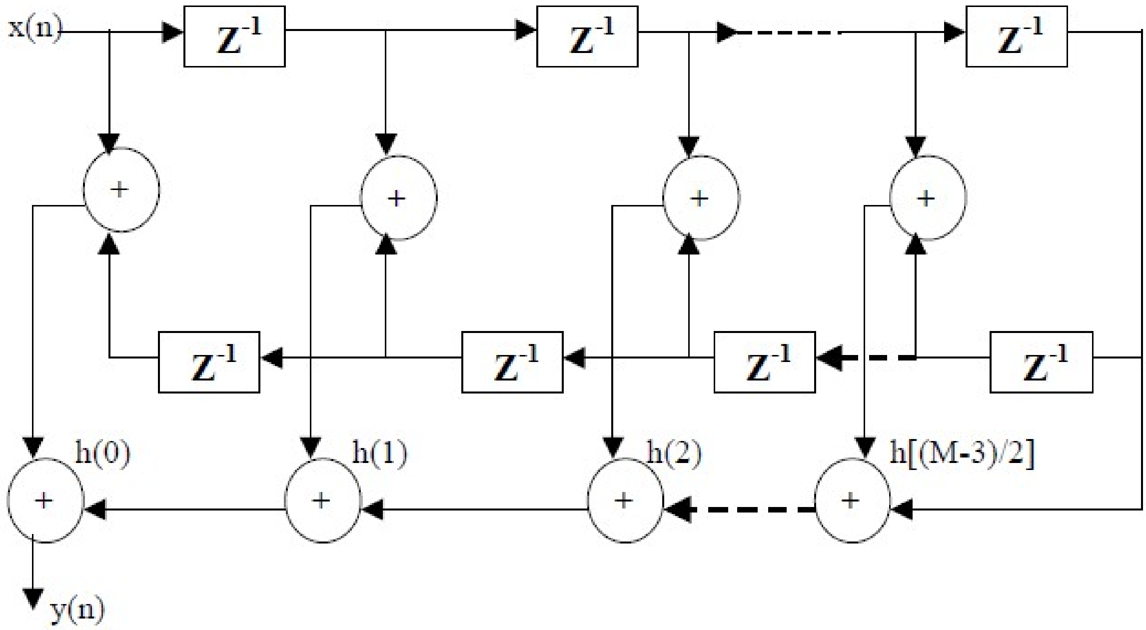
The structure has M-1 memory location for storing M-1 previous inputs. The structure has M complex multiplication and M-1 additions. The output is the weighted linear combination of M-1 past input and the weighted current value of the input.

Direct Form (N = odd)

When the FIR system has linear phase, the unit sample response of the system satisfies either the symmetry or asymmetry condition

$h(n) = \pm h(M-1-n)$ 7

for such system, the number of multiplication is reduced from M to M/2 for even M and M to (M-1)/2 for odd M.



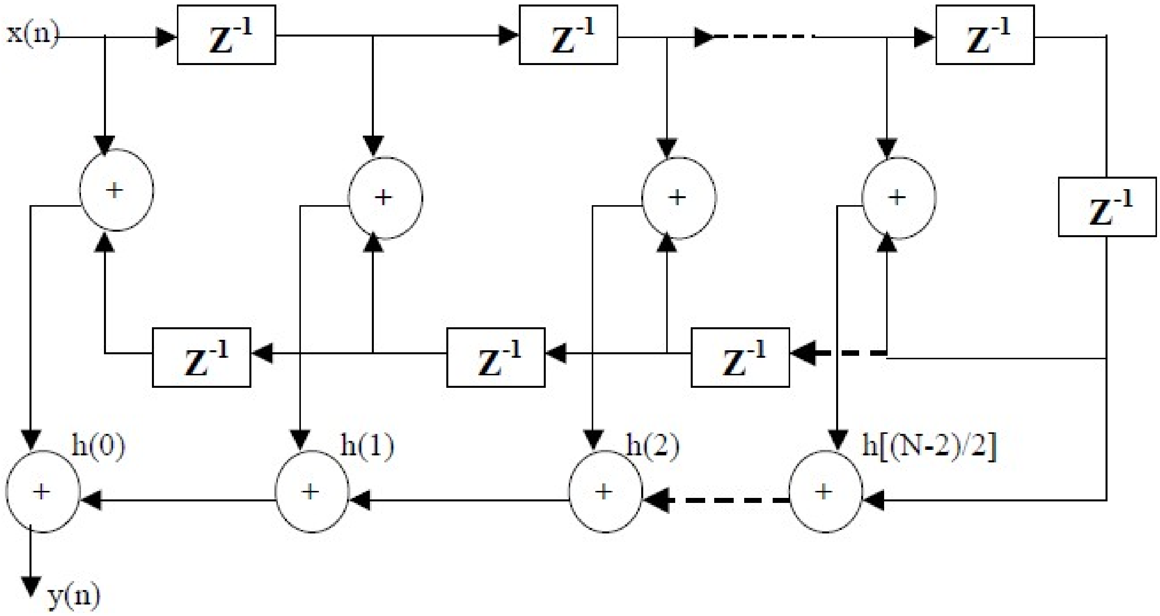
For M = odd,

$$H(Z) = \sum_{n=0}^{(M-3)/2} h(n)\, Z^{-n} + \sum_{n=(M+1)/2}^{(M-1)} h(n)\, Z^{-n} + h\left(\frac{M-1}{2}\right) Z^{-\left(\frac{M-1}{2}\right)}$$

$$H(Z) = \sum_{n=0}^{(M-3)/2} h(n)\, [Z^{-n} + Z^{-(M-1-n)}] + h\left(\frac{M-1}{2}\right) Z^{-\left(\frac{M-1}{2}\right)} \qquad 8$$

The number of multiplier required are $\left(\frac{N+1}{2}\right)$

Direct Form (N = even)



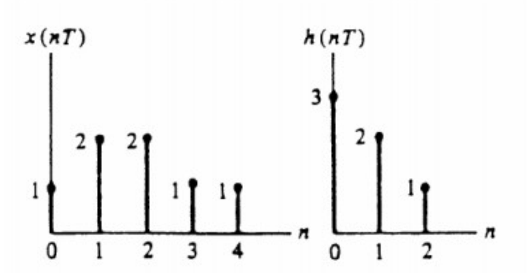
For M = even,

$$H(Z) = \sum_{n=0}^{(N-2)/2} h(n)\, Z^{-n} + \sum_{n=N/2}^{(N-1)} h(n)\, Z^{-n}$$

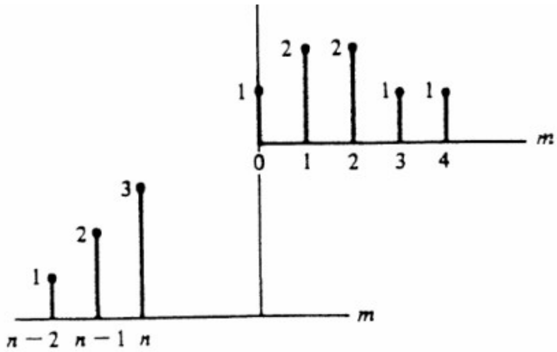
$$H(Z) = \sum_{n=0}^{(N-2)/2} h(n)\, [Z^{-n} + Z^{-(N-1-n)}] \qquad 9$$

The number of multiplier required are $\left(\frac{N+1}{2}\right)$

PROBLEMS: ON CONVOLUTION



(a) Input and unit pulse response



(b) Functions for computing convolution sum

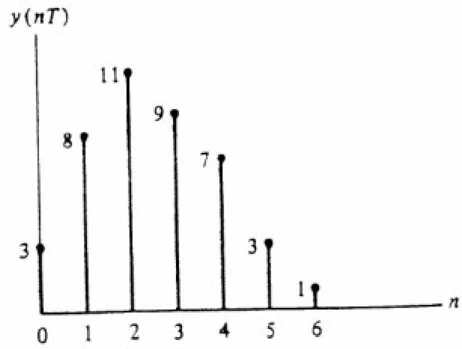
(b) Functions for computing convolution sum

$h(-m)$

$h(1-m)$

	Samples $x(mT)$							
	0	0	1	2	2	1	1	0
$n = 0$	1	2	3	0	0	0	0	0
$n = 1$	0	1	2	3	0	0	0	0
$n = 2$	0	0	1	2	3	0	0	0
$n = 3$	0	0	0	1	2	3	0	0
$n = 4$	0	0	0	0	1	2	3	0
$n = 5$	0	0	0	0	0	1	2	3
$n = 6$	0	0	0	0	0	0	1	2

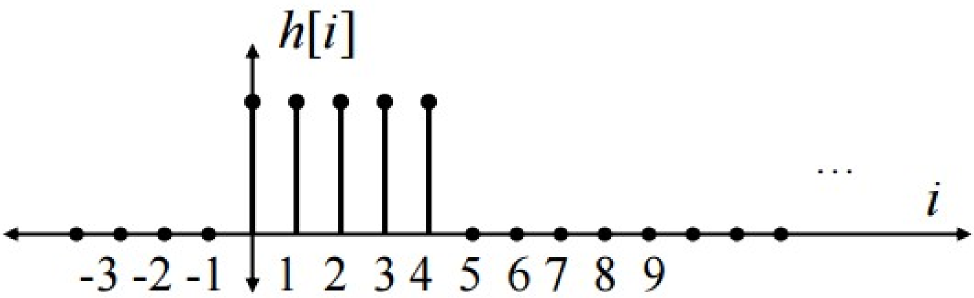
(c) Table for evaluating summation

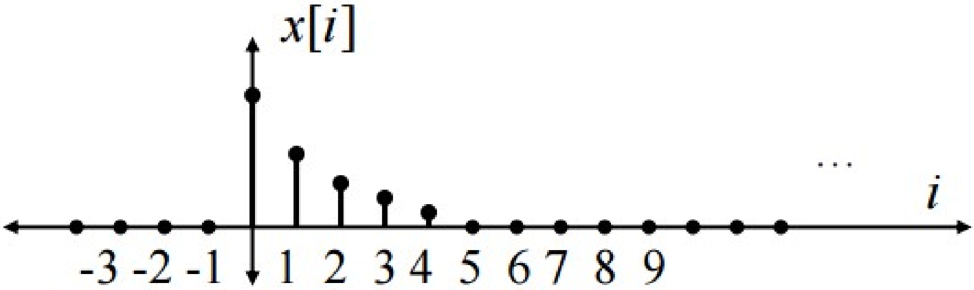


(d) Output

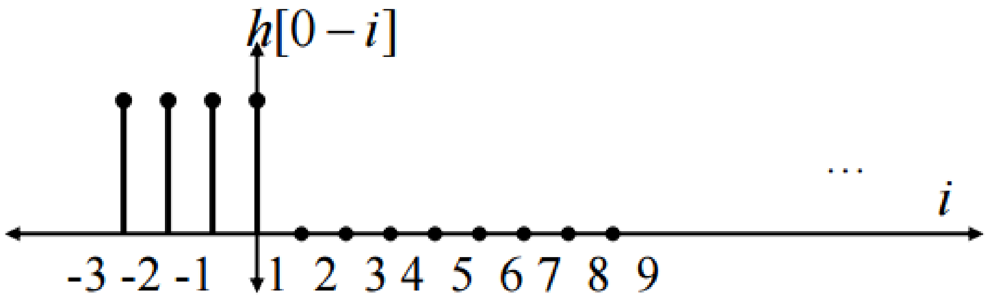
D-T Convolution Examples

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] \qquad h[n] = u[n] - u[n-4]$$

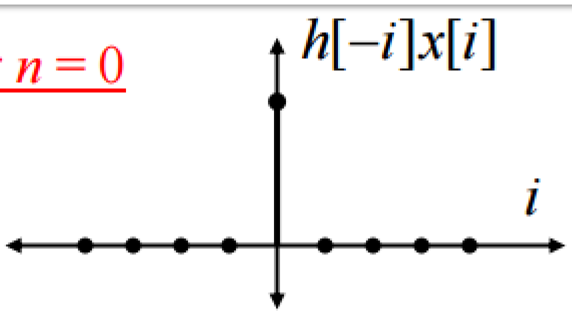




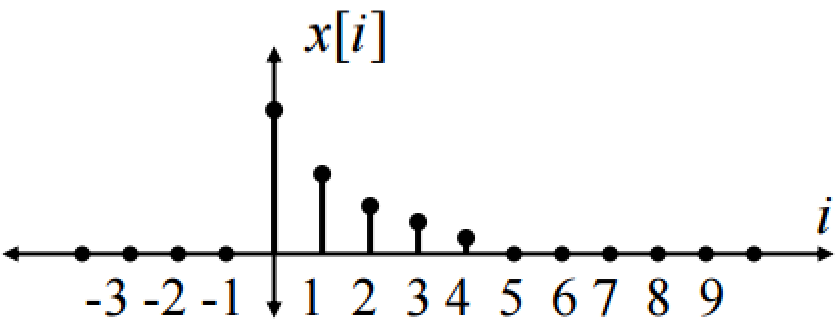
Choose to flip and slide $h[n]$ This shows $h[n\text{-}i]$ for $n = 0$



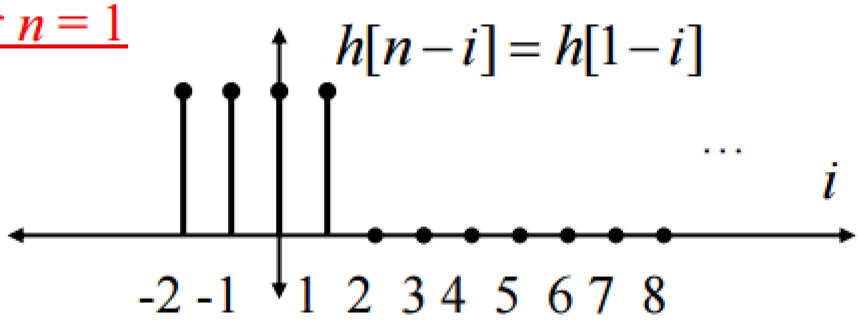
<u>For $n = 0$</u>



$\Rightarrow y[0] = 1$



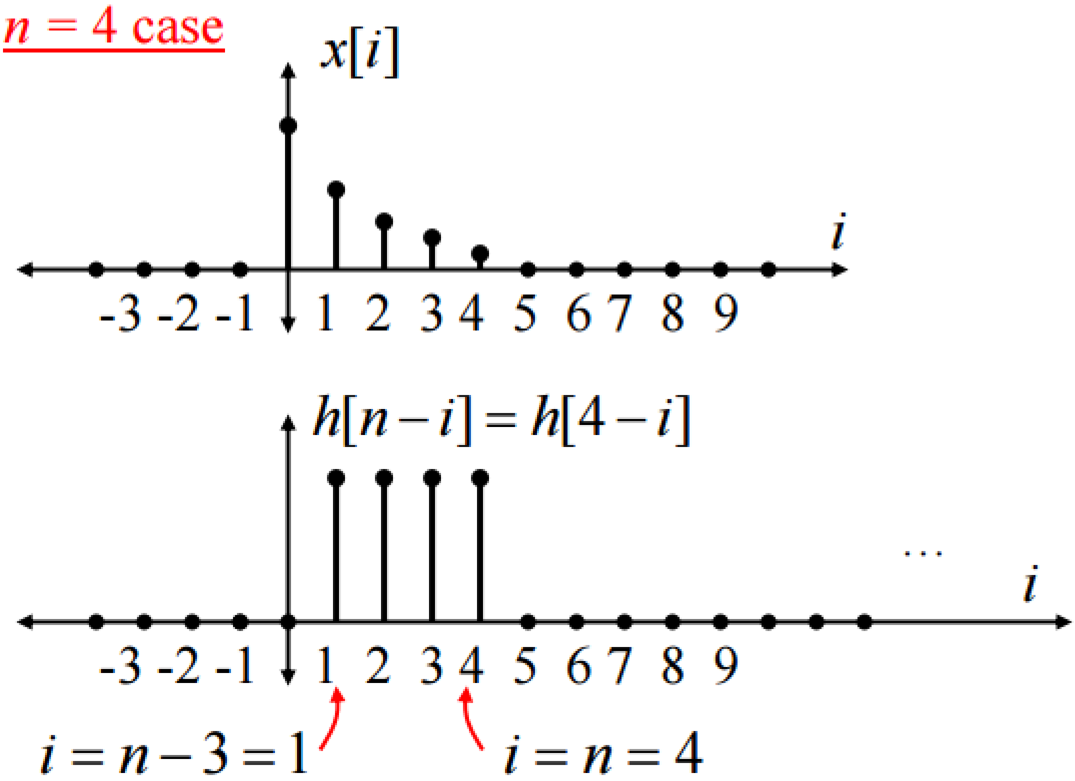
<u>For $n = 1$</u>



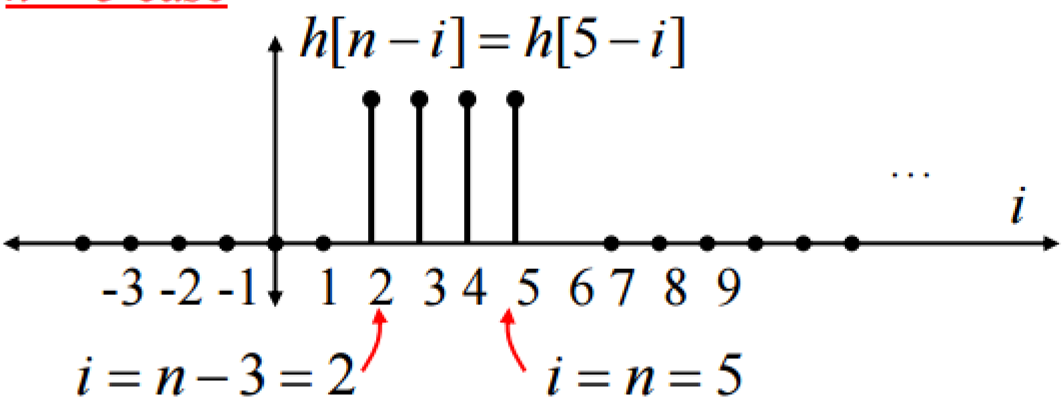
$\boxed{\Rightarrow y(1) = 1 + ½ = 3/2}$

Now for $n = 4$, $n = 5$, …

$n = 4$ case



$n = 5$ case



Notice that: for $n = 4, 5, 6, \ldots$

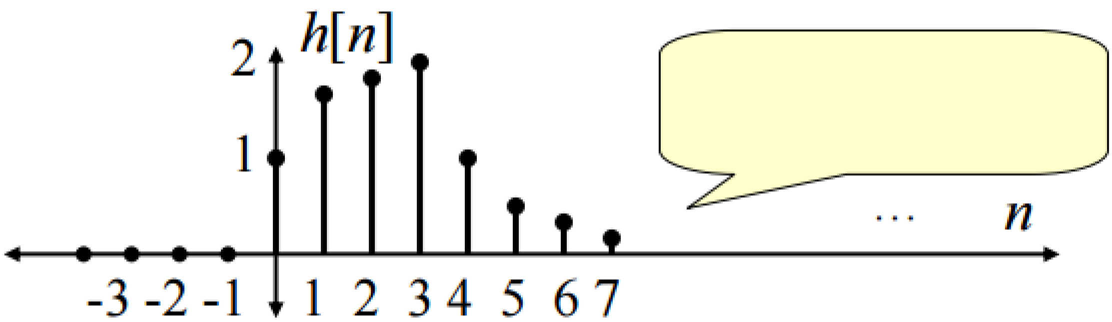
$$y[n] = \sum_{i=n-3}^{n} \left(\tfrac{1}{2}\right)^{i} \quad for\ n = 4,5,6,\ldots$$

$$= \frac{\left(\tfrac{1}{2}\right)^{n-3} - \left(\tfrac{1}{2}\right)^{n+1}}{1-\tfrac{1}{2}} \qquad then\ simplify!$$

Then we can write out the solution as:

Then we can write out the solution as:

$$y[n] = \begin{cases} 0, & n < 0 \\ 2\left[1-(1/2)^{n+1}\right], & n = 0,1,2,3 \\ 2\left[(1/2)^{n-3}-(1/2)^{n+1}\right], & n = 4,5,6,\ldots \end{cases}$$



PROBLEMS

1. The input and output of an LTI system are relatedby

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]$$

Note that without further information such as the initial condition, this equation does not uniquelyspecify $y[n]$ when $x[n]$ is given. Taking z-transform of this equation and using the time shifting property, weget

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z)$$

and the transfer function can be obtained

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}} = \frac{1}{1-\frac{1}{2}z^{-1}}(1+\frac{1}{3}z^{-1})$$

Note that the causality and stability of the system is not provided by this equation, unlessthe ROCof this $H(z)$ is specified. Consider these two possible ROCs:

- If ROCis $|z| > 1/2$, it is outsidethepole $z_p = 1/2$ and includes the unit circle. The system is causal andstable:

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

- If ROCis $z| < 1/2$, it is inside thepole $z_p = 1/2$ and does not include the unit circle. The system is anti-causal andunstable:

$$h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[-n]$$

First order system

The first order discrete system is described by

$$y[n] - ay[n-1] = x[n]$$

Theimpulse response $h[n]$ can be found by solving thefollowing

$$h[n] - ah[n-1] = \delta[n]$$

to be

$$h[n] = a^n u[n]$$

Alternatively, we can take z-transform of the DE and get

$$Y(z) - az^{-1}Y(z) = (1 - az^{-1})Y(z) = X(z)$$

and the transfer function of the system (assumed causal)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - az^{-1}} \qquad |z| > |a|$$

$H(z)$ has azeroat $\underline{z=0}$ and apole $z_p = a$ and its ROC istheregion $|z| > |a|$ outside thepole.If $|a| < 1$, then theunitcircle $|z| = r = 1$ can be included in the ROC, the Fourier transform exists and the system is stable. The impulse response (unit sample response) of the systemis

$$h[n] = \mathcal{Z}^{-1}[H(z)] = a^n u[n]$$

Although $h[n]$ seems to have a form different from the typical impulse response incontinuous case $\underline{h(t) = e^{-t/\tau}u(t)}$, they are essentially thesame as $h(t)$ can be rewrittenas

$$h(t) = e^{-t/\tau}u(t) = (e^{-1/\tau})^t u(t) = a^t u(t)$$

where

$$a \triangleq e^{-1/\tau}$$

Letting $z = e^{j\omega}$ in $H(z)$, we get the frequency response function of thesystem

$$H(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}} = \frac{e^{j\omega}}{e^{j\omega}-a} = \frac{e^{j\omega}-0}{e^{j\omega}-a} = \frac{u}{v}$$

where u and v are two vectors in z-plane definedas

$$u \triangleq e^{j\omega} - 0 = e^{j\omega}, \quad v \triangleq e^{j\omega} - a$$

Foranyfrequency $-\pi \le \omega \le \pi$ represented byapoint $z = e^{j\omega}$ on the unit circle, the magnitude and phase angle of the frequency response function can be represented in the z-plane as

$$|H(e^{j\omega})| = \frac{|u|}{|v|}$$

and

$$\angle H(e^{j\omega}) = \angle u - \angle v$$

which can be evaluated graphically in the z-plane asthe frequency ω changes in the range $-\pi \le \omega \le \pi$. Ifweassume $0 < a < 1$, then when $\omega = 0$, the denominator reaches itsminimumof $1-a$, and $|H(e^{j\omega})|$ is maximizedto be $1/(1-a)$; andwhen $\omega = \pi$,the denominator reaches its maximumof $|(-1-a)| = 1+a$, and $|H(e^{j\omega})|$ is minimized to be $1/(1+a)$. Thephaseangle of $H(e^{j\omega}) = \angle u - \angle v$ is zero when $\omega = 0$ or $\omega = \pi$, and isnegativefor $0 < \omega < \pi$ and positive for $-\pi < \omega < 0$.

System Algebra and Block Diagram

Z-transform converts time-domain operations such as difference and convolution into algebraic operations in z-domain. Moreover, the behavior of complex systems composed of a set of interconnected LTI systems can also be easily analyzed in z-domain. Some simple interconnections of LTI systems are listed below.

- **Parallel systems:** If the system is composed of two LTI systems with $h_1[n]$ and $h_2[n]$ connected in parallel, its impulse response is

$$h[n] = h_1[n] + h_2[n]$$

or in s-domain

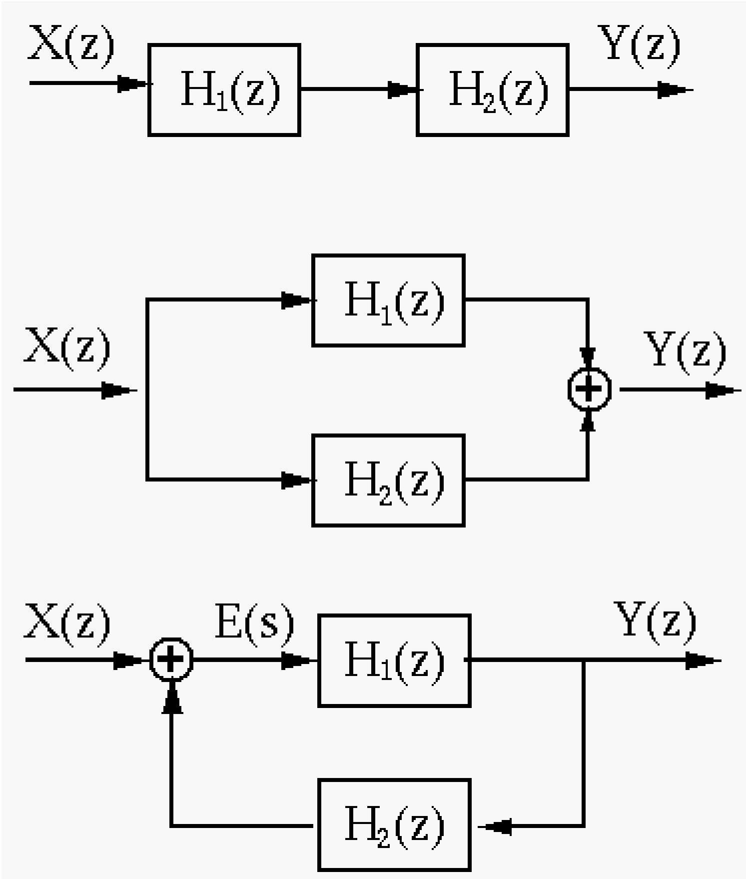
$$H(z) = H_1(z) + H_2(z)$$

- **Serial or cascade system:** If the system is composed of two LTIsystems with $h_1[n]$ and $h_2[n]$ connected in series, its impulse responseis

$$h[n] = h_1[n] * h_2[n] = h_2[n] * h_1[n]$$

or in s-domain

$$H(z) = H_1(z)H_2(z) = H_2(z)H_1(z)$$



- **Feedback system:** If the system is composed of an LTIsystemwith $h_1[n]$ in aforward path and anotherLTIsystem $h_2[n]$ in a feedback path,its output $y[n]$ can be implicitly found in timedomain

$$y[n] = h_1[n] * e[n] = h_1[n] * [x[n] + h_2[n] * y[n]]$$

or in s-domain

$$Y(z) = H_1(z)E(z) = H_1(z)[X(z) + H_2(z)Y(z)]$$

While it is difficult to solve the equation in time domain to find an explicit expressionfor output $y[n] = h[n] * x[n]$, it is easy to solve the algebraic equation in z-domain to find $Y(z)$

$$Y(z)[1 - H_1(z)H_2(z)] = H_1(z)X(z)$$

and the transfer function can be obtained

$$H(z) = \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 - H_1(z)H_2(z)}$$

The feedback could be either positive or negative. For the latter, there will be anegative sign infront of $h_2(t)$ and $H_2(s)$ of the feedback pathso that $e[n] = x[n] - h_2[n] * y[n]$ and

$$H(z) = \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$

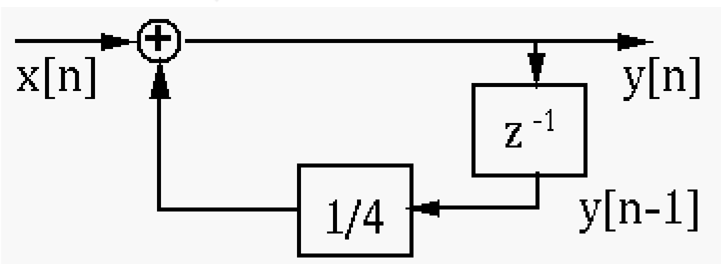
Example 0: The transfer function of a first order LTI system

$$y[n] - \frac{1}{4}y[n-1] = x[n], \qquad Y(z)[1 - \frac{1}{4}z^{-1}] = X(z)$$

is

$$H(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}$$

Comparingthis $H(z)$ with the transfer function of the feedback system, we see that a first order system can be represented as a feedbacksystemwith $H_1(z) = 1$ in the forward path, and $H_2(z)$ for theproduct of $1/4$ and z^{-1} (a delay elementwithinput $x[n]$ and output $y[n] = x[n-1]$) in the negative feedbackpath.



Example 1:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 2z^{-1}}{1 - \frac{1}{4}z^{-1}}$$

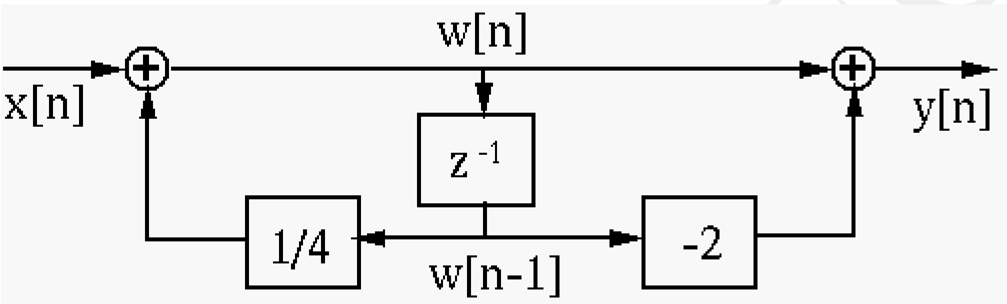
This equation can be rewritten as:

$$Y(z) = (1 - 2z^{-1})\frac{1}{1 - \frac{1}{4}z^{-1}}X(z) = (1 - 2z^{-1})W(z)$$

where

$$W(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}X(z)$$

can be obtained the same way as in previous example. Once $W(z)$ and $W(z)z^{-1}$ are available, we can easily obtain $Y(z)$:



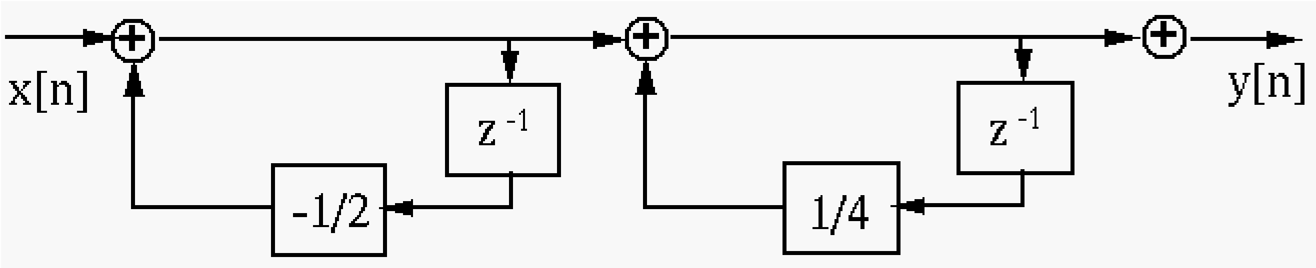
Example 2: Consider a second order system with transfer function

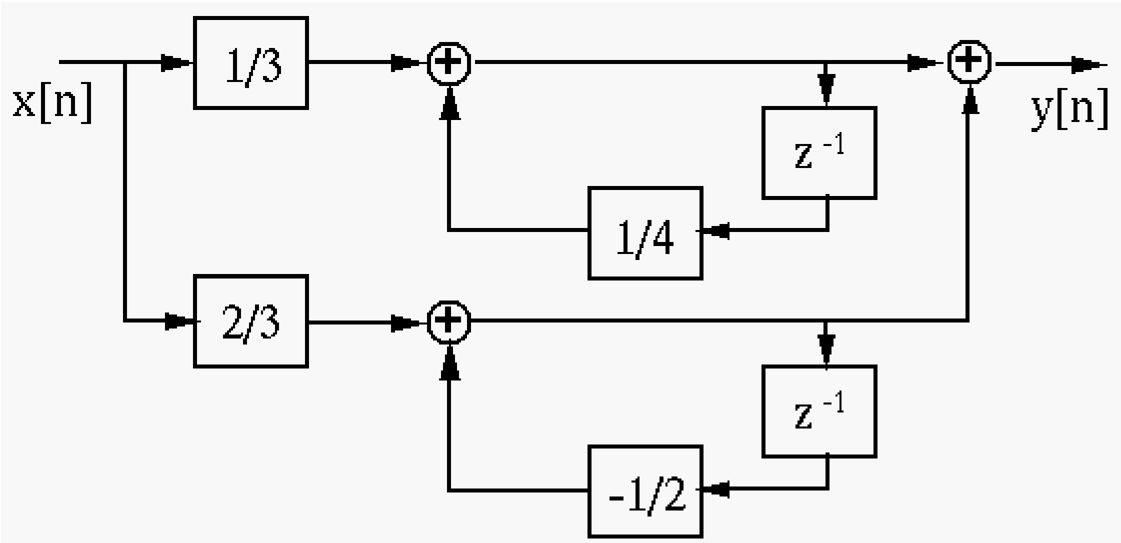
$$H(z) = \frac{1}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}} = \frac{1}{(1 + \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})} = \frac{2/3}{1 + \frac{1}{2}z^{-1}} + \frac{1/3}{1 - \frac{1}{4}z^{-1}}$$

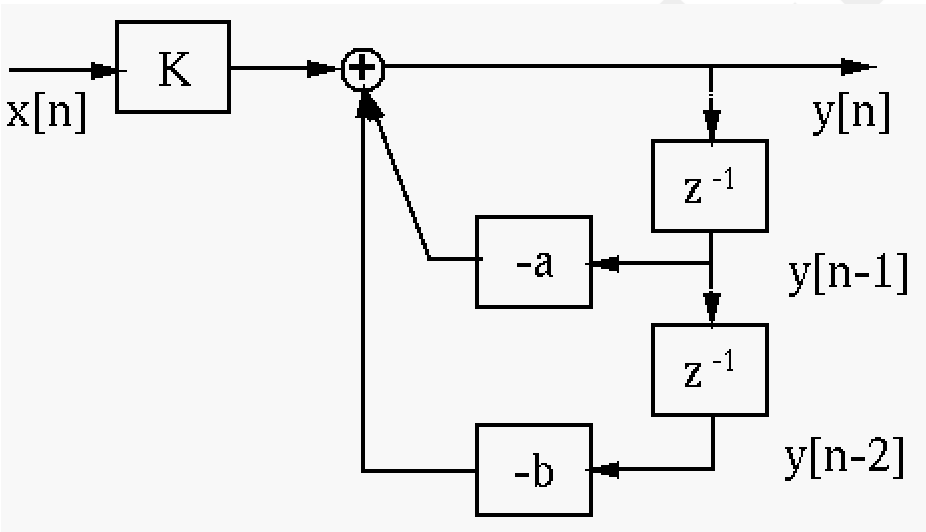
These three expressions of this $H(z)$ correspond to three different block diagram representations of the system. The last two expressions are, respectively, the cascade and the parallel representations (same as the corresponding cases in Laplace transform), while the first one is the direct representation, as shown below. We first consider a general 2nd order system

$$Y(z) + asY(z)z^{-1} + bY(z)z^{-2} = X(z), \quad \text{or} \quad Y(z) = X(z) - aY(z)z^{-1} - bY(z)z^{-2}$$

Weseethat $Y(z)$ is the linear combination of the delayed versions of itself and the input $X(z)$ which can be represented as a feedback system with two feedback paths of $-az^{-1}$ and $-bz^{-2}$. In thisparticular system, $a = -1/4$ and $b = 1/8$.







Example 3: A second order system with transfer function

$$H(z) = K\frac{1+cz^{-1}+dz^{-2}}{1+az^{-1}+bz^{-2}} = \frac{K}{1+az^{-1}+bz^{-2}}(1+cz^{-1}+dz^{-2})$$

This system can be represented as a cascade of two systems

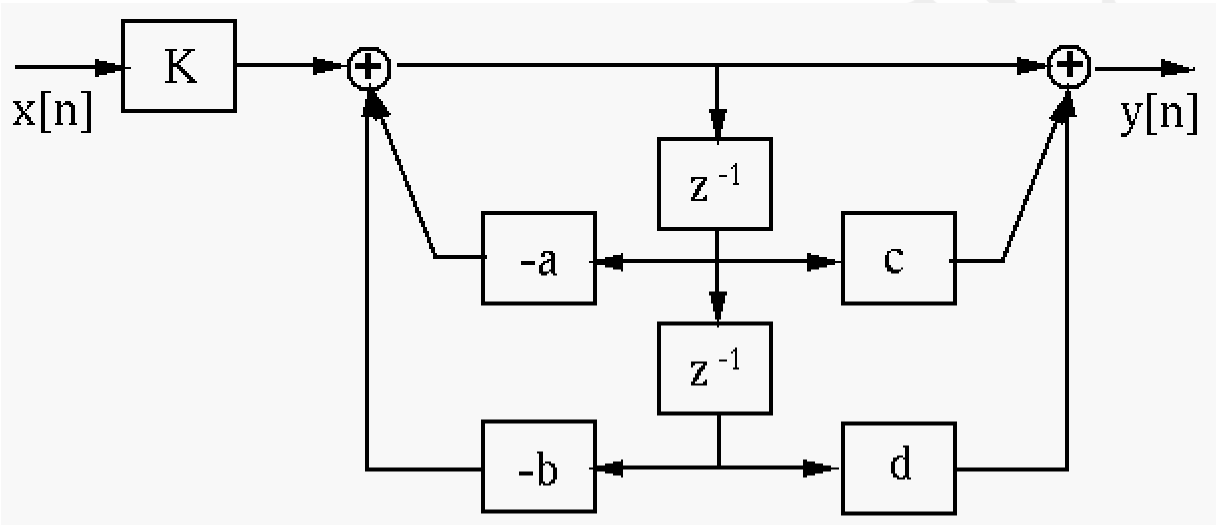
$$W(z) = H_1(z)X(z) = \frac{K}{1+az^{-1}+bz^{-2}}X(z)$$

and

$$Y(z) = H_2(z)W(z) = (1+cz^{-1}+dz^{-2})W(z)$$

Thefirstsystem $H_1(z)$ can be implemented by two delay elements with proper feedback paths as shown in the previous example, and the second system is a linearcombination of $W(z)$, $W(z)z^{-1}$ and $W(z)z^{-2}$, all of which are available along the feedback path of the first system. The over all system can therefore be represented as shown. Obviously theblock diagram of this example can be generalized to represent any system with a rational transfer function

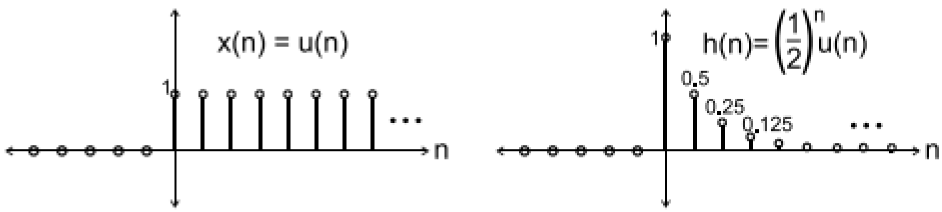
$$H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \quad (M \le N)$$



EXAMPLES OF DT CONVOLUTION

Say we are given the following signal $x(n)$ and system impulse response $h(n)$.

$$x(n) = u(n) \quad \text{and} \quad h(n) = \left(\frac{1}{2}\right)^n u(n)$$



We wish to find the step response $s(n)$ of the system (i.e. the response of the system to the unit step input $x(n) = u(n)$. This is shown below.

$$s(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

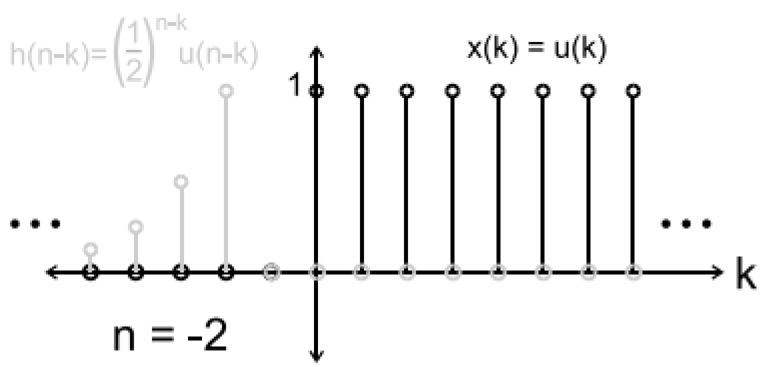
Thus the step response is as follows, found by substituting our actual signals into the general convolution sum.

$$s(n) = \sum_{k=-\infty}^{\infty} u(k)\left(\frac{1}{2}\right)^{n-k} u(n-k)$$

Let's look at this step response in smaller ranges to see what happens.

- First, consider the case where $n < 0$.

Here, $s(n) = 0$. This is because $u(n-k)$ (and the associated exponential) will be starting at a point less than 0 in the k domain, and will extend to $-\infty$, whereas $u(k)$ starts at 0 and extends to $+\infty$. We can visualize this, say for a value of $n = -2$.



Notice that there is no non-zero overlap of $x(k)$ and $h(n-k)$. Since they are multiplied together, the zero part of one signal cancels out the non-zero part of the other, and vice versa. Thus, $s(n) = 0$ for $n < 0$.

- The more interesting case is when $n \geq 0$.

  Recall the convolution sum we are using to determin $s(n)$.

$$s(n) = \sum_{k=-\infty}^{\infty} u(k) \left(\frac{1}{2}\right)^{n-k} u(n-k)$$

Note that $u(k)$ means we know the summation will be 0 for all values of $k < 0$, so we can change the lower limit of the summation to 0. Similarly, the $u(n-k)$ term means that the summation for all values of $k > n$ will be 0, since that unit step is flipped and extends toward $-\infty$. So, we can change the upper limit of the summation to n. In the range $0 \leq k \leq n$, both of the unit steps will have a value of 1. This is shown below.

$$\begin{aligned}
s(n) &= \sum_{k=-\infty}^{\infty} u(k) \left(\frac{1}{2}\right)^{n-k} u(n-k) \\
&= \sum_{k=0}^{n} 1 \cdot \left(\frac{1}{2}\right)^{n-k} \cdot 1 \\
&\quad \text{We can pull out any terms only in } n \\
&\quad \text{since that is not the summation variable.} \\
&= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{n} \left(\frac{1}{2}\right)^{-k} \\
&= \left(\frac{1}{2}\right)^{n} \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{-k} \\
&= \left(\frac{1}{2}\right)^{n} \sum_{k=0}^{n} 2^{k}
\end{aligned}$$

Now we have a form consistent with a geometric series. We can use that to solve.

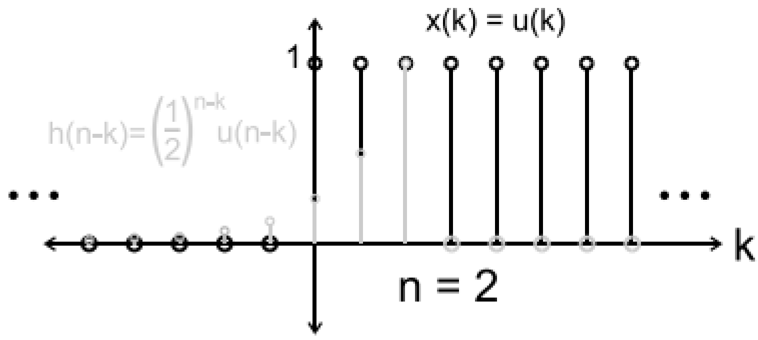
$$\text{Recall} \quad \sum_{k=0}^{n} 2^k = \frac{1-2^{n+1}}{1-2} = 2^{n+1} - 1$$

So we have $s(n)$ as follows.

$$s(n) = \left(\frac{1}{2}\right)^{n} \left(2^{n+1} - 1\right)$$

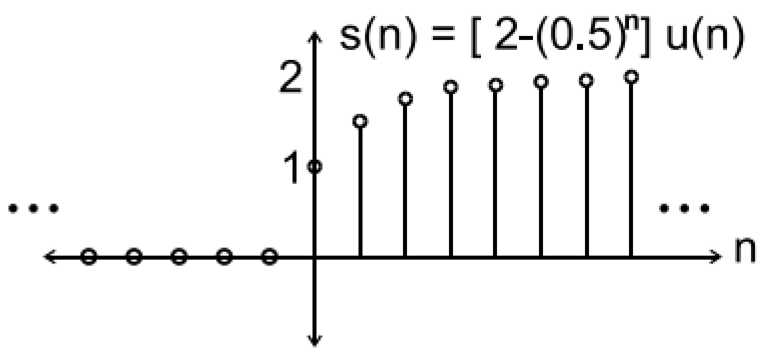
$$
\begin{aligned}
&= \left(\frac{1}{2}\right)^n (2 \cdot 2^n - 1) \\
&= \left(\frac{1}{2}\right)^n \left(2 \cdot \left(\frac{1}{2}\right)^{-n} - 1\right) \\
&= 2 \cdot \left(\frac{1}{2}\right)^{-n} \left(\frac{1}{2}\right)^n - 1 \cdot \left(\frac{1}{2}\right)^n \\
s(n) &= 2 - \left(\frac{1}{2}\right)^n
\end{aligned}
$$

We can visualize this, say for $n = 2$, as shown below. Note how the system output comes from the overlap of the input signal and the shifted and flipped impulse response.



So, overall, we have the following step response.

$$s(n) = \left[2 - \left(\frac{1}{2}\right)^n\right] u(n)$$



The u(n) comes from our first case above since s(n) = 0 for n < 0, and obviously the other part comes from the expression found in the second case above.

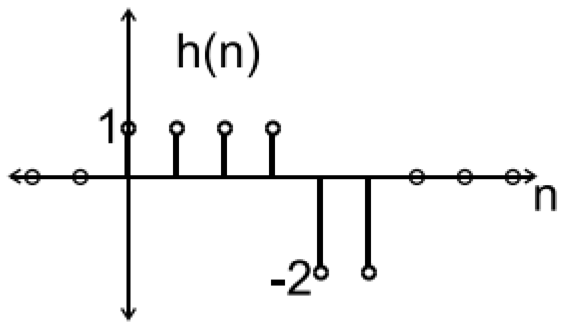
3. In the convolution sum, the impulse response is written as h(n−k), meaning that in the k domain, the impulse response is shifted by n and flipped around thatpoint

   We can visualize the convolution operation as that shifted-and-flipped impulse response sliding along the k axis from −∞ to ∞ as the summation occurs.

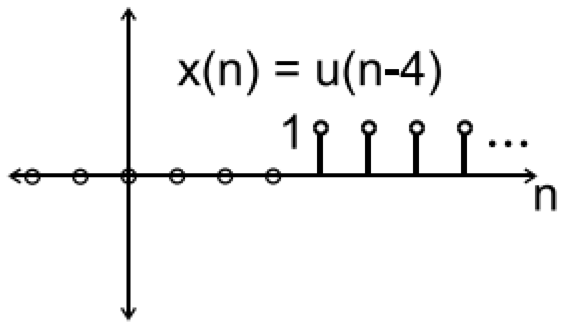
   Whenever there is some non-zero overlap between this shifted-and-flipped impulse response and the input signal, the system output will be non-zero (unless the non-zero overlaps cancel each other).

We are given the impulse response shown below.

$$h(n) = \begin{cases} 0 & \text{for} \quad n < 0 \\ 1 & \text{for} \quad 0 \leq n \leq 3 \\ -2 & \text{for} \quad 4 \leq n \leq 5 \\ 0 & \text{for} \quad n > 5 \end{cases}$$



Let $x(n) = u(n-4)$.



We want to determine

$$\begin{aligned} y(n) = x(n) * h(n) &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\ &= \sum_{k=-\infty}^{\infty} u(k-4)h(n-k) \end{aligned}$$

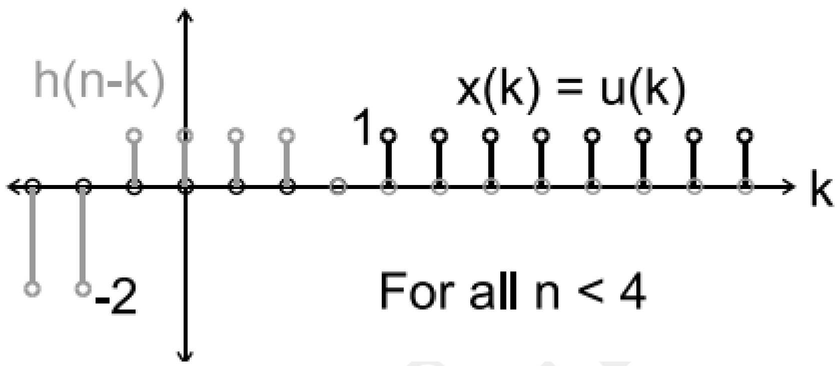
We can use $u(k-4)$ to change the summation limits, but it doesn't help much.

$$= \sum_{k=4}^{\infty} 1 \cdot h(n-k)$$

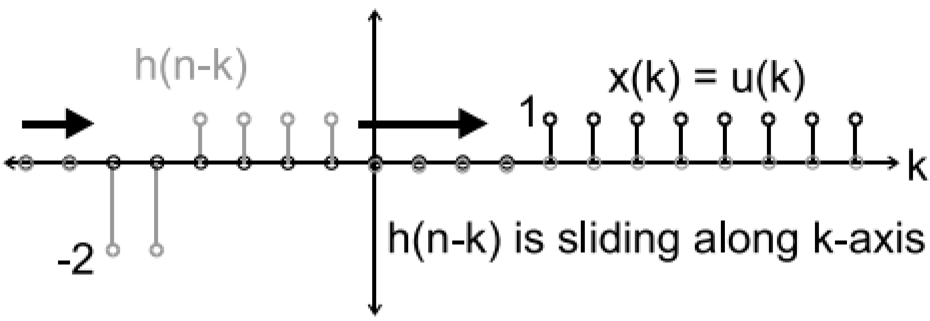
but we have no convenient functional representation of $h(n)$ to allow us to solve.

Consider the solution in a piece-wise fashion.

For $n < 4$, $y(n) = 0$ since there is only zero overlap between the two signals. This is illustrated below.

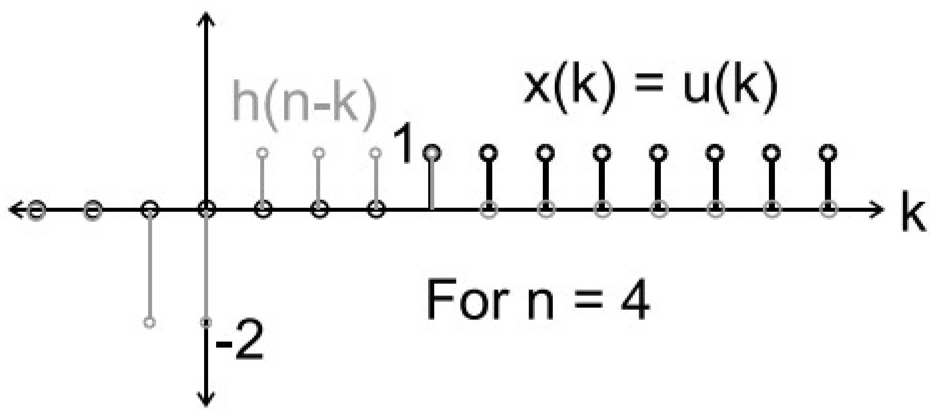


For $n \geq 4$, we need to visualize what is happening in order to determine the value of $y(n)$. In the figure below, we can see how $h(n-k)$ slides along the k axis and overlaps with $x(k)$.

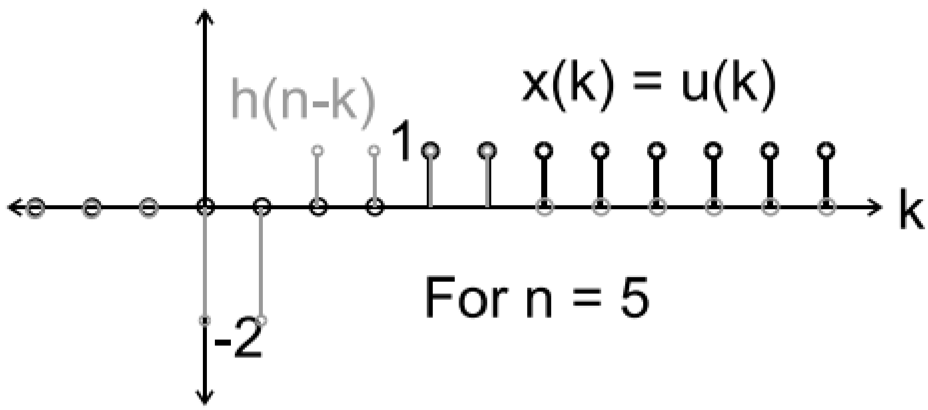


So we have to determine the value of $y(n)$ for specific values of n. We start at the first point of non-zero overlap, when $n = 4$.

When $n = 4$, $y(4) = 1$ since only one point of the two signals overlaps, and $1 \cdot 1 = 1$. This is shown in the figure below.

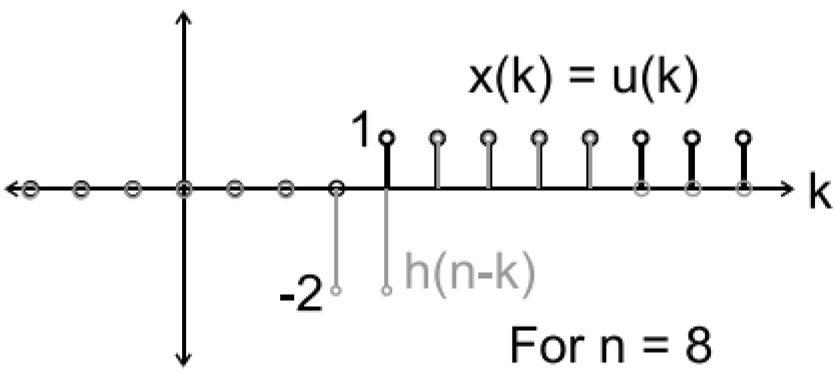


When $n = 5$, $y(5) = 2$ since it is the sum of the two overlapping points. This is shown in the figure below.

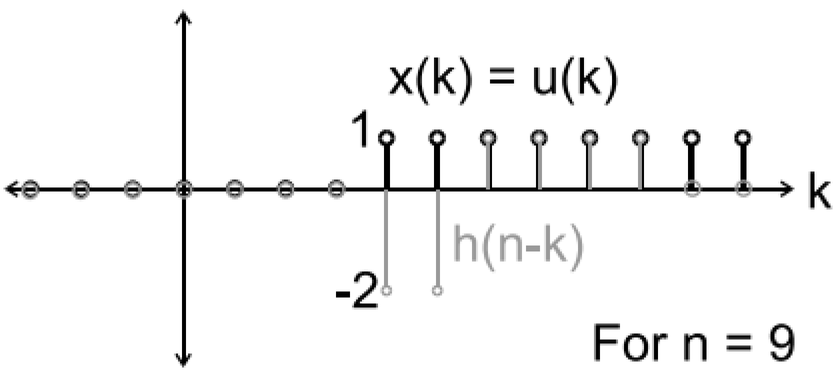


Similarly, $y(6) = 3$ and $y(7) = 4$.

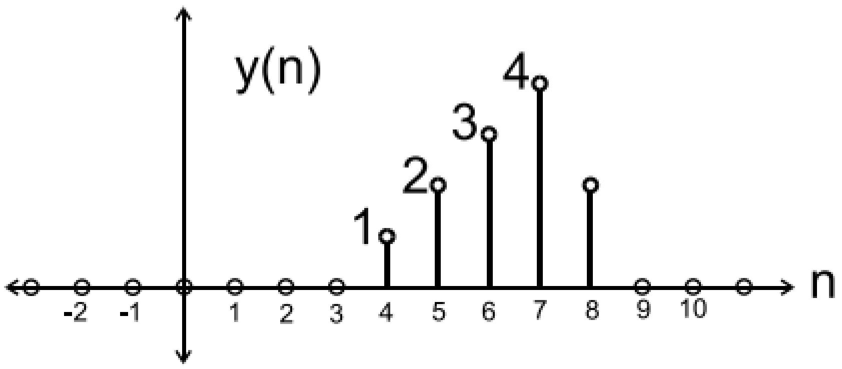
Note that when $n = 8$, we have a negative overlap, and so $y(8) = 2$. This is shown below.



For the case where $n \geq 9$, $y(n) = 1$ since it is summing over the entire length of the impulse response. This is shown in the figure below.



Thus, we can plot our overall $y(n)$ as shown here.



Thus, we have evaulated the convolution some graphically by taking advantage of this shifting and flipping behaviour.

4. Finding the impulse response of a diffeq system. Find the impulse response of thesystem described by the followingdiffeq:
   y[n] = 4 3 y[n − 1] − 7 12 y[n − 2] + 1 12 y[n − 3] + x[n] − x[n − 3] .

   Step 0: Find the system function. (linearity, shift property)

$$Y(z) = \frac{4}{3}z^{-1}Y(z) - \frac{7}{12}z^{-2}Y(z) + \frac{1}{12}z^{-3}Y(z) + X(z) - z^{-3}X(z)$$

$$\left[1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}\right]Y(z) = \left[1 - z^{-3}\right]X(z)$$

so (by convolution property):

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-3}}{1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}}.$$

Step 1: Decompose system function into proper form + polynomial.

In this case we can see by comparing the coefficients of the z^{-3} terms that the coefficient for the 0th-order term will be 1/(1/12) = 12.

$$H(z) = 12 + \left[\frac{1 - z^{-3}}{1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}} - 12\right] = 12 + P(z)$$

where

$$\begin{aligned} P(z) &= \frac{1 - z^{-3}}{1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}} - 12 \\ &= \frac{1 - z^{-3} - 12\left[1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}\right]}{1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}} \\ &= \frac{-11 + 16z^{-1} - 7z^{-2}}{1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}}. \end{aligned}$$

Note that $P(z)$ is a proper rational function.
Since $H(z) = 12 + P(z)$, we see that $h[n] = 12\,\delta[n] + p[n]$.
We now focus on finding $p[n]$ from $P(z)$ by PFE.

Step 2: Find poles (roots of denominator).

The MATLAB command `roots([1 -4/3 7/12 -1/12])` returns `0.5 0.5 0.33`, so we check and verify that the denominator can be factored:

$$1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3} = \left(1 - \frac{1}{2}z^{-1}\right)^2 \left(1 - \frac{1}{3}z^{-1}\right),$$

so in factored form:

$$P(z) = \frac{-11 + 16z^{-1} - 7z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)^2 \left(1 - \frac{1}{3}z^{-1}\right)}.$$

Step 3: Find PFE

Since there is one repeated root, the PFE form is

$$P(z) = \frac{r_{1,1}}{1 - \frac{1}{2}z^{-1}} + \frac{r_{1,2}}{\left(1 - \frac{1}{2}z^{-1}\right)^2} + \frac{r_2}{1 - \frac{1}{3}z^{-1}}.$$

For a single pole at $z = p_k$, we find the residue using this formula:

$$r_k = \left(1 - p_k z^{-1}\right) P(z)\Big|_{z=p_k}.$$

Thus for the single pole at $z = 1/3$:

$$r_2 = \left(1 - \frac{1}{3}z^{-1}\right) P(z)\Big|_{z=1/3} = \left.\frac{-11 + 16z^{-1} - 7z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)^2}\right|_{z=1/3} = -104.$$

For a double pole at $z = p_k$, the residues are given by

$$r_{k,1} = \frac{1}{-p_k}\frac{\mathrm{d}}{\mathrm{d}z^{-1}}(1 - p_k z^{-1})^2 P(z)\Big|_{z=p_k}, \quad \text{and} \quad r_{k,2} = (1 - p_k z^{-1})^2 P(z)\Big|_{z=p_k}.$$

Thus for the double pole at $z = 1/2$:

$$\begin{aligned} r_{1,1} &= \frac{1}{-1/2}\frac{d}{dz^{-1}}(1 - \frac{1}{2}z^{-1})^2 P(z)\Big|_{z=1/2} = -2\frac{d}{dz^{-1}}\left.\frac{-11 + 16z^{-1} - 7z^{-2}}{1 - \frac{1}{3}z^{-1}}\right|_{z=1/2} \\ &= -2\left.\frac{(1 - \frac{1}{3}z^{-1})(16 - 14z^{-1}) - (-11 + 16z^{-1} - 7z^{-2})(-\frac{1}{3})}{(1 - \frac{1}{3}z^{-1})^2}\right|_{z=1/2} = 114, \end{aligned}$$

and

$$r_{1,2} = (1 - \frac{1}{2}z^{-1})^2 P(z)\Big|_{z=1/2} = \left.\frac{-11 + 16z^{-1} - 7z^{-2}}{(1 - \frac{1}{3}z^{-1})}\right|_{z=1/2} = -21.$$

Substituting in these residues into equation (3-2):

$$P(z) = \frac{114}{1 - \frac{1}{2}z^{-1}} + \frac{-21}{(1 - \frac{1}{2}z^{-1})^2} + \frac{-104}{1 - \frac{1}{3}z^{-1}}.$$

Step 4: Inverse z-transform ______ (using table lookup)

$$p[n] = 114\left(\frac{1}{2}\right)^n u[n] - 21(n+1)\left(\frac{1}{2}\right)^n u[n] - 104\left(\frac{1}{3}\right)^n u[n].$$

Substituting into proper form decomposition above yields our final answer:

$$\boxed{h[n] = 12\,\delta[n] + \left[(114 - 21(n+1))\left(\frac{1}{2}\right)^n - 104\left(\frac{1}{3}\right)^n\right] u[n].}$$

Response of systems with rational system functions

$X(z) \to \boxed{H(z)} \to Y(z)$. Goal: characterize $y[n]$

Assume

- $H(z)$ is a pole-zero system, *i.e.*, $H(z) = B(z)\,/\,A(z)$.
- Input signal has a rational z-transform of the form $X(z) = N(z)/Q(z)$.

Then

$$Y(z) = H(z)\,X(z) = \frac{B(z)\,N(z)}{A(z)\,Q(z)}.$$

So the output signal also has a rational z-transform.

How do we find $y[n]$? Since $Y(z)$ is rational, we use PFE to find $y[n]$.

Assume

- Poles of system $p_1, \ldots, p_N$ are unique
- Poles of input signal $q_1, \ldots, q_L$ are unique
- Poles of system and input signal are all different
- Zeros of system and input signal differ from all poles (so no pole-zero cancellation)
- Proper form
- Causal input sequence and causal LTI system

Then

$$X(z) = \sum_{k=1}^{L} \frac{\alpha_k}{1 - q_k z^{-1}} \xrightarrow{\mathcal{T}} Y(z) = \sum_{k=1}^{N} \frac{r_k}{1 - p_k z^{-1}} + \sum_{k=1}^{L} \frac{s_k}{1 - q_k z^{-1}}$$

so (assuming a causal system) the response is:

$$y[n] = \underbrace{\sum_{k=1}^{N} r_k p_k^n\, u[n]}_{\text{natural}} + \underbrace{\sum_{k=1}^{L} s_k q_k^n\, u[n]}_{\text{forced}}.$$

The output signal for a causal pole-zero system with input signal having rational z-transform is a weighted combination of geometric progression signals.

If there are repeated poles, then of course the PFE has terms of the form $np^n\, u[n]$ etc.

The output signal has two parts

- The p_k terms are the **natural response** $y_{\mathrm{nr}}[n]$ of the system. (The input signal affects only the residues r_k). Each term of the form $p_k^n\, u[n]$ is called a **mode** of the system.
- The q_k terms are the **forced response** $y_{\mathrm{fr}}[n]$ of the system. (The system affects "only" the residues s_k.)

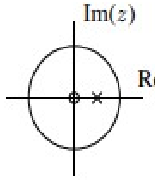
Transient and steady-state response

Define $y_{\mathrm{nr}}[n]$ to be the natural response of the system, *i.e.*, $y_{\mathrm{nr}}[n] = \sum_{k=1}^{N} r_k p_k^n \, u[n]$.

- If all the poles have magnitude less than unity, then this response decays to zero as $n \to \infty$.
- In such cases we also call the natural response the **transient response**.
- Smaller magnitude poles lead to faster signal decay. So *the closer the pole is to the unit circle, the longer the transient response.*

The forced response has the form $y_{\mathrm{fr}}[n] = \sum_{k=1}^{L} s_k q_k^n \, u[n]$.

- If all of the input signal poles are within the unit circle, then the forced response will decay towards zero as $n \to \infty$.
- If the input signal has a pole *on the unit circle* then there is a persistent sinusoidal component of the input signal. The forced response to such a sinusoid is also a persistent sinusoid.
- In this case, the forced response is also called the **steady-state** response.



Example. System (initially relaxed) described by diffeq: $y[n] = \frac{1}{2} y[n-1] + x[n]$.

What are the poles of the system? At $p = 0.5$. $H(z) = \dfrac{1}{1 - \frac{1}{2} z^{-1}}$.

Signal: $x[n] = (-1)^n u[n]$. Pole at $q = -1$. $X(z) = \dfrac{1}{1 + z^{-1}}$.

$$Y(z) = H(z)\,X(z) = \frac{1}{(1 - \frac{1}{2}z^{-1})(1 + z^{-1})} = \frac{1}{1 + \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2}} = \frac{1/3}{1 - \frac{1}{2}z^{-1}} + \frac{2/3}{1 + z^{-1}}$$

where I found the PFE using `[r p k] = residuez(1, [1 1/2 -1/2])`. So

$$y[n] = \underbrace{\frac{1}{3}\left(\frac{1}{2}\right)^n u[n]}_{\text{natural / transient}} + \underbrace{\frac{2}{3}(-1)^n u[n]}_{\text{forced / steady state}} .$$

Causality and stability

We previously described six system properties: **linearity**, **invertibility**, **stability**, **causality**, **memory**, **time-invariance**.

- We first described these properties in general.
- We then characterized these properties in terms of the impulse response $h[n]$ of an LTI system, because any LTI system is described completely by its impulse response $h[n]$.
  - **causality**: $h[n] = 0 \; \forall \, n < 0$.
  - **stability**: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$.
- Now we characterize these properties in the z-domain.
  If it exists, the system function $H(z)$ (including its ROC) also describes completely an LTI system, since we can find $h[n]$ from $H(z)$, *i.e.*, we can determine the output $y[n]$ for any input signal $x[n]$ if we know $H(z)$ and its ROC.

Skill: ***Examine conditions for causality, stability, invertibility, memory in the z-domain.***

Why Are Z Transforms Used?

You should know that Laplace transform methods are widely used for analysis in linear systems. Laplace transform methods are used when a system is described by a linear differential equation, with constant coefficients. However:

- There are numerous systems that are described by difference equations - notdifferential equations - and those systems are common and different from those described by differentialequations.
- Systems that satisfy difference equations include thingslike:
  - Computer controlled systems - systems that take measurements with digital I/O boards or GPIB instruments, calculate an output voltage and output thatvoltage

digitally. Frequently these systems run a program loop that executes in a fixed interval of time.

- Other systems that satisfy difference equations are those systems with Digital Filters - which are found anywhere digital signal processing - digital filtering is done. That includes:
    - Digital signal transmission systems like the telephonesystem.
    - Systems that process audio signals. For example, a CD contains digital signal information, and when it is read off the CD, it is initially a digital signal thatcan be processed with a digitalfilter.

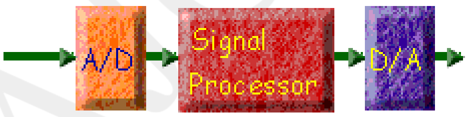
At this point, there are an incredible number of systems we use every day that have digital components which satisfy difference equations.

In continuous systems Laplace transforms play a unique role. They allow system and circuit designers to analyze systems and predict performance, and to think in different terms - like frequency responses - to help understand linear continuous systems. They are a very powerful tool that shapes how engineers think about those systems. Z-transforms play the rolein sampled systems that Laplace transforms play in continuoussystems.

- In continuous systems, inputs and outputs are related by differential equations and Laplace transform techniques are used to solve those differentialequations.
- In sampled systems, inputs and outputs are related by difference equations and Z-transform techniques are used to solve those differentialequations.

In continuous systems, Laplace transforms are used to represent systems withtransfer functions, while in sampled systems, Z-transforms are used to represent systems with transfer functions.

There are numerous sampled systems that look like the one shown below.



- An analog signal is converted to a digital form in anA/D.
- The digital signal is processedsomehow.
- The processed digital signal is converted to an analog signal for use in the analogworld.

The processing can take many forms.

- In a voice transmission situation, the processing might be to band-limit the signal and filter noise from thesignal.
- In a control situation, a measurement might be processed to calculate a signal to controla system.
- And there are many othersituations.

Goals

In sampled systems you will deal with sequences of samples, and you will need to learn Z-transform techniques to deal with those signals. In this lesson many of your goals relate to basic understanding and use of Z-transform techniques. In particular, work toward these goals.

- Given a sequence of samples intime,
    - Be able to calculate the Z-transform of the sequence for simplesequences.
- Given a Z-transform,

  - Be able to determine the poles and zeroes of theZ-transform.
  - Be able to locate and plot the poles and zeroes in thez-plane.

Later you will need to learn about transfer functions in the realm of sampled systems. As you move through this lesson, there are other things you should learn.

- Given a Z-transform of a signal, and the pole locations,
  - Be able to relate distance from the origin to decayrate.
  - Be able to relate angle off the horizontal to the number of samples in a cycleof signaloscillation.

What Is A ZTransform?

You will be dealing with sequences of sampled signals. Let us assume that we have a sequence, y_k. The subscript "k" indicates a sampled time interval and that $y_k$is the value of y(t) at the k^{th}sampleinstant.

- $y_k$could be generated from a sample of a time function. Forexample:
  - y_k= y(kT), where y(t) is a continuous time function, and T is the sampling interval.
  - We will focus on the index variable, k, rather than the exact time, kT, in allthat we do in thislesson.

It's easy to get a sequence of this sort if a computer is running an A/D board, and measuring some physical variable like temperature or pressure at some prescribed interval, T seconds. A sampled sequence like this plays the same role that a continuous signal plays in a continuous system. It carries information just like a continuous signal.

The Z transform, Y[z], of a sequence, $y_k$is defined as:

$$\sum_{k=0}^{\infty} y_k z^{-k}$$

We will use the following notation. A large "z" denotes the operation of taking a Z-transform (i.e., performing the sum above) and the result is usually denoted with an upper-case version of the variable used for the sampled time function, y_k.

- Z[y_k] =Y[z]

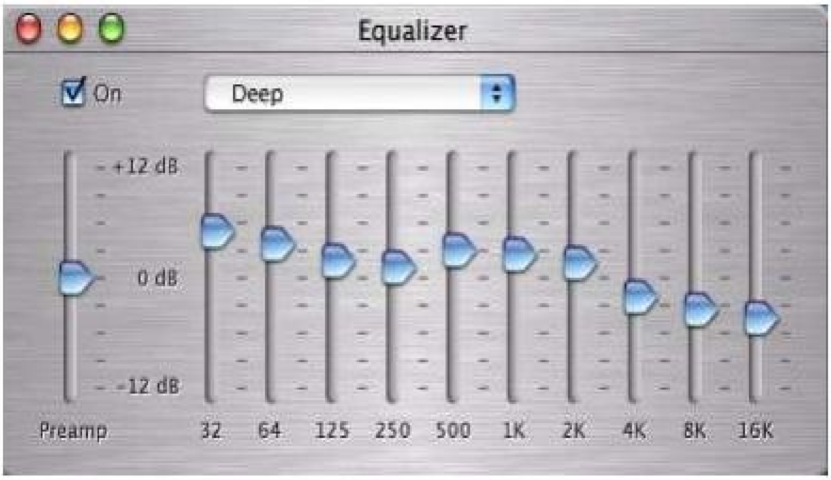
The definition is simple. Take the sequence, and multiply each term in the sequence by a negative power of z. Then sum all of the terms to infinity. That's it.

HINTS

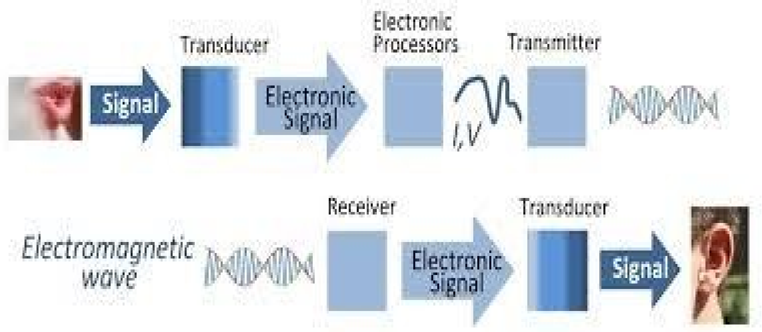
Some important examples of systems include filters, encryption and control. •
Time domain vs. frequency domain: We are used to viewing signals in the time domain — that is, as a function of time. however, there is another domain — namely, the frequency domain, that is critically important in understanding, analyzing and transforming signals. In lecture, we introduced the concept of frequency, and the frequency domain. . The process of sampling a continuous-time signal changes the signal, and it is important to understand how exactly the sampling process changes the information content of that signal. In due time, we will carefully look at the implications of sampling, and how choosing an appropriate sampling rate is critically important to preserving the information contained in a signal.

The frequency domain tells us the relative contribution of sinusoids of different frequencies to the overall signal. Consider the figure below, which illustrates a simple example of the transformation between the time and frequency domains. On the left, we plot the function :

Once we understand that we can represent any time-varying signal in the frequency domain, the concept of an equalizer, as present in many audio systems and mp3 players begins to make sense. Consider for example the equalizer of the *iTunes* mp3 player pictured below:



Note the numbers at the bottom of each of the slider bars (e.g. 32, 64, 125, etc...). These represent frequency bands centered around 32Hz, 64Hz, 125Hz, ..., respectively. By increasing the slider value corresponding to one of the frequency bands, we are essentially saying, "transform the signal to amplify those frequency components of the signal (i.e. mp3 file)." Different combinations of gains achieve different effects, such as "Bass Boosting" (amplifying low-frequency values) or "Treble Boosting" (amplifying high-frequency values). This kind of signal modification would be very difficult to achieve if the designers of the equalizer did not have a very good understanding of the frequency domain, and the frequency representation of time-varying signals. (Note that the equalizer above is a very familiar example of a *system*.)


Transducer
Electronic Processors
Transmitter
Signal
Electronic Signal
I,V
Electromagnetic wave
Receiver
Electronic Signal
Transducer
Signal

ALL THE BEST