PROBABILITY AND COMPLEX FUNCTION SUB CODE: MA3303

Complex Anlaysis

Liouville’s Theorem and the Maximum Modulus Theorem

…they’re not just for polynomials!

Recall. A function is said to be *analytic* if it has a power series representation. A function of a complex variable, $f(z)$, is analytic (and therefore has a power series representation) at point z_0 if f is continuously differentiable at z_0.

Theorem. "Cauchy's Formula"

If f is analytic in a neighborhood of z_0 and C is a positively oriented simple closed curve in the neighborhood with z_0 as an interior point, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}.$$

Augustin Cauchy (1789-1857)

Note. If we let C be a circle of radius R with center z_0, then Cauchy's Formula allows us to put a bound on derivatives of f:

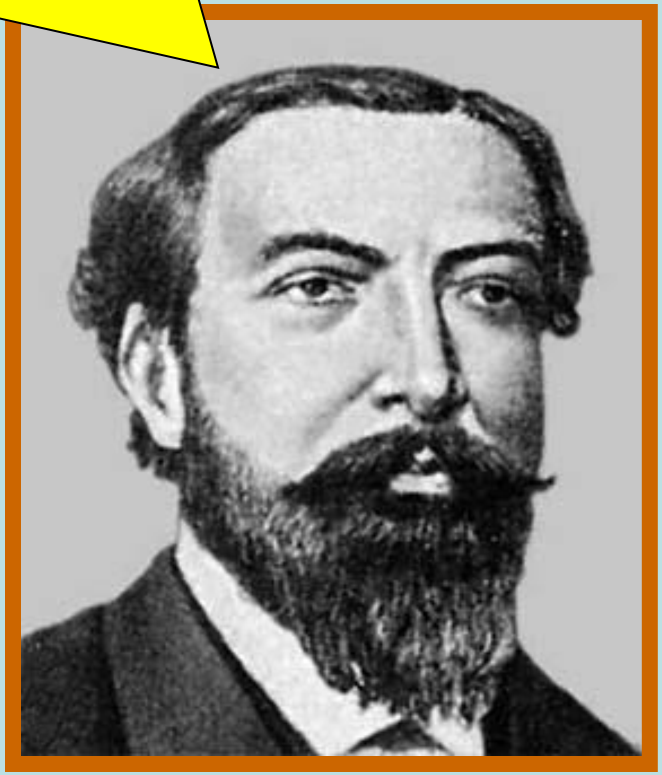
$$\left|f^{(n)}(z_0)\right| = \left|\frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{\left(z-z_0\right)^{n+1}}\right|$$

$$\leq \left|\frac{n!}{2\pi i}\right|\int_C \left|\frac{f(z)\,dz}{\left(z-z_0\right)^{n+1}}\right| \leq \frac{n!}{2\pi|i|}\int_C \frac{|f(z)|\,|dz|}{|z-z_0|^{n+1}}$$

$$\leq \frac{n!}{2\pi}\frac{M_R\, 2\pi R}{R^{n+1}} = \frac{n!\,M_R}{R^n},$$

where M_R is an upper bound of $|f(z)|$ over C. This is called *Cauchy's Inequality.*

Theorem. Liouville's Theorem. If f is a function analytic in the entire complex plane (f is called an *entire* function) which is bounded in modulus, then f is a constant function.



Joseph Liouville (1809-1882)

Proof. Let M be bound on $|f(z)|$. Then by Cauchy's Inequality with $n = 1$,

$$|f'(z_0)| \le \frac{M}{R}.$$

This is true for any z_0 and for any R since f is entire. Since we can let $R \to \infty$ we see that

$|f'(z_0)| = 0$ and f must be constant. ■

The only bounded entire functions of a complex variable are constant functions. What do you make of that, Puddin' Head? Do your beloved functions of a real variable behave like that?

Soitenly not! Sine and Cosine are bounded functions analytic on the whole real line! *Nyuk, nyuk, nyuk!*

Theorem. Maximum Modulus Theorem.

Let G be a bounded open set in $\mathbb{C}$ and suppose f is continuous on the closure of G, $\text{cl}(G)$, and analytic in G. Then

$$\max\left\{\left|f(z)\right| : z \in \text{cl}(G)\right\} = \max\left\{\left|f(z)\right| : z \in \partial G\right\}.$$

Also, if

$$\max\left\{\left|f(z)\right| : z \in G\right\}$$

$$= \max\left\{\left|f(z)\right| : z \in \partial G\right\},$$

then f is constant.

Theorem. The Maximum Modulus Theorem for Unbounded Domains (Simplified).

Let D be an open disk in the complex plane. Suppose f is analytic on the complement of D, continuous on the boundary of D, $| f(z) |$ % M on the boundary of D, and $\lim_{|z|\to\infty} f(z) = L$ for some complex L. Then $| f(z) |$ % M on the complement of D.

Fundamental Theorem of Algebra

Theorem. Fundamental Theorem of Algebra.
Any polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0 \qquad (a_n \neq 0)$$

of degree n ($n \geq 1$) has at least one zero. That is, there exists at least one point z_0 such that $P(z_0) = 0$.

Note. It follows that P can be factored into n (not necessarily distinct) linear terms:

$$P(z) = a_n (z - z_1)(z - z_2) \cdots (z - z_{n-1})(z - z_n),$$

Where the zeros of P are $z_1, z_2, \ldots, z_n$.

Proof. Suppose not. Suppose P is never zero. Define the function

$$f(z) = \frac{1}{P(z)}.$$

Then f is an entire function. Now $|P(z)| \to \infty$ as $|z| \to \infty$.

Therefore $\dfrac{1}{P(z)} \to 0$ as $|z| \to \infty$.

So for some R, $\dfrac{1}{|P(z)|} \leq 1$ for $|z| \geq R$.

Next, by the Extreme Value Theorem, for some M,

$$\left|\frac{1}{P(z)}\right| \le M \text{ for } |z| \le R.$$

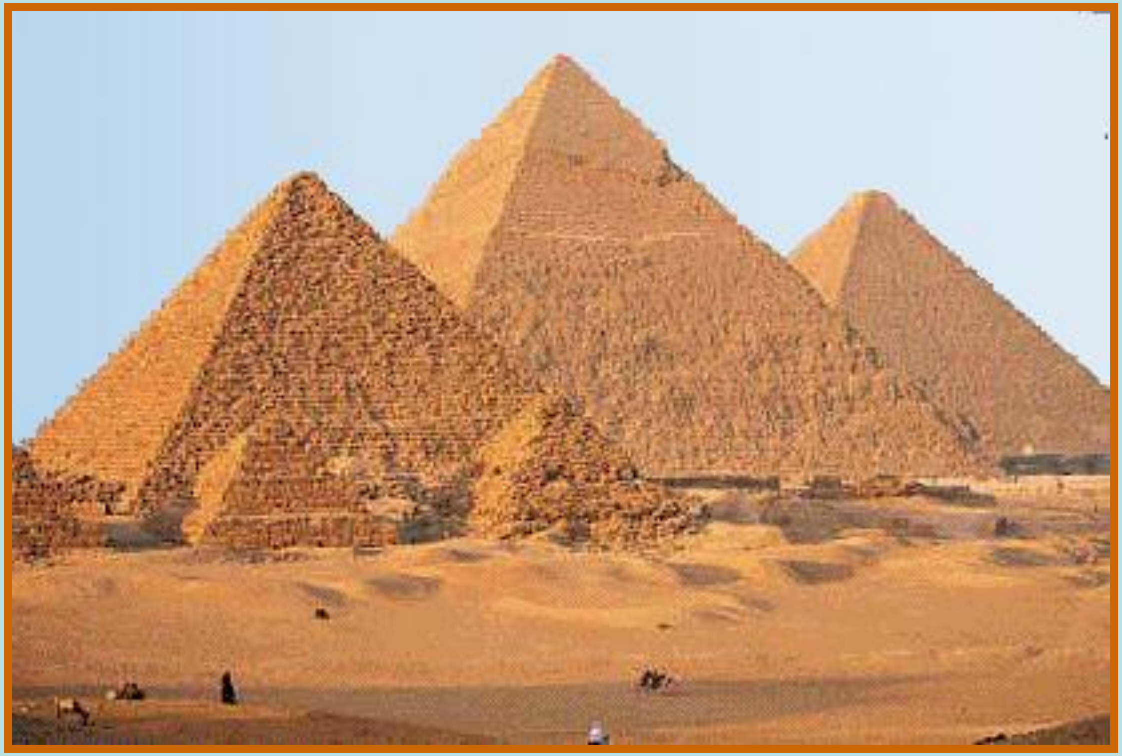
But then, P is bounded for all z by max{1, R}. So by Louisville's Theorem, P is a constant, contradicting the fact that it is a polynomial. ■

Nope! **Gauss** was the first to prove the Fundamental Theorem of Algebra, but his proof and all proofs since then have required some result from analysis!

That seems too easy, Old Bob. Can't you present a purely algebraic proof? After all, it's the Fundamental Theorem of *Algebra* !

1777-1855

The "Centroid Theorem"



Theorem. The Centroid Theorem.
The centroid of the zeros of a polynomial is the same as the centroid of the zeros of the derivative of the polynomial.

Proof. Let polynomial P have zeros z_1, z_2, …, z_n. Then

$$P(z) = \sum_{k=0}^{n} a_k z^k = a_n \prod_{k=1}^{n} (z - z_k).$$

Multiplying out, we find that the coefficient of z^{n-1} is

$$a_{n-1} = -a_n (z_1 + z_2 + \cdots + z_n).$$

Therefore the centroid of the zeros of P is

$$\frac{z_1 + z_2 + \cdots + z_n}{n} = \left(\frac{1}{n}\right)\left(\frac{-a_{n-1}}{a_n}\right) = \frac{-a_{n-1}}{na_n}.$$

Let the zeros of P' be $w_1, w_2, \ldots, w_n$. Then

$$P'(z) = \sum_{k=1}^{n} k a_k z^{k-1} = n a_n \prod_{k=1}^{n-1} (z - w_k).$$

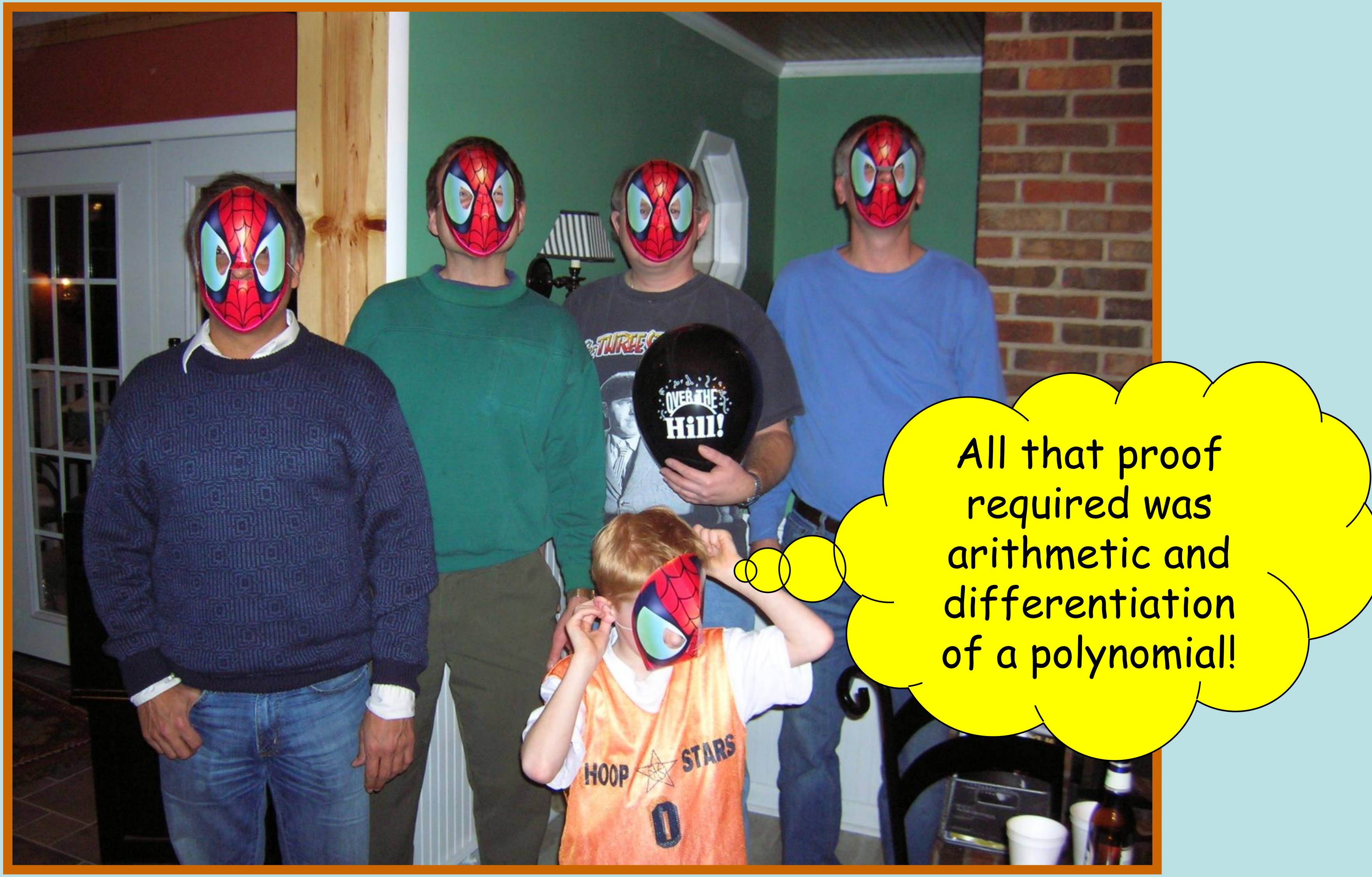
Multiplying out, we find that the coefficient of $z^{n\text{-}2}$ is

$$(n-1)a_{n-1} = -n a_n (w_1 + w_2 + \cdots + w_{n-1}).$$

Therefore the centroid of the zeros of P' is

$$\frac{w_1 + w_2 + \cdots + w_{n-1}}{n-1} = \left(\frac{1}{n-1}\right)\left(\frac{-(n-1)a_{n-1}}{n a_n}\right) = \frac{-a_{n-1}}{n a_n}.$$

Therefore the centroid of the zeros of P' is the same as the centroid of the zeros of P. ■



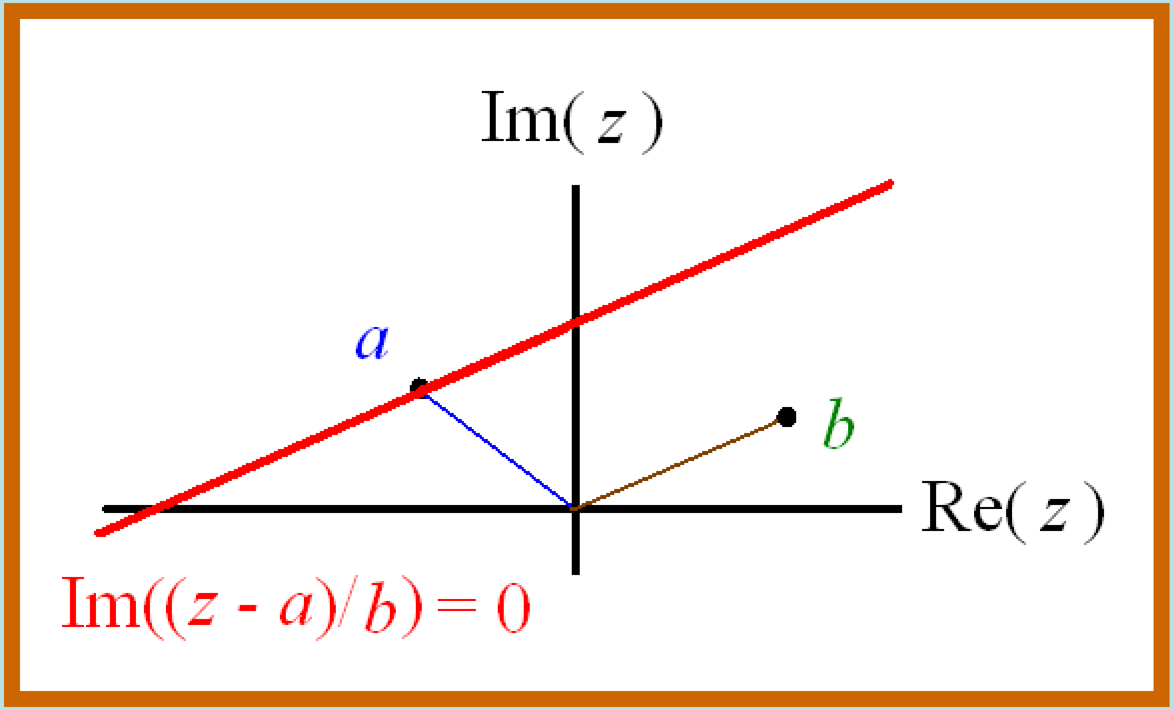
Three ETSU department chairs, a research astrophysicist, and a young Spider-man.

The Lucas Theorem

Note. Recall that a line in the complex plane can be represented by an equation of the form

$$\mathrm{Im}((z-a)/b)=0$$

where the line is "parallel" to the vector b and translated from the origin by an amount a (here we are knowingly blurring the distinction between vectors in $\mathbb{R}^2$ and numbers in $\mathbb{C}$).



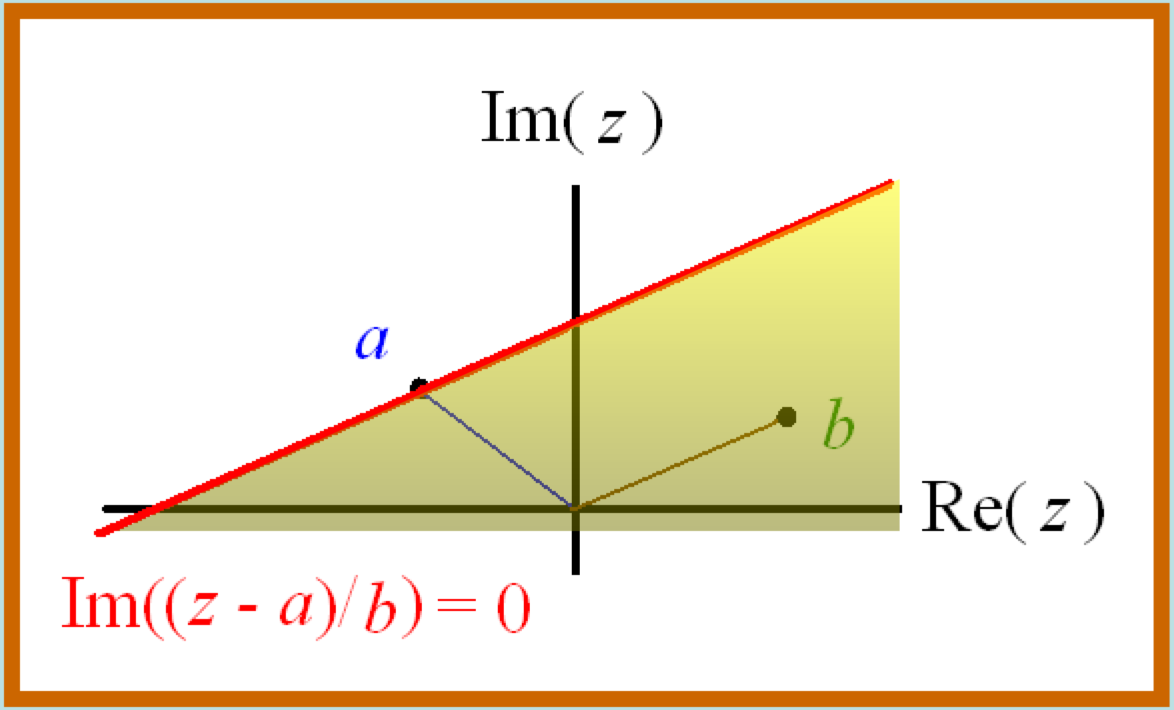
Note. We can represent a closed half plane with the equation

$$\mathrm{Im}((z-a)/b) \leq 0.$$

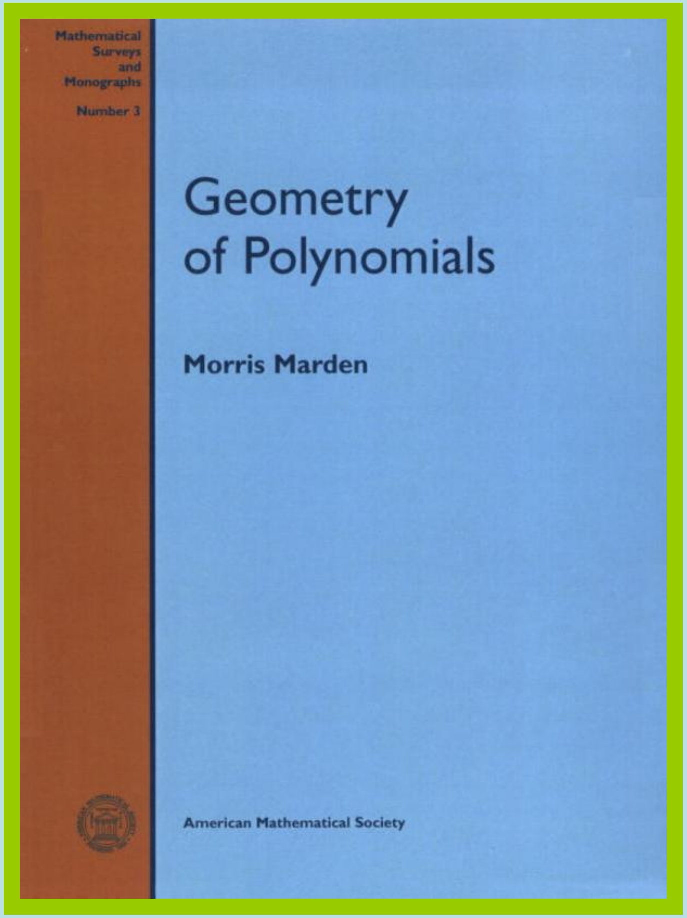
This represents the half plane to the right of the line

$$\mathrm{Im}((z-a)/b) = 0$$

when traveling along the line in the "direction" of *b*.



Note. According to Morris Marden's *Geometry of Polynomials* (A.M.S. 1949, revised 1985), F.A.E. Lucas proved the "Lucas Theorem" in 1874. It was independently proved later by several others.



Francois A.E. Lucas (1842-1891)

Theorem. The Lucas Theorem. If all the zeros of a polynomial $P(z)$ lie in a half plane in the complex plane, then all the zeros of the derivative $P'(z)$ lie in the same half plane.

Proof. By the Fundamental Theorem of Algebra, we can factor P as

$$P(z) = a_n(z - r_1)(z - r_2)\cdots(z - r_n).$$

So

$$\log P(z) = \log a_n + \log(z - r_1) + \log(z - r_2) + \cdots + \log(z - r_n)$$

and differentiating both sides gives

$$\frac{P'(z)}{P(z)} = \frac{1}{z - r_1} + \frac{1}{z - r_2} + \cdots + \frac{1}{z - r_n} = \sum_{k=1}^{n} \frac{1}{z - r_k}. \qquad (1)$$

Suppose the half plane H that contains all the zeros of $P(z)$ is described by

$$\operatorname{Im}((z-a)/b) \leq 0.$$

Then

$$\operatorname{Im}((r_1-a)/b) \leq 0, \quad \operatorname{Im}((r_2-a)/b) \leq 0, \quad \ldots, \quad \operatorname{Im}((r_n-a)/b) \leq 0.$$

Now let z^* be some number not in H. We want to show that $P'(z^*)$ s 0 (this will mean that all the zeros of $P'(z)$ are in H). Well,

$$\operatorname{Im}((z^*-a)/b) > 0.$$

Let r_k be some zero of P. Then

$$\operatorname{Im}\left(\frac{z^*-r_k}{b}\right)=\operatorname{Im}\left(\frac{z^*-a-r_k+a}{b}\right)=\operatorname{Im}\left(\frac{z^*-a}{b}\right)-\operatorname{Im}\left(\frac{r_k-a}{b}\right)>0.$$

(Notice that $\operatorname{Im}((z^*-a)/b)>0$ since z^* in not in H and $-\operatorname{Im}((r_n-a)/b)\geq 0$ since r_k is in H.) The imaginary parts of reciprocal numbers have opposite signs, so

$$\operatorname{Im}(b/(z^*-a))<0.$$

Recall:

$$\frac{P'(z)}{P(z)} = \frac{1}{z-r_1} + \frac{1}{z-r_2} + \cdots + \frac{1}{z-r_n} = \sum_{k=1}^{n} \frac{1}{z-r_k}. \qquad (1)$$

Applying (1),

$$\operatorname{Im}\left(\frac{bP'(z^*)}{P(z^*)}\right) = \sum_{k=1}^{n} \operatorname{Im}\left(\frac{b}{z^*-r_k}\right) < 0.$$

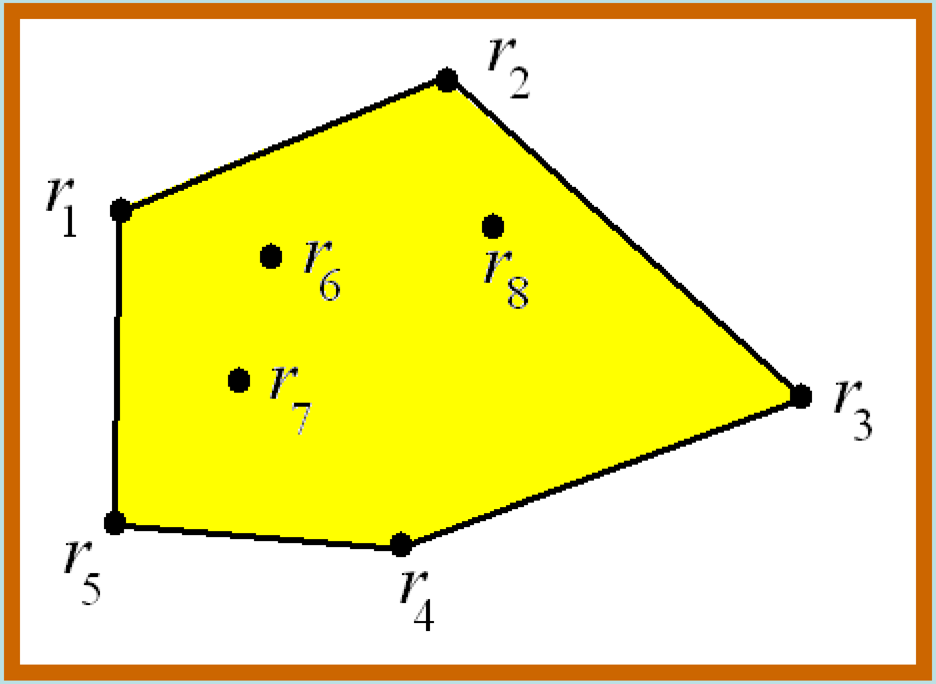
So $\frac{P'(z^*)}{P(z^*)} \neq 0$ and $P'(z^*) \neq 0.$ Therefore if

$P'(z) = 0$ then $z \in H.$ ■

Note. With repeated application of the Gauss-Lucas Theorem, we can prove the following corollary.

Corollary 1. The convex polygon in the complex plane which contains all the zeros of a polynomial P, also contains all the zeros of P'.

Note. For example, if P has eight zeros, then the convex polygon containing them might look like this:



Dr. Beeler, does
Theorem hold for p
a real vari

No!

UGH

Bernstein's Inequality

Definition. For a polynomial $p(z)$, define the *norm*:

$$\|p\| = \max_{|z|=1} |p(z)|.$$

Note. This is sometimes called the "sup norm" or "infinity norm."

If p is a polynomial of degree n, then

$$\|p'\| \le n\|p\|.$$

Equality holds if and only if $p(z)=\lambda z^n$ for some λ.

Sergei N. Bernstein (1880-1968)

Note. Bernstein's original result (in 1926) concerned trigonometric polynomials, which are of the form $\sum_{v=-n}^{n} a_v e^{iv\theta}$. The version presented here is a special case of Bernstein's general result. There is a lengthy history of the so-called "Bernstein's Inequality" (there is also a different result in statistics with the same name). We give a proof from scratch for our favored version.

Lemma. If P is a polynomial of degree n with no zeros in $|z| < 1$ and

$$Q(z) = z^n \overline{P\left(1/z \right)}$$

then

$$|P'(z)| \le |Q'(z)| \text{ for } |z| \ge 1.$$

Proof. Let $P(z)$ be a polynomial of degree n such that P is nonzero in $|z| < 1$, and define Q as

$$Q(z) = z^n \overline{P\left(1/z \right)} = \overline{a_0} z^n + \overline{a_1} z^{n-1} + \cdots + a_n .$$

Then on $|z| = 1$, we have $|P(z)| = |Q(z)|$. Also Q has all its zeros in $|z|$ # 1. This implies that

$$\frac{P(z)}{Q(z)} \text{ is analytic in } |z| \geq 1.$$

Since

$$\left|\frac{P(z)}{Q(z)}\right| = 1 \text{ on } |z| = 1 \text{ and } \lim_{z\to\infty}\left|\frac{P(z)}{Q(z)}\right| = \left|\frac{a_n}{\overline{a_0}}\right|,$$

then by the Maximum Modulus Theorem for Unbounded Domains,

$$\left|\frac{P(z)}{Q(z)}\right| \leq 1 \text{ for } |z| \geq 1.$$

Therefore $|P'(z)| \leq |Q'(z)|$ for $|z| \geq 1$. $\blacksquare$

Proof of Bernstein's Theorem. Let p be any polynomial. Define

$$P(z) = p(z) - \lambda\|p\|, \text{ where } |\lambda| = 1.$$

Then by the Maximum Modulus Theorem, P has no zero in $|z| < 1$, and we can apply the Lemma to P. First,

$$Q(z) = z^n \overline{P\left(1/\bar{z}\right)} = z^n \left\{ \overline{p\left(1/\bar{z}\right)} - \bar{\lambda}\|p\| \right\}$$

$$= z^n \overline{p\left(1/\bar{z}\right)} - \bar{\lambda} z^n \|p\| = q(z) - \bar{\lambda} z^n \|p\|.$$

So $Q'(z) = q'(z) - \bar{\lambda} n z^{n-1} \|p\|.$

Then by the Lemma, we have for $|z|$P 1:

$$|P'(z)| \leq |Q'(z)|, \text{ or}$$

$$|p'(z)| \leq \left| q'(z) - \bar{\lambda} n z^{n-1} \|p\| \right|$$

$$= |\lambda| n |z|^{n-1} \|p\| - |q'(z)|$$

for the correct choice of arg(λ). Rearranging we have:

$$|p'(z)| + |q'(z)| \le n|z|^{n-1}\|p\| \text{ for } |z| \ge 1.$$

In particular, when |z| = 1, we have

$$|p'(z)| + |q'(z)| \le n\|p\|. \quad (**)$$

Dropping the q' term and taking a maximum over all |z| = 1 yields

$$\|p'\| \le n\|p\|. \qquad \blacksquare$$

Note. Since equality holds in Bernstein's Theorem if and only if all of the zeros of p lie at the origin, the bound can be improved by putting restrictions on the location of the zeros of p. The first such result was posed by Paul Erdos and proved by Peter Lax.

Paul Erdos (1913-1996)

Peter Lax (1926-)

The Erdos-Lax Theorem. If p has no zeros in $|z| < 1$, then

$$\|p'\| \leq \frac{n}{2}\|p\|.$$

Proof. The proof of the Erdos-Lax Theorem is easy, given what we already know.

From the Lemma we have

$$|p'(z)| \le |q'(z)| \ \text{ for } \ |z| \ge 1.$$

From (**) we have

$$|p'(z)| + |q'(z)| \le n\|p\|.$$

Combining these, we get the result

$$2|p'(z)| \le |p'(z)| + |q'(z)| \le n\|p\|. \qquad \blacksquare$$

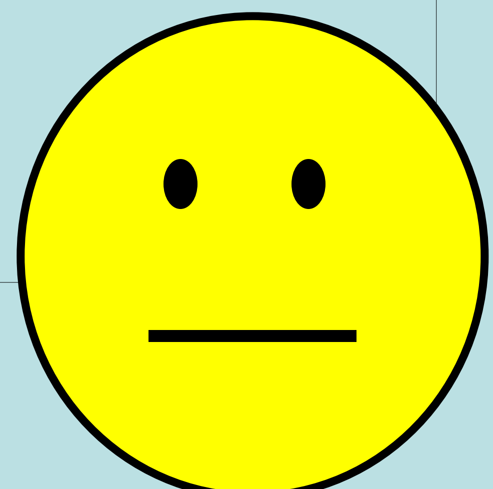
An anonymous member of the Fall 2011 graduate Complex Analysis 1 class.

De Bruijn’s Theorem

Nicolaas de Bruijn (1918-)

Theorem. If p has no zeros in $|z| < 1$, then

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left|p'\left(e^{i\theta}\right)\right|^\delta d\theta\right)^{1/\delta} \le n\left(\frac{2\pi}{\int_0^{2\pi}\left|1+e^{i\theta}\right|^\delta d\theta}\right)^{1/\delta}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|p\left(e^{i\theta}\right)\right|^\delta d\theta\right)^{1/\delta}.$$



Note. Of course, all this integral stuff looks like an L^p norm. So we define (using δ instead of p):

$$\|p\|_\delta = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| p\left(e^{i\theta}\right) \right|^\delta d\theta \right)^{1/\delta}.$$

Then Bernstein can express himself more clearly…

Theorem. If p has no zeros in $|z| < 1$, then

$$\|p'\|_\delta \le \frac{n}{\|1+z\|_\delta}\|p\|_\delta.$$

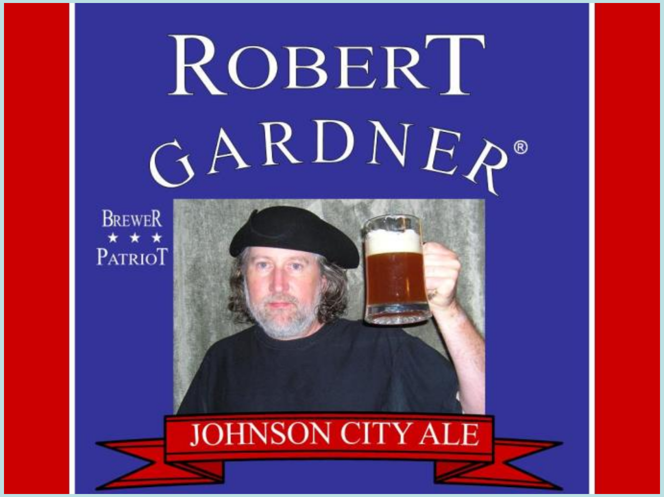
When δ →N, then my result implies the Erdos-Lax Theorem.


I suppose, as chair, I should use my **giant administrative salary** to buy more Michelob Ultra for Old Bob…
POOR RICHARD'S
★GREAT FOOD & SPIRITS★
WE I.D.
…but does this guy ever *do* any complex analysis research of his own, or does he *just talk* about other people's research?

Theorem. Let $P(z) = a_n \prod_{v=1}^{n} (z - z_v),\ a_n \neq 0,$ be a polynomial of degree n. If $|z_v| \geq K_v \geq 1,$ $1 \leq v \leq n,$ then for $0 \leq \delta \leq \infty$

$$\|P'\|_\delta \leq \frac{n}{\|t_0 + z\|_\delta} \|P\|_\delta,$$

where $t_0 = 1 + n \Big/ \sum_{v=1}^{n} \left(1/(K_v - 1)\right).$



Note. Some observations on this result:

- When δ = N, this implies:

$$\|P'\| \le \frac{n}{2}\left\{1 - \frac{1}{1+(2/n)\sum_{v=1}^{n}\left(1/(K_v - 1)\right)}\right\}\|P\|.$$

This is a result of Govil and Labelle (1987). Notice that this is a refinement of both Bernstein's Theorem and the Erdos-Lax Theorem.

•When some $K_v = 1$, the result reduces to de Bruijn's Theorem.

• The result even holds for $0 \# \delta < 1$, even though the resulting integral does not determine a norm (it fails the triangle inequality).

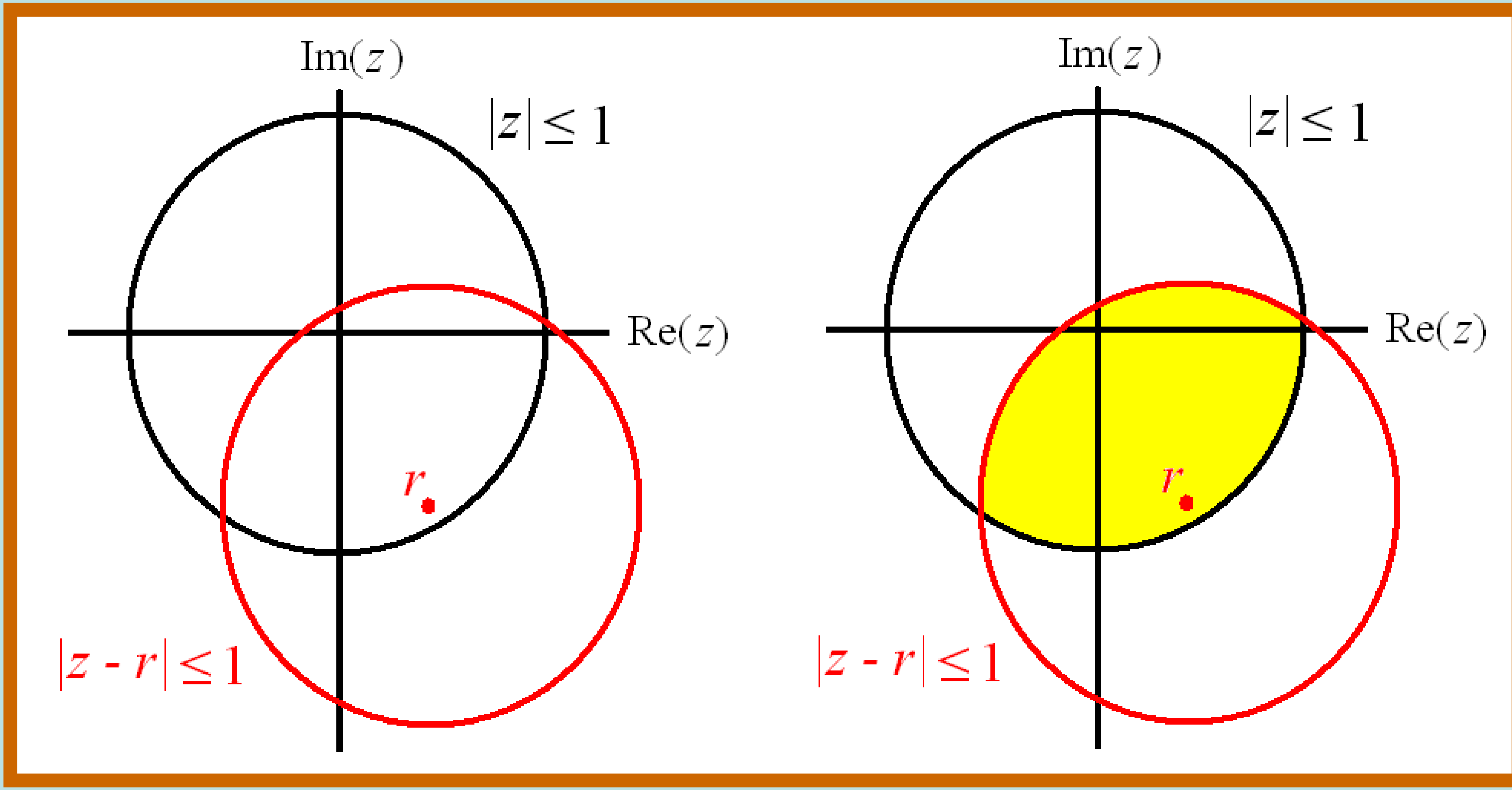
These results appeared (with N.K. Govil) in the ***Journal of Mathematical Analysis and its Applications***, **179**(1) (1993), 208-213 and **193** (1995), 490-496/**194** (1995), 720-726. A generalization appeared (with Amy Weems) in **219** (1998), 472-478.

The Illief-Sendov Conjecture

Note. The conjecture of interest is known variously as the *Ilieff Conjecture*, the *Ilieff-Sendov Conjecture*, and the *Sendov Conjecture* (making it particularly difficult to search for papers on the subject). It was originally posed by Bulgarian mathematician Blagovest Sendov in 1958 (sometimes the year 1962 is reported), but often attributed to Ilieff because of a reference in Hayman's *Research Problems in Function Theory* in 1967.

Conjecture. *The Ilieff-Sendov Conjecture.*

If all the zeros of a polynomial P lie in $|z|$ % 1 and if r is a zero of P, then there is a zero of P' in the circle $|z-r|$ %1.



Note. According to a 2008 paper by Michael Miller, there have been over 80 papers written on the conjecture. As a result, it has be demonstrated in many special cases. Some of the special cases are:

1. 3rd and 4th degree polynomials,

2. 5th degree polynomials,

3. polynomials having a root of modulus 1,

4. polynomials with real and non-positive coefficients,

5. polynomials with at most three distinct zeros,

6. polynomials with at most six distinct zeros,

7. polynomials of degree less than or equal to 6,

8. polynomials of degree less than or equal to 8,

and

9. the circle $|z - r|$ % 1.08331641.


All I want for Christmas is a proof of the Illief-Sendov Conjecture!
I'll see what I can do! I have an idea that the Centroid Theorem might be useful… HO, HO, HO!
…probably a lump of coal is more appropriate for this one…

Photographic Sources

The photographs of famous mathematicians are from The MacTutor History of Mathematics archive at:

http://www-groups.dcs.st-and.ac.uk/~history/

The other photographs are from the private collection of Dr. Bob and are results of the last several years of math picnics, Christmas parties, birthdays, and outings to local establishments.

Thanking you…..!!!!!