EC 8451- ELECTROMAGNETIC FIELDS

UNIT-1 -INTRODUCTION

CONTENT

- Electromagnetic model,
- Units and constants,
- Review of vector algebra,
- Rectangular, cylindrical and spherical coordinate systems,
- Line, surface and volume integrals,
- Gradient of a scalar field,
- Divergence of a vector field,
- Divergence theorem,
- Curl of a vector field,
- Stoke's theorem,
- Null identities,
- Helmholtz's theorem

INTRODUCTION ABOUT EMF

- Electromagnetics is the study of the effect of charges at rest and charges in motion.
- Some special cases of electromagnetics:
  - Electrostatics: charges at rest
  - Magnetostatics: charges in steady motion (DC)
  - Electromagnetic waves: waves excited by charges in time-varying motion
- Both *positive* and negative charges are sources of an electric field, moving charges produce a current which gives rise to a magnetic field
- A *field* is a spatial distribution of a quantity; which may or may not be a function of time
- Application field of EMF,

  Atom smashers, CRO, Radar, satellite Communication, television reception, remote Sensing

ELECTROMAGNETIC MODEL

- Two approaches in the development of a scientific subject
  1. Inductive (one follows the historical development of the subject, stating with the observation of some simple experiments and inferring from them laws and theorems)
  2. Deductive (postulates a fundamental relations for an idealized model)
- Building a theory on an idealized model three steps are needed,
  1. Some basic quantities to the subject of study are defined
  2. Rules of operation of these quantities are specified
  3. Some fundamental relations are postulated
- Example: built on a circuit model of ideal sources and pure R,L,C
  1. Basic quantities:V,I,R,L,C
  2. Algebra, ordinary differential equation, Laplace transformation
  3. KVL,KCL
- Electromagnetic model quantities has,
  1. Source quantities
  2. Field quantities

- *Electric charge* is a fundamental property of matter and exists only in positive or negative integral multiples of the charge on an electron

  $e = 1.60 \times 10^{-19}\ C$

- the principle of conservation of electric charge must be satisfied at all times and under any circumstances (equation of continuity)
- Volume charge density $\rho = \lim_{\Delta v \to 0}\left(\frac{\Delta q}{\Delta v}\right)$ C/m^3
- Surface charge density $\rho_s = \lim_{\Delta s \to 0}\left(\frac{\Delta q}{\Delta s}\right)$ C/m^2
- Line charge density $\rho_l = \lim_{\Delta l \to 0}\left(\frac{\Delta q}{\Delta l}\right)$ C/m
- Current is the rate of charge with respect to time $I = \frac{dq}{dt}$ C/s or A
- Volume current density J, which measures the amount of current flowing through a unit area normal to the direction of current flow (A/m^2)
- Surface current density J_s which is the current per unit width on the conductor surface normal to the direction of current flow (A/m)
- There are 4 fundamental vector field quantities in electro magnetics

Symbols and units for field quantities	Field quantities	Symbol	unit
Electric	Electric field intensity	E	V/m
	Electric flux density	D	C/m^2
Magnetic	Magnetic flux density	B	T
	Magnetic field intensity	H	A/m

Units and constants

- Measurements of any physical quantity must be expressed as a number followed by a unit.
- The SI (International System of Units) is an MKSA system

Quantity	Unit	abbreviation
Length	Meter	m
Mass	Kilogram	kg
Time	Second	s
Current	Ampere	A

- Constants :
    1. Velocity of electromagnetic wave in free space $c = 3 \times 10^8$ m/s
    2. Permittivity of free space ε_0
    3. Permeability of free space μ_0
- ε_0 is the proportionality constant between the electric flux density D and the electric field intensity E in free space

$$D = \varepsilon_0 E$$

- μ_0 is the proportionality constant between the Magnetic flux density B and the Magnetic field intensity H in free space

$$H = \frac{1}{\mu_0} B$$

- the value of ε_0 and μ_0 are determined by the choice of unit systems.

Units and constants Conti....

$$\mu_0 = 4\pi \times 10^{-7} \quad \text{H/m (henry / meter)}$$

- The relationship b/w three constants are

$$C = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \text{ m/s}$$

$$E_0 = \frac{1}{C^2 \mu_0} = \frac{1}{36\pi} \times 10^{-9} \text{ F/m} \quad \text{(farad / meter)}$$

Review of vector algebra

- A *scalar* is a quantity having only an amplitude

  Examples: voltage, current, charge, energy, temperature

- A *vector* is a quantity having direction in addition to amplitude

  Examples: velocity, acceleration, force

- a vector A can be written as $\mathbf{A} = a_A A$ where A is the magnitude of **A**,

  $A = |\boldsymbol{A}|$ and a_A is a dimensionless unit vector

$$a_A = \frac{\boldsymbol{A}}{|\boldsymbol{A}|}$$

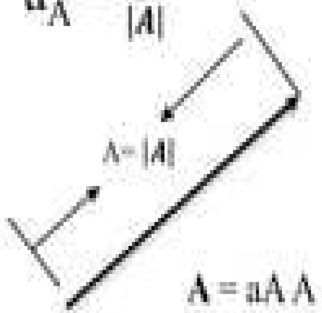


Fig: graphical representation of vector **A**

Vector Addition

- Two vectors **A** and **B** which are not in the same direction nor in opposite directions .their sum is another vector **C** in the plane **C**= **A** + **B** can be obtained graphically in two ways.
  1. By the parallelogram rule: the resultant **C** is the diagonal vector of the parallelogram formed by A and B drawn from the point
  2. By the head to tail rule:the head of the **A** connects to the tail of **B**.their Sum C is the vector drawn from the tail of **A** to the head of B and vectors a, **B** and **C** form a triangle

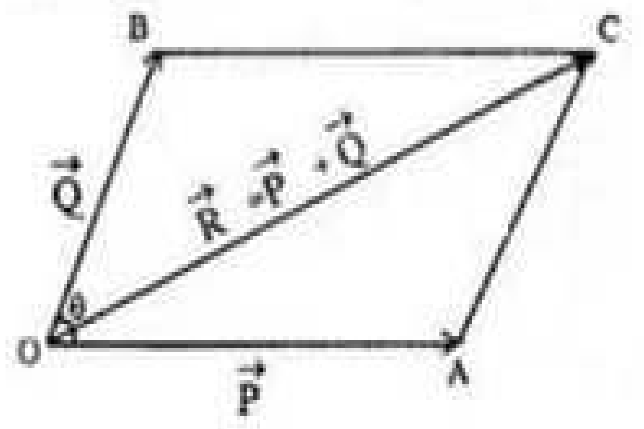
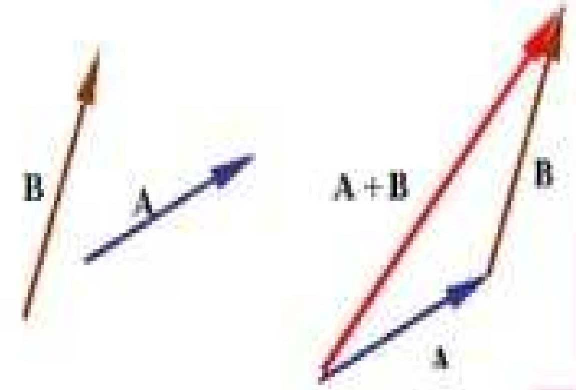


Fig. Parallelogram Law of vectors



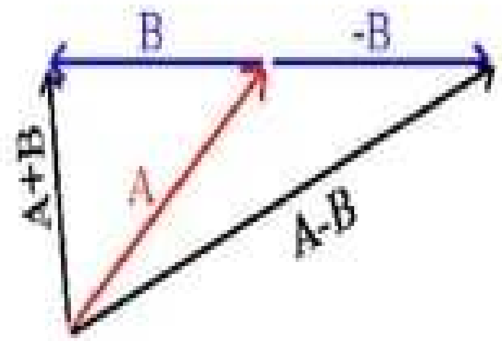
- Commutative law :A + B = B + A
- Associative law :A + (B + C) = (A + B) + C

Vector subtraction

- Vector subtraction can be defined in terms of vector addition in the following way:

$$A - B = A + (-B)$$

- Where -**B** is the negative of vector **B**



Products of vectors

- Multiplication of vector **A** by positive scalar k changes the magnitude of **A** by k times with out changing its direction

$$kA = a_A (kA)$$

- Types of products vectors
  1. Scalar or dot products
  2. Vector or cross products
- Scalar or dot product : dot product of two vectors A and B denoted by **A . B** is a scalar which equals the product of the magnitudes of A and B the cosine of the angle between them

$$A \cdot B \triangleq AB \cos \theta_{AB}$$

- Where $\triangleq$ *signifies equal by definition and θ_{AB} smaller angle b/w* ***A*** *and* ***B is less than π radians (180)***
- ***The dot product of two vectors***
  1. *is less than or equal to the product of their magnitudes*
  2. *Can be either a positive or a negative quantity, depending on whether the angle between them is smaller or larger than π/2 radians(90)*
  3. *is equal to the product of the magnitude of one vector and the projection of the other vector upon the first one*
  4. *is zero when the vectors are perpendicular to each other*
- $A.A = A^2$
  - *commutative law : A . B = B . A*
  - *Distributive law :A . (B + C) = A . B + A . C*

- Vector or cross product : Cross product of two vectors A and B, A x B is a vector perpendicular to the plane containing A and B. its magnitude is AB $\sin\theta_{AB}$ where θ_{AB} is the smaller angle b/w A and B

$$\mathbf{A} \times \mathbf{B} \triangleq \mathbf{a}_n |AB \sin\theta_{AB}|$$

Where $\sin\theta_{AB}$ is the height of the parallelogram formed by the vectors A and B

- B x A = - A x B (cross product is not commutative)
- A x (B + C) = A x B + A x C (distributive)
- A x (B x C) ≠ (A x B) x C (associative)
- Product of three vectors: 1.scalar triple product 2. vector triple product
- 1.scalar triple product A . (B x C) = B. (C x A) = C .(A x B)
- A.(B x C) = - A. (C X B) = - B. (A X C) = - C. (B X A)
- 2. vector triple product : A X (B X C) = B(A.C) - C(A.B) (back-cab rule)
- Division by a vector is not defined

ORTHOGONAL COORDINATE SYSTEMS

- To determine the electric field at a certain point in space, to describe the position of the source and the location of this point in a coordinate systems
- Three dimensional space a point an be located as the intersection of three surfaces.(u_1,u_2,u_3-constant)
- when these 3 surfaces are mutually perpendicular to one another its orthogonal coordinate systems
- Let a_{u1},a_{u2},a_{u3} be the unit vectors in the 3 coordinate directions. they are called the base vectors.
- right handed, orthogonal, curvilinear system the base vector are arranged in such a way
- $a_{u1} \times a_{u2} = a_{u3}$, $a_{u2} \times a_{u3} = a_{u1}$, $a_{u3} \times a_{u1} = a_{u2}$,
- $a_{u1} \cdot a_{u2} = a_{u2} \cdot a_{u3} = a_{u3} \cdot a_{u1} = 0$
- $a_{u1} \cdot a_{u1} = a_{u2} \cdot a_{u2} = a_{u3} \cdot a_{u3} = 1$

ORTHOGONAL COORDINATE SYSTEMS conti

- Any vector **A** can be written as the sum of its components in the three orthogonal directions

$$\mathbf{A} = a_{u1} A_{u1} + a_{u2} A_{u2} + a_{u3} A_{u3}$$

The magnitude of $A = |A| = (A^2_{u1} + A^2_{u2} + A^2_{u3})^{1/2}$

- To express the differential length change corresponding to a differential change in one of the coordinates

$$dl_i = h_i du_i$$

- h_i –metric coefficient its function of u_1, u_2, u_3
- Sum of the component length change

$$dl = a_{u1} d_{l1} + a_{u2} d_{l2} + a_{u3} d_{l3}$$

or

$$dl = a_{u1}(h_1 du1) + a_{u2}(h_2 du2) + a_{u3}(h_3 du3)$$

- Magnitude of dl is

$$dl = [dl_1^2 + dl_2^2 + dl_3^2]^{1/2}$$

or

$$[(h_1 du_1)^2 + (h_2 du_2)^2 + (h_3 du_3)^2]^{1/2}$$

- Differential volume $dl_1 dl_2 dl_3$ or $dv = h_1 h_2 h_3 du_1 du_2 du_3$

- Differential area a vector with a direction normal to the surface

$$ds = a_n ds$$

- The differential area *ds1* normal to the unit vector a_{u1},

$$ds_1 = dl_2\, dl_3$$

or

$$ds_1 = h_2 h_3 du_2 du_3$$

- Iily a_{u2} , a_{u3} ;

$$ds_2 = h_1 h_3 du_1 du_3 ;$$

$$ds_3 = h_1 h_2 du_1 du_2$$

Common orthogonal coordinate systems

1. Cartesian (or rectangular) coordinates
2. Cylindrical coordinates
3. Spherical coordinates

Cartesian (or rectangular) coordinates

$$(u_1, u_2, u_3) = (x, y, z)$$

- A point $p(x_1, y_1, z_1)$ in Cartesian coordinates is the intersection of three planes specified by $x = x_1, y = y_1, z = z_1$ it's a right handed system with base vector

$$a_x \times a_y = a_z,\ a_y \times a_z = a_x,\ a_z \times a_x = a_y$$

- Algebra 11 - Cartesian Coordinates in Three Dimensions.mp4
- The position vector to the point $P(x_1, y_1, z_1)$ is $\vec{OP} = a_x x_1 + a_y y_1 + a_z z_1$
- A vector **A** in Cartesian coordinates can be written as

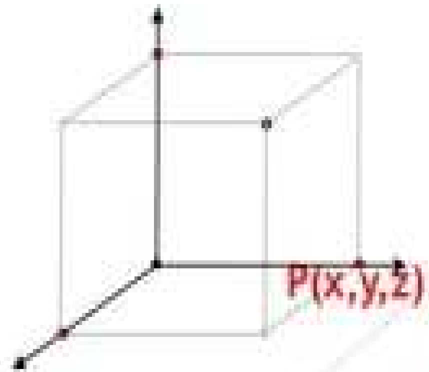
$$\mathbf{A} = a_x A_x + a_y A_y + a_z A_z$$

- The dot product of two vectors **A** and **B** ,

$$\mathbf{A}.\mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} a_x & a_y & a_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

- All three metric coefficients are unity $h_1 = h_2 = h_3 = 1$



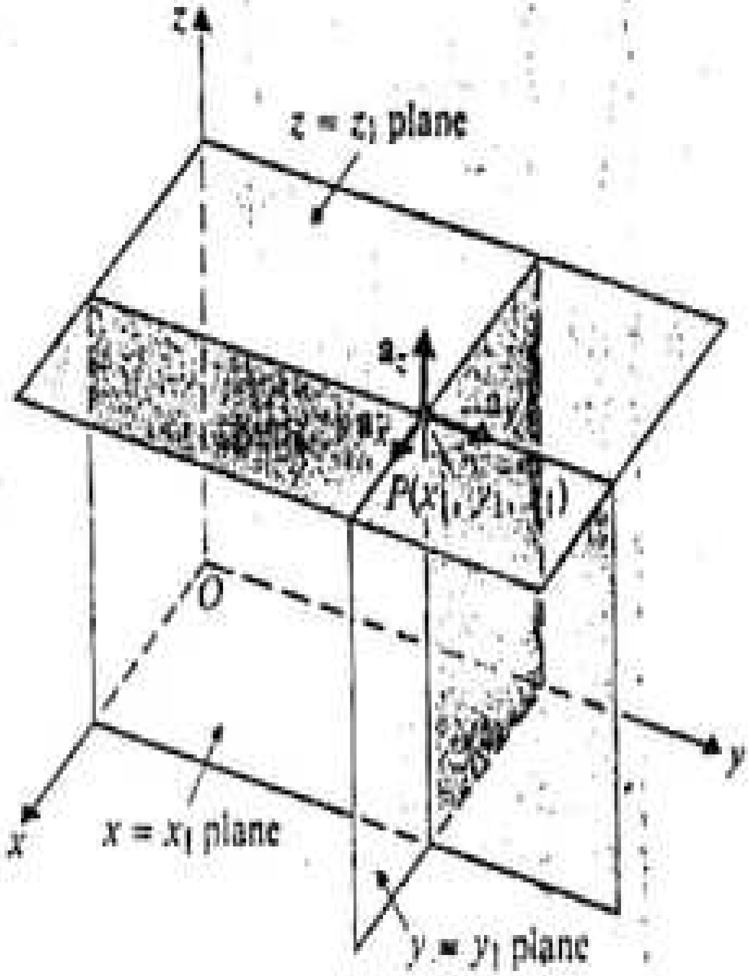


Fig. 2-9 Cartesian coordinates.

- Differential length,

$$dl = a_x d_x + a_y d_y + a_z d_z$$

- Differential length

$$ds_x = dydz \ ;$$

$$ds_y = dxdz \ ;$$

$$ds_z = dxdy$$

- differential surface areas normal to the directions a_x, a_y, a_z
- Differential volume

$$dv = dxdydz$$

Cylindrical Co-ordinate system

$$(u_1, u_2, u_3) = (r, ø, z)$$

- The surfaces used to define the cylindrical cocordinate system are
- 1.Plane of constant z which is parallel to xy plane 2. a cylinder of radius r with z axis as the axis of cylinder 3. a half plane perpendicular to xy plane and at an angle ø with respect to xy plane. The angle ø is called azimuthal angle. Ranges (0 < r < infinity ;0< ø < 2π ; -infinty < z < infinity)

$$a_r \times a_ø = a_z,\ a_ø \times a_z = a_r,\ a_z \times a_r = a_ø$$

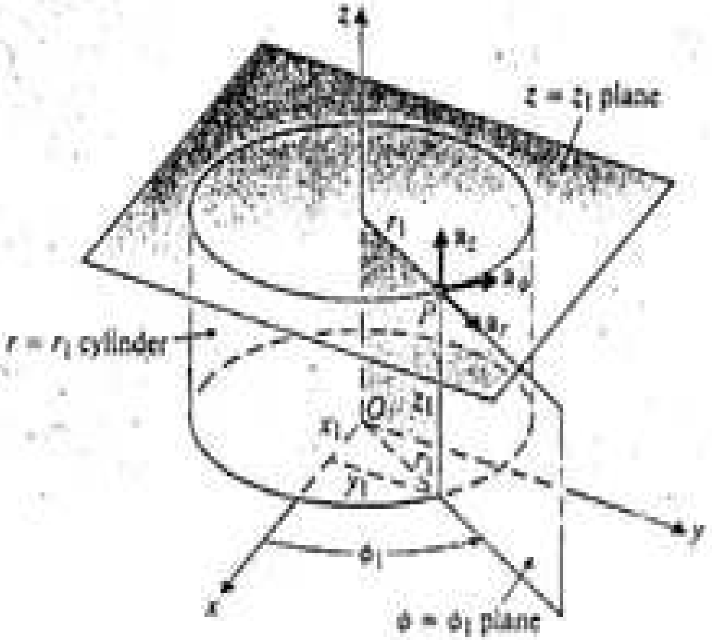


Fig. 2-11 Cylindrical coordinates.

- A vector in cylindrical is written as

$$A = a_r A_r + a_ø A_ø + a_z A_z$$

- The dot product of two vectors

$$A.B = A_r B_r + A_ø B_ø + A_z B_z$$

- The vector product of two vectors

$$A \times B = \begin{vmatrix} a_r & a_ø & a_z \\ A_r & A_ø & A_z \\ B_r & B_ø & B_z \end{vmatrix}$$

- The differential length (h$_1$=h$_3$=1;h$_2$= 1)

$$dl = a_r d_r + a_ø r d_ø + a_z d_z ;$$

$$\text{magnitude } |dl| = (dr^2 + (rdø)^2 + dz^2)^{1/2}$$

- The differential surface are,

$$ds_r = rdødz\ a_r ;\ ds_ø = drdz\ a_ø ;\ ds_z = rdrdø\ a_z$$

- The differential volume

$$dv = rdrdødz$$

Relationship between Cartesian and cylindrical systems;

- Cylindrical to Cartesian

$x = r\cos Ø$

$y = r \sin Ø$

$z = z$

- Cartesian to cylindrical

$r = \sqrt{x^2 + y^2}$

$Ø = \tan^{-1} y/x$

$z = z$

Note:

1.r is positive or zero, hence positive sign of square root must be consider.

2.While calculating Ø make sure the sign of x and y.

both are positive is positive in the first quadrant.

If x is negative y is positive then the point is in the second quadrant. Hence add 10 degree to the Ø calculation.

if x is positive y is negative Ø fourth quadrant.

if x is negative y is negative third quadrant, hence 180 degree subtracted from Ø

Spherical coordinate systems

$$(u_1, u_2, u_3) = (R, \theta, \varphi)$$

- Sphere of radius r origin as the entre of the sphere
- A right circular cone with its apex at the origin and its axis as z axis. Its half angle is θ.it rotates about z axis and θ varies from 0 to 180 degree
- A half plane perpendicular to xy plane containing z axis making an angle ø with the xy plane.

- $0 < r <$ infinity; $0 < \varnothing < 2\pi$; $0 < \theta < \pi$

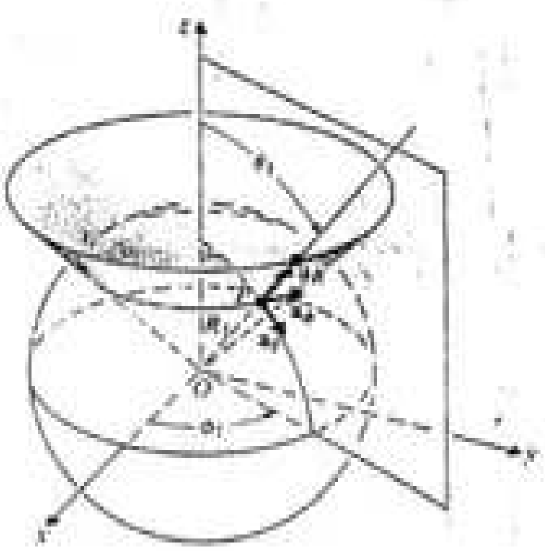


Fig. 2–16 Spherical coordinates.

$a_R \times a_\theta = a_\emptyset$; $a_\theta \times a_\emptyset = a_R$; $a_\emptyset \times a_R = a_\theta$

- A vector spherical coordinates is written as

$$A = a_R A_R + a_\theta A_\theta + a_\emptyset A_\emptyset$$

- The dot and cross product of spherical vector are,

$$A.B = A_R B_R + A_\theta B_\theta + A_\emptyset B_\emptyset$$

$$A \times B = \begin{matrix} a_R & a_\theta & a_\emptyset \\ A_R & A_\theta & A_\emptyset \\ B_R & B_\theta & B_\emptyset \end{matrix}$$

- **In spherical coordinates only R is lenghth,the other coordinates θ , Ø** are angles. Differential volume element $h_2 = R$ and $h_3 = R \sin\theta$
- **The differential vector length**

$$dl = a_R\, dR + a_\theta\, R d\theta + a_\emptyset\, R \sin\theta\, d\emptyset$$

- Magnitude of differential length vector is $|dl| = ((dR)^2 + (R\, d\theta)^2 + (R \sin\theta\, d\emptyset)^2)^{1/2}$
- **The differential surface areas**

$$ds_R = R^2 \sin\theta d\theta d\emptyset\ a_r\ ; \qquad ds_\theta = R \sin\theta dR\, d\emptyset\ a_\theta; \qquad ds_\emptyset = R dR\, d\theta\ a_\emptyset$$

- **The differential volume is**

$$dv = R^2 \sin\theta\, dR\, d\theta\, d\emptyset$$

Relationship between Cartesian and spherical systems

- Spherical coordinate to Cartesian coordinate system

$$x = R \sin\theta \cos\phi$$
$$y = R \sin\theta \sin\phi$$
$$z = R \cos\theta.$$

- Cartesian to spherical coordinate system

$$R = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z}$$
$$\phi = \tan^{-1} \frac{y}{x}$$

- Transformation of vectors
- Cartesian to cylindrical

$$\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

- Cylindrical to Cartesian

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\varphi \\ A_z \end{bmatrix}$$

- Cartesian to spherical

$$\begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

- Spherical to Cartesian

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\phi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix}$$

- Spherical to cylindrical

$$\begin{bmatrix} A_r \\ A_\varphi \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_R \\ A_\theta \\ A_\phi \end{bmatrix}$$

Line, surface and volume integrals

- In electromagnetic a charge can exist in point form, line form, surface form or volume form.

Line integral:

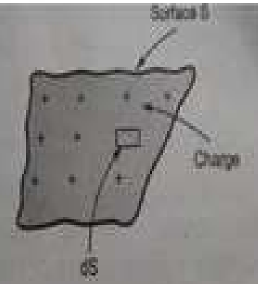
- Line can exist as a straight line or it can be a distance travelled along a curve.ie a line is a curved path in a space.
- Consider a vector field F, the curved path shown in the field is p-r.This is called path of integration and corresponding integral can be defined as

$$\int_L \vec{F}.\vec{dl} = \int_p^r |F| dl \cos\theta$$; dl = elementary length

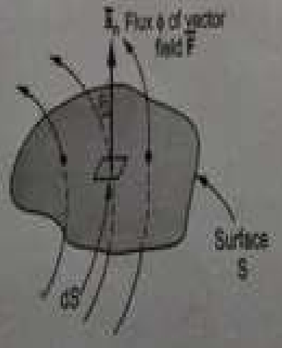
- This is called line integral of F around the curved path.
- Open path as p-r and closed path as p-q-r-s-p (contour integral or closed or circular integral)

Surface integral:

- A flux Ø may pass through a surface, while doing analysis of such cases an integral is required called surface integral, to be carried out over a surface related to vector field
- For a charge distribution in fig a, e can write for the total charge existing on the surface as,
- Note :the surface integral involve the double integration procedure mathamatically.



(a) Surface charge



b) Flux crossing a surface

$$Q = \int_S \rho_S \, dS \qquad \dots (1.14.4)$$

where ρ_S = Surface charge density in C/m^2

dS = Elementary surface

- Similarly for the Fig. 1.14.3 (b), the total flux crossing the surface S can be expressed as,

$$\phi = \int_S \bar{F} \cdot d\bar{S} = \int_S |\bar{F}| \, dS \cos\theta = \int_S \bar{F} \cdot \bar{a}_n \, dS \qquad \dots (1.14.5)$$

where $\bar{a}_n$ = Unit vector normal to the surface S

- Both the equations (1.14.4) and (1.14.5) represent the **surface integrals** and mathematically it becomes a **double integration** while solving the problems.
- If the surface is closed, then it defines a volume and corresponding surface integral is given by,

$$\phi = \oint_S \bar{F} \cdot d\bar{S} \qquad \dots (1.14.6)$$

- This represents the net outward flux of vector field $\bar{F}$ from surface S.

- Volume integral
- If the charge distribution exists in a three dimensional volume form as in fig.,then the volume integral is required to calculate the total charge

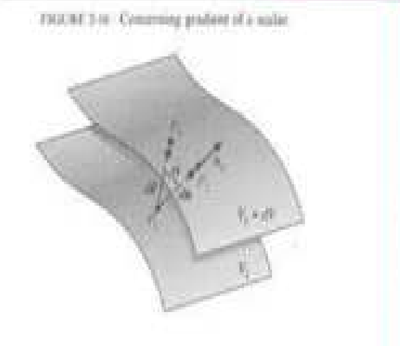
$$Q=\int_{v} \rho_{v}\, dv$$

GRADIENT OF A SCALAR FIELD

- Electromagnetic quantities that depend on both time and position. Let us consider a scalar function of space coordinates V(u1,u2,u3).
- The magnitude of V depends on the position of the Point in space, but it may be constant along certain lines or surfaces.
- *The vector that represents both the magnitude and the direction of the maximum space rate of increa of a scalar as the gradient of that scalar.*

$$\text{grad } V \triangleq a_n \frac{dV}{dn};\text{------------}(1)$$

- $\nabla V \triangleq a_n \frac{dV}{dn}$ (dv is positive)----------(2)



- (if dv is negative -decrease in V from P_1 to P_2) ∇V will be negative in the a_n direction)

$$\nabla V = a_{u1}\frac{\partial V}{h_1 du_1} + a_{u2}\frac{\partial V}{h_2 du_2} + a_{u3}\frac{\partial V}{h_3 du_3}$$

- For Cartesian coordinate system: $\nabla V = a_x \frac{\partial V}{dx} + a_y \frac{\partial V}{dy} + a_z \frac{\partial V}{dz}$
- For cylindrical system: $\nabla V = a_r \frac{\partial V}{dr} + a_\phi \frac{1}{r}\frac{\partial V}{d\phi} + a_z \frac{\partial V}{dz}$
- For spherical system : $\nabla V = a_R \frac{\partial V}{dR} + a_\theta \frac{1}{r}\frac{\partial V}{d\theta} + a_\phi \frac{1}{R\sin\theta}\frac{\partial V}{d\phi}$

- Properties of gradient of scalar field

  1.the gradient ∇V gives the max.rate of change of V per unit distance

  2.the gradient ∇V alays indicate the direction of the max.rate of change of V

  3.the grdient ∇V at any point is perpendicular to the constant V surface,which passes through the point.

  4.the directional derivative of V along the unit vector a is $\nabla V.a$ which is projection of ∇V in the direction of unit vector a

  5.$\nabla(V + U) = \nabla V + \nabla U$

  6.$\nabla(UV) = U \nabla V + V \nabla U$

  7. $\nabla(U/V) = V \nabla U - U \nabla V / V^2$

Divergence of a vector field

- It is seen that $\oint F.ds$ gives the flux flowing across the surface S. Then mathematically divergence is defined as the net outward flow of the flux per unit volume over a closed incremental surface.

$$\text{Div } \vec{F} \triangleq \lim_{\nabla v \to 0} = \frac{\oint_s \vec{F}.d\vec{s}}{\nabla v} \text{ (divergence of } \vec{F}\text{)}$$

- Divergence of vector field F at a point P is the outward flux per unit volume as the volume shrinks about point P i.e. $\nabla v \to 0$ representing differential volume element at point P.

$$\nabla . F = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 F_1)}{\partial u_1} + \frac{\partial(h_1 h_3 F_2)}{\partial u_2} + \frac{\partial(h_1 h_2 F_3)}{\partial u_3}\right]$$

- $\nabla . F$ = divergence of F (∇ = vector operator $\nabla = a_x \frac{\partial}{dx} + a_y \frac{\partial}{dy} + a_z \frac{\partial}{dz}$) ($F = a_x F_x + a_y F_y + a_z F_z$)

$$\nabla . F = \frac{\partial F_x}{dx} + \frac{\partial F_y}{dy} + \frac{\partial F_z}{dz} = \text{div } F \text{ (this is divergence of F in Cartesian system)}$$

- Lly

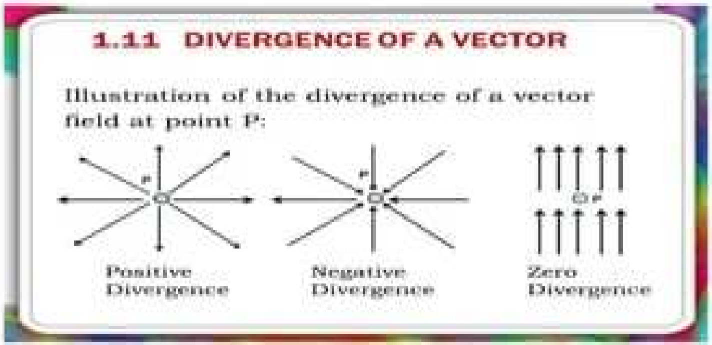
$$\nabla . F = \frac{1}{r}\frac{\partial r F_r}{dr} + \frac{1}{r}\frac{\partial F_\theta}{d\theta} + \frac{\partial F_z}{dz} \text{(cylindrical)}$$

- (spherical)
- Divergence at a point indicate ho much that vector field diverges from that point

- Consider a solenoid i.e. Electromagnet obtained by winding a coil around the core. hence current passes through it, flux is produced around it. such a flux is produced around it. such a flux completes a closed path through the solenoid

  $\nabla \cdot A = 0$ for A to be solenoidal

- The vector field strength is measured by the no. of flux lines passing through a unit surface normal to the vector



- Properties of divergence

1. The divergence produces a scalar field as the dot product is involved in the operation
2. Divergence of a scalar has no meaning.
3. $\nabla \cdot (\hat{A} + \hat{B}) = \nabla \cdot \hat{A} + \nabla \cdot B$

Divergence theorem

- W.k.t , $\nabla \cdot F \triangleq \lim_{\nabla v \to 0} = \frac{\oint_S F \cdot ds}{\nabla v}$ (divergence of vector)
- We can write that $\oint_S F \cdot ds = \int_V \nabla \cdot F \, dv$ (divergence theorem or gauss ostrogradsky theorem)
- It states that, the integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by that closed surface.
- The divergence theorem converts the surface integral into a volume integral provided that the closed surface encloses certain volume.
- The divergence theorem is advantages in EMF as volume integrals are more easy to evaluate than the surface integrals
- Proof Of Divergence Theorem:
- Let the closed surface encloses certain volume v. subdivided this volume v into a large number of subsections called cells. let the vector field associated with the surface S is F.
- Then if i^{th} cell has the volume ∇v_i and is bounded by the surface s_i then ,

$$(\nabla \cdot F)_i \nabla v_i = \oint_{s_i} F \cdot ds$$

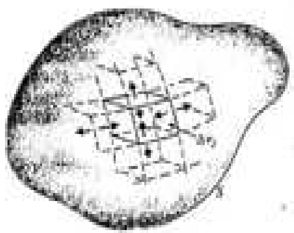


Fig. 3-20 Subdivided volume for proof of divergence theorem.

- Let us now combine the contributions of all these differential volumes to both sides,

$$\lim_{\Delta V_i \to 0} \left[\sum_{i=1}^{N} (\nabla \cdot F)_i \, \Delta V_i\right] = \lim_{\Delta V_i \to 0} \left[\sum_{i=1}^{N} \oint_{S_i} F \cdot ds\right]$$

- In left side, volume integral, $\lim_{\Delta V_i \to 0} \left[\sum_{i=1}^{N} (\nabla \cdot F)_i \, \Delta V_i\right] = \int_V (\nabla \cdot F) dv$
- The surface integrals on the right side are summed over all the faces of all the differential volume elements. the contribution from the internal surfaces of adjacent elements ill cancel each other. hence the net contribution of the right side is due only to that of the external surface S bounding the volume V.
- $\lim_{\Delta V_i \to 0} \left[\sum_{i=1}^{N} \oint_{S_i} F \cdot ds\right] = \oint_S F \cdot dsi$
- Hence, $\oint_S F \cdot ds = \int_V \nabla \cdot F$ dv

Curl of a vector field

- The circulation of a vector field around a closed path is given by curl of vector.

$$\text{Curl of } \bar{F} = \lim_{\Delta S_N \to 0} \frac{\oint \bar{F} \bullet d\bar{l}}{\Delta S_N} \qquad \text{... (1.181)}$$

where ΔS_N = Area enclosed by the line integral in normal direction

- Thus maximum circulation of $\bar{F}$ per unit area as area tends to zero whose direction is normal to the surface is called curl of $\bar{F}$.
- Symbolically it is expressed as,

$$\nabla \times \bar{F} = \text{Curl of } \bar{F} \qquad \text{... (1.182)}$$

> **Key Point :** Curl indicates the rotational property of vector field. If curl of vector is zero, the vector field is irrotational. If $\nabla \times \bar{F} = 0$ then $\bar{F}$ is irrotational.

- In various co-ordinate systems, the curl of $\bar{F}$ is given by,

$$\nabla \times \bar{F} = \left[\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right]\bar{a}_x + \left[\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right]\bar{a}_y + \left[\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right]\bar{a}_z$$

i.e.

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \quad \text{Cartesian} \qquad \text{... (1.183)}$$

$$\nabla \times \bar{F} = \left[\frac{1}{r}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right]\bar{a}_r + \left[\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right]\bar{a}_\phi + \left[\frac{1}{r}\frac{\partial (r F_\phi)}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \phi}\right]\bar{a}_z$$

i.e.

$$\nabla \times \bar{F} = \frac{1}{r}\begin{vmatrix} \bar{a}_r & r\bar{a}_\phi & \bar{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_r & rF_\phi & F_z \end{vmatrix} \quad \text{Cylindrical} \qquad \text{... (1.184)}$$

$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \mathbf{a}_{u_1} h_1 & \mathbf{a}_{u_2} h_2 & \mathbf{a}_{u_3} h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

Key Point : In $\frac{\partial (r F_\phi)}{\partial r}$, r cannot be taken outside as differentation is with respect to r.

$$\nabla \times \bar{F} = \frac{1}{r \sin\theta}\left[\frac{\partial F_\phi \sin\theta}{\partial\theta} - \frac{\partial F_\theta}{\partial\phi}\right]\bar{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial (r F_\phi)}{\partial r}\right]\bar{a}_\theta + \frac{1}{r}\left[\frac{\partial (r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right]\bar{a}_\phi$$

i.e.

$$\nabla \times \bar{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \bar{a}_r & r\bar{a}_\theta & r\sin\theta\,\bar{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ F_r & r F_\theta & r\sin\theta\, F_\phi \end{vmatrix} \quad \text{Spherical}$$

... (

- The curl is closed line integral per unit area .
- If curl of vector field exists then the field is called rotational. For irrotational vector field the curl is zero

Stoke's theorem

- It relates the line integral to a surface integral.
- The line integral of F around a closed path L is equal to the integral of curl of F over the open surface S enclosed by the closed path L

$$\oint_L F \cdot dL = \int_S (\nabla X F) \cdot dS \;;$$

where dL= perimeter of total surface S

- This theorem is applicable only when F and $\nabla X F$ are continuous on the surface S

Proof of stoke's theorem:

- consider a surface S which is splitted into no of incremental surfaces. each incremental surface is having area ΔS
- Applying definition of the curl to any of these incremental surfaces

$$(\nabla X F)_N = \oint_L F \cdot dL_{\Delta S} / \Delta S$$

- where N = normal to ΔS according to right hand rule, $dL_{\Delta S}$ = perimeter of the incremental surface ΔS
- The curl of F in the normal direction is the dot product of curl of F with a_N (unit vector) normal to the surface ΔS

$$(\nabla X F)_N = (\nabla X F).\vec{a_N}$$

$$\oint_L F \cdot dL_{\Delta S} = (\nabla \times F) \cdot \overrightarrow{a_N} \Delta S$$

- $\oint_L F \cdot dL_{\Delta S} = (\nabla \times F) \cdot \overrightarrow{\Delta S}$
- To obtain total curl for every incremental surface add the closed line integrals for each ΔS.from the fig it can be seen that at a common boundary b/w the two surfaces the line integral is getting canced as the boundary traced in opposite direction
- Hence summation of all closed line integrals for each and every ΔS ends up in a single losed line integral to be obtained for the outer boundary of the total surface.

$$\oint_L F \cdot dL = \int_S (\nabla \times F) \cdot dS$$

Two null identities

- Identity I
- The curl of the gradient of any scalar field is identically zero

$$\nabla \times (\nabla V) \equiv 0$$

- If a vector field is curl free then it can be expressed as the gradient of a scalar field.

$$\nabla \times E = 0;\ E = -\nabla V$$

- Identity II
- The divergence of the curl of any vector field is identically zero.

$$\nabla \cdot (\nabla \times A) \equiv 0$$

- If a vector field is divergence less , then it can be expressed as the curl of another vector field.

$$\nabla \cdot B = 0\ ;\ B = \nabla \times A$$

HELMHOLTZ'S THEOREM

- Types of vector field F
- 1 solenoidal and irrotational ∇.F = 0 and ∇ x F = 0 (static electric field in a charge free region)
- 2.solenoidal but not irrotational if ∇.F = 0 and ∇ x F $\neq$ 0 (steady magnetic field in a current carrying conductor)
- 3.irrotational but not solenoidal ∇ x F = 0 and ∇.F $\neq$ 0 (static electric field in a charged region)
- 4.neither solenoidal nor irrotational ∇.F $\neq$ 0 and ∇ x F $\neq$ 0
- Helmholtz theorem: a vector field is determined to within an additive constant if both its divergence and its curl are specified everywhere.

$$F = - \nabla.V + \nabla \times A$$

- In an unbounded region, divergence and curl of the vector field at infinity. bounded by surface ,it is determined if its divergence and curl throughout the region.