Unit I - Introduction

Introduction

Digital Signal Processing (DSP) refers to processing of signals by digital systems like Personal Computers (PC) and systems designed using digital Integrated Circuits (ICs), microprocessors and microcontrollers. rapid advancement in computers and IC fabrication technology leads to complete domination of DSP systems in both real-time and non-real-time applications in all fields of engineering and technology.

The basic components of a DSP system are shown in fig 1.1. The **DSP system** involves conversion of analog signal to digital signal, then processing of the digital signal by a digital system and then conversion of the processed digital signal back to analog signal

Fig 1.1 : Basic components of a DSP system.

The real-world signals are analog, and only for processing by digital systems, the signals are converted to digital. For conversion of signals from analog to digital, an ADC (Analog to Digital Converter) is employed. The various steps in analog to digital conversion process are sampling and quantization of analog signals, and then converting the quantized samples to suitable binary codes. The digital signals in the form of binary codes are fed to digital system for processing, and after processing, it generates an output digital signal in the form of binary codes. The output analog signal is constructed from the output binary codes using a DAC (Digital to Analog Converter).

Advantages of Digital Signal Processing

Some of the advantages of digital processing of signals are,
1. The digital hardware are compact, reliable, less expensive, and programmable.

2 Since the DSP systems are programmable, the performance of the system can be easily upgraded/ modified.

3 By employing high speed, sophisticated digital hardware higher precision can be achleved in processing of signals.

4. The digital signals can be permanently stored in magnetic media so that they are transportable and can be processed in non-real-time or off-line.

Applications of Digital Signal Processing

The digital processing of signal plays a vital role in almost every field of Science and Engineering. Some of the applications of digital processing of signals in various field of Science and Engineering arc listed here.

Biomedical

ECG is used to predict heart diseases.
EEG is used to study normal and abnormal behaviour of the brain.
EMG is used to study the condition of muscles
X-ray images are used to predict the bone fractures and tuberculosis.
Ultrasonic scan images of kidney and gall bladder is used to predict stones.
Ultrasonic scan images of foetus is used to predict abnormalities in a baby. MRI scan is used to study minute inner details of any part of the human body.

Speech Processing

Speech compression and decompression to reduce memory requirement of storage systems
Speech compression and decompression for effective use of transmission channels
Speech recognition for voice operated systems and voice based security systems.
Speech recognition for conversion of voice to text
Speech synthesis for various voice based warnings or announcements.

Audio and Video Equipment

The analysis of audio signals will be useful to design systems for special effects in audio systems like stereo, woofer, karoke, equalizer, attenuator, etc.
Music synthesis and composing using music keyboards
Audio and video compression for storage in DVDs

Communication

The spectrum analysis of modulated signals helps to identify the information bearing frequency component that can be used for transmission.

The analysis of signals received from radars are used to detect flying objects and their velocity.

Generation and detection of DTMF signals in telephones

Echo and noise cancellation in transmission channels

Power Electronics

The spectrum analysis of the output of converters and inverters will **reveal** the harmonics present in the output, which in turn helps to design suitable filter to eliminate the harmonics.

The analysis of switching currents and voltages in power devices will help to reduce losses.

Image processing

Image compression and decompression to reduce memory requirement of storage systems

Image compression and decompression for effective use of transmission channels Image recognition for security systems.

Filtering operations on images to extract the features or hidden information

Geology

The seismic signals are used to determine the magnitude of earthquakes and volcanic eruptions.

The seismic signals are also used to predict nuclear explosions.

The seismic noises are also used to predict the movement of earth layers (tectonic plates).

Astronomy

The analysis of light received from a star is used to determine the condition of the star.

The analysis of images of various celestial bodies gives vital information about them.

Signals, Systems, and Signal Processing

A ***signal*** is defined as any physical quantity that varies with time, space, or any other independent variable or variables. Mathematically, we describe a signal as a function of one or more independent variables. For example, the functions

$$Sl(t)=5t$$
$$S2(t) = 20t2$$

describe two signals, one that varies linearly with the independent variable *t* (time) and a second that varies quadratically with *t.* As another example, consider the function

$$s(x. y) = 3x + 2xy + y$$

This function describes a signal of two independent variables *x* and *y* that could represent the two spatial coordinates in a plane. Speech, electrocardiogram, and electroencephalogram signals are examples of information-bearing signals that evolve as functions of a single independent variable (viz) time. An example of a signal that is a function of two independent variables
is an image signal.

System

A **system** may also be defined as a physical device that performs an operation on a signal. For example a filter used to reduce the noise and interference corrupting a desired information-bearing signal is called a system. In this case the filter performs some operation(s) on the signal, which has the effect of reducing (filtering) the noise and interference from the desired information-bearing signal. In digital processing of signals on a digital computer the operations performed on a signal consist of a number of mathematical operations as specified by a software program. In this case, the program represents an implementation of the system in *software.* Thus we have a system that is realized on a digital computer by means of a sequence of mathematical operations; that is, we have a digital signal processing system realized in software. For example a digital computer can be programmed to perform digital filtering. Alternatively the digital processing on the signal may be performed by digital *hardware* (logic circuits) configured to perform the desired specific operations. In such a realization we have a physical dcvice that performs the specified operations. In a broader sense a digital system can bc implemented as a combination of digital hardware and software each of which performs its own set of specified operations.

Classification of Signals

1.Multichannel and Multidimensional Signals are signals are generated by multiple sources or multiple sensors. Such signals, in turn, can be represented in vector form. If sdl), *k* = 1,2, 3, denotes the electrical signal from the *kth* sensor as a function of time. the set of p = 3 signals can be represented by a vector S(I), where

$$\mathbf{S}_3(t) = \begin{bmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \end{bmatrix}$$

We refer to such a vector of signals as a **multichannel signal**. In electrocardiography for example, 3-lead and 12-lead electrocardiograms (ECG) are often used in practice, which result in 3-channel and 12-channel signals. if the signal is a function of a single independent variable, the signal is called a one-dimensional signal. On the other hand, a signal is called *M -dimensional* if its value is a function of *M* independent variables. A Still picture is an example of a two-dimensional signal. Since the intensity or brightness *I (x, y)* at each point is a function of two independent variables. On the other hand, a black-and-white television picture may be represented as *I(x, y. t)* since the brightness is a function of time. Hence the TV picture may be treated as a three-dimensional signal. In contrast, a color TV picture may be described by three intensity functions of the form *I,. (x. y, t), I* K *(x, y. t),* and I" *(x , y. t).*
corresponding to the brightness of the three principal colors (red. green. blue) as functions of time. Hence the color TV picture is a three-channel. three-dimensional signal, which can be represented by the vector

$$\mathbf{I}(x, y, t) = \begin{bmatrix} I_r(x, y, t) \\ I_g(x, y, t) \\ I_b(x, y, t) \end{bmatrix}$$

Signals can be further classified into four different categories depending on the characteristics of the time (independent) variable and the values they take.

2. Continuous time (Analog)signals and Discrete Time Signals

Continuous Time signals or analog signals are defined for every value of time and they take on values in the continuous interval *(a, b),* where *a* can be - ꝏ and *b* can be ꝏ. Mathematically, these signals can be described by functions of a continuous variable.

Discrete-time(DT) signals are defined only at certain specific values of time. These time instants need not be equidistant, but in practice they are usually taken at equally spaced intervals. we use the index II of the discrete time instants as the independent variable, the signal value becomes a function of an integer variable (i.e., a sequence of numbers). Thus a discrete-time signal can be represented mathematically by a sequence of real or complex numbers. In discrete time signal, the time is divided uniformly using the relation t = nT, where T is the sampling time period. (The sampling time period is the inverse of sampling frequency). The discrete time signal is denoted by x(n) or x(nT).

$$x(n) = \begin{cases} 0.8^n, & \text{if } n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

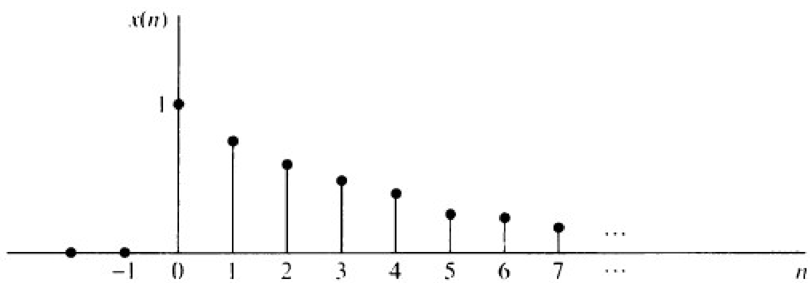


Figure 1.2.3 Graphical representation of the discrete time signal $x(n) = 0.8^n$ for $n > 0$ and $x(n) = 0$ for $n < 0$.

3. Continuous-Valued and Discrete-Valued Signals

The values of a continuous-time or discrete-time signal can be continuous or discrete. If a signal takes on all possible values on a finite or an infinite range it is said to be a continuous-valued signal. Alternatively, if the signal takes on values from a finite set of possible values it is said to be a discrete-valued signal. Usually, these values are equidistant and hence can be expressed as an integer multiples of the distance between two successive values. A discrete-time signal having a set of discrete values is called a *digital signal.* Figure 1.2.5 shows a digital signal that takes on one of four possible values.

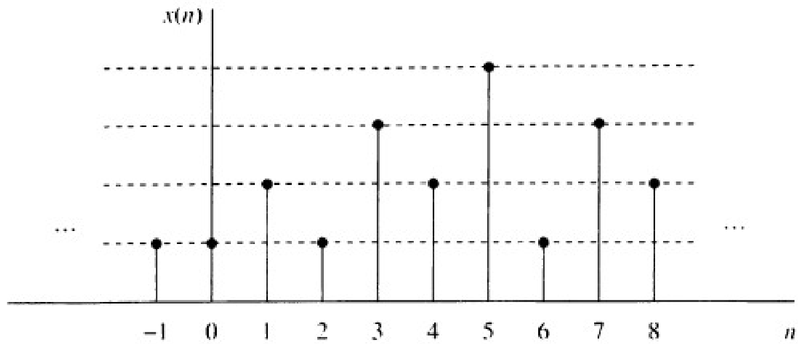


Figure 1.2.5 Digital signal with four different amplitude values.

4.Deterministic and Random Signals

Any signal that can be uniquely described by an explicit mathematical expression, a table of data, or a well-defined rule is called *deterministic.* All past, present, and future values of the signal are known precisely, without any uncertainty signals that either cannot be described to any reasonable degree of accuracy by explicit mathematical formulas. or such a description is too complicated to be of any practical use We refer to these signals as *random. The* output of a noise generator, the seismic signal, speech signal are examples of random signals.

Representation of Discrete time Signals

The discrete time signal can be represented by the following methods.

1. Functional representation, such as

$$x(n) = \begin{cases} 1, & \text{for } n = 1, 3 \\ 4, & \text{for } n = 2 \\ 0, & \text{elsewhere} \end{cases} \tag{2.1.1}$$

2. Tabular representation, such as

n	$\cdots$	-2	-1	0	1	2	3	4	5	$\cdots$
$x(n)$	$\cdots$	0	0	0	1	4	1	0	0	$\cdots$

3. Sequence representation

An infinite-duration signal or sequence with the time origin ($n = 0$) indicated by the symbol $\uparrow$ is represented as

$$x(n) = \{\ldots 0, \underset{\uparrow}{0}, 1, 4, 1, 0, 0, \ldots\} \tag{2.1.2}$$

A sequence $x(n)$, which is zero for $n < 0$, can be represented as

$$x(n) = \{\underset{\uparrow}{0}, 1, 4, 1, 0, 0, \ldots\} \tag{2.1.3}$$

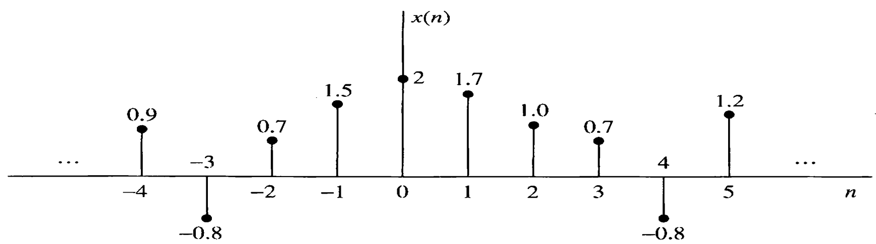


Figure 2.1.1 Graphical representation of a discrete-time signal.

4.Graphical Representation

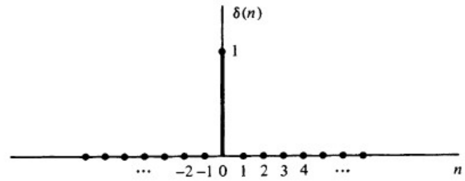
In graphical representation, the signal is represented in a two-dimensional plane. The independent variable is represented in the horizontal axis and the value of the signal is represented in the vertical axis as shown in fig 2.1.1

Some Elementary Discrete-Time Signals

1. The **unit sample sequence** *or* **Unit impulse signal** is denoted as *8(n)* and is defined as

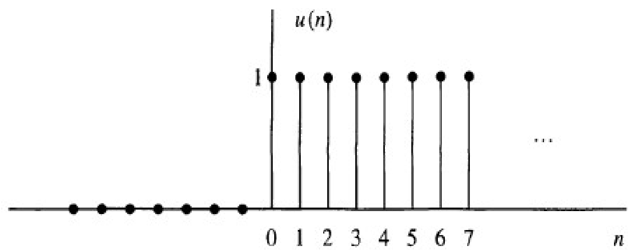
$$\delta(n) \equiv \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0 \end{cases}$$

The unit sample sequence is a signal that is zero everywhere, except at $n = 0$ where its value is unity. This signal is sometimes referred to as a unit impulse



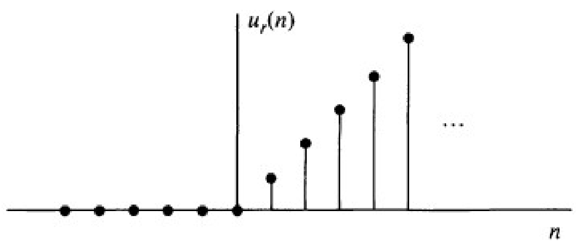
2.The **Unit step signal** is denoted as *u(n)* and is defined as

$$u(n) \equiv \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$



3.The **unit ramp signal** is denoted as *ur(n)* and is defined as

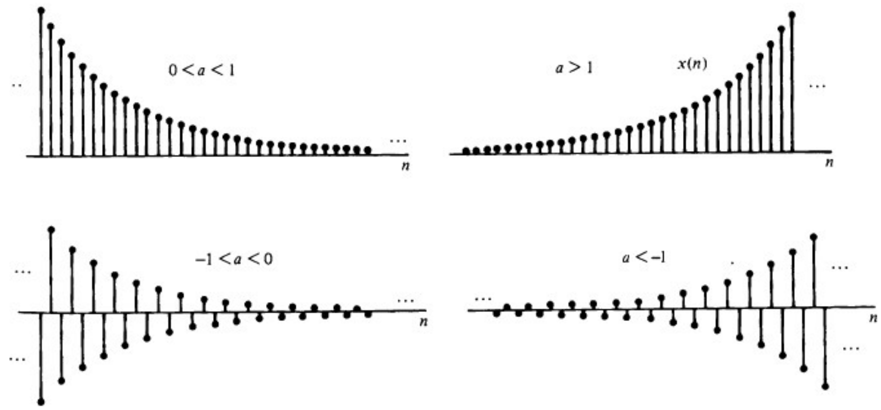
$$u_r(n) \equiv \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$



4. The **exponential signal** is a sequence of the form

$$x(n) = a^n \qquad \text{for all } n$$

If the parameter *a* is real, then *x (n)* is a real signal. Figure shown below illustrates *x (n)* for various values of the parameter *a.*



Classification of Discrete Time Signals

The discrete time signals are classified depending on their characteristics. Some ways of classifying discrete time signals are,

1.Periodic and aperiodic signals

2.Symmetric and antisymmetric signals

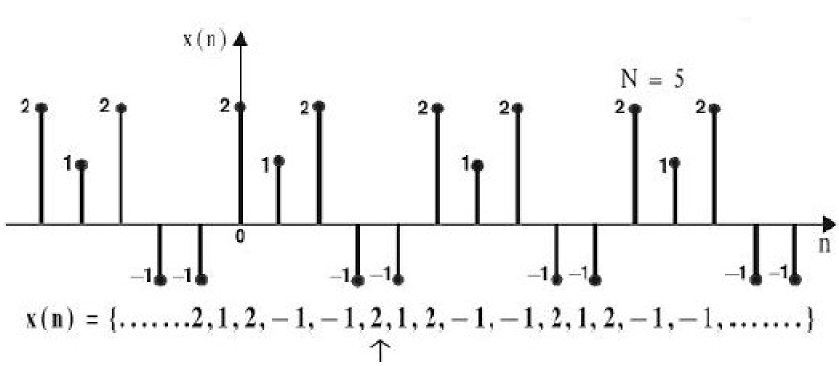
3.Energy and power signals

4.Causal and noncausal signals

1.**Periodic signals and aperiodic signals.** A signal *x(n)* is periodic with period *N(N* > 0) if and only if

$$x(n+N) = x(n) \text{ for all } n$$

The smallest value of *N* which satisfies the above equation is called the (fundamental) period.
If there is no value of *N* that satisfies, the signal is called ***nonperiodic*** **or** ***aperiodic.***



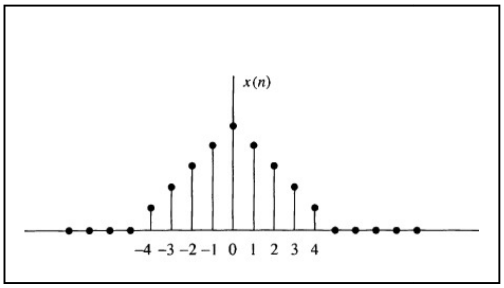
2 **Symmetric (even) and antisymmetric (odd) signals**
A real-valued signal *x(n)* is called symmetric (even) if

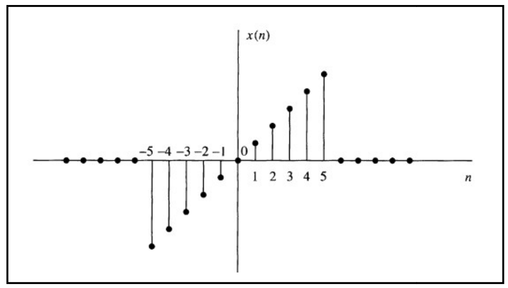
$$x(-n) = x(n)$$

On the other hand, a signal $x(n)$ is called antisymmetric (odd) if

$$x(-n) = -x(n)$$

We note that if *x(n)* is odd, then *x(0)* = 0.





Any arbitrary signal can be expressed as the sum of two signal components, one of which is even and the other odd. The even signal component is formed by adding *x(n)* to *x(-n)* and dividing by 2, that is,

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

Similarly, we form an odd signal component *xo(n)* according to the relation

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

3. **Energy signals and power signals.**

The energy *E* of a signal *x(n)* is defined as

$$E \equiv \sum_{n=-\infty}^{\infty} |x(n)|^2$$

The energy of a signal can be finite or infinite. If *E* is finite (i.e., 0 < *E* < 00), then *x(n)* is called an *energy signal.*

The average power of a discrete-time signal *x(n)* is defined as

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

If power P of a discrete time signal is finite and non-zero, then the discrete time signal is called **a power signal.** The periodic signals are examples of power signals.

For energy signals, the energy will be finite and average power will be zero.

For power signals the average power is finite and energy will be infinite.

\ For energy signal, $0 < E < \infty$ and $P = 0$
For power signal, $0 < P < \infty$ and $E = \infty$

4.**Causal, Non-causal and Anticausal signals**

A discrete time signal is said to be ***causal***, if it is defined for n ³ 0. Therefore if $x(n)$ is causal, then $x(n) = 0$ for $n < 0$.

A discrete time signal is said to be ***noncausal***, if it is defined for either $n \leq 0$, or for both $n \leq 0$ and $n > 0$. Therefore if $x(n)$ is noncausal, then $x(n) \neq 0$ for $n < 0$. A noncausal signal can be converted to causal signal by multiplying the noncausal signal by a unit step signal, $u(n)$.

When a noncausal discrete time signal is defined only for $n \leq 0$, it is called an ***anticausal signal***.

$$\left.\begin{array}{l} x(n) = \{\underline{1}, -1, 2, -2, 3, -3\} \\ x(n) = \{\underline{2}, 2, 3, 3, \ldots\ldots\ldots\ldots\} \end{array}\right\} \text{Causal signals}$$

$$\left.\begin{array}{l} x(n) = \{1, -1, 2, -2, 3, \underline{-3}\} \\ x(n) = \{\ldots\ldots\ldots\ldots, 2, 2, 3, \underline{3}\} \end{array}\right\} \text{Anticausal signals}$$

$$x(n) = \{2, 3, 4, \underline{5}, 4, 3, 2\}$$

$$x(n) = \{\ldots\ldots, 2, 3, 4, \underline{5}, 4, 3, 2, \ldots\ldots\}$$

Noncausal signals

Classification of Discrete Time Systems

Discrete time systems are classified based on their characteristics. Some of the classifications of discrete time systems are,

1. Static and dynamic systems
2. Time invariant and time variant systems
3. Linear and nonlinear systems
4. Causal and noncausal systems
5. Stable and unstable systems
6. FIR and IIR systems

1.Static and Dynamic systems.

A discrete-time system is called **static or memoryless** if its output at any instant *n* depends at most on the input sample at the same time, but not on past or future samples of the input. In any other case, the system is said to be **dynamic** or to have memory. If the output of a system at time *n* is completely determined by the input samples in the interval from *n* - *N* to *n(N* :::: 0), the system is said to have *memory* of duration *N.* If *N* = 0, the system is static. If 0< *N* < OO, the system is said to have *finite memory,* whereas if *N* = OO, the system is said to have *infinite memory.*

The systems described by the following input-output equations

$$y(n) = ax(n)$$

$$y(n) = nx(n) + bx\backslash n)$$

are both static or memory less. Note that there is no need to store any of the past inputs or outputs in order to compute the present output. On the other hand, the systems described by the following input-output relations

$$y(n) = x(n) + 3x(n - 1)$$

is a dynamic system or systems with memory.

2 Time-invariant versus time-variant systems.

A system is called time-invariant if its input-output characteristics do not change with time.

Definition. A relaxed system is **time invariant or shift invariant** if and only if

$$x(n) \xrightarrow{\mathcal{T}} y(n)$$

$$x(n-k) \xrightarrow{\mathcal{T}} y(n-k)$$

for every input signal *x(n)* and every time shift *k*.

3. **Linear and Non-linear systems.**

A linear system is one that satisfies the *superposition principle.* Simply stated, the principle of superposition requires that the response of the system to a weighted sum of signals be equal to the corresponding weighted sum of the responses (outputs) of the system to each of the individual input signals.

Definition. A system is linear if and only if

$$\mathcal{T}[a_1x_1(n) + a_2x_2(n)] = a_1\mathcal{T}[x_1(n)] + a_2\mathcal{T}[x_2(n)]$$

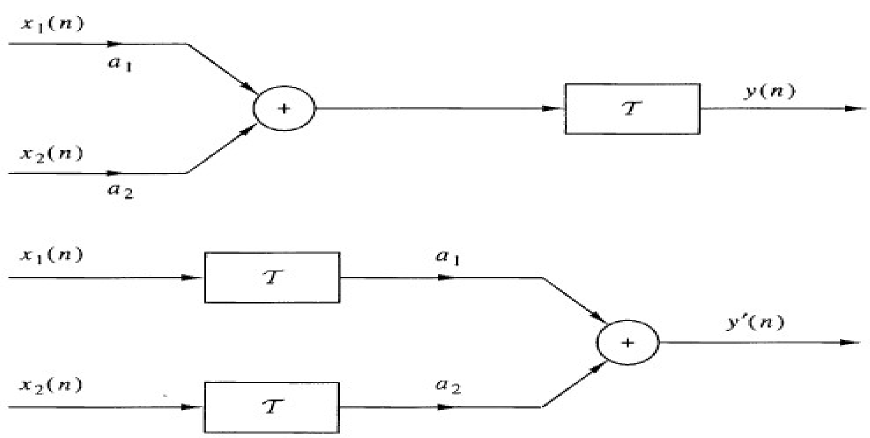


Figure 2.2.9 Graphical representation of the superposition principle. $\mathcal{T}$ is linear if and only if $y(n) = y'(n)$.

4. Causal and Non-causal systems.

Definition. A system is said to be *causal* if the output of the system at any time *n* [i.e., *y(n)]* depends only on present and past inputs [i.e., *x(n), x(n* -1), *x(n* - 2), ...], but does not depend on future inputs [i.e., *x(n* +1), *x(n*+2), ...].

In mathematical form the output of a causal system satisfies an equation of the form

$$y(n) = F[x(n), x(n-1), x(n-2), \ldots]$$

If a system does not satisfy this definition, it is called **non-causal.** Such a system has an output that depends not only on present and past inputs but also on future inputs.

5.Stable versus unstable systems.

Definition. An arbitrary relaxed system is said to be bounded input-bounded output (BIBO) stable if and only if every bounded input produces a bounded output.The condition that the input sequence *x(n)* and the output sequence *yen)* are bounded is translated mathematically to mean that there exist some finite numbers, say *Mx* and *My,* such that

$$|x(n)| \leq M_x < \infty, \qquad |y(n)| \leq M_y < \infty$$

for all *n.* If, for some bounded input sequence *x (n),* the output is unbounded (infinite), the system is classified as unstable.

6. FIR and IIR systems

In FIR *system* (Finite duration Impulse Response system), the impulse response consists of finite number of samples.

In IIR system (Infinite duration Impulse Response system), the impulse response has infinite number of samples.

Sampling Techniques

Most signals of practical interest, such as speech, biological signals, seismic signals, radar signals, sonar signals, and various communications signals such as audio and video signals, are analog. To process an analog signals by digital means, it is first necessary to convert the m into digital form , that is, to convert them to a sequence of numbers having finite precision. This procedure is called analog-to-digital (A /D) conversion , and the corresponding devices are called A / D converters (A D C s) .Conceptually , we view A /D con version as a three step process. This process is illustrated in Fig. 1.14.

1. Sampling. This is the conversion of a continuous-time signal in to a discrete time signal obtained by taking “ sample s’" of the continuous-time signal at disc rete-time instants. Thus, if x a(t) is the input to the sampler, the output is $x_a(nT) = x(n)$, w here T is called the sampling interval.

2. Quantization. This is the conversion of a discrete time continuous value signal into a discrete-time, discrete-valued (digital) signal. T h e value of each signal sample is represented by a value selected from a finite set of possible values. The difference between the unquantized sample x(n) and the quantized output xq(n) is called the quantization error.

Quantization error e(n)= xq(n)- x(n)

3. Coding. In the coding process, each discrete value xq{n) is represented by a b-bit binary sequence.

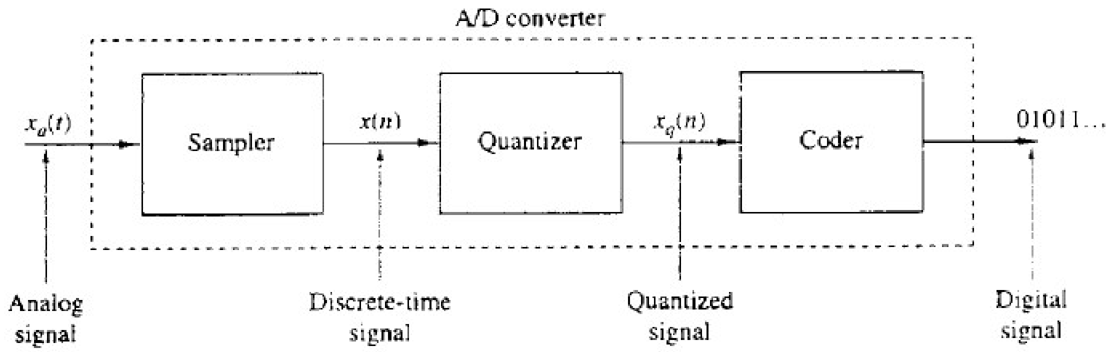


Figure 1.14 Basic parts of an analog-to-digital (A/D) converter.

In order to develop the sampling theorem, we need a convenient way in which to represent the sampling of a continuous-time signal at regular intervals. A useful way to do this is through the use of a periodic impulse train multiplied by the continuous-time signal *x(t)* that we wish to sample. This mechanism, known as *impulse-train sampling,* is depicted in Figure 7 .2. The periodic impulse train *p(t)* is referred to as the *sampling function,* the period *T* as the *sampling period,* and the fundamental frequency of *p(t)*, $\omega_s = 2\pi/T$, as the *sampling frequency.* In the time domain,

$$x_p(t) = x(t)p(t),$$

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT).$$

multiplying *x(t)* by a unit impulse samples the value of the signal at the point at which the impulse is located; i.e., $x(t)\delta(t- to) = x(t_0)\delta(t- t_0)$. Applying this to eq. (7.1), we see, as illustrated in Figure 7.2, that $x_p(t)$ is an impulse train with the amplitudes of

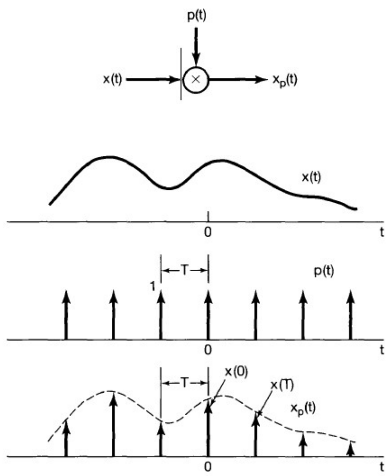


Figure 7.2 Impulse-train sampling.

the impulses equal to the samples of $x(t)$ at intervals spaced by T; that is,

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT).$$

$$X_p(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)P(j(\omega - \theta))d\theta.$$

Since convolution with an impulse simply shifts a signal [i.e., $X(j\omega) * \delta(\omega - \omega_0) = X(j(\omega - \omega_0))$], it follows that

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X(j(\omega - k\omega_s)).$$

That is, $X_p(j\omega)$ is a periodic function of ω consisting of a superposition of shifted replicas of $X(j\omega)$, scaled by $1/T$, as illustrated in Figure 7.3. In Figure 7.3(c), $\omega_M < (\omega_s - \omega_M)$, or equivalently, $\omega_s > 2\omega_M$, and thus there is no overlap between the shifted replicas of $X(j\omega)$, whereas in Figure 7.3(d), with $\omega_s < 2\omega_M$, there is overlap. For the case illustrated in Figure 7.3(c), $X(j\omega)$ is faithfully reproduced at integer multiples of the sampling frequency. Consequently, if $\omega_s > 2\omega_M$, $x(t)$ can be recovered exactly from $x_p(t)$ by means of

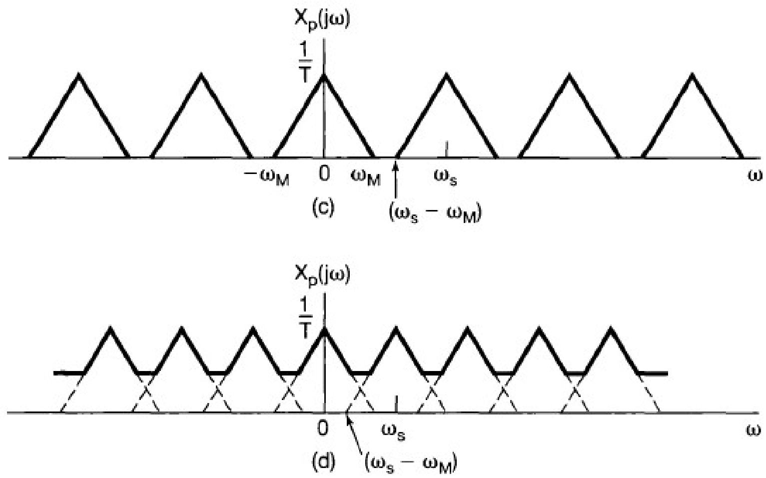


Figure 7.3 *Continued* (c) spectrum of sampled signal with $\omega_s > 2\omega_M$; (d) spectrum of sampled signal with $\omega_s < 2\omega_M$.

a lowpass filter with gain T and a cutoff frequency greater than ω_M and less than $\omega_s - \omega_M$, as indicated in Figure 7.4. This basic result, referred to as the *sampling theorem,* can be stated as follows: [1]

Sampling Theorem:
Let $x(t)$ be a band-limited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n = 0, \pm 1, \pm 2, \ldots$, if

$$\omega_s > 2\omega_M,$$

where

$$\omega_s = \frac{2\pi}{T}.$$

1. **Test the causality of the following systems**

(i) **y (n) = x(n) + 3x (n+4).**

(ii) **y (n) = x(-n)**

(iii) **y (n) = x(n^2)**

Solution:

Sol:-

(i) $y(n) = x(n) + 3x(n+4)$

For $n=-1$; $y[-1] = x[-1] + 3x[3]$

$n=0$; $y[0] = x[0] + 3x[4]$

$n=1$; $y[1] = x[1] + 3x[5]$

* For all the values of 'n', the output depends on present and Future inputs. So, the system is Non-Causal.

(ii) $y[n] = x[-n]$

For $n=-1$; $y[-1] = x[1]$

$n=0$; $y[0] = x[0]$

$n=1$; $y[1] = x[-1]$

* For all the values of 'n', the output depends on Present, Past and Future inputs. So the system is Non-Causal.

(iii) $y[n] = x[n^2]$

For $n=-1$; $y[-1] = x[1]$

$n=0$; $y[0] = x[0]$

$n=1$; $y[1] = x[1]$

$n=2$; $y[2] = x[4]$

* For all the values of 'n', the output depends on Present and Future inputs. So the system is Non-Causal.

---

2. Test the following systems for linearity.

(i) y(n) =x(n)+c

(ii) y(n) =nx^2(n)

(iii) y(n)=a$^{x(n)}$

Solution:

(i) Given : $\underline{\underline{y[n] = x[n] + C}}$.

For two input sequences $x_1[n]$ and $x_2[n]$ the corresponding outputs are,

$$y_1[n] = T[x_1[n]] = x_1[n] + C$$

$$y_2[n] = T[x_2[n]] = x_2[n] + C$$

The output due to weighted sum of inputs is

$$y_3'[n] = T[a_1 x_1[n] + a_2 x_2[n]]$$

$$= a_1 x_1[n] + a_2 x_2[n] + C \rightarrow ①$$

on the other hand, the linear combination of two output is,

$$y_3[n] = a_1 y_1[n] + a_2 y_2[n] = a_1 x_1[n] + a_1 C + a_2 x_2[n] + a_2 C \rightarrow ②$$

Equation ① & ② are not equal, superposition Principle is not satisfied. So the system is Non-Linear.

(ii) Given: $\underline{\underline{y[n] = n\, x^2[n]}}$.

* For two input sequences $x_1[n]$ and $x_2[n]$ the corresponding outputs are,

$$y_1[n] = T[x_1[n]] = n\, x_1^2[n]$$

$$y_2[n] = T[x_2[n]] = n\, x_2^2[n].$$

* The output due to weighted sum of input is,

$$y_3[n] = T\left[a_1 x_1[n] + a_2 x_2[n]\right]$$

$$= \left[a_1 n\, x_1[n] + a_2 n\, x_2[n]\right]^2$$

$$= a_1^2 n^2 x_1^2[n] + a_2^2 n^2 x_2^2[n] + 2a_1 a_2 n^2 x_1[n]\, x_2[n] \longrightarrow (1)$$

* On the other hand, the linear combination of two output is,

$$y_3'[n] = a_1 y_1[n] + a_2 y_2[n]$$

$$= a_1 n\, x_1^2[n] + a_2 x_2^2[n] \longrightarrow (2)$$

* From equation (1) & (2) are not equal, superposition principle is not satisfied. So, the system is Non-Linear.

(iii) Given: $y[n] = a^{x[n]}$.

* For two input sequences $x_1[n]$ and $x_2[n]$ the corresponding outputs are,

$$y_1[n] = T[x_1[n]] = a^{x_1[n]}$$

$$y_2[n] = T[x_2[n]] = a^{x_2[n]}$$

* The output due to weighted sum of input is,

$$y_3[n] = T\left[a_1 x_1[n] + a_2 x_2[n]\right] = a^{a_1 x_1[n] + a_2 x_2[n]} \longrightarrow (1).$$

* On the otherhand, the Linear combination of two output is,

$$y_3'[n] = a_1 a^{x_1[n]} + a_2 a^{x_2[n]} \longrightarrow (2)$$

* From equation (1) & (2) are not equal, superposition principle is not satisfied. So the system is Non-Linear.

3. Determine the impulse response h (n) for the system described by the second order difference equation, y (n) – 4 y (n-1) + 4y (n-2) = x (n-1).

Sol:- Given: $y[n] - 4y[n-1] + 4y[n-2] = x[n-1] \longrightarrow (1)$

Let

$$Z[y[n]] = Y(z)$$

$$Z[y[n-1]] = z^{-1}Y(z)$$

$$Z\{y[n-2]\} = z^{-2}Y(z)$$

$$Z\{x[n-1]\} = z^{-1}X(z).$$

Now Take Z-Transform on equation (1), we have

$$Z\{y[n] - 4y[n-1] + 4y[n-2]\} = Z\{x[n-1]\}$$

By Linearity Property,

$$Z\{y[n]\} - 4Z\{y[n-1]\} + 4Z\{y[n-2]\} = Z\{x[n-1]\}$$

$$Y(z) - 4z^{-1}Y(z) + 4z^{-2}Y(z) = z^{-1}X(z)$$

$$Y(z)[1 - 4z^{-1} + 4z^{-2}] = z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - 4z^{-1} + 4z^{-2}}$$

$$H(z) = \frac{z^{-1} \times z^2}{(1 - 4z^{-1} + 4z^{-2}) \times z^2} = \frac{z}{z^2 - 4z + 4}$$

$$\frac{H(z)}{z} = \frac{1}{z^2 - 4z + 4} = \frac{1}{(z-2)^2}$$

$$H(z) = \frac{z}{(z-2)^2}$$

$$\boxed{\text{W.K.T } z^{-1}\left(\frac{z}{(z-a)^2}\right) = n a^{n-1} u(n)}$$

Take Inverse Z-Transform,

$$h(n) = n(2)^{n-1} u(n)$$

---

4. The impulse response of a LTI system is given by h (n) = 0.6 ^{n} u (n). Find the frequency response.

Given:- $h(n) = (0.6)^n u[n]$.

Sol:-

Take Z-Transform on both sides,

$$Z[h[n]] = Z\{(0.6)^n u[n]\}$$

$$H(z) = \frac{z}{z - 0.6}$$

Put $z = e^{-j\omega}$

$$H(\omega) = \frac{e^{-j\omega}}{e^{-j\omega} - 0.6}$$

$$H(\omega) = \frac{e^{-j\omega}}{e^{-j\omega} - 0.6} \times \frac{e^{j\omega}}{e^{j\omega}}$$

$$\therefore H(\omega) = \frac{1}{1 - 0.6e^{j\omega}}$$

5. Test the following systems for time invariance.

(i) y (n) = x(n) – x (n-1)

(ii) y (n) = x(n)

(iii) y (n) = x(-n)

(iv) y (n) = x(n) – b x (n-1)

Solution:

Sol:-

(i) Given : $y[n] = x[n] - x[n-1]$

* If the input is delayed by K units in time, we have

$$y[n,K] = T[x(n-K)] = x[n-K] - x[n-K-1] \rightarrow ①$$

* If we delay the output by K units in time, then

$$y[n-K] = x[n-K] - x[n-K-1] \rightarrow ②$$

$\therefore$ ① = ②

So the System is time invariant.

(ii) Given : $y[n] = x[n]$.

* If the input is delayed by 'k' units in time and we have

$$y[n,k] = T[x[n-k]] = x[n-k] \rightarrow ①$$

* If we delay the output by 'k' units in time, then

$$y[n-k] = x[n-k] \rightarrow ②.$$

Here, ① = ②

So, the system is time - invariant.

(iii) Given : $y[n] = x[-n]$

* If the input is delayed by 'k' units in time, we have,

$$y[n,k] = T[x[n-k]] = x[-n-k] \rightarrow ①$$

* If we delay the output by 'k' units in time then,

$$y[n-k] = x[-(n-k)] = x[-n+k] \rightarrow ②$$

Here ① ≠ ②

So, the system is time - variant.

(iv) Given : $y[n] = x[n] - bx[n-1]$.

* If the input is delayed by 'k' units in time, we have

$$y[n,k] = T[x(n-k)] = x[n-k] - bx[n-k-1] \rightarrow ①$$

* If we delay the output by 'k' units in time,

$$y[n-k] = x[n-k] - bx[n-k-1] \rightarrow ②$$

Here ① = ②,

So, the system is time - invariant.

6. Determine the step response of a LTI system whose impulse response h (n) is given by

h (n) = a^{-n} u (-n) ; o < a < 1.

<u>Solution:</u>

Given: $x[n] = u[n]$

$h[n] = a^{-n} u(-n)$.

Take z-Transform.

$$X(z) = z[x(n)] = \sum_{n=0}^{\infty} 1 z^{-n} = \frac{z}{z-1}$$

$$H(z) = z[h(n)] = \sum_{n=-\infty}^{0} a^{-n} u(-n)$$

$$= \sum_{n=0}^{\infty} a^{n} u(n)$$

$$= \frac{z}{z-a}.$$

$$Y(z) = H(z) \times X(z)$$

$$Y(z) = \frac{z}{z-1} \times \frac{z}{z-a} = \frac{z^2}{(z-1)(z-a)}$$

$$Y(z) = \frac{z}{(z-1)(z-a)}$$

By partial Fraction,

$$\frac{z}{(z-1)(z-a)} = \frac{A}{z-1} + \frac{B}{z-a} \Rightarrow \frac{z}{\cancel{(z-1)(z-a)}} = \frac{A(z-a)+B(z-1)}{\cancel{(z-a)(z-1)}}$$

$$z = A(z-a) + B(z-1)$$

Put $z = a$ $z = 1$

$a = B(a-1)$ $1 = A(1-a)$

$B = \frac{a}{a-1}$ $A = \frac{1}{1-a}$.

$$Y(z) = \frac{\frac{1}{1-a} z}{z-1} + \frac{\frac{a}{a-1} z}{z-a}$$

Taking Inverse z-Transform,

$$y(n) = \frac{1}{1-a} (1)^n u(n) + \frac{a}{a-1} (a)^n u(n).$$

7. Determine the steady state response for the system with impulse function.

h (n) = (j/2)^{n} u (n) for an input x (n) = [Cos πn] u (n)

Given: $h[n] = (j/2)^n u(n)$

$x[n] = (\cos \pi n)\, u[n] = (-1)^n u[n]$

Take z-Transform ~~of~~ on above expressions,

$$H(z) = \frac{z}{z - j/2}$$

$$X(z) = \frac{z}{z+1}.$$

$$Y(z) = X(z) \cdot H(z) = \frac{z}{z - j/2} \cdot \frac{z}{z+1}.$$

$$\frac{Y(z)}{z} = \frac{z}{(z+1)(z - j/2)}$$

By partial Fraction,

$$\frac{z}{(z+1)(z-j/2)} = \frac{A}{z+1} + \frac{B}{z - j/2}$$

$$\frac{z}{(z+1)\cancel{(z-j/2)}} = \frac{A(z - j/2) + B(z+1)}{(z+1)\cancel{(z-j/2)}}$$

$$z = A(z - j/2) + B(z+1)$$

Sub $z = j/2$

$$j/2 = 0 + (j/2 + 1) B \Rightarrow j/2 = \left(\frac{j+2}{2}\right) B$$

$$B = \frac{j}{j+2} = \frac{j(j-2)}{j^2 - 2^2} = \frac{j^2 - 2j}{-1-4} = \frac{-1-2j}{-5}$$

Sub $z = -1$

$$-1 = A(-1 - j/2) + 0.$$

$$-1 = \left(\frac{-2-j}{2}\right)A$$

$$\frac{-2}{-2-j} = A. \Rightarrow A = \frac{2}{2+j} = \frac{2}{2+j} \times \frac{2-j}{2-j}$$

$$A = \frac{4-2j}{4-j^2} = \frac{4-2j}{4+1} = \frac{4-2j}{5}.$$

Now,

$$\frac{Y(z)}{z} = \frac{\frac{4-2j}{5}}{z+1} + \frac{\frac{1+2j}{5}}{z-j/2}$$

$$Y(z) = \frac{4-2j}{5} \cdot \frac{z}{z+1} + \frac{1+2j}{5} \frac{z}{z-j/2}$$

Take Inverse z-Transform,

$$y(n) = \left[\frac{4-2j}{5}(-1)^n + \frac{1+2j}{5}\left(\frac{j}{2}\right)^n\right]u[n].$$

8. Determine which of the following signals are periodic and determine the fundamental period also.

$(i) x(t) = 20\sin 25\pi t$ $\quad$ $(ii) x(t) = 20\sin\sqrt{5}\, t$

$(iii) x(t) = 10\cos 10\pi t$ $\quad$ $(iv) x(t) = 3\cos(5t + \frac{\pi}{6})$

Given:

(i) $x(t) = 20\sin 25\pi t$.

Here $\omega_0 = 25\pi$, The fundamental frequency is multiple of π. Therefore, the signal is periodic.

$$\text{Fundamental Period } (N) = 2\pi\left(\frac{m}{\omega_0}\right)$$

$$= 2\pi\left(\frac{m}{25\pi}\right)$$

The Minimum value of 'm' for which N is integer is 25.

$$N = 2\cancel{\pi}\left(\frac{25}{25\cancel{\pi}}\right) = 2.$$

Therefore the fundamental period = 2.

(ii) $x(t) = 20\sin\sqrt{5}\, t$.

Here $\omega_0 = \sqrt{5}$, which is not a multiple of 'π'. Therefore, the signal is aperiodic.

(iii) $x(t) = 10\cos 10\pi t$.

Here $\omega_0 = 10\pi$, which is multiple of 'π'. Therefore, the signal is periodic.

Fundamental period $(N) = 2\pi\,(m/\omega_0)$

$= 2\pi\,(m/10\pi)$

The minimum value of 'm' for which 'N' is integer is 5.

$N = 2\pi\left(\frac{5}{10\pi}\right) = 1$.

Therefore the fundamental period $= 1$.

(iv) $x(t) = 3\cos(5t + \pi/6)$.

Here $\omega_0 = 5$, which is not a multiple of 'π'. Therefore, the signal is aperiodic.

9.Determine the even and odd parts of the following:

$(i)x(n) = A\sin\omega n + b\sin\omega n$

$(ii)x(n) = 3\cos\omega n + 5$

Sol:-

$$X_e(n) = \frac{x(n) + x(-n)}{2} \quad \& \quad x_o(n) = \frac{x(n) - x(-n)}{2}$$

(i) $x(n) = A\sin\omega n + B\sin\omega n$

$$X_e(n) = \frac{A\sin\cancel{\omega n} + B\sin\cancel{\omega n} - A\cancel{\sin}\omega n - B\cancel{\sin}\omega n}{2} = 0.$$

$$x_o(n) = \frac{A\sin\omega n + B\sin\omega n + A\sin\omega n + B\sin\omega n}{2}$$

$$X_o(n) = A\sin\omega n + B\sin\omega n = x(n)$$

(ii) $x(n) = 3\cos\omega n + 5$

$$x_e(n) = \frac{3\cos\omega n + 5 + 3\cos\omega n + 5}{2} = 3\cos\omega n + 5.$$

$$x_o(n) = \frac{3\cos\cancel{\omega}n + \cancel{5} - 3\cancel{\cos}\omega n - \cancel{5}}{2} = 0.$$

10. Discuss whether the following are energy or power signals.(Nov/Dec'11)

(i) $x(n) = \left(\frac{3}{2}\right)^n u(n)$

(ii) $x(n) = Ae^{jwon}$

Given: $x[n] = (3/2)^n u[n]$.

The Energy of the signal, $E = \sum_{n=-\infty}^{\infty} |x[n]|^2$

$$= \sum_{n=0}^{\infty} |(3/2)^n|^2$$

$$= \sum_{n=0}^{\infty} |(3/2)^2|^n = \sum_{n=0}^{\infty} (9/4)^n .$$

$$= \frac{1}{1 - 9/4} = \frac{1}{\frac{4-9}{4}}$$

W.K.T $\left[\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}\right]$

$\sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}$

$$\boxed{E = \frac{-4}{5}}.$$

The power of the signal, $P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} (1/9)^n$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \frac{1-(1/9)^{N+1}}{1-\frac{1}{9}}$$

$$\boxed{P = 0}$$

∴ The Energy is finite and power is zero.

Hence the signal is Energy signal.

(ii) Given : $x[n] = Ae^{j\omega_0 n}$.

$$E = \sum_{n=-\infty}^{\infty} |Ae^{j\omega_0 n}|^2$$

$$= A^2 \sum_{n=-\infty}^{\infty} |e^{2j\omega_0 n}|$$

$$= A^2 \sum_{n=-\infty}^{\infty} [1] \quad , \text{ where } \omega_0 \text{ is a multiple of } \pi$$

$$= \infty .$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} A^2 |e^{2j\omega_0 n}|.$$

$$= A^2 \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$

$|e^{2j\omega_0 n}| = 1$ where, ω_0 is a multiple of 'π'

$$= A^2 \underset{N\to\infty}{Lt} \frac{2N+1}{2N+1} = A^2 .$$

The Energy is infinite and the power is finite. Therefore, the signal is a Power signal.

11. Explain the concept of quantization. (Nov/Dec '11)

* The process of converting a discrete time continuous signal $x(n)$ into a discrete time discrete amplitude signal $x_q(n)$ is known as quantization.

* This is done by rounding off each sample in $x(n)$ to nearest quantization level.

* Then each sample in $x_q(n)$ is represented by a finite number of digits using a order.

* If a signal with amplitude range 'R' is represented by an b+1 bit word (including sign bit) then the number of values, or quantization levels, that can be represented is 2^{b+1}.

* The difference b/w adjacent levels, or the quantization step interm of the range of the signal is,

$$q = \frac{\text{Range of signal}}{\text{Number of quantization Level}} = \frac{R}{2^{b+1}}.$$

* with fixed point Representation of Fractional number, if the range of signal exceeds ± 1, it is necessary to scale the signal.

* The process of quantization is shown given below.

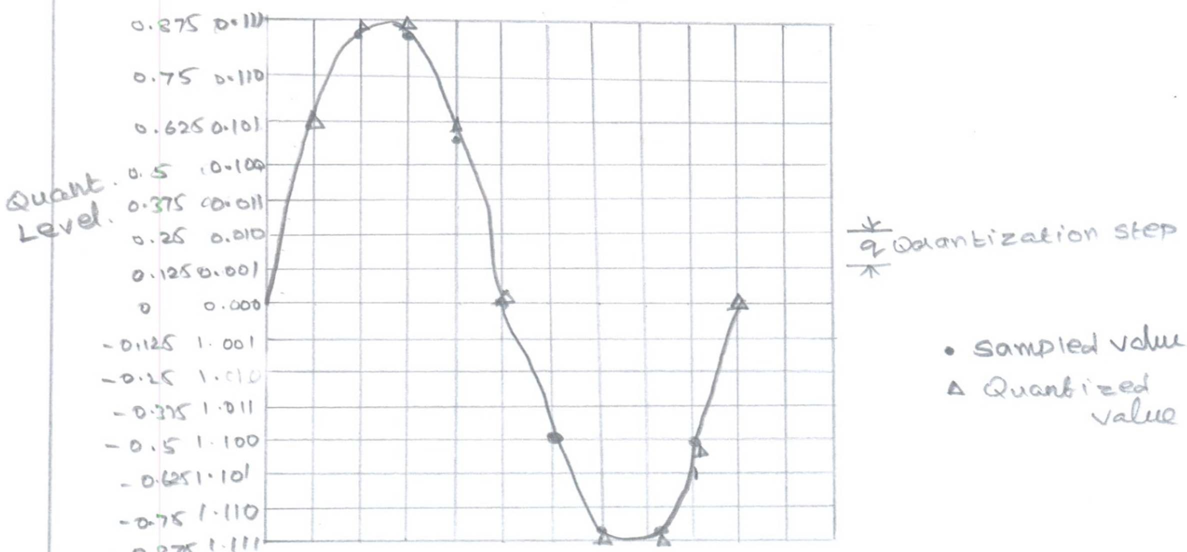


Fig: Illustration of quantization.

* The time axis of the discrete time signal is labelled with sample number ($n = 0, 1, 2, \ldots$).

* Corresponding to different values of sample number 'n', the discrete time continuous amplitude signal shown in above Fig.

* We can represent the sample values by a sequence,

$$x[n] = \{0, 0.620, 0.85, 0.85, 0.575, -0.03, -0.625, -0.85, -0.85, -0.575, 0\}$$

* Let a b+1 bit ADC is used to represent the above sequence. With b+1 binary digits, 2^{b+1} quantization level can be obtained and the input can be resolved to one part in 2^{b+1}.

* If the input signal has a range of 2V, then the quantization step size is equal to,

$$q = \frac{2}{2^{b+1}} = 2^{-b}.$$

* If b+1 is equal to 4, the quantization step size is equal to 0.125. Thus the input signal must change almost 0.125 inorder to produce a change in output.

* The process of converting $x[n]$ to finite number of digits introduces an error known as quantization Noise.

* It is a sequence $e(n)$ defined as the difference between the quantized value and the actual sample value. Thus,

$$e(n) = x_q[n] - x[n].$$

Table : Illustration of Quantization Using Rounding.

n	Sampled value $x[n]$	Binary Representation	Rounding	Quatized Value $x_q[n]$	Quantization Noise $e(n) = x_q[n] - x[n]$.
0	0	0.00000000	0.000	0	0
1	0.620	0.10011110	0.101	0.625	0.005
2	0.85	0.1101100I	0.111	0.875	0.025
3	0.85	0.1101100I	0.111	0.875	0.025
4	0.575	0.10010011	0.101	0.625	0.05
5	-0.03	0.00000111	1.00	0	0.03
6	-0.625	1.10100000	1.101	-0.625	0
7	-0.85	1.11011001	1.111	-0.875	-0.025
8	-0.85	1.11011001	1.111	-0.875	-0.025
9	-0.575	1.10010011	1.101	-0.625	-0.05
10	0	0.00000000	0.000	0	0

12. Check whether following are linear, time invariant, causal and stable.
(i)y(n) = x(n)+nx(n+1)
(ii) y(n) = cos x(n)
(iii) y(n)= x(-n=5) (Nov/Dec '11) (May/June'12)

Given:- $y[n] = x[n] + n\,x[n-1]$

* For two input sequences $x_1[n]$ and $x_2[n]$ the Corresponding outputs are,

$$y_1[n] = T[x_1[n]] = x_1[n] + n\,x_1[n-1]$$

$$y_2[n] = T(x_2[n]) = x_2[n] + n\,x_2[n-1]$$

The output due to weighted sum of input is

$$y_3[n] = T\left[a_1 x_1[n] + a_2 x_2[n]\right]$$

$$= a_1 x_1[n] + a_2 x_2[n] + n\left[a_1 x_1[n-1] + a_2 x_2[n-1]\right] \rightarrow ①$$

On the otherhand, the linear combination of two outputs is,

$$y_3'[n] = a_1 y_1[n] + a_2 y_2[n] = a_1 x_1[n] + a_1 n x_1[n-1] + a_2 x_2[n] + a_2 n x_2[n-1] \rightarrow ②$$

Here, ① = ②, superposition principle is satisfied.

The System is linear.

* If the input is delayed by 'k' units in time, we have,

$$y[n,k] = x[n-k] + n\, x[n-k-1] \rightarrow ①$$

If we delay the output by 'k' units in time, then

$$y[n-k] = x[n-k] + (n-k)\, x[n-k-1] \rightarrow ②$$

$① \neq ②$

So, the system is time-variant.

* put $n=-1$; $y[-1] = x[-1] + (-1)\, x[-2]$

$n=0$; $y[0] = x[0] + (0)\, x[-1]$

$n=1$; $y[1] = x[1] + (1)\, x[0]$

For all the values of 'n', the output depends on present and post inputs, so the system is Causal.

Given: $y[n] = x[-n+5]$

* For two input sequences $x_1[n]$ and $x_2[n]$ the Corresponding outputs are,

$$y_1[n] = x_1[-n+5]$$

$$y_2[n] = x_2[-n+5].$$

The output due to weigthed sum of input is,

$$y_3[n] = T[a_1 x_1[n] + a_2 x_2[n]]$$

$$= a_1 x_1[-n+5] + a_2 x_2[-n+5] \rightarrow ①$$

On the otherhand, the Linear Combination of two output is

$$y_3'[n] = a_1 y_1[n] + a_2 y_2[n]$$

$$= a_1 x_1[-n+5] + a_2 x_2[-n+5] \rightarrow ②$$

Here, $① = ②$, Superposition principle is satisfied. The system is linear.

* If the input is delayed by 'k' units in time, we have

$$y[n,k] = x[-n-k+5] \rightarrow ①$$

If we delay the output by k' units in time, then

$$y[n-k] = x[-(n-k)+5] \rightarrow ②$$

Here ① ≠ ②, so the system is time-variant.

* put $n=-1$; $y[-1] = x[6]$

$n=0$; $y[0] = x[5]$

$n=1$; $y[1] = x[4]$

For all the value's of 'n', the output depends Future and past inputs, so the system is Non-Causal.

*

13. What is causality and stability of a system? Derive the necessary and sufficient condition on the impulse response of the system for causality and stability. (Nov/Dec 12)

*Causality :- Causal system is one whose output depends on past or/and present values of input.

Derivation for causality :-

Using convolution sum, we have,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \longrightarrow ①$$

$$= \sum_{k=-\infty}^{-1} h[k]\, x[n-k] + \sum_{k=0}^{\infty} h[k]\, x[n-k].$$

$$= \cdots + \underbrace{h[-2]\, x[n+2] + h[-1]\, x[n+1]}_{\text{depends on future inputs}} + \underbrace{h[0]\, x[n]}_{\text{present input}}$$

$$+ \underbrace{h[1]\, x[n-1]}_{\text{Past inputs}} + \cdots \longrightarrow ②$$

* From equ ①, we find that the output depends on the past an present values of the input if the index $k \geq 0$. If $k<0$ then the output depends on the future values of input.

* Therefore for a causal system whose output does not depends on the future values of the input, the limits on the summation changes as

$$y[n] = \sum_{k=0}^{\infty} h[k]\, x[n-k] \longrightarrow ③$$

* From equ ③, we find that for causal system $h[k]$ should be zero for $k<0$. That is,

$$h[k] = 0, \quad k<0$$

* An LTI system is Causal if and only its impulse response is zero for negative value of 'n'.

* The Limits in the convolution sum can be modified according to the type of sequence and system.

* For the causal system, the impulse response $h[n]=0$ for $n<0$. Therefor the limit of convolution sum is modified as,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=0}^{\infty} h[k]\, x[n-k]$$

* If the input to the causal system is a causal sequence (ie. $x[n]=0$ for $n<0$) the limit in the convolution sum is modified as,

$$y[n] = \sum_{k=0}^{n} x[k]\, h[n-k] = \sum_{k=0}^{\infty} h[k]\, x[n-k].$$

Stability: An LTI System is stable if it produces a bounded output sequence for every bounded input sequence.

Derivation for Stability :-

If, for some bounded input sequence $x[n]$, the output is unbounded (infinite), the system is defined as unstable.

* Let $x[n]$ be a bounded input sequence, $h[n]$ be the impulse response of the system and $y[n]$ be the output sequence. Taking the magnitude of the output, we have

$$|y[n]| = \left| \sum_{k=-\infty}^{k=\infty} h[k]\, x[n-k] \right|$$

* We know that the magnitude of the sum of terms is less than or equal to sum of the magnitudes, Hence,

$$|y[n]| \le \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]|$$

* Let the bounded value of the input is equal to 'M', above equation can be written as,

$$|y[n]| \le M \sum_{k=-\infty}^{\infty} h[k]$$

* The above equation (condition) will be satisfied when,

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

So, the necessary and sufficient condition for stability is,

$$\sum_{n=-\infty}^{n=\infty} |h[n]| < \infty .$$

14. What is meant by energy and power signal? Determine whether the following signals are energy or power or neither energy nor power signals.

$x_1(n) = \left(\frac{1}{2}\right)^n u(n)$ $x_2(n) = \sin\left(\frac{\pi}{6}n\right)$, $x_3(n) = e^{j\left(\frac{\pi n}{3}+\frac{\pi}{6}\right)}$, $x_4(n) = e^{2n}u(n)$ (Nov/Dec 12)

Sol:-

[i] Given: $x_1[n] = (1/2)^n u[n]$.

The energy of the signal,

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= \sum_{n=0}^{\infty} \left|\left(1/2\right)^n\right|^2 \qquad \therefore u[n] = \begin{cases} 1, & \text{for } n \ge 0 \\ 0, & \text{for } n < 0 \end{cases}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \qquad \left(1 + a + a^2 + \cdots + \infty = \frac{1}{1-a}\right)$$

$$= \frac{1}{1 - 1/4} = \frac{1}{\frac{4-1}{4}} = 4/3 \quad .(\text{Finite})$$

The power of the signal,

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left|\left(\frac{1}{2}\right)^n\right|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{4}\right)^n \qquad \left[\cancel{1 + \tfrac{1}{2} + \cdots (\tfrac{1}{2})^N = \frac{1-(\tfrac{1}{2})^{N+1}}{1-\tfrac{1}{2}}}\right]$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \; \frac{1 - (1/4)^{N+1}}{1 - 1/4} \qquad 1 + a + \cdots + a^N = \frac{1-a^{N+1}}{1-a}$$

$$= 0 .$$

$\therefore$ The Energy is finite and power is zero. Therefore, the signal is energy signal.

(ii) Given: $x_2[n] = \sin(\frac{\pi}{6}n)$.

$$E = \sum_{n=-\infty}^{\infty} \left|\sin^2(\pi/6\, n)\right| = \sum_{n=-\infty}^{\infty} \left(\frac{1-\cos \pi/3\, n}{2}\right) = \infty.$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|\sin^2(\pi/6\, n)\right|$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1-\cos \pi/3\, n}{2} = \frac{1}{2} \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$

$$= 1/2. \qquad \left[\sum_{n=-N}^{N} 1 = 2N+1\right]$$

$\therefore$ The Energy is infinite and the power is finite. Therefore the signal is power signal.

(iii) Given: $x_3[n] = e^{j\left(\frac{n\pi}{3} + \pi/6\right)}$.

$$E = \sum_{n=-\infty}^{\infty} \left|e^{j(\pi/3 n + \pi/6)}\right|^2 = \sum_{n=-\infty}^{\infty} 1 \qquad \left[\because e^{j(\omega+\theta)} = 1\right]$$

$$= \infty.$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|e^{j(\pi/3 n + \pi/4)}\right|^2$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} 1 = \underset{N\to\infty}{Lt} \frac{1}{2N+1}\, 2N+1 = 1.$$

$\therefore$ The energy is infinite and power is finite. Therefore, the signal is power signal.

(iv) Given: $x_4[n] = e^{2n} u(n)$

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=0}^{\infty} (e^{2n})^2 = \sum_{n=0}^{\infty} (e^4)^n = 1 + e^4 + e^8 + \dots + \infty$$
$$= \infty.$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x_4[n]|^2 = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} e^{4n} = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \left(\frac{e^{4(N+1)}-1}{e^4-1}\right)$$
$$= \infty.$$

The signal is neither power nor energy signal.

Definition of Power singnal and Energy signal

For a discrete time signal $x[n]$, the energy 'E' is defined as

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

The average power of a discrete-time signal $x[n]$ is defined as,

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

* A signal is an energy signal, if and only if the total energy of the signal is finite and power of the signal is zero. [E = finite & $P=0$]

* Similarly the signal is said to be power signal if the average power of the signal is finite and Energy is infinite. [P = finite & $E=\infty$]

* The signal that do not satisfy above properties are neither energy nor power signal.

15. A discrete time systems can be (i) Static or Dynamic, (ii) Linear or Non-Linear, (iii) Time invariant or time varying & (iv) Stable or Unstable. Examine the Following system with respect to the properties above y(n)= x(n)+ nx(n+1) (Nov/Dec 13)

Given: $y[n] = x[n] + n\,x[n+1]$

(i) ~~Linear or~~ Static (or) Dynamic

put $n=-1$; $y[-1] = x[-1] - 1\,x[0]$

$n=0$; $y[0] = x[0] + 0$

$n=1$; $y[1] = x[1] + x[2]$

The output $y[n]$ depends on present and past input. Therefore the system is dynamic or to have memory.

(ii) Linear (or) Non-Linear:

* For two input sequences $x_1[n]$ and $x_2[n]$ the corresponding output are,

$$y_1[n] = T[x_1[n]] = x_1[n] + n\,x_1[n+1]$$

$$y_2[n] = T[x_2[n]] = x_2[n] + n\,x_2[n+1]$$

The output due to weighted sum of input is,

$$y_3[n] = T[a_1 x_1(n) + a_2 x_2[n]]$$

$$= a_1 x_1[n] + a_2 x_2[n] + a_1 n x_1[n+1] + a_2 n x_2[n+1] \rightarrow (1)$$

On the other hand, the linear combination of two output is,

$$y_3'[n] = a_1 y_1[n] + a_2 y_2[n]$$

$$= a_1 x_1[n] + a_1 x_1[n+1] + a_2 x_2[n] + a_2 x_2[n+1] \rightarrow (2)$$

Here, (1) = (2) and the superposition principle is satisfied. So the system is Linear.

(iii) Time invariant (or) Time varying

If the input is delayed by 'k' units in time, we have

$$y[n,k] = T[x[n-k]] = x[n-k] + n\,x[n-k+1] \rightarrow (1)$$

If we delay the output by 'k' units in time, then

$$y[n-k] = x[n-k] + (n-k)\,x[n-k+1] \rightarrow (2)$$

Here, (1) ≠ (2), Therefore the system is time variant.

16. Given y[n]= x[n^2]. Determine whether the system is linear, time invariant, memoryless and causal. (May/Jun 13)

Sol:- Given: $y[n] = x[n^2]$

* For $n=-1$; $y[-1] = x[1]$

$n=0$; $y[0] = x[0]$

$n=1$; $y[1] = x[1]$

For all values of 'n', (except n=0 and n=1), the system depends on future inputs. So, the system is Non-Causal.

* For two input sequence $x_1[n]$ and $x_2[n]$ the corresponding outputs are,

$$y_1[n] = T[x_1[n]] = x_1[n^2]$$

$$y_2[n] = T[x_2[n]] = x_2[n^2]$$

The output due to weighted sum of input is,

$$y_3[n] = T[a_1 x_1[n] + a_2 x_2[n]] = a_1 x_1[n^2] + a_2 x_2[n^2] \rightarrow (1)$$

on the otherhand, the linear combination of the two output is, $y_3'[n] = a_1 y_1[n] + a_2 y_2[n]$

$$y_3'[n] = a_1 x_1[n^2] + a_2 x_2[n^2] \longrightarrow (2)$$

Hence, (1) = (2), the superposition principle is satisfied. so, the system is ~~Non~~ linear.

* If the input is delayed by 'k' units in time, we have

$$y[n,k] = T[x(n-k)] = x[(n-k)^2] \longrightarrow (1)$$

If we delay the output by 'k' units in time, then,

$$y[n-k] = x[(n-k)^2] \longrightarrow (2)$$

Here, (1) = (2), Therefor the system is time-variant.

* The output $y[n]$ depends on future inputs (except $n=0$ & $n=1$), so the system is Dynamic or to have memory.

17. Determine whether the following is an energy signal or power signal.

(1) $x_1[n] = 6\cos\left(\frac{\pi}{2}n\right)$

(2) $x_2[n] = 3(0.5)^n\ u(n)$. **(May/Jun 13)**

Sol:-

Given:- (1) $x_1[n] = 6\cos(\pi/2\, n)$.

$$E = \sum_{n=-\infty}^{\infty} |x_1[n]|^2 = \sum_{n=-\infty}^{\infty} 36\cos^2 \frac{\pi}{2} n \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= 36 \sum_{n=-\infty}^{\infty} \left(\frac{1+\cos \pi n}{2}\right) = \infty.$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 36 \cos^2 \frac{\pi}{2} n.$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 36 \left(\frac{1+\cos\pi n}{2}\right) = 1.$$

The Energy is infinite and Power is finite. Therefore the signal is a power signal.

Given: $x_2[n] = 3(0.5)^n\, u(n)$

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=0}^{\infty} 3^2 (0.5^2)^n = 3 \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^{2n}.$$

$$= 3 \sum_{n=0}^{\infty} \left(\tfrac{1}{4}\right)^n = 3^2 \frac{1}{1-\frac{1}{4}} = 3^2 \cdot \frac{1}{\frac{4-1}{4}} = 3^2 \times \frac{4}{3}$$

$$= \frac{4}{3}.$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} 9 \cdot \left(\frac{1}{2}\right)^{2n}$$

$$= 9 \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{4}\right)^n = 9 \lim_{N\to\infty} \frac{1}{2N+1} \frac{1-\left(\frac{1}{4}\right)^{N+1}}{1-\frac{1}{4}}$$

$$= 0$$

$\therefore$ The energy is finite and power is zero. Therefore, the signal is an energy signal.

UNIT-2

DISCRETE TIME SYSTEM ANALYSIS

Z-TRANSFORM.

The Z-transform of $x(n)$ will convert the time domain signal $x(n)$ into z-domain signal $X(z)$, where the signal becomes a function of complex variable z.

The complex variable z is defined as,

$$z = u + jv = re^{j\omega}$$

where $u \rightarrow$ real part of z

$v \rightarrow$ imaginary part of z.

$$r = \sqrt{u^2+v^2} = \text{magnitude of } z.$$

$$\omega = \tan^{-1}\left(\frac{v}{u}\right) = \text{Phase (or) Argument of } z.$$

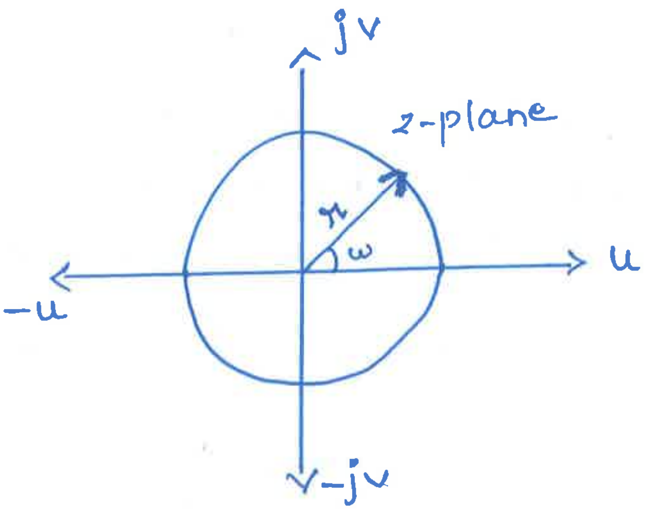


fig 1: z-plane

2) Properties - (2)

3) Inverse - 1 problem unsolved

4) Stability + ROC

Definition of z-transform :-

Let $x(n)$ = Discrete time signal

$X(z)$ = Z-transform of $x(n)$.

The Z-transform of a discrete time signal, $x(n)$ is defined as,

$$X(z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

(Two sided Z-transform)

$$X(z) = Z[x(n)] = \sum_{n=0}^{\infty} x(n) z^{-n} \quad \text{(One sided z-transfm)}$$

Region of convergence (ROC) :-

The computation of $X(z)$ involves summation of infinite terms which are functions of z. Hence it is possible that the infinite series may not converge to finite values of z. Therefore, for every $X(z)$, there will be a set of values of z for which $X(z)$ can be computed. Such set of values will lie in a particular region of z-plane called as Region of Convergence (ROC).

Inverse z-transform:- It is defined as,

$$\boxed{x(n) = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz}$$

Region of Convergence:- The ROC for following six types of signals are given below.

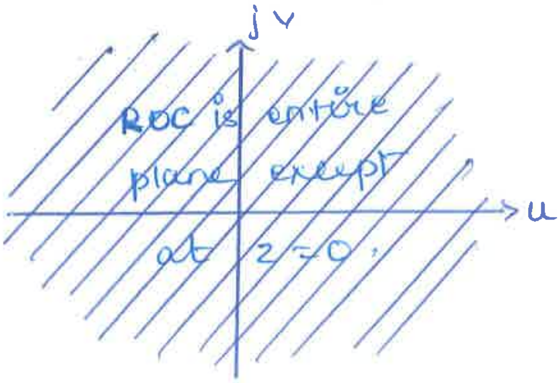
Case (i) Finite duration, right sided (causal signal):-

→ $x(n)$ ranges from 0 to $N-1$

$$X(z) = \sum_{n=0}^{N-1} x(n)\, z^{-n}$$

$$= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \cdots \frac{x(N-1)}{z^{N-1}}$$

In the above summation, when $z=0$, all terms become ∞ except $x(0)$. So $X(z)$ exists for all values of z except at $z=0$. ∴ ROC is entire z-plane except at $z=0$.



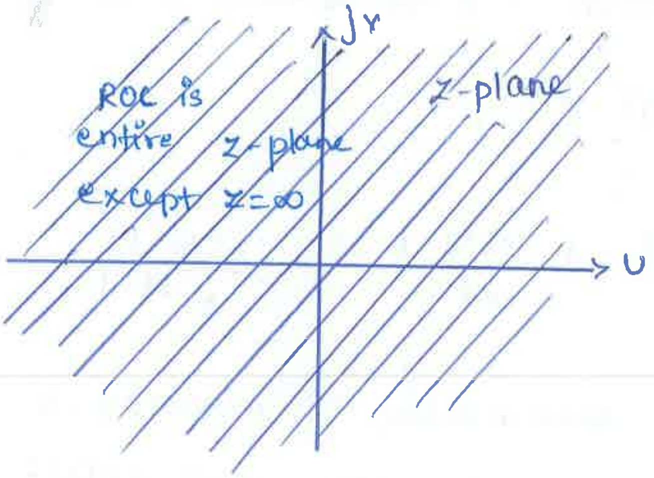
Case(ii) - Finite duration, left-sided (anticausal) s/g :-

→ x(n) ranges from $-(N-1)$ to 0.

$$X(z) = \sum_{n=-(N-1)}^{0} x(n) z^{-n}$$

$$= x(-(N-1)) z^{N-1} + \cdots\; x(-2) z^{2} + x(-1) z + x(0).$$

In above summation if $z=\infty$, all terms becomes ∞ except $x(0)$. So ROC ~~exa~~ of $X(z)$ is entire z-plane except $z=\infty$.



Case-(iii) Finite duration, two sided (non-causal) s/g :-

→ x(n) ranges from $\frac{-(N-1)}{2}$ to $\frac{N-1}{2}$

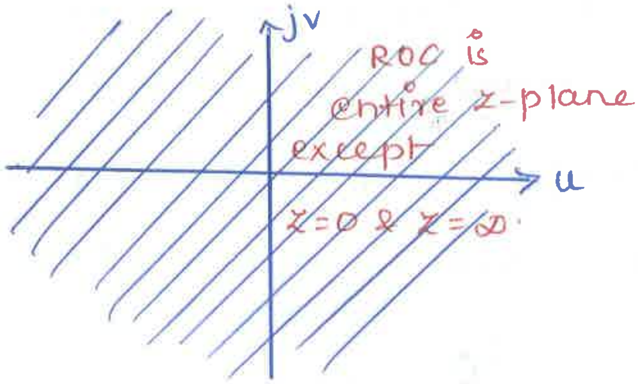
Let $\frac{N-1}{2} = M$

∴ x(n) ranges from $-M$ to M.

$$X(z) = \sum_{n=-M}^{M} x(n) z^{-n}$$

$$= x(-M)z^{M} + \dots x(-2)z^{2} + x(-1)z + x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \dots \frac{x(M)}{z^M}$$

In the above summation, ROC exists for all values of z except at $z=0$ and $z=\infty$.



Case(iv) Infinite Duration, right sided (causal) s/g:

Let $x(n) = a^n$; $n \geq 0$.

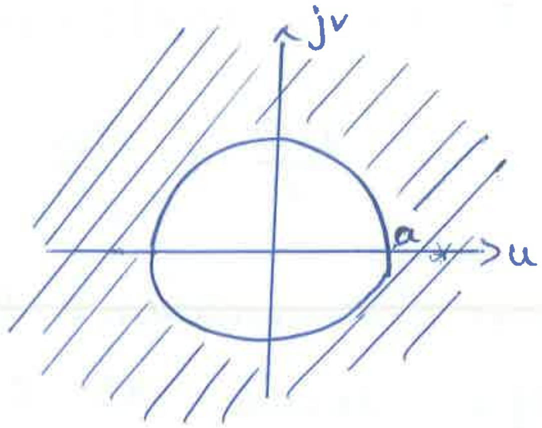
$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n.$$

If $0 < |az^{-1}| < 1$, then $\sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$

Hence the condition to be satisfied here for convergence of $X(z)$ is,

$$0 < |az^{-1}| < 1$$

$$\therefore |az^{-1}| < 1 \implies \left|\frac{a}{z}\right| < 1 \implies |z| > |a|$$



The term $|a|$ represents a circle of radius a in z-plane. From the above analysis, we say $X(z)$ converges for all points external to the circle of radius a in z-plane.

$\therefore$ ROC of $X(z)$ is exterior of the circle of radius 'a' in z-plane.

Case (v) : Infinite duration, left sided (anticausal) s/g

$$x(n) = a^n \; ; \; n \leq 0$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{0} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} a^{-n} z^{n} = \sum_{n=0}^{\infty} (a^{-1}z)^n$$

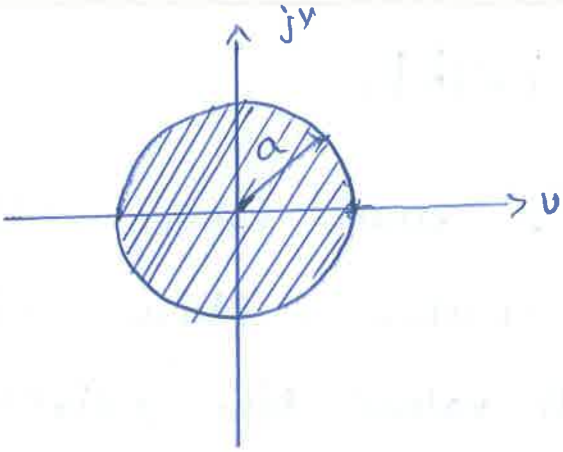
If $0 < |a^{-1}z| < 1$, then $\sum_{n=0}^{\infty} (a^{-1}z)^n = \frac{1}{1-a^{-1}z}$.

Here the condition to be satisfied for convergence is $0 < |a^{-1}z| < 1$.

$$|a^{-1}z| < 1 \implies \left|\frac{z}{a}\right| < 1$$

$$\boxed{|z| < |a|}$$

$\therefore$ ROC is interior of the circle of radius 'a'.



Case (vi) Infinite duration, two sided (non causal) s/g:-

Let $x(n) = a^n u(n) + b^n u(-n)$

$$X(z) = \sum_{n=-\infty}^{\infty} \left[a^n u(n) + b^n u(-n)\right] z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=-\infty}^{0} b^n z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=0}^{\infty} b^{-n} z^{n} \Rightarrow \sum_{n=0}^{\infty} (az^{-1})^n + \sum_{n=0}^{\infty} (b^{-1}z)^n$$

$$\boxed{X(z) = \frac{1}{1-az^{-1}} + \frac{1}{1-b^{-1}z}}$$

The term $\sum_{n=0}^{\infty} (az^{-1})^n$ converges if

$$0 < |az^{-1}| < 1 \implies \left|\frac{a}{z}\right| < 1 \implies |z| > |a|$$

The term $\sum_{n=0}^{\infty}(b^{-1}z)^n$ converges if,

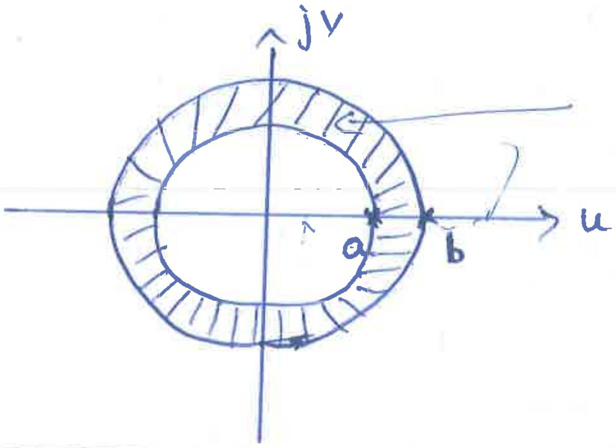
$$0 < |b^{-1}z| < 1 \Rightarrow$$

$$|z/b| < 1$$

$$|z| < |b|.$$

If $|b| > |a|$, then there will be a region b/w 2 circles. Now $X(z)$ will converge for all values b/w 2 circles. (ie)

$$\boxed{|a| < |z| < |b|}$$



ROC of
$x(n) = a^n u(n) + b^n u(-n)$
if $|b| > |a|$.

PROPERTIES OF ROC:-

1. The ROC is a ring (or) disk in z-plane centered at origin
2. It cannot contain any poles.
3. ROC must be a connected region.

4. If $x(n)$ is :-

(a) finite & causal → ROC is entire z-plane except at $z=0$.

(b) finite & anticausal → ROC is entire z-plane except at $z=\infty$.

(c) finite & non-causal (Two sided) → ROC is entire z-plane except at $z=0$ & $z=\infty$.

5. If $x(n)$ is :-

(a) infinite & causal → ROC is exterior of circle

(b) infinite + anticausal → ROC is interior of circle

(c) infinite + non-causal (two sided) → ROC is region b/w 2 circles

Properties of z- transform :-

1. Linearity :-

It states that z-transform of a weighted sum of 2 signals is equal to the weighted sum of individual z- transforms.

If $X_1(z) = Z[x_1(n)]$ and $X_2(z) = Z[x_2(n)]$,

then $Z[a x_1(n) + b x_2(n)] = a X_1(z) + b X_2(z)$.

Proof :-

$$Z[a x_1(n) + b x_2(n)] = \sum_{n=-\infty}^{\infty} [a x_1(n) + b x_2(n)] z^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} a x_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} b x_2(n) z^{-n}$$

$$= a \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + b \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}.$$

$$\boxed{Z[a x_1(n) + b x_2(n)] = a X_1(z) + b X_2(z)}$$

2. Time shifting :- If $X(z) = Z[x(n)]$ & initial condition for $x(n)$ are zero, then $Z[x(n-k)] = z^{-k} X(z)$.

Proof :- $Z[x(n-k)] = \sum_{n=-\infty}^{\infty} x(n-k) z^{-n}$.

Put $n-k = m$,

$$= \sum_{m=-\infty}^{\infty} x(m) z^{-(m+k)}.$$

$$= \sum_{m=-\infty}^{\infty} x(m) z^{-m} . z^{-k}.$$

$$= z^{-k} \left\{ \sum_{m=-\infty}^{\infty} x(m) z^{-m} \right\} = z^{-k} X(z).$$

$$\boxed{Z[x(n-k)] = z^{-k} X(z)}$$

3. **Time Reversal** :- If,

$$x(n) \xleftrightarrow{z} X(z) \quad ROC: r_1 < |z| < r_2$$

then, $$x(-n) \xleftrightarrow{z} X(z^{-1}) \quad ROC: \frac{1}{r_1} < |z| < \frac{1}{r_2}$$

Proof : Let

$$Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

Sub $-n = m$

$$Z[x(-n)] = \sum_{m=\infty}^{-\infty} x(m) z^{m}$$

$$= \sum_{m=-\infty}^{\infty} x(m) z^{-m} = \sum_{m=-\infty}^{\infty} x(m) (z^{-1})^{m}$$

$$\boxed{Z[x(-n)] = X(z^{-1})}$$

∴ According to time reversal property, folding the signal in time domain is equivalent to replacing z by z^{-1}. Further, ROC of $X(z)$ is $r_1 < |z| < r_2$ which becomes $r_1 < |z^{-1}| < r_2$ (or) $\frac{1}{r_1} |z| < \frac{1}{r_2}$.

4. **Multiplication by n**

If $Z[x(n)] = X(z)$, then

$$Z[n\, x(n)] = -z \frac{d}{dz} X(z)$$

Proof :- Let $$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z[n\,x(n)] = \sum_{n=-\infty}^{\infty} n\,x(n)\,z^{-n}$$

Mul right side by $z \cdot z^{-1}$,

$$Z[n\,x(n)] = z \sum_{n=-\infty}^{\infty} n\,x(n)\,z^{-n-1}$$

$$= z \sum_{n=-\infty}^{\infty} x(n)\left(n\,z^{-n-1}\right)$$

$$= z \sum_{n=-\infty}^{\infty} x(n)\left\{\frac{-d}{dz}\left(z^{-n}\right)\right\}$$

$$= -z\,\frac{d}{dz}\left[\sum_{n=-\infty}^{\infty} x(n)\,z^{-n}\right]$$

$$\boxed{Z[n\,x(n)] = -z\,\frac{d}{dz}X(z)}$$

(5) Multiplication by an Exponential

If $Z[x(n)] = X(z)$, then

$$Z[a^n x(n)] = X[a^{-1}z]$$

Proof:-

$$Z[a^n x(n)] = \sum_{n=-\infty}^{\infty} a^n x(n)\,z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\left(a^{-1}z\right)^{-n}$$

$$Z[a^n x(n)] = X[a^{-1}z] \;//$$

$\therefore$ ROC is $r_1 < |a^{-1}z| < r_2$

In general, In $X(z)$, z is replaced by $a^{-1}z$ (or) z/a.

Parseval's theorem / Parseval's relation :-

If $Z[x_1(n)] = X_1(z)$ and $Z[x_2(n)] = X_2(z)$,

Then Parseval's relation states that,

$$\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j} \oint_c X_1(z)\, X_2^*\left(\frac{1}{z^*}\right) z^{-1}\, dz$$

Proof :- By definition of Inverse Z-transform,

$$x_1(n) = \frac{1}{2\pi j} \oint_c X_1(z)\, z^{n-1}\, dz = \frac{1}{2\pi j} \oint_c X_1(v)\, (v)^{n-1}\, dv \rightarrow (1)$$

$\boxed{\text{Let } z = v.}$

Now, by definition of z-transform,

$$Z[x_2(n)] = \sum_{n=-\infty}^{\infty} x_2(n)\, z^{-n} \longrightarrow (2).$$

$$\therefore Z[x_1(n)\, x_2^*(n)] = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n)\, z^{-n} \longrightarrow (3).$$

Sub (1) in (3),

$$\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n)\, z^{-n} = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi j} \oint X_1(v)\, v^{n-1}\, dv\right] x_2^*(n)\, z^{-n}$$

Interchanging the order of summation and integration.

$$= \frac{1}{2\pi j} \oint X_1(v) \left[\sum_{n=-\infty}^{\infty} x_2^*(n)\, z^{-n}\, v^{n}\right] v^{-1}\, dv$$

$$= \frac{1}{2\pi j} \oint X_1(v) \left[\sum_{n=-\infty}^{\infty} x_2^*(n) \left(\frac{z^*}{v^*}\right)^{-n}\right] v^{-1}\, dv$$

$$= \frac{1}{2\pi j} \oint X_1(v) \left[\sum_{n=-\infty}^{\infty} x_2^*(n) \left(\frac{z^*}{v^*}\right)^{-n} \right] v^{-1}\, dv$$

$$= \frac{1}{2\pi j} \oint X_1(v)\; X_2^*\left(\frac{z^*}{v^*}\right) v^{-1}\, dv$$

Taking $\underset{z \to 1}{Lt}$ in above eqn,

$$\underset{z\to 1}{Lt} \sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n)\, z^{-n} = \underset{z\to 1}{Lt} \frac{1}{2\pi j} \oint_C X_1(v)\; X_2^*\left(\frac{z^*}{v^*}\right) v^{-1}\, dv$$

$$\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(v)\; X_2^*\left(\frac{1}{v^*}\right) v^{-1}\, dv$$

$$\boxed{\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(z)\; X_2^*\left(\frac{1}{z^*}\right) z^{-1}\, dz}$$

$\boxed{\text{Let } v = z}$

⑥ **Time expansion** :- If $Z[x(n)] = X(z)$, then.

$$Z[x_k(n)] = X(z^k).$$

Proof :-

$$Z[x_k(n)] = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{k}\right) z^{-n}$$

where n is a multiple of 'k'. sub $\frac{n}{k} = l$.

$$Z[x_k(n)] = \sum_{l=-\infty}^{\infty} x(l)\, z^{-kl}$$

$$= \sum_{l=-\infty}^{\infty} x(l) (z^k)^{-l}$$

$$\boxed{Z[x_k(n)] = X(z^k).}$$

⑦ **Convolution Theorem** :-

If $y(n) = x(n) * h(n)$, then

$$Y(z) = X(z)\, H(z).$$

Proof :-

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$Y(z) = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \right] z^{-n}$$

Mul by $z^k \cdot z^{-k}$ $\Longrightarrow$ $\sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \right] z^{-n} \cdot z^k \cdot z^{-k}$

$$= \sum_{k=-\infty}^{\infty} x(k) z^{-k} \cdot \sum_{n=-\infty}^{\infty} h(n-k) z^{-(n-k)}$$

Sub $(n-k) = l$,

$$Y(z) = \sum_{k=-\infty}^{\infty} x(k) z^{-k} \sum_{l=-\infty}^{\infty} h(l) z^{-l}$$

$$\boxed{Y(z) = H(z) X(z)}$$

---

(8) **Initial Value Theorem** :- If $X(z) = Z[x(n)]$,

where $x(n) \longrightarrow$ causal, then,

$$x[0] = \lim_{z \to \infty} X(z)$$

Proof :- For a causal signal $x(n)$,

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} + \cdots$$

Taking $z \longrightarrow \infty$ on both sides,

$$\lim_{z \to \infty} X(z) = \lim_{z \to \infty} \left\{ x(0) + x(1) z^{-1} + x(2) z^{-2} + \cdots \right\}$$

$$\boxed{\lim_{z \to \infty} X(z) = x(0).}$$ //.

(9) **Final value theorem** :- If $Z[x(n)] = X(z)$, where $x(n)$ is a causal signal and ROC of $X(z)$ has no poles on (or) outside the unit circle, then.

$$x(\infty) = \underset{z\to 1}{Lt} \left\{\frac{z-1}{z}\right\} X(z) \text{ (OR) } \underset{z\to 1}{Lt} (1-z^{-1})X(z)$$

Proof :- By definition of one-sided z-transform,

$$Z[x(n)] = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$\therefore \quad Z[x(n-1) - x(n)] = \sum_{n=0}^{\infty} [x(n-1) - x(n)] z^{-n}$$

LHS

$$LHS = Z[x(n-1) - x(n)]$$

$$= Z[x(n-1)] - Z[x(n)]$$

$$= z^{-1} X(z) + x(-1) - X(z)$$

$$= x(-1) - (1-z^{-1}) X(z)$$

$$= \underset{z\to 1}{Lt} \left\{ x(-1) - (1-z^{-1}) X(z) \right\}$$

(Taking limit $z \to 1$)

$$LHS = x(-1) - \underset{z\to 1}{Lt} (1-z^{-1}) X(z) \rightarrow (1)$$

RHS

$$RHS = \sum_{n=0}^{\infty} [x(n-1) - x(n)] z^{-n}$$

$$= \underset{z\to 1}{Lt} \sum_{n=0}^{\infty} [x(n-1) - x(n)] z^{-n}$$

$$= \sum_{n=0}^{\infty} [x(n-1) - x(n)] (1)^{-n}$$

$$= \underset{p\to\infty}{Lt} \sum_{n=0}^{p} [x(n-1) - x(n)]$$

$$= \underset{p\to\infty}{Lt} \left\{ [x(-1) - x(0)] + [x(0) - x(1)] + \dots + [x(p-2) - x(p-1)] + [x(p-1) - x(p)] \right.$$

$$= \underset{p\to\infty}{Lt} [x(-1) - x(p)]$$

$$RHS = x(-1) - x(\infty) \rightarrow (2)$$

Equating (1) + (2)

$$\cancel{x(-1)} - \underset{z\to 1}{Lt} (1-z^{-1}) X(z) = \cancel{x(-1)} - x(\infty)$$

$$\boxed{x(\infty) = \underset{z\to 1}{Lt} (1-z^{-1}) X(z)}$$

Problems :-

① $x(n) = u(n-n_0)$.

$$u(n) \xleftrightarrow{z} \frac{z}{z-1}$$

$$Z[u(n-n_0)] = z^{-n_0} \cdot \frac{z}{z-1} = \frac{z^{-n_0+1}}{z-1}$$

② $x(n) = a^{n+1} u(n+1)$

$$a^n u(n) \xleftrightarrow{z} \frac{z}{z-a}$$

By applying time shifting (left shifted) property.

$$Z[a^{n+1} u(n+1)] = z^{1} \cdot \frac{z}{z-a} = \frac{z^2}{z-a}.$$

③ $x(n) = a^{n-1} u(n-1)$

$$a^n u(n) \xrightarrow{z} \frac{z}{z-a}$$

By applying time shifting (right shifted) property.

$$Z[a^{n-1} u(n-1)] = z^{-1} \cdot \frac{z}{z-a}$$

$$X(z) = \frac{1}{z-a}$$

② Determine Z-transform & ROC

Practice:

1. $x(n) = (2/3)^n u(n) + (-1/2)^n u(n)$

③ $x(n) = \{ \underset{\uparrow}{2}, -1, 0, 3, 4 \}$.

(Arrow denotes $n=0$)

$$X(z) = \sum_{n=0}^{4} x(n) z^{-n}$$

$$= x(0)z^{0} + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4}$$

$$= 2 - z^{-1} + 0 + 3z^{-3} + 4z^{-4}$$

$$\boxed{X(z) = 2 - z^{-1} + 3z^{-3} + 4z^{-4}}$$

ROC is $|z| > 0$

④ $x(n) = \{ 1, -2, 3, -1, \underset{\uparrow}{2} \}$

$$X(z) = \sum_{n=-4}^{0} x(n) z^{-n}$$

$$= z^4 - 2z^3 + 3z^2 - z + 2.$$

ROC is $|z| < \infty$.

⑤ $x(n) = u(n)$

$$X(z) = \sum_{n=0}^{\infty} z^{-n}$$

$$= 1 + 1/2 + 1/2^2 + \cdots$$

$$= \frac{1}{1 - 1/z}$$

$\therefore$ ROC is $|1/z| < 1$

$|z| > 1$

$$= \frac{z}{z-1}$$

⑥ $x(n) = u(-n)$

$$X(z) = \sum_{n=-\infty}^{0} z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{n}$$

$$= 1 + z + z^2 + \cdots = \frac{1}{1-z}$$

ROC: $|z| < 1$.

⑦ $x(n) = a^{|n|}$; $a < 1$

$$x(n) = a^n u(n) + a^{-n} u(-n-1)$$

$$Z[a^n u(n)] = \frac{z}{z-a} \quad \text{ROC}: |z| > a \longrightarrow ①$$

$$Z[a^{-n} u(-n-1)] = \sum_{n=-\infty}^{-1} a^{-n} z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (az)^{-n}$$

$$= \sum_{n=1}^{\infty} (az)^{n}$$

$$= az + az^2 + az^3 + \cdots$$

$$= az\left[1 + az + az^2 + \cdots\right]$$

$$= az\left[\frac{1}{1-az}\right] \Rightarrow \text{ROC}: |az| < 1$$

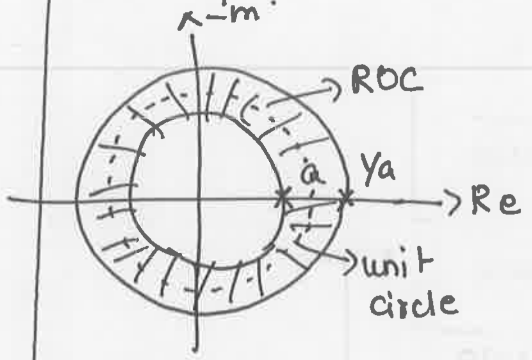
$$|z| < 1/a$$

$$\boxed{X(z) = \frac{-z}{z - 1/a}} \longrightarrow ②$$

Now add ① + ②

$$X(z) = \frac{z}{z-a} - \frac{z}{z-1/a}$$

$\therefore$ ROC is $a < |z| < 1/a$



Since given as $a<1$.

8 $x(n) = e^{j\omega_0 n} u(n)$

$$X(z) = \sum_{n=0}^{\infty} e^{j\omega_0 n} z^{-n}$$

$$= \sum_{n=0}^{\infty} \left[\frac{e^{j\omega_0}}{z}\right]^n$$

$$= \frac{1}{1 - \frac{e^{j\omega_0}}{z}}$$

$$\boxed{X(z) = \frac{z}{z-e^{j\omega_0}}}$$

$\therefore$ ROC is $\left|\frac{e^{j\omega_0}}{z}\right| < 1$

$|e^{j\omega_0}| < |z|$

$\boxed{|z| > 1}$

$\left\{\because |e^{j\omega_0}| = 1\right\}$.

Always

(9) $x(n) = \cos\omega_0 n \; u(n).$

$$= \frac{1}{2}\left[e^{j\omega_0 n} + e^{-j\omega_0 n}\right].$$

$$Z\left[e^{j\omega_0 n}\right] = \frac{z}{z-e^{j\omega_0}} \quad \& \quad Z\left[e^{-j\omega_0}\right] = \frac{z}{z-e^{-j\omega_0}}$$

$$X(z) = \frac{1}{2}\left[\frac{z}{z-e^{j\omega_0}} + \frac{z}{z-e^{-j\omega_0}}\right]$$

$$= \frac{z}{2}\left[\frac{z-e^{-j\omega_0} + z - e^{j\omega_0}}{z^2 - ze^{-j\omega_0} - ze^{j\omega_0} + 1}\right]$$

$$= \frac{z}{2}\left[\frac{2z-(e^{j\omega_0}+e^{-j\omega_0})}{z^2 - z(e^{j\omega_0}+e^{-j\omega_0})+1}\right]$$

$$= \frac{z}{2}\left[\frac{2z - 2\cos\omega_0}{z^2 - 2z\cos\omega_0 + 1}\right]$$

$$= \frac{z}{\cancel{2}}\left[\frac{\cancel{2}(z-\cos\omega_0)}{z^2 - 2z\cos\omega_0 + 1}\right]$$

(Taking z^2 as common from denominator)

$$\boxed{X(z) = \frac{1 - z^{-1}\cos\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}}$$

$$X(z) = \frac{z(z-\cos\omega_0)}{z^2 - 2z\cos\omega_0 + 1}$$

ROC :- $|e^{j\omega_0} z^{-1}| < 1 \implies |e^{j\omega_0}| < |z| = \boxed{|z| > 1}$

If the Question is $x(n) = a^n \cos\omega n \; u(n) \rightarrow$ find $Z[\cos\omega n \; u(n)]$ and replace z by $a^{-1}z$ in it.

(10) Find $Z[\sin wn\, u(n)]$?

$x(n) = \sin wn \; u(n)$.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sin wn \; u(n)\; z^{-n}$$

$$\boxed{\because \sin wn = \frac{e^{jwn} - e^{-jwn}}{2j}}$$

$$= \sum_{n=0}^{\infty} \left[\frac{e^{jwn} - e^{-jwn}}{2j}\right] z^{-n}$$

$$= \frac{1}{2j}\left\{\sum_{n=0}^{\infty} e^{jwn} z^{-n} - \sum_{n=0}^{\infty} e^{-jwn} z^{-n}\right\}$$

$$= \frac{1}{2j}\left\{\sum_{n=0}^{\infty} \left(e^{jw} z^{-1}\right)^n - \sum_{n=0}^{\infty} \left(e^{-jw} z^{-1}\right)^n\right\}$$

$$= \frac{1}{2j}\left\{\frac{1}{1-e^{jw}z^{-1}} - \frac{1}{1-e^{-jw}z^{-1}}\right\}$$

$$= \frac{1}{2j}\left\{\frac{z}{z-e^{jw}} - \frac{z}{z-e^{-jw}}\right\}$$

$$= \frac{1}{2j}\left\{\frac{\cancel{z^2} - ze^{-jw} - \cancel{z^2} + ze^{jw}}{z^2 - ze^{-jw} - ze^{jw} + e^{jw}\cdot e^{-jw}}\right\}$$

$$\boxed{\because e^{jw}\cdot e^{-jw} = 1}$$

$$= \frac{z\left[e^{jw} - e^{-jw}\right]/2j}{z^2 - z\left(e^{jw} + e^{-jw}\right) + 1} = \frac{z \sin w}{z^2 - 2z\cos w + 1}.$$

$$\boxed{X(z) = \frac{z \sin w}{z^2 - 2z\cos w + 1}}$$

ROC :-

$$\left| e^{j\omega} z^{-1} \right| < 1$$

$$\left| \frac{e^{j\omega}}{z} \right| < 1 \qquad \because \left| e^{j\omega} \right| = 1$$

$$\left| e^{j\omega} \right| < |z|$$

$$\boxed{|z| > 1}$$

Note:- If the question is given as $x(n) = a^n \sin \omega n \, u(n)$, find $Z[\sin \omega n \, u(n)]$ & replace z by $a^{-1}z$ in it.

(11) $x(n) = \delta(n) + \frac{1}{2}\delta(n+1) + \delta(n-3)$

$$\delta(n) \xrightarrow{Z} 1$$

$$\delta(n+1) \xrightarrow{Z} z \cdot (1)$$

$$\delta(n-3) \xrightarrow{Z} z^{-3} \cdot (1).$$

$$\therefore Z\left[\delta(n) + \frac{1}{2}\delta(n+1) + \delta(n-3)\right] = 1 + \frac{1}{2} \cdot z + z^{-3}.$$

$$\boxed{X(z) = 1 + \frac{z}{2} + z^{-3}}$$

(12) $x(n) = n\,u(n-1)$

$$u(n-1) \xrightarrow{Z} z^{-1}\left(\frac{z}{z-1}\right) = \frac{1}{z-1}.$$

$$n\,u(n-1) \xrightarrow{Z} -z \cdot \frac{d}{dz}\left\{X(z)\right\} = -z\frac{d}{dz}\left\{\frac{1}{z-1}\right\}.$$

$$\boxed{Z[n\,u(n-1)] = \frac{z}{(z-1)^2}}$$

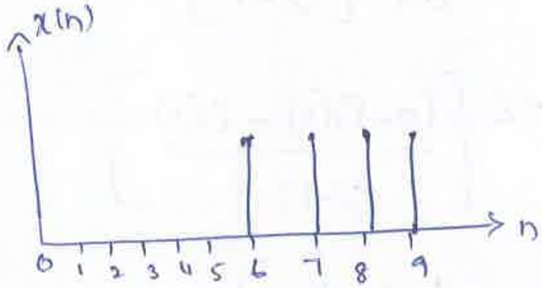
(4) $x(n) = \left(\frac{1}{2}\right)^n u(-n)$

$$u(-n) \xleftrightarrow{z} \frac{1}{1-z}.$$

By multiplication property, replace z by $(1/2)^{-1} z$.

$$X(z) = \frac{1}{1-2z}.$$

(5) $x(n) = u(n-6) - u(n-10)$



$X(z) \Rightarrow$

$$z[u(n-6)] = z^{-6} \cdot \left(\frac{z}{z-1}\right) \rightarrow (1)$$

$$z[u(n-10)] = z^{-10} \left(\frac{z}{z-1}\right) \rightarrow (2)$$

(1) - (2)

$$X(z) = \left(z^{-6} - z^{-10}\right)\left[\frac{z}{z-1}\right]$$

$$\boxed{X(z) = \frac{z^{-5} - z^{-9}}{z-1}}$$

6 $x(n) = n u(n)$

$$Z[u(n)] = \frac{z}{z-1}$$

Applying differentiation property in z,

$$Z[n u(n)] = -z \frac{d}{dz} X(z)$$

$$= -z \frac{d}{dz}\left[\frac{z}{z-1}\right]$$

$$= -z \left[\frac{(z-1)(1) - z(1)}{(z-1)^2}\right]$$

$$= -z\left[\frac{\cancel{z} - 1 - \cancel{z}}{(z-1)^2}\right] = \frac{z}{(z-1)^2}$$

---

7 Show that $u(n) * u(n-1) = n u(n)$

$$Z[u(n)] = \frac{z}{z-1}$$

$$Z[u(n-1)] = z^{-1}\left(\frac{z}{z-1}\right) = \frac{1}{z-1}$$

Convolution in time domain = multiplication in ~~freq~~ z domain

$$\therefore Z[u(n)] \cdot Z[u(n-1)] = \frac{z}{(z-1)^2}$$

(or)

$$u(n) * u(n-1) = \frac{z}{(z-1)^2}$$

RHS is nothing but $Z[nu(n)]$.

$\therefore \quad u(n) * u(n-1) = nu(n)$.

(8) Find $X(z)$, $x(n) = \left[(1/2)^n - (1/4)^n\right] u(n)$. Plot poles and zeros.

$$x_1(n) = (1/2)^n u(n) \xleftrightarrow{Z} \frac{z}{z - 1/2}$$

$$x_2(n) = (1/4)^n u(n) \xleftrightarrow{Z} \frac{z}{z - 1/4}$$

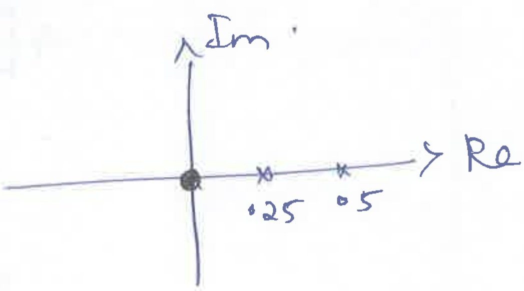
$$x(n) = x_1(n) - x_2(n)$$

$$X(z) = X_1(z) - X_2(z)$$

$$= \frac{z}{(z-0.5)} - \frac{z}{(z-0.25)}$$

$$= \frac{z(z-0.25) - z(z-0.5)}{(z-0.5)(z-0.25)}$$

$$\boxed{X(z) = \frac{0.25\,z}{(z-0.5)(z-0.25)}}$$



No zeros

Poles at $z = 0.5$ & 0.25

(9) Find initial & final values of following functions:

(i) $X(z) = \dfrac{z}{4z^2-5z+1}$ ROC: $|z|>1$

Initial value:

$$x[0] = \underset{z\to\infty}{Lt} X(z)$$

$$= \underset{z\to\infty}{Lt} \left\{ \frac{z}{4z^2-5z+1} \right\}$$

$$= \underset{z\to\infty}{Lt} \left\{ \frac{\cancel{z}}{z^{\cancel{2}}\left[4-\frac{5}{z}+\frac{1}{z^2}\right]} \right\}$$

$$\boxed{x(0) = 0}$$

Final value: First check if poles are on (or) inside the unit circle. Only then Final value theorem can be valid.

To find poles of :- $4z^2-5z+1$

$$4\left[z^2-\tfrac{5}{4}z+\tfrac{1}{4}\right] = 4\left[z^2-z-\tfrac{1}{4}z+\tfrac{1}{4}\right]$$

$$= 4\left[z(z-1)-\tfrac{1}{4}(z-1)\right]$$

$$= 4\left[(z-1)(z-\tfrac{1}{4})\right].$$

$\therefore$ The poles are at $z=1$, and $z=1/4$

Both lie inside (or) on unit circle. So finalvalue theorem can be applied.

Final value theorem:-

$$\boxed{x(\infty) = \underset{z\to 1}{Lt} \left(\frac{z-1}{z}\right) X(z)} \quad \text{(or)} \quad \underset{z\to 1}{Lt} (1-z^{-1})\, X(z)$$

$$x(\infty) = \underset{z\to 1}{Lt} \frac{(\cancel{z-1})}{\cancel{z}} \cdot \frac{\cancel{z}}{4(\cancel{z-1})(z-1/4)}$$

$$= \frac{1}{4(1-1/4)} = 1/3$$

$$\boxed{x(\infty) = 1/3}$$

Inverse Z-transform.

If $X(z)$ is given, the sequence $x(n)$ is determined by following methods,

(i) Partial fraction method

(ii) Power series Expansion (or) Long division method

(iii) Residue method. or Contour Integration method.

Partial fraction method

(i) Find inverse z-transform of

$$X(z) = \frac{1 - \frac{1}{3}z^{-1}}{(1-z^{-1})(1+2z^{-1})} \qquad ROC: |z| > 2$$

$$X(z) = \frac{z\left(z - \frac{1}{3}\right)}{(z-1)(z+2)}$$

$$\frac{X(z)}{z} = \frac{A_1}{(z-1)} + \frac{A_2}{(z+2)}$$

$$\frac{\left(z - \frac{1}{3}\right)}{\cancel{(z-1)(z+2)}} = \frac{A_1(z+2) + A_2(z-1)}{\cancel{(z-1)(z+2)}}$$

$$\left(z - \frac{1}{3}\right) = A_1(z+2) + A_2(z-1) \longrightarrow (1)$$

Sub $z=1$, in eqn (1) $\boxed{A_1 = 2/9}$

Sub $z=-2$, in eqn (1) $\boxed{A_2 = 7/9}$

$$\frac{X(z)}{z} = \frac{2}{9(z-1)} + \frac{7}{9(z+2)}$$

$$X(z) = \frac{1}{9}\left\{\frac{2z}{(z-1)} + \frac{7z}{(z+2)}\right\}$$

Taking inverse transform,

$$\boxed{x(n) = \frac{1}{9}\left[2(1)^n + 7(-2)^n\right]u(n)}$$

② $$X(z) = \frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}$$

Solution :-
$$X(z) = \frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}} \times \frac{z^2}{z^2}$$

$$= \frac{z^2+z}{z^2-z+0.5} = \frac{z(z+1)}{z^2-z+0.5}$$

$$\therefore \frac{X(z)}{z} = \frac{z+1}{\{z-(0.5+0.5j)\}\{z-(0.5-0.5j)\}}$$

$$= \frac{A}{\{z-(0.5+j0.5)\}} + \frac{B}{\{z-(0.5-j0.5)\}}$$

To find poles of denominator

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad \begin{array}{l} a=1 \\ b=-1 \\ c=0.5 \end{array}$$

$$= \frac{1 \pm \sqrt{1-4(1/2)^2}}{2}$$

$$= \frac{1 \pm \sqrt{-1}}{2} = \frac{1 \pm j}{2}$$

$$= 0.5 \pm 0.5j \quad (\text{since } \sqrt{-1} = j)$$

$$\frac{z+1}{\cancel{\{z-(0.5+0.5j)\}\{z-(0.5-0.5j)\}}} = \frac{A(z-(0.5-j0.5)) + B(z-(0.5+j0.5))}{\cancel{\{z-(0.5+0.5j)\}\{z-(0.5-0.5j)\}}}$$

$\boxed{\text{Sub } z = 0.5+j0.5}$

$$0.5+j0.5+1 = A\{\cancel{0.5}+j0.5-\cancel{0.5}+j0.5\} + 0$$

$$1.5+j0.5 = jA \Rightarrow \boxed{A = 0.5-1.5j}$$

Now sub $z = 0.5 - 0.5j$

$$0.5 - 0.5j + 1 = 0 + B\left[\cancel{0.5} - 0.5j - \cancel{0.5} - 0.5j\right]$$

$$1.5 - 0.5j = B[-j]$$

$$jB = -1.5 + 0.5j$$

$$\boxed{B = 0.5 + 1.5j}$$

$$\frac{X(z)}{z} = \frac{0.5 - j1.5}{z - (0.5 + j0.5)} + \frac{0.5 + 1.5j}{z - (0.5 - j0.5)}.$$

$$X(z) = (0.5 - j1.5)\frac{z}{z - (0.5 + j0.5)} + (0.5 + 1.5j)\frac{z}{z - (0.5 - j0.5)}$$

$$\boxed{x(n) = (0.5 - j1.5)(0.5 + j0.5)^n u(n) + (0.5 + 1.5j)(0.5 - j0.5)^n u(n)}$$

---

(2) Long Division Method/Power Series Expansion

① By using long division method, find inverse z-transform.

$$X(z) = \frac{z + 0.2}{(z + 0.5)(z - 1)}$$

$$= \frac{z + 0.2}{z^2 + z + 0.5z - 0.5} = \frac{z + 0.2}{z^2 - 0.5z - 0.5}$$

$$
\begin{array}{r|l}
 & z^{-1} + 0.7z^{-2} + 0.85z^{-3} \\
\hline
z^2 - 0.5z - 0.5 & \cancel{z} + 0.2 \\
 & \underset{(-)}{\cancel{z}} \underset{(+)}{- 0.5} \underset{(+)}{- 0.5z^{-1}} \\
\hline
 & 0.\cancel{7} + 0.5z^{-1} \\
 & \underset{(-)}{\cancel{0.7}} \underset{(+)}{- 0.35z^{-1}} \underset{(+)}{- 0.35z^{-2}} \\
\hline
 & 0.85\cancel{z^{-1}} + 0.35z^{-2} \\
 & \underset{(-)}{\cancel{0.85z^{-1}}} \underset{(+)}{- 0.425z^{-2}} \underset{(+)}{- 0.425z^{-3}} \\
\hline
 & 0.775z^{-2} + 0.425z^{-3}
\end{array}
$$

$$\therefore X(z) = z^{-1} + 0.7z^{-2} + 0.85z^{-3} + \cdots\cdots$$

$$= \sum_{n=0}^{\infty} x(n) z^{-n}.$$

$\therefore x(0) = 0, \quad x(1) = 1, \quad x(2) = 0.7, \quad x(3) = 0.85$ & so on.

---

③ Residue method

The Z-transform of a sequence $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-1}$$

We can find $x(n)$ by,

$$x(n) = \sum_{\substack{\text{all poles} \\ \text{of } X(z)}} \text{Residues of} \left(X(z) z^{n-1} \right)$$

$$\text{Residue} = \frac{1}{(m-1)!} \lim_{z \to a} \left[\frac{d^{m-1}}{dz^{m-1}} (z-a)^m X(z)\, z^{n-1} \right]$$

$m =$ order of pole at $z = a$.

Problem (1). Find $x(n)$ using residue method, if

$$X(z) = \frac{z}{(z-1)(z-2)}$$

It has poles at $z=1$ and $z=2$

Residue at $z=1$, $m=1$

$$\text{Residue} = \frac{1}{(m-1)!} \lim_{z \to a} \left[\frac{d^{m-1}}{dz^{m-1}} (z-a)^m X(z)\, z^{n-1} \right]$$

$$= \lim_{z \to 1} \left[\cancel{(z-1)} \frac{z}{\cancel{(z-1)}(z-2)} z^{n-1} \right]$$

$$= \frac{1}{(1-2)} (1)^{n-1} = -1 (1)^{n-1}$$

$$= -1 (1)^n (1)^{-1} = -1^n$$

Residue at $z=2$, $m=1$

$$\text{Residue} = \frac{1}{0!} \lim_{z \to 2} \left[\frac{d^0}{dz^0} \cancel{(z-2)} \frac{z}{(z-1)\cancel{(z-2)}} \cdot z^{n-1} \right]$$

$$= \frac{2}{1} (2)^{n-1} = 2^n$$

$$\boxed{\text{Sum of residues} = [(-1)^n + 2^n]\, u(n)} \Rightarrow x(n)$$

→ Find inverse z-transform using Partial fraction method.

① $$X(z) = \frac{7z-23}{(z-3)(z-4)}$$

$$\frac{7z-23}{(z-3)(z-4)} = \frac{A}{z-3} + \frac{B}{z-4}$$

$$7z-23 = A(z-4) + B(z-3).$$

Put $z=4$,

$$28-23 = B(1)$$

$$\boxed{B=5}$$

Put $z=3$,

$$21-23 = A(-1)$$

$$\boxed{A=2}$$

$$X(z) = \frac{2}{z-3} + \frac{5}{z-4}$$

$$\boxed{x(n) = \left[2(3)^{n-1} + 5(4)^{n-1}\right]u(n-1)}$$

$$\boxed{\because \frac{1}{z-a} = a^{n-1}u(n-1)}$$

② $$X(z) = \frac{z(z^2+z-30)}{(z-2)(z-4)^3}$$

$$\frac{X(z)}{z} = \frac{z^2+z-30}{(z-2)(z-4)^3}$$

$$\frac{z^2+z-30}{(z-2)(z-4)^3} = \frac{A_1}{z-2} + \frac{A_2}{(z-4)^3} + \frac{A_3}{(z-4)^2} + \frac{A_4}{(z-4)}$$

$$z^2+z-30 = A_1(z-4)^3 + A_2(z-2) + A_3(z-4)(z-2) + A_4(z-2)(z-4)^2$$

Sub $z=2, \Rightarrow A_1 = 3$

sub $z=4, \Rightarrow A_2 = -5$

Sub A_1 & A_2 in above equation,

$$z^2+z-30 = 3(z-4)^3 - 5(z-2) + A_3(z-4)(z-2) + A_4(z-2)(z-4)^2$$

$$z^2+z-30 = 3(z^3-12z^2+48z-64) - 5(z-2) + A_3(z^2-6z+8)$$
$$+ A_4(z^3-10z^2+32z-32).$$

Compare coefficients of z^2,

$$1 = -36 + A_3 - 10A_4$$

$$A_3 - 10A_4 = 37 \longrightarrow (1)$$

Compare coefficients of z,

$$1 = 144 - 5 - 6A_3 + 32A_4$$

$$6A_3 - 32A_4 = 138 \longrightarrow (2)$$

Solving (1) & (2), $A_3 = 7$, $A_4 = -3$

$$\therefore X(z) = \frac{3z}{z-2} - \frac{5z}{(z-4)^3} + \frac{7z}{(z-4)^2} - \frac{3z}{z-4}$$

$$\boxed{\begin{array}{lll} 1.\ \dfrac{z}{z-a} & \xrightarrow{z^{-1}} & a^n u(n) \\ 2.\ \dfrac{z}{(z-a)^2} & \xrightarrow{z^{-1}} & n a^{n-1} u(n) \\ 3.\ \dfrac{z}{(z-a)^3} & \xrightarrow{z^{-1}} & \dfrac{n(n-1)a^{n-2}u(n)}{2!} \end{array}}$$

From the above table,

$$x(n) = 3(2)^n u(n) - 5\,\frac{n(n-1)}{2\times 1}(4)^{n-2}u(n) + 7n(4)^{n-1}u(n) - 3(4)^n u(n)$$

$$\boxed{x(n) = \left\{3(2)^n - \frac{5}{32}n(n-1)(4)^n + \frac{7}{4}n(4)^n - 3(4)^n\right\}u(n).}$$

---

(3) $$X(z) = \frac{5z^3 - 29z^2 + 8z + 61}{z^2 - 7z + 10}.$$

This is the case with irrational system function. So, dividing the numerator polynomial by the denominator polynomial,

$$\begin{array}{r|l} & \quad 5z+6 \\ \hline z^2-7z+10 & 5z^3 - 29z^2 + 8z + 61 \\ & \underset{(-)}{5z^3} \underset{(+)}{-} 35z^2 + \underset{(-)}{50z} \\ \hline & \qquad 6z^2 - 42z + 61 \\ & \underset{(-)}{6z^2} \underset{(+)}{-42z} \underset{(-)}{+60} \\ \hline & \qquad\qquad 1 \end{array}$$

$$X(z) = 1 + \frac{5z+6}{z^2-7z+10} = 1 + \frac{5z+6}{(z-5)(z-2)}.$$

$$x(n) = Z^{-1}\left[1 + \frac{5z+6}{(z-5)(z-2)}\right]$$

$$= Z^{-1}(1) + Z^{-1}\left[\frac{5z+6}{(z-5)(z-2)}\right]$$

$$\frac{z}{z} \times \frac{5z+6}{(z-5)(z-2)} = \frac{A}{z} + \frac{B}{z-5} + \frac{C}{z-2}$$

$$\frac{X(z)}{z} = \frac{5z+6}{z(z-5)(z-2)}$$

$$5z+6 = A(z-5)(z-2) + Bz(z-2) + Cz(z-5)$$

Sub $z=0$,

$$6 = A(-5)(-2) \Rightarrow \boxed{A = 3/5}$$

Sub $z=5$, $5(5)+6 = B(5)(3) \Rightarrow \boxed{B = \frac{31}{15}}$

Sub $z=2$, $5(2)+6 = C(2)(-3) \Rightarrow \boxed{C = -8/3}$

$$\frac{X(z)}{z} = \frac{3/5}{z} + 31/15 \cdot \frac{1}{z-5} - \frac{8/3}{z-2}$$

$$X(z) = 3/5 + 31/15\,\frac{z}{z-5} - 8/3\,\frac{z}{z-2}$$

$$\boxed{x(n) = \delta(n) + \frac{3}{5}\delta(n) + \frac{31}{15}(5)^n u(n) - \frac{8}{3}(2)^n u(n)}$$

Residue method

1. Find $x(n)$ using residue method, if

$$X(z) = \frac{z^{-1}}{1-10z^{-1}+24z^{-2}} \quad ; \quad 4<|z|<6$$

mul & div by z^2, $X(z) = \dfrac{z}{z^2-10z+24} = \dfrac{z}{(z-4)(z-6)}$.

Residue at $z=4$, $m=1$

$$\text{Residue} = \frac{1}{(m-1)!} \lim_{z\to a} \left[\frac{d^{m-1}}{dz^{m-1}} (z-a)^m X(z)\, z^{n-1} \right].$$

$$= \lim_{z\to 4} \left[\cancel{(z-4)} \frac{z}{\cancel{(z-4)}(z-6)} \, z^{n-1} \right]$$

$$= \frac{4(4)^{n-1}}{4-6} = \frac{-1}{2}(4^n)$$

Residue at $z=6$, $m=1$

$$\text{Residue} = \frac{1}{(m-1)!} \lim_{z\to a} \left[\frac{d^{m-1}}{dz^{m-1}} (z-a)^m X(z)\, z^{n-1} \right]$$

$$= \lim_{z\to 6} \left[\cancel{(z-6)} \frac{z}{(z-4)\cancel{(z-6)}} \cdot z^{n-1} \right].$$

$$= \frac{6\,(6)^{n-1}}{(6-4)} = \frac{6}{2}(6)^{n-1}$$

$$= \frac{1}{2}(-6)^n u(-n-1)$$

$$\boxed{\because \frac{z}{z-a} \xrightarrow{z^{-1}} (-a)^n u(-n-1) \quad \text{if } |z|<|a|}$$

Adding both the residues,

$$\boxed{x(n) = \frac{-1}{2}(4)^n u(n) + \frac{1}{2}(-6)^n u(-n-1)}$$

---

(5) $$X(z) = \frac{z}{(z-1/2)^2}$$

$m = 2, \quad a = 1/2.$

Residue at $z = 1/2$ $$= \frac{1}{(m-1)!} \lim_{z\to a} \left\{ \frac{d^{m-1}}{dz^{m-1}} (z-a)^m X(z)\, z^{n-1} \right\}.$$

$$= \frac{1}{(2-1)!} \lim_{z\to 1/2} \left\{ \frac{d}{dz} \cancel{(z-1/2)^2} \frac{z}{\cancel{(z-1/2)^2}} z^{n-1} \right\}$$

$$= \lim_{z\to 1/2} \left[\frac{d}{dz} z\, z^{n-1} \right]$$

$$= \lim_{z\to 1/2} \left\{ \frac{d}{dz}(z^n) \right\} = \lim_{z\to 1/2} n z^{n-1}$$

$$= n\left(\frac{1}{2}\right)^{n-1} \Rightarrow \boxed{x(n) = 2n\left(\frac{1}{2}\right)^n u(n)}$$

③ $X(z) = \dfrac{1+z^{-1}}{1+8z^{-1}+15z^{-2}}$; $|z| > 5$

mul & div by z^2,

$$= \frac{z(z+1)}{z^2+8z+15} = \frac{z(z+1)}{(z+3)(z+5)}$$

Residue at $z=-3$, $m=1$

$$\text{Residue} = \frac{1}{(m-1)!} \lim_{z\to a} \left\{ \frac{d^{m-1}}{dz^{m-1}} (z-a)^m X(z)\, z^{n-1} \right\}$$

$$= \lim_{z\to -3} \left\{ \cancel{(z+3)} \cdot \frac{z(z+1)}{\cancel{(z+3)}(z+5)}\, z^{n-1} \right\}$$

$$= \frac{-3(-3+1)}{(-3+5)} (-3)^{n-1}$$

$$= \frac{-\cancel{3}(-\cancel{2})}{\cancel{2}} \frac{(-3)^n}{\cancel{(-3)}} = -(-3)^n u(n).$$

Residue at $z=-5$,

$$= \frac{1}{(m-1)!} \lim_{z\to a} \left\{ \frac{d^{m-1}}{dz^{m-1}} (z-a)^m X(z)\, z^{n-1} \right\}$$

$$= \lim_{z\to -5} \left\{ \cancel{(z+5)} \cdot \frac{z(z+1)}{(z+3)\cancel{(z+5)}}\, z^{n-1} \right\}.$$

$$= \frac{(-5)(-4)(-5)^{n-1}}{(-2)}$$

$$= 2(-5)^{n} \cdot u(n)$$

$$\boxed{x(n) = \{-(-3)^n + 2(-5)^n\}u(n)}$$

Difference Equation – Solution By Z-transform.

① Determine the impulse response and frequency response of given linear constant coefficient difference equation.

$$y(n) - \frac{5}{6} y(n-1) + \frac{1}{6} y(n-2) = x(n) - \frac{1}{2} x(n-1)$$

Taking z-transform on both sides,

$$Y(z) - \frac{5}{6} Y(z) z^{-1} + \frac{1}{6} Y(z) z^{-2} = X(z) - \frac{1}{2} z^{-1} X(z)$$

$$Y(z)\left[1 - 5/6\, z^{-1} + 1/6\, z^{-2}\right] = X(z)\left[1 - 1/2\, z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 1/2\, z^{-1}}{1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}} \times \frac{z^2}{z^2}$$

System function

$$= \frac{z^2 - 1/2\, z}{z^2 - 5/6\, z + 1/6} = \frac{z\cancel{(z - 1/2)}}{\cancel{(z - 1/2)}(z - 1/3)}$$

$$= \frac{z}{z - 1/3} \xrightarrow{IZT} (1/3)^n u(n).$$

$$\boxed{h(n) = \left(\frac{1}{3}\right)^n u(n)}$$

(2) Using z-transform, determine response $y(n)$ for $n \geq 0$ if, $y(n) = \frac{1}{2} y(n-1) + x(n)$ (May 2013).

where $x(n) = \left(\frac{1}{3}\right)^n u(n)$ & $y(-1) = 1$.

Soln :-

$$y(n) = \frac{1}{2} y(n-1) + \left(\frac{1}{3}\right)^n u(n).$$

Taking z-transform on Both sides,

$$Y(z) = \frac{1}{2}\left\{z^{-1} Y(z) + y(-1)\right\} + \frac{z}{z - 1/3} \quad X(z).$$

$$Y(z) = \frac{1}{2}\left[z^{-1} Y(z) + 1\right] + \frac{z}{z - 1/3}$$

$$Y(z) - 0.5 z^{-1} Y(z) - 0.5 = \frac{z}{z - 1/3}.$$

$$Y(z)\left[1 - 0.5 z^{-1}\right] = \frac{z}{z - 1/3} + 0.5$$

$$Y(z)\left[\frac{z - 0.5}{z}\right] = \left(\frac{z}{z - 1/3}\right)\left(\frac{z - 0.5}{z - 0.5}\right) + 0.5\left(\frac{z}{z - 0.5}\right)$$

$$Y(z) = \frac{z^2}{(z - 1/3)(z - 0.5)} + \frac{0.5 z}{z - 0.5}.$$

$$\frac{Y(z)}{z} = \frac{z}{(z - 1/3)(z - 0.5)} + \frac{0.5}{(z - 0.5)}.$$

Taking Partial fraction,

$$\frac{z}{(z-1/3)(z-0.5)} = \frac{A}{z-1/3} + \frac{B}{z-0.5}$$

$$z = A(z-0.5) + B(z-1/3)$$

Put $z = 0.5$, $\quad 0.5 = B\left(\frac{1}{2} - \frac{1}{3}\right)$

$$0.5 = B\left[\frac{3-2}{6}\right]$$

$$\frac{1}{\cancel{2}} \times \frac{\cancel{6}^{3}}{1} = B \quad \Rightarrow \boxed{B = 3}$$

Put $z = 1/3$, $\quad \frac{1}{3} = A\left[\frac{1}{3} - \frac{1}{2}\right] + 0$

$$\frac{1}{3} = A\left[\frac{2-3}{6}\right]$$

$$\frac{1}{\cancel{3}} \times (-\cancel{6}^{2}) = A \quad \Rightarrow \boxed{A = -2}$$

$$\frac{Y(z)}{z} = \frac{-2}{z-1/3} + \frac{3}{z-0.5} + \frac{0.5}{z-0.5}$$

$$\boxed{y(n) = \left\{-2\left(\frac{1}{3}\right)^n + 3(0.5)^n + 0.5(0.5)^n\right\} u(n)}$$

---

(5) Determine the response of the system, whose linear constant coeff difference eqn is given by,

$$y(n) - 0.1\,y(n-1) - 0.12\,y(n-2) = x(n) - 0.4\,x(n-1)$$

if $y(-1) = y(-2) = 2$ and $x(n) = (0.4)^n u(n)$

Solution :- If initial conditions are given, use them in solution, otherwise take them as '0'.

Taking z-transform on either sides,

$$Y(z) - 0.1\left[Y(z)z^{-1} + y(-1)\right] - 0.12\left[Y(z)z^{-2} + y(-1)z^{-1} + y(-2)\right]$$

$$= X(z) - 0.4\left[X(z)z^{-1} + x(-1)\right]$$

Take $x(-1)=0$, since its not given.

$$Y(z) - 0.1\left[Y(z)z^{-1} + 2\right] - 0.12\left[Y(z)z^{-2} + 2z^{-1} + 2\right] = X(z) - 0.4z^{-1}X(z)$$

$$Y(z) - 0.1z^{-1}Y(z) - 0.2 - 0.12Y(z)z^{-2} - 0.24z^{-1} - 0.24 = X(z)\left[1 - 0.4z^{-1}\right]$$

Given $x(n) = (0.4)^n u(n)$

$$\therefore X(z) = \frac{z}{z-0.4} = \frac{1}{1-0.4z^{-1}}$$

Sub $X(z)$ in above eqn,

$$Y(z)\left[1 - 0.1z^{-1} - 0.12z^{-2}\right] - 0.24z^{-1} - 0.44 = \frac{1}{\cancel{(1-0.4z^{-1})}}\cancel{(1-0.4z^{-1})}$$

$$Y(z)\left[1 - 0.1z^{-1} - 0.12z^{-2}\right] = 1 + 0.24z^{-1} + 0.44$$

$$Y(z) = \frac{1.44 + 0.24z^{-1}}{1 - 0.1z^{-1} - 0.12z^{-2}} \times \frac{z^2}{z^2}$$

$$Y(z) = \frac{1.44z^2 + 0.24z.}{z^2 - 0.1z - 0.12}$$

$$\frac{Y(z)}{z} = \frac{0.24 + 1.44z}{(z-0.4)(z+0.3)}$$

$$\frac{0.24+1.44z}{(z-0.4)(z+0.3)} = \frac{A}{(z-0.4)} + \frac{B}{(z+0.3)}$$

$$0.24 + 1.44z = A(z+0.3) + B(z-0.4)$$

Put $z = 0.3$

$$0.24 + 1.44(0.3) = 0 + B(-0.3 - 0.4).$$

$$\boxed{B = 0.27}$$

Put $z = 0.4$, $\boxed{A = 1.17}$

$$Y(z) = \frac{1.17z}{z-0.4} + \frac{0.27z}{z+0.3}.$$

$$\boxed{y(n) = 0.27(-0.3)^n u(n) + 1.17(0.4)^n u(n)}$$

---

(4) Evaluate the frequency response of the system described by system function

$$H(z) = \frac{1}{1-0.5z^{-1}}$$

$$H(z) = \frac{1}{1-0.5z^{-1}} = \frac{z}{z-0.5}$$

Freq response, Sub $z = e^{j\omega}$

$$H(e^{j\omega}) = \frac{1}{1-0.5e^{-j\omega}} = \frac{1}{1-\frac{0.5}{e^{j\omega}}} = \frac{e^{j\omega}}{e^{j\omega}-0.5}$$

$$\boxed{H(e^{j\omega}) = \frac{\cos\omega + j\sin\omega}{\cos\omega + j\sin\omega - 0.5}}$$

CONVOLUTION

Let $x_1(n)$ + $x_2(n)$ are two finite duration sequence. Then convolution of 2 sequence is given by $x_3(n)$.

$$\boxed{x_3(n) = x_1(n) * x_2(n)}$$

2 types → Linear Convolution

→ Circular Convolution.

Linear Convolution

① $x_1(n) = \{1, 2, 3, 4\}$; $x_2(n) = \{4, 3, 2, 1\}$.

Length of o/p sequence = $L + m - 1$

where $L \rightarrow$ length of first sequence - $x_1(n)$

$m \rightarrow$ " " second " - $x_2(n)$

$= 4 + 4 - 1 = 7$

$x_1(n)$ \ $x_2(n)$	4	3	2	1
1	4	3	2	1
2	8	6	4	2
3	12	9	6	3
4	16	12	8	4

$$y(n) = \{\underset{\uparrow}{4}, 11, 20, 30, 20, 11, 4\}.$$

In this problem, in $x_1(n)$ & $x_2(n)$, no arrow is given. So take first value as $x_1(0)$ (or) $x_2(0)$.

---

② Find linear convolution of

$$x(n) = \{1, 2, \underset{\uparrow}{3}, 4\} \quad \& \quad h(n) = \{2, -4, 6, \underset{\uparrow}{-8}\}$$

Length of o/p sequence $= 4 + 4 - 1 = 7$.

Width of first sequence $x(n)$ to the left, $N_1 = 2$

Width of 2nd sequence $h(n)$ to the left, $N_2 = 3$

$\therefore$ In o/p sequence, the no. of elements to the left is given by, $N_1 + N_2 = 2 + 3 = 5$.

	2	−4	6	−8
1	2	−4	6	−8
2	4	−8	12	−16
3	6	−12	18	−24
4	8	−16	24	−32

$$y(n) = \{2, 0, 4, 0, -14, \underset{\uparrow}{0}, -32\}$$

Circular Convolution :-

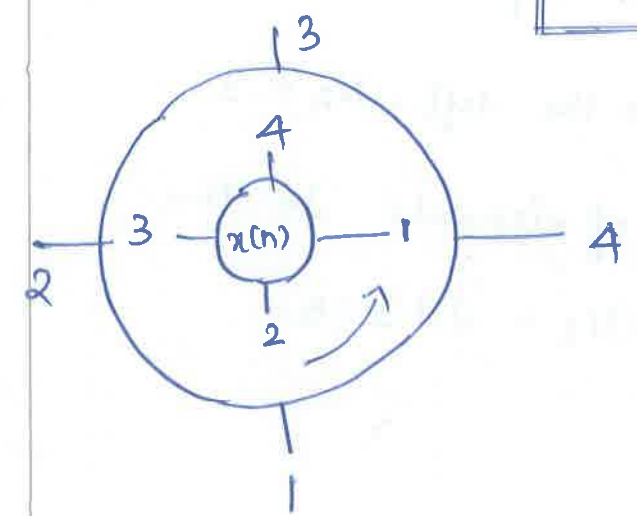
- → Circle method
- → Matrix method

① Find circular convolution of sequence

$x(n) = \{1, 2, 3, 4\}$ $h(n) = \{4, 3, 2, 1\}$

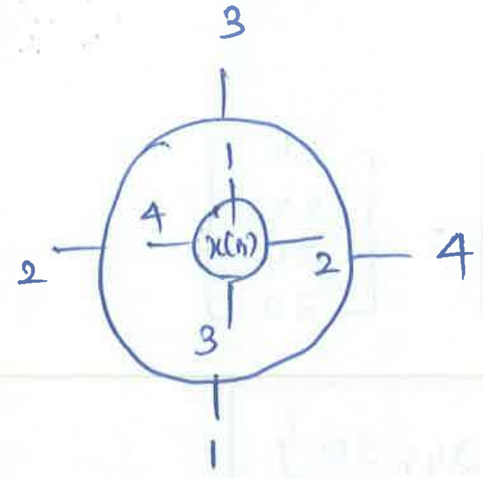
→ Circle method :-

> When length of the sequences are unequal, add zeros to make it equal.
> ↳ zero Padding.



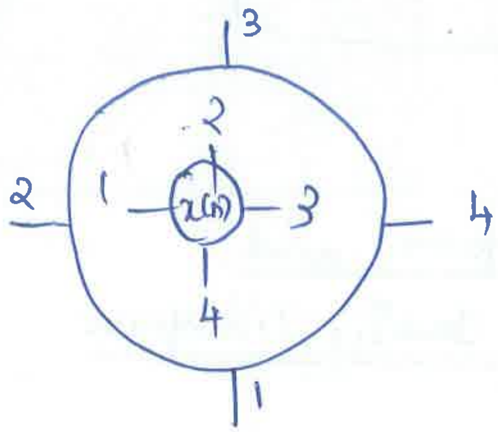
$$y(0) = (1\times 4) + (2\times 1) + (3\times 2) + (4\times 3)$$

$$\boxed{y(0) = 24}$$



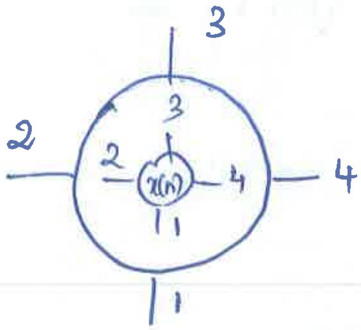
$$y(1) = (2\times 4) + (3\times 1) + (2\times 4) + (3\times 1)$$

$$\boxed{y(1) = 22}$$



$$y(2) = (3\times 2) + (4\times 3) + (4\times 1) + (2\times 1)$$

$$\boxed{y(2) = 24}$$



$$y(3) = (4\times 4) + (3\times 3) + (2\times 2) + (1\times 1)$$

$$\boxed{y(3) = 30}$$

$$y(n) = \{24, 22, 24, 30\}.$$

→ <u>Matrix method</u>.

$$x(n) = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

$$h(n) = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

$y(n) = x(n) \circledast h(n)$.

$$= \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 24 \\ 22 \\ 24 \\ 30 \end{bmatrix}$$

$$\boxed{y(n) = \{24, 22, 24, 30\}}$$

Relation between Z-transform and Discrete Time Fourier Transform

Discrete time Fourier transform (DTFT) is given by,

$$\boxed{X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn}} \longrightarrow (1)$$

Z-transform of $x(n)$ is given by,

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}} \longrightarrow (2)$$

If ROC of $X(z)$ contains unit circle, then $X(e^{jw})$ equals $X(z)$ evaluated on unit circle. That is

$$\boxed{X(e^{jw}) = X(z)\Big|_{z=e^{jw}}} \longrightarrow (3)$$

Using eqn ③, Fourier Transform is found by substituting $z = e^{jw}$ provided $x(n)$ is summable.

Discrete Time Fourier Transform (DTFT)

DTFT is given by,

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}}$$

Inverse DTFT is given by,

$$\boxed{x(n) = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega}$$

<u>Note</u> :- If you replace z by $e^{j\omega}$ in z-transform, you get DTFT.

<u>Properties of DTFT</u> :-

1. <u>Linearity</u> :- If $x_1(n) \xleftrightarrow{DTFT} X_1(e^{j\omega})$,

$x_2(n) \xleftrightarrow{DTFT} X_2(e^{j\omega})$

then $A x_1(n) + B x_2(n) \xleftrightarrow{DTFT} A\, X_1(e^{j\omega}) + B\, X_2(e^{j\omega})$

<u>Proof</u> :- $X_1(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_1(n)\, e^{-j\omega n}$.

$X_2(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_2(n)\, e^{-j\omega n}$

$\{A_1 x_1(n) + B_2 x_2(n)\} \xleftrightarrow{DTFT} \sum_{n=-\infty}^{\infty} A_1\, x_1(n)\, e^{-j\omega n} + \sum_{n=-\infty}^{\infty} B\, x_2(n)\, e^{-j\omega n}$

$$= A_1 X_1(e^{j\omega}) + B X_2(e^{j\omega}).$$

$$\{A_1 x_1(n) + B x_2(n)\} \xleftrightarrow{DTFT} A_1 X_1(e^{j\omega}) + B X_2(e^{j\omega})$$

---

② **Time Shifting Property** :- If $x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$,

then $x(n-n_0) \xleftrightarrow{DTFT} e^{-j\omega n_0} X(e^{j\omega})$.

Proof :-

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$x(n-n_0) \xleftrightarrow{DTFT} \sum_{n=-\infty}^{\infty} x(n-n_0) e^{-j\omega n}.$$

Let $(n-n_0) = m$,

$$x(n-n_0) \xleftrightarrow{DTFT} \sum_{m=-\infty}^{\infty} x(m) e^{-j\omega(m+n_0)}$$

$$= \sum_{m=-\infty}^{\infty} x(m) e^{-j\omega m} e^{-j\omega n_0}$$

$$\boxed{x(n-n_0) \xleftrightarrow{DTFT} e^{-j\omega n_0} X(e^{j\omega})}$$

---

③ **Frequency Shifting** :-

If $x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$

then $e^{j\omega_0 n} x(n) \xleftrightarrow{DTFT} X(e^{j(\omega-\omega_0)})$

$$e^{j\omega_0 n} x(n) \xleftrightarrow{DTFT} \sum_{n=-\infty}^{\infty} x(n) e^{j\omega_0 n} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-jn(\omega-\omega_0)}.$$

$$\boxed{e^{j\omega_0 n} x(n) \xleftrightarrow{DTFT} X(e^{j(\omega-\omega_0)}).}$$

④ **Time Reversal** :- If $x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$

then $x(-n) \xleftrightarrow{DTFT} X(e^{-j\omega})$

Proof :-

$$x(-n) \xleftrightarrow{DTFT} \sum_{n=-\infty}^{\infty} x(-n)\, e^{-j\omega n}$$

Sub $-n = m$

$$x(-n) \xleftrightarrow{DTFT} \sum_{m=-\infty}^{\infty} x(m)\, e^{j\omega m}$$

$$= \sum_{m=-\infty}^{\infty} x(m)\, e^{-j(-\omega)m}$$

$$\boxed{x(-n) \xleftrightarrow{DTFT} X(e^{-j\omega})}$$

⑤ **Periodicity** :- The DTFT $X(e^{j\omega})$ is periodic in ω with period 2π, satisfying the following condition,

$$X(e^{j\omega}) = X\left[e^{j(\omega+2k\pi)}\right]$$

for any integer k.

⑥ **Time Scaling :-** If $x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$,

then $x(an) \xleftrightarrow{DTFT} X(e^{j\frac{\omega}{a}})$

$$x(an) \xleftrightarrow{DTFT} \sum_{n=-\infty}^{\infty} x(an)\, e^{-j\omega n}$$

Sub $an = p$,

$$= \sum_{p=-\infty}^{\infty} x(p)\, e^{-\frac{jp\omega}{a}}$$

$$= \sum_{p=-\infty}^{\infty} x(p)\, e^{-j\left(\frac{\omega}{a}\right)p}$$

$$\boxed{x(an) \xleftrightarrow{DTFT} X(e^{j\frac{\omega}{a}})}$$

⑦ **Multiplication by n,**

If $x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$, then

$$n\,x(n) \xleftrightarrow{DTFT} j\frac{d}{d\omega} X(e^{j\omega})$$

WKT: $X(e^{j\omega}) \longleftrightarrow \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$

Differentiating on Both sides wrt ω,

$$\frac{d}{d\omega} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} -jn\, x(n)\, e^{-j\omega n}$$

$$\left(\frac{1}{-j}\right)\frac{d}{d\omega} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} n\,x(n)\, e^{-j\omega n}$$

$$\boxed{j\frac{d}{d\omega} X(e^{j\omega}) \xleftrightarrow{DTFT} \{n\,x(n)\}}$$

(8) Conjugation :- If $x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$, then

$$x^*(n) \xleftrightarrow{DTFT} X^*(e^{-j\omega})$$

Proof :-

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$X^*(e^{j\omega}) = \left[\sum_{n=-\infty} x(n) e^{-j\omega n}\right]^*$$

$$= \sum_{n=-\infty}^{\infty} x^*(n) e^{j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x^*(n) e^{-j(-\omega)n}$$

$$\boxed{x^*(n) = X^*(e^{-j\omega})}$$

(9) Time Convolution :- If $x_1(n) \xleftrightarrow{DTFT} X_1(e^{j\omega})$

$x_2(n) \xleftrightarrow{DTFT} X_2(e^{j\omega})$,

then

$$x_1(n) * x_2(n) \xleftrightarrow{DTFT} X_1(e^{j\omega}) X_2(e^{j\omega})$$

Proof :- When 2 signals are convolved,

$$y(n) = x_1(n) * x_2(n)$$

$$Y(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x_1(m) \sum_{n=-\infty}^{\infty} x_2(n-m) e^{-j\omega n}$$

Let $p = n-m$,

$$Y(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x_1(m) \sum_{p=-\infty}^{\infty} x_2(p) \cdot e^{-j\omega(p+m)}$$

$$= \sum_{m=-\infty}^{\infty} x_1(m)\, e^{-j\omega m} \sum_{p=-\infty}^{\infty} x_2(p)\, e^{-j\omega p}$$

$$\boxed{x_1(n) * x_2(n) = X_1(e^{j\omega})\, X_2(e^{j\omega})}$$

(10) **Parseval's theorem :-**

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 \xleftrightarrow{DTFT} \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2\, d\omega .$$

The above relation states that average power in a DT periodic signal is equal to the squared magnitude of $X(e^{j\omega})$

Proof :- WKT, $X^*(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x^*(n)\, e^{j\omega n}$

Now

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x^*(n)\, x(n)$$

$$= \sum_{n=-\infty}^{\infty} x^*(n) \left[\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega\right]$$

$$= \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) \left[\sum_{n=-\infty}^{\infty} x^*(n)\, e^{j\omega n}\, d\omega\right]$$

$$= \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, X^*(e^{j\omega})\, d\omega .$$

$$\boxed{\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2\, d\omega}$$ //.

(11) Modulation Property :-

If $x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$, then

$$x(n)\cos\omega_0 n \xleftrightarrow{DTFT} \frac{1}{2}\left\{X[e^{j(\omega+\omega_0)}]+X[e^{j(\omega-\omega_0)}]\right\}$$

Proof :-

$$x(n)\cos\omega_0 n \xleftrightarrow{DTFT} \sum_{n=-\infty}^{\infty} x(n)\cos\omega_0 n\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\left\{\frac{e^{j\omega_0 n}+e^{-j\omega_0 n}}{2}\right\}e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \frac{x(n)e^{j\omega_0 n}e^{-j\omega n}}{2} + \sum_{n=-\infty}^{\infty} \frac{x(n)e^{-j\omega_0 n}e^{-j\omega n}}{2}$$

$$= \frac{1}{2}\left\{\sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega-\omega_0)n} + \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega+\omega_0)n}\right\}$$

$$= \frac{1}{2}\left\{X(e^{j(\omega-\omega_0)}) + X(e^{j(\omega+\omega_0)})\right\}.$$

Problem :- (1) Find DTFT of $x(n) = \left(\frac{1}{2}\right)^n u(n)$ & plot the magnitude & phase spectrum.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{2} e^{-j\omega}\right)^n$$

$$X(e^{j\omega}) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

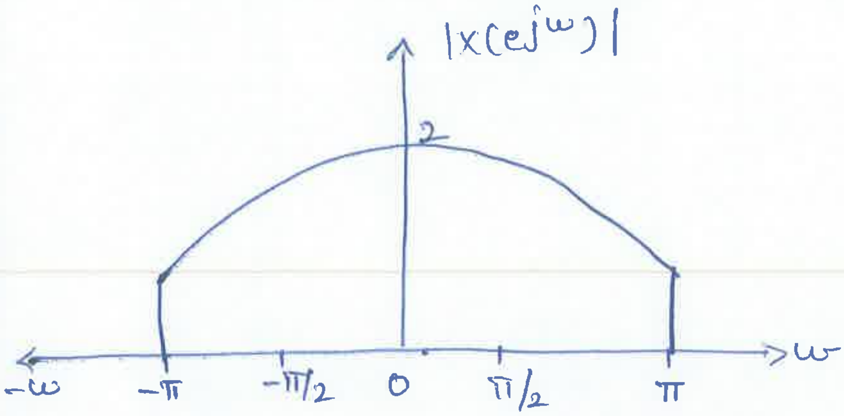
$$= \frac{1}{1 - \frac{1}{2}(\cos\omega - j\sin\omega)} = \frac{1}{1 - \frac{\cos\omega}{2} + j\frac{\sin\omega}{2}}$$

Mangitude $\Rightarrow \sqrt{\left(1 - \frac{\cos\omega}{2}\right)^2 + \sin^2\omega} = |X(e^{j\omega})|$

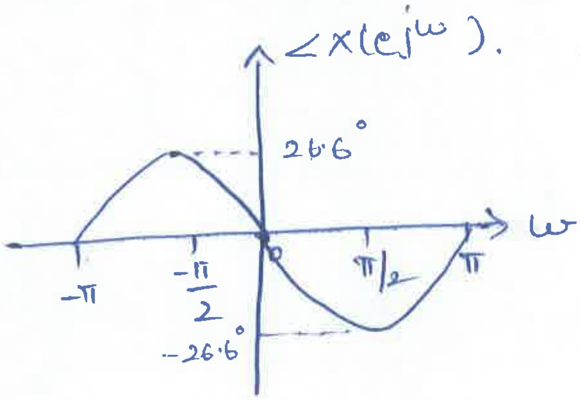
phase $\Rightarrow -\tan^{-1}\left\{\frac{\sin\omega/2}{1 - \frac{\cos\omega}{2}}\right\} = \angle X(e^{j\omega})$.

ω	$-\pi$	$-\pi/2$	0	$\pi/2$	π
$\lvert X(e^{j\omega})\rvert$	0.667	0.894	2	0.894	0.667
$\angle X(e^{j\omega})$	0	26.6°	0	-26.6°	0

Magnitude spectrum



Phase spectrum



Relation Between Causality, Stability & ROC

Let $h(n)$ be impulse response & $H(z)$ be system function of an LTI sly.

→ For a causal system,

(a) $H(z)$ should be rational

(ie) degree of numerator polynomial should be less than denominator polynomial.

(b) ROC should be outside the outermost pole.

→ For a stable system,

ROC should include the unit circle.

(ie) $|z| = 1$.

So for a system to be both causal and stable, if and only if all poles of $H(z)$ lie inside the unit circle.

(i) consider the following system function,

$$H(z) = \frac{\left(2 - \frac{13}{4} z^{-1}\right)}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-3z^{-1}\right)}$$

Determine causality + stability of the s/y for foll. cases.

(a) ROC : $|z| > 3$

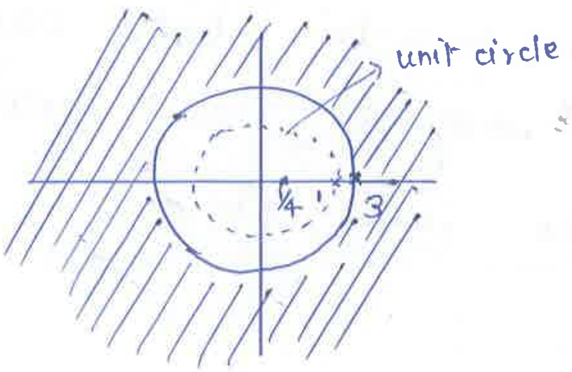
(b) ROC : $|z| < 1/4$

(c) ROC : $\frac{1}{4} < |z| < 3$.

Solution :-

$$H(z) = \frac{z(2z - 13/4)}{(z-1/4)(z-3)}$$

(a) ROC : $|z| > 3$.

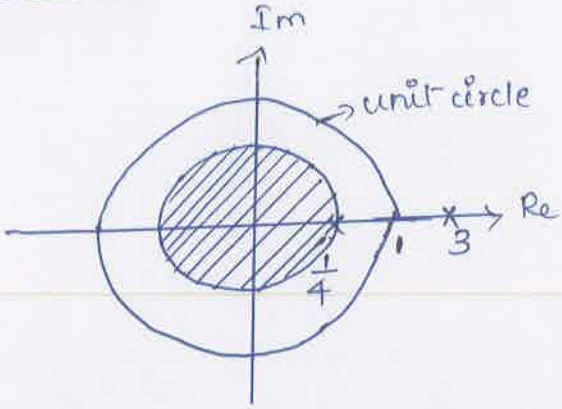


→ H(z) is rational & ROC is outside outermost pole. So system is causal.

→ ROC doesn't include unit circle. So system is unstable.

$\boxed{\text{Causal + Unstable s/y}}$

(b) ROC : $|z| < 1/4$

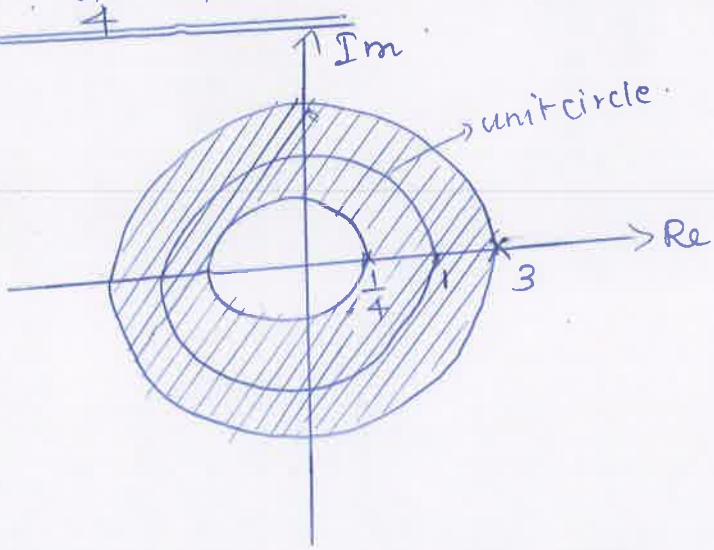


→ H(z) is rational but ROC is inside the innermost pole. So system is non-causal.

→ ROC doesnt include unit circle. So unstable s/y.

$\boxed{\text{s/y is unstable \& non-causal}}$

(c) ROC : $\frac{1}{4} < |z| < 3$



→ H(z) is rational. but ROC is left of outermost pole & to right of innermost pole.

So s/y is non-causal.

→ ROC includes unit circle. So s/y is stable.

$\boxed{\text{Stable \& Non-causal s/y}}$

Discrete Fourier Transform:

The Discrete Fourier Transform (DFT) is a powerful computation tool which allows us to evaluate the Fourier Transform $X(e^{j\omega})$ on a digital computer or specially designed hardware. Unlike DTFT, which is defined for finite and infinite sequences, DFT is defined only for sequences of finite length. Since $X(e^{j\omega})$ is continuous and periodic, DFT is obtained by sampling one period of the Fourier Transform at a finite number of frequency points. DFT plays an important role in the implementation of many signal processing algorithms. Apart from determining the frequency content of a signal, DFT is used to perform linear filtering operations in the frequency domain.

Definition of DFT:

It is a finite duration discrete frequency sequence which is obtained by sampling one period of fourier transform. Sampling is done at 'N' equally spaced points, over the period extending from $\omega = 0$ to $\omega = 2\pi$.

Mathematical Equations:

The DFT of discrete sequence x(n) is denoted by X(k). It is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} , \quad k=0,1,\ldots N-1.$$

Since this summation is taken for 'N' points; it is called as 'N' point DFT.

We can obtain discrete sequence x(n) from its DFT. It is called as IDFT. It is given by,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi nk/N} , \quad n=0,1,\ldots N-1$$

This is called as 'N' point IDFT.

Now we will define the new term 'W' as,

$$W_N = e^{-j2\pi/N}$$

This is called twiddle factor. Twiddle factor makes the computation of DFT a bit easy and fast.

Using twiddle factor we can write equations of DFT and IDFT as follows:

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} , \quad K=0,1\ldots N-1$$

and

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} , \quad n=0,1\ldots N-1$$

1. Find the DFT of a sequence $x(n) = \{1, 1, 0, 0\}$. and ~~find the~~

Soln

Let us assume $N = L = 4$

$L \rightarrow$ length of the sequence $x(n)$.

$N \rightarrow$ No. of sample points.

We have $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}$. $k = 0, 1, \ldots, N-1$

if $k = 0$

$$X(0) = \sum_{n=0}^{4-1} x(n) e^{-j2\pi n(0)/N} = \sum_{n=0}^{3} x(n) e^{0} = \sum_{n=0}^{3} x(n)$$

$$= x(0) + x(1) + x(2) + x(3)$$

$$= 1 + 1 + 0 + 0 = 2 \quad \boxed{X(0) = 2}$$

if $k = 1$

$$X(1) = \sum_{n=0}^{4-1} x(n) e^{-j2\pi n(1)/4} = \sum_{n=0}^{3} x(n) e^{-j\pi n/2}$$

$$= x(0) e^{0} + x(1) e^{-j\pi/2} + x(2) e^{-j\pi} + x(3) e^{-j3\pi/2}$$

$$= x(0) + x(1)\left[\cos \pi/2 - j \sin \pi/2\right] \qquad \left[\because e^{-j\theta} = \cos\theta - j\sin\theta\right.$$

$$+ x(2)\left[\cos\pi - j\sin\pi\right] + x(3)\left[\cos 3\pi/2 - j\sin 3\pi/2\right]$$

$$= 1 + 1(0 - j) = 1 - j \quad \boxed{X(1) = 1 - j}$$

if $k = 2$

$$X(2) = \sum_{n=0}^{4-1} x(n) e^{-j2\pi n(2)/4} = \sum_{n=0}^{3} x(n) e^{-j\pi n}$$

$$= x(0) + x(1)\left[\cos\pi - j\sin\pi\right] + x(2)\left[\cos 2\pi - j\sin 2\pi\right]$$

$$+ x(3)\left[\cos 3\pi - j\sin 3\pi\right]$$

$$= 1 + 1(-1 - 0) = 1 - 1 = 0 \quad \boxed{X(2) = 0}$$

$$X(3) = \sum_{n=0}^{4-1} x(n) e^{-j2\pi n(3)/4} = \sum_{n=0}^{3} x(n) e^{-j3\pi n/2}$$

$$= x(0) + x(1)\left[\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right] +$$

$$x(2)\left[\cos 3\pi - j\sin 3\pi\right] + x(3)\left[\cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2}\right]$$

$$= 1 + 1(0 + j) = 1 + j \qquad \boxed{X(3) = 1 + j}$$

$$X(K) = \{X(0), X(1), X(2), X(3)\}$$

$$\boxed{X(K) = \{2, 1-j, 0, 1+j\}}$$

2. Find the DFT of a sequence $x(n) = \begin{cases} 1 & \text{for } 0 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$

for (i) $N=4$ (ii) $N=8$. Plot $|X(K)|$ and $\angle X(K)$. Comment on the result.

Soln

Given $L=3$. For $N=4$, the periodic extension of $x(n)$ shown in fig(1) can be obtained by adding one zero (i.e. $N-L$ zeros). We have

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nK/N}, \quad k = 0, 1, \dots N-1$$

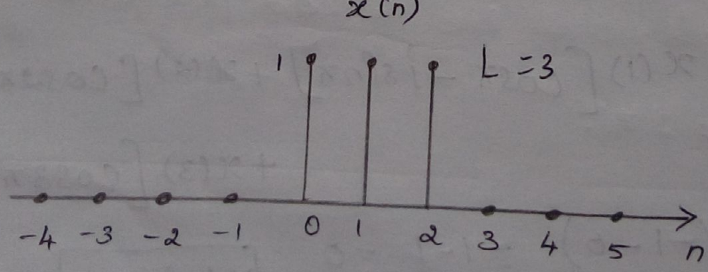
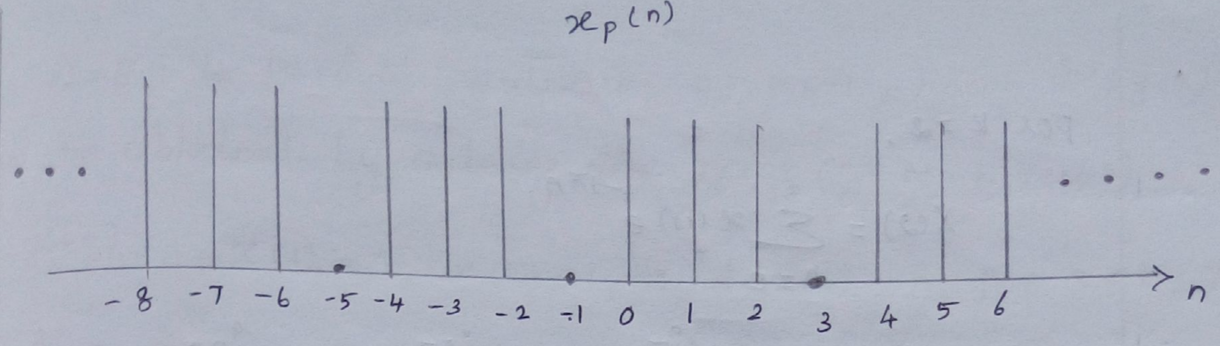


fig (1) a. The sample given in example 2.



(b)

fig(1) b. periodic extension of the sequence for $N = 4$.

From fig(b) we find

$$x(0) = 1, \quad x(1) = 1, \quad x(2) = 1, \quad x(3) = 0$$

For $N = 4$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}, \quad k = 0, 1 \ldots N-1$$

$$X(k) = \sum_{n=0}^{3} x(n) e^{-j2\pi nk/4}, \quad k = 0, 1, 2, 3$$

for $k = 0$,

$$X(0) = \sum_{n=0}^{3} x(n) e^{-j2\pi n(0)/4} = \sum_{n=0}^{3} x(n)$$

$$= x(0) + x(1) + x(2) + x(3) = 1 + 1 + 1 + 0$$

$\boxed{X(0) = 3}$ Therefore, $|X(0)| = \sqrt{(3)^2} = 3$. $\angle X(0) = 0$.

For $k = 1$,

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j\pi n/2}$$

$$= x(0) + x(1) e^{-j\pi/2} + x(2) e^{-j\pi} + x(3) e^{-j3\pi/2}$$

$$= 1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + \cos\pi - j\sin\pi + 0$$

$$= 1 - j - 1 = -j \quad \boxed{X(1) = -j}$$

Therefore,

$$|X(1)| = 1, \quad \angle X(1) = -\frac{\pi}{2}$$

For $k=2$,

$$X(2) = \sum_{n=0}^{3} x(n)\, e^{-j\pi n}$$

$$= x(0) + x(1)\, e^{-j\pi} + x(2)\, e^{-j2\pi} + x(3)\, e^{-j3\pi}$$

$$= 1 + \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi + 0$$

$$= 1 - 1 + 1 = 1 \qquad \boxed{X(2) = 1}$$

Therefore,

$$|X(2)| = 1, \quad \angle X(2) = 0$$

For $k=3$,

$$X(3) = \sum_{n=0}^{3} x(n)\, e^{-j3\pi n/2}$$

$$= x(0) + x(1)\, e^{-j3\pi/2} + x(2)\, e^{-j3\pi} + x(3)\, e^{-j9\pi/2}$$

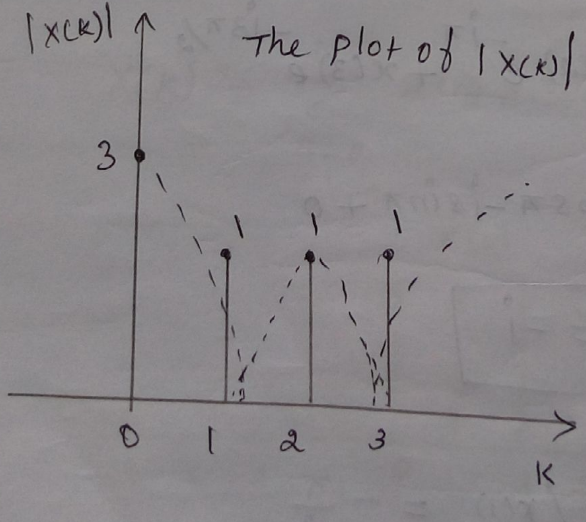
$$= 1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} + \cos 3\pi - j\sin 3\pi + 0$$

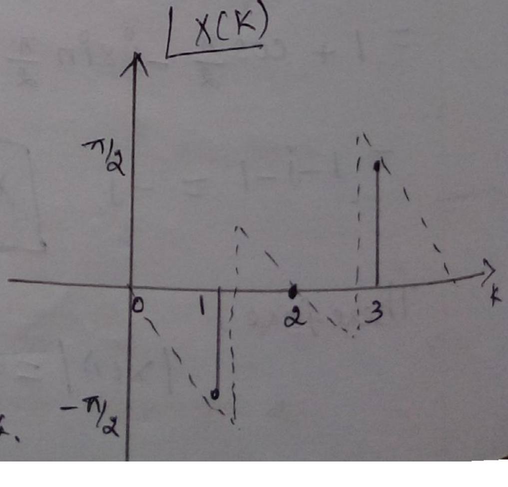
$$= 1 + j - 1 = j \qquad \boxed{X(3) = j}$$

Therefore $|X(3)| = 1$, $\angle X(3) = \tan^{-1}(1/0) = \pi/2$.

$$|X(k)| = \{3, 1, 1, 1\}, \quad \angle X(k) = \{0, -\pi/2, 0, \pi/2\}.$$

The plot of $|X(k)|$ and $\angle X(k)$ for $N=4$ is shown in fig(2).

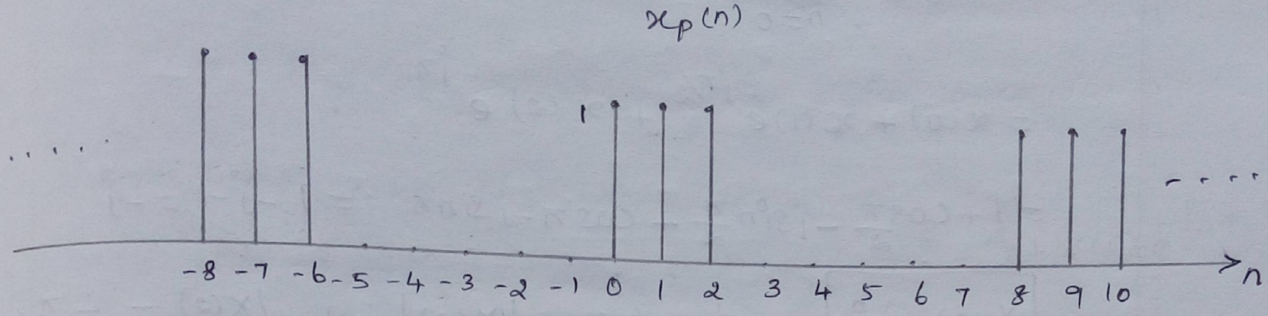




fig(2) freq response of $x(n)$ for $N=4$.

For $N=8$ the periodic extension of $x(n)$ shown in fig(3) can be obtained by adding five zeros ($\because$ N-L zeros).

$x(0)=1$, $x(1)=1$, $x(2)=1$, and $x(n)=0$ for $3 \le n \le 7$.



fig(3) periodic extension of the sequence $x(n)$ for $N=8$.

We have
$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}, \quad k=0,1,\dots N-1$$

For $N=8$.
$$X(k) = \sum_{n=0}^{8-1} x(n) e^{-j2\pi nk/8}, \quad k=0,1\dots 8-1$$

$$X(k) = \sum_{n=0}^{7} x(n) e^{-j\pi nk/4}, \quad k=0,1\dots 7$$

For $k=0$

$$X(0) = \sum_{n=0}^{7} x(n) e^{0} = \sum_{n=0}^{7} x(n) = x(0)+x(1)+x(2)+x(3)+x(4)+x(5)+x(6)+x(7)$$

$$= 1+1+1+0+0+0+0+0 = 3 \qquad \boxed{X(0)=3}$$

Therefore $|X(0)| = 3$, $\angle X(0) = 0$

For $k=1$,

$$X(1) = \sum_{n=0}^{7} x(n) e^{-j\pi n/4} = x(0) + x(1)e^{-j\pi/4} + x(2)e^{-j\pi/2} + 0+0+0+0+0$$

$$= 1+0.707-j0.707+0-j$$

$$\boxed{X(1) = 1.707 - j1.707}$$

Therefore, $|X(1)| = 2.414$, $\angle X(1) = -\pi/4$.

For $k=2$.

$$X(2) = \sum_{n=0}^{7} x(n)\, e^{-j\pi n/2}$$

$$= x(0) + x(1) e^{-j\pi/2} + x(2)\, e^{-j\pi}$$

$$= 1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + \cos\pi - j\sin\pi \quad = 1 - j - 1 = -j$$

$\boxed{X(2) = -j}$ Therefore, $|X(2)| = 1$, $\angle X(2) = -\pi/2$.

For $k=3$

$$X(3) = \sum_{n=0}^{7} x(n)\, e^{-j3\pi n/4}$$

$$= x(0) + x(1) e^{-j3\pi/4} + x(2)\, e^{-j3\pi/2}$$

$$= 1 + \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}$$

$$= 1 - 0.707 - j0.707 + j \quad = 0.293 + j0.293$$

Therefore, $|X(3)| = 0.414$, $\angle X(3) = \pi/4$

For $k=4$

$$X(4) = \sum_{n=0}^{7} x(n)\, e^{-j\pi n} \quad = x(0) + x(1)\, e^{-j\pi} + x(2).e^{-j2\pi}$$

$$= 1 + \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi$$

$$= 1 - 1 + 1 = 1$$

Therefore $|X(4)| = 1$, $\angle X(4) = 0$

For $k=5$,

$$X(5) = \sum_{n=0}^{7} x(n)\, e^{-j5\pi n/4}$$

$$= x(0) + x(1) e^{-j5\pi/4} + x(2)\, e^{-j5\pi/2}$$

$= \cancel{1 + \cos\pi - j\sin\pi} \ldots$

$= 1 + \cos\frac{5\pi}{4} - j\sin\frac{5\pi}{4} + \cos\frac{5\pi}{2} - j\sin\frac{5\pi}{2}$

$= 1 - 0.707 + j0.707 - j \quad = 0.293 - j0.293$

$$\boxed{X(5) = 0.293 - j0.293}$$

$|X(5)| = 0.414, \quad \angle X(5) = -\pi/4.$

For $k = 6$

$$X(6) = \sum_{n=0}^{7} x(n) e^{-j3\pi n/2} = x(0) + x(1) e^{-j\frac{3\pi}{2}} + x(2) e^{-j3\pi}$$

$= 1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} + \cos 3\pi - j\sin 3\pi$

$= 1 + j - 1 = j \quad \boxed{X(6) = j} \quad |X(6)| = 1, \quad \angle X(6) = \pi/2.$

For $k = 7$

$$X(7) = \sum_{n=0}^{7} x(n) e^{-j7\pi n/4} = 1 + e^{-j7\pi/4} + e^{-j7\pi/2}$$

$= 1 + \cos 7\pi/4 - j\sin\frac{7\pi}{4} + \cos\frac{7\pi}{2} - j\sin\frac{7\pi}{2}$

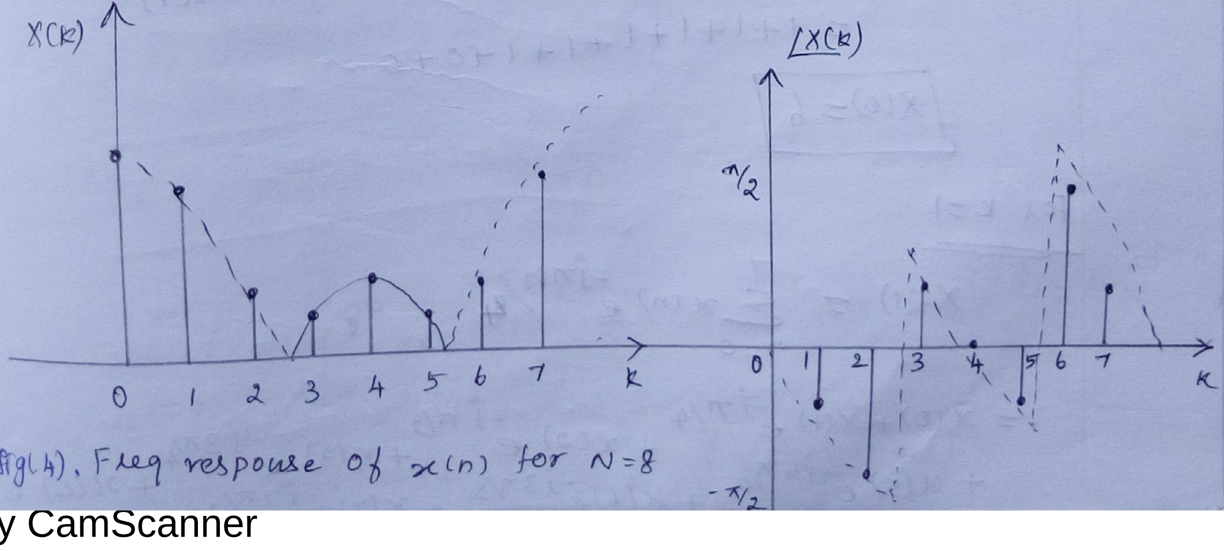
$= 1 + 0.707 + j0.707 + j$

$\boxed{X(7) = 1.707 + j1.707} \quad |X(7)| = 2.414 \quad \angle X(7) = \pi/4$

$|X(k)| = \{3, 2.414, 1, 0.414, 1, 0.414, 1, 2.414\}$

$\angle X(k) = \{0, -\pi/4, -\pi/2, \pi/4, 0, -\pi/4, \pi/2, \pi/4\}.$

The plot of $|X(k)|$ and $\angle X(k)$ Vs k, for $N=8$ is shown in fig(4)



fig(4). Freq response of $x(n)$ for $N=8$

Comments: Based on the fig (2) and fig (4) we can observe the following.

From fig (2) we can observe that, with $N=4$, it is difficult to extrapolate the entire frequency spectrum. For low values of N, the spacing between successive samples is high, which results in poor resolution. On the other hand when $N=8$, from fig (4). we can observe that it is possible to extrapolate the frequency of spectrum. That is zero padding gives a high density spectrum and provides a better displayed version for plotting.

3. Determine 8-point DFT of the sequence

$x(n) = \{1,1,1,1,1,1,0,0\}$.

Soln

We know $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}$, $k=0,1,\ldots N-1$

For $N=8$

$$X(k) = \sum_{n=0}^{7} x(n) e^{-j2\pi nk/8}, \quad k=0,1,\ldots N$$

For $k=0$

$$X(0) = \sum_{n=0}^{7} x(n) = x(0)+x(1)+x(2)+x(3)+x(4)+x(5)+x(6)+x(7)$$

$$= 1+1+1+1+1+1+0+0$$

$$\boxed{X(0) = 6}$$

For $k=1$

$$X(1) = \sum_{n=0}^{7} x(n) e^{-j\pi n/4}$$

$$= x(0)+x(1)e^{-j\pi/4}+x(2)e^{-j\pi/2}+x(3)e^{-j3\pi/4}+x(4)e^{-j\pi}+x(5)e^{-j5\pi/4}+x(6)e^{-j3\pi/2}+x(7)e^{-j7\pi/2}$$

$= 1 + 1(\cos\pi/4 - j\sin\pi/4) + 1(\cos\pi/2 - j\sin\pi/2) +$
$1(\cos 3\pi/4 - j\sin 3\pi/4) + 1(\cos\pi - j\sin\pi) + x(5)(\cos 5\pi/4$
$- j\sin 5\pi/4) + 0 + 0$

$= 1 + 0.707 - j0.707 - j - 0.707 - j0.707 - 1 - 0.707 + j0.707$

$$\boxed{X(1) = -0.707 - j1.707}$$

For $k = 2$

$$X(2) = \sum_{n=0}^{7} x(n) e^{-j\pi n/2}$$

$= x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + x(3)e^{-j3\pi/2} + x(4)e^{-j2\pi} +$
$x(5)e^{-j5\pi/2} + x(6)e^{-j3\pi} + x(7)e^{-j7\pi/2}$

$= 1 + 1(\cos\pi/2 - j\sin\pi/2) + 1(\cos\pi - j\sin\pi) + 1(\cos 3\pi/2 -$
$j\sin 3\pi/2) + 1(\cos 2\pi - j\sin 2\pi) + 1(\cos 5\pi/2 - j\sin 5\pi/2) + 0 + 0$

$= 1 - j - 1 + j + 1 - j$

$$\boxed{X(2) = 1 - j}$$

For $k = 3$

$$X(3) = \sum_{n=0}^{7} x(n) e^{-j3n\pi/4}$$

$= x(0) + x(1)e^{-j3\pi/4} + x(2)e^{-j3\pi/2} + x(3)e^{-j9\pi/4} + x(4)e^{-j3\pi}$
$+ x(5)e^{-j15\pi/4} + x(6)e^{-j9\pi/2} + x(7)e^{-j21\pi/4}$

$$\boxed{X(3) = 0.707 + j0.293}$$

For $k = 4$

$$X(4) = \sum_{n=0}^{7} x(n) e^{-j\pi n}$$

$= x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi} + x(4)e^{-j4\pi} + x(5)e^{-j5\pi}$
$+ x(6)e^{-j6\pi} + x(7)e^{-j7\pi}$

$X(4) = 1 - 1 + 1 - 1 + 1 - 1 = 0$ $\quad \boxed{X(4) = 0}$

For $k=5$

$$X(5) = \sum_{n=0}^{7} x(n)\, e^{-j5\pi n/4}$$

$$= x(0) + x(1)e^{-j5\pi/4} + x(2)e^{-j5\pi/2} + x(3)e^{-j15\pi/4} +$$

$$x(4)e^{-j5\pi} + x(5)e^{-j25\pi/4} + x(6)e^{-j15\pi/2} + x(7)e^{-j35\pi/4}$$

$$= 1 - 0.707 + j0.707 - j + 0.707 + j0.707 - 1 + 0.707 - j0.707$$

$$\boxed{X(5) = 0.707 - j0.293}$$

For $k=6$

$$X(6) = \sum_{n=0}^{7} x(n)\, e^{-j3\pi n/2}$$

$$= x(0) + x(1)e^{-j3\pi/2} + x(2)e^{-j3\pi} + x(3)e^{-j9\pi/2} + x(4)e^{-j6\pi}$$

$$+ x(5)e^{-j15\pi/2} + x(6)e^{-j9\pi} + x(7)e^{-j21\pi/2}$$

$$= 1 + j - 1 - j + 1 + j \qquad \boxed{X(6) = 1 + j}$$

For $k=7$

$$X(7) = \sum_{n=0}^{7} x(n)\, e^{-j7\pi n/4}$$

$$= x(0) + x(1)e^{-j7\pi/4} + x(2)e^{-j7\pi/2} + x(3)e^{-j21\pi/4} +$$

$$x(4)e^{-j7\pi} + x(5)e^{-j35\pi/4} + x(6)e^{-j21\pi/2} + x(7)e^{-j49\pi/4}$$

$$= 1 + 0.707 + j0.707 + j - 0.707 + j0.707 - 1 - 0.707 - j0.707$$

$$= -0.707 + j1.07 \qquad \boxed{X(7) = -0.707 + j1.07}$$

$$X(k) = \{6, -0.707 - j1.707, 1-j, 0.707 + j0.293, 0,$$
$$0.707 - j0.293, 1+j, -0.707 + j1.707\}.$$

4. Find IDFT of sequence $Y(k) = \{1, 0, 1, 0\}$.

Soln

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) e^{j2\pi nk/N}, \quad n = 0, 1, \dots N-1$$

For $n=0$

$$y(0) = \frac{1}{4} \sum_{k=0}^{3} Y(k) = \frac{1}{4}\left[Y(0) + Y(1) + Y(2) + Y(3)\right]$$

$$= \frac{1}{4}\left[1+0+1+0\right] = 2/4 = 0.5 \quad \boxed{y(0) = 0.5}$$

For $n=1$

$$y(1) = \frac{1}{N} \sum_{k=0}^{3} Y(k) e^{j\pi k/2}$$

$$= \frac{1}{4}\left[Y(0) + Y(1) e^{+j\pi/2} + Y(2) e^{j\pi} + Y(3) e^{j3\pi/2}\right]$$

$$= \frac{1}{4}\left[1 + 0 + 1(\cos\pi - j\sin\pi) + 0\right] = \frac{1}{4}\left[1-1\right] = 0$$

$$\boxed{y(1) = 0}$$

For $n=2$

$$y(2) = \frac{1}{N} \sum_{k=0}^{3} Y(k) e^{kj\pi 2/2} = \frac{1}{N} \sum_{k=0}^{3} Y(k) e^{j\pi k}$$

$$= \frac{1}{4}\left[Y(0) + Y(1) e^{j\pi} + Y(2) e^{j2\pi} + Y(3) e^{j3\pi}\right]$$

$$= \frac{1}{4}\left[1+1\right] = 2/4 = 0.5 \quad \boxed{y(2) = 0.5}$$

For $n=3$

$$y(3) = \frac{1}{N} \sum_{k=0}^{3} Y(k) e^{jk3\pi/2}$$

$$= \frac{1}{4}\left[Y(0) + Y(1) e^{j3\pi/2} + Y(2) e^{j3\pi} + Y(3) e^{j9\pi/2}\right]$$

$$= \frac{1}{4}\left[1+0-1+0\right] = 0 \quad \boxed{y(3) = 0} \quad \boxed{y(n) = \{0.5, 0, 0.5, 0\}}$$

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5. Find IDFT of the sequence $X(K) = \{5, 0, 1-j, 0, 1, 0, 1+j, 0\}$

Soln

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi nk/N} , \quad n = 0, 1, \dots N-1$$

For N = 8

$$x(n) = \frac{1}{8} \sum_{k=0}^{N-1} X(k) e^{j\pi kn/4} , \quad n = 0, 1, \dots 7$$

For n = 0

$$x(0) = \frac{1}{8} \left[\sum_{k=0}^{7} X(k) \right] = \frac{1}{8} \left[x(0) + x(1) + x(2) + x(3) + x(4) + x(5) + x(6) + x(7) \right]$$

$$= 5 + 0 + 1 - j + 0 + 1 + 0 + 1 + j + 0$$

$$\boxed{x(0) = 1}$$

For n = 1

$$x(1) = \frac{1}{8} \left[\sum_{k=0}^{7} x(k) e^{j\pi k/4} \right]$$

$$= \frac{1}{8} \left[5 + (1-j)(j) + 1(-1) + (1+j)(-j) \right] = \frac{1}{8}(6)$$

$$\boxed{x(1) = 0.75}$$

For n = 2

$$x(2) = \frac{1}{8} \left[\sum_{k=0}^{7} x(K) e^{j\pi k/2} \right]$$

$$\boxed{x(2) = 0.5}$$

For n = 3

$$x(3) = \frac{1}{8} \left[\sum_{k=0}^{7} X(k) e^{j3\pi k/4} \right]$$

$$\boxed{x(3) = 0.25}$$

For n = 4

$$x(4) = \frac{1}{8} \left[\sum_{k=0}^{7} X(k) e^{j\pi k} \right] \qquad \boxed{x(4) = 1}$$

for $n=5$

$$x(5) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k)\, e^{j5\pi k/4}\right]$$

$$\boxed{x(5) = 0.75}$$

for $n=6$

$$x(6) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k)\, e^{j3\pi k/2}\right]$$

$$\boxed{x(6) = 0.5}$$

for $n=7$

$$x(7) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k)\, e^{j7\pi k/4}\right]$$

$$\boxed{x(7) = 0.25}$$

$$x(n) = \{1, 0.75, 0.5, 0.25, 1, 0.75, 0.5, 0.25\}$$

6. Compute 4-point DFT of causal three sample sequence given by $x(n) = \begin{cases} 1/3, & 0 \le n \le 2 \\ 0, & \text{else.} \end{cases}$ plot the magnitude and phase spectrum.

Soln

N-point DFT of $x(n)$ is

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N}, \quad k = 0, 1, \ldots N-1$$

Here $N=4$.

$$X(k) = \sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi nk}{4}} = \sum_{n=0}^{3} x(n)\, e^{-j\pi nk/2}, \quad k = 0,1,2,3$$

$$x(n) = \{1/3, 1/3, 1/3, 0\}.$$

FOR $k=0$

$$X(0) = \sum_{n=0}^{3} x(n) e^{-j2\pi n(0)/4} = \sum_{n=0}^{3} x(n)$$

$$= \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + 0\right) = 3/3 = 1$$

$$\boxed{X(0) = 1} \qquad |X(0)| = 1, \quad \angle X(0) = 0$$

FOR $k=1$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j2\pi n(1)/4} = \sum_{n=0}^{3} x(n) e^{-jn\pi/2}$$

$$= \frac{1}{3} + \frac{1}{3}\left(\cos\pi/2 - j\sin\pi/2\right) + 1/3\left(\cancel{e}\ \cos\pi - j\sin\pi\right) + 0$$

$$= \frac{1}{3} + \frac{1}{3}(0 - j) + 1/3(-1) = \frac{1}{3} - \frac{1}{3}j - 1/3$$

$$= \frac{1}{3}(1 - j - 1) = -\frac{1}{3}j \qquad \boxed{X(1) = -\frac{1}{3}j} \qquad |X(1)| = 1/3, \quad \angle X(1) = -\pi/2$$

for $k=2$

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j2\pi n(2)/4} = \sum_{n=0}^{3} x(n) e^{-j\pi}$$

$$= \frac{1}{3} + \frac{1}{3}\left(\cos\pi - j\sin\pi\right) + \frac{1}{3}\left(\cos 2\pi - j\sin 2\pi\right) + 0$$

$$= \frac{1}{3} + \frac{1}{3}(-1) + \frac{1}{3}(1) = \frac{1}{3} - \frac{1}{3} + \frac{1}{3}$$

$$= \frac{1}{3} \qquad \boxed{X(2) = \frac{1}{3}} \qquad |X(2)| = 1/3$$

$$\angle X(2) = 0$$

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For $k=3$

$$X(3) = \sum_{n=0}^{3} x(n)\, e^{-j2\pi n(3)/4} = \sum_{n=0}^{3} x(n)\, e^{-j3\pi n/2}$$

$$= x(0) + x(1)\left(\cos 3\pi/2 - j\sin 3\pi/2\right) + x(2)\left(\cos 3\pi - j\sin 3\pi\right)$$

$$= \frac{1}{3} + \frac{1}{3}(j) + \frac{1}{3}(-1) = \frac{1}{3}(1+j-1) = \frac{1}{3}j$$

$$\boxed{X(3) = \frac{1}{3}j} \qquad |X(3)| = \frac{1}{3}$$

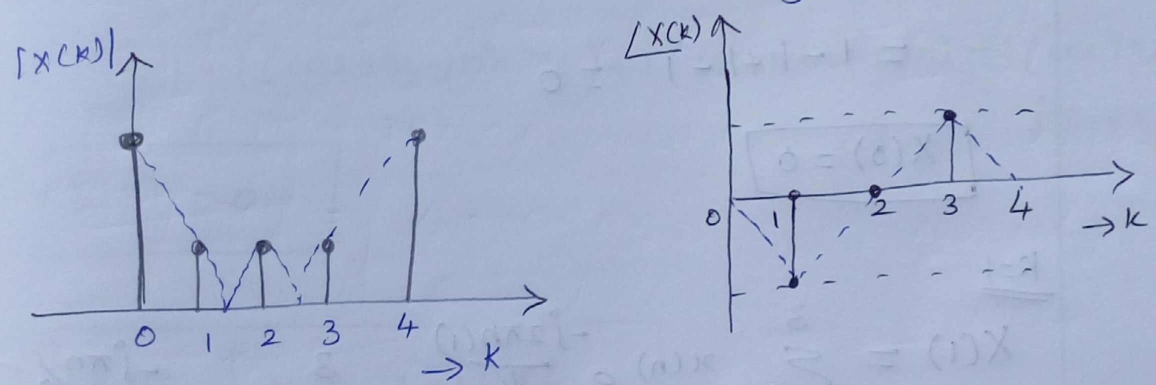
$$\angle X(3) = \pi/2$$

Therefore,

$$X(k) = \left\{1, -\frac{1}{3}j, \frac{1}{3}, \frac{1}{3}j\right\}$$

$$|X(k)| = \left\{1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$$

$$\angle X(k) = \left\{0, -\pi/2, 0, \pi/2\right\}$$



Magnitude & phase spectrum.

7. Find 4-point DFT of the sequence $x(n) = \cos \pi n$.

Let us assume $n = 0, 1, 2, 3$.

$x(0) = \cos(0) = 1$

$x(1) = \cos(1(\pi)) = -1$

$x(2) = \cos(2\pi) = 1$

$x(3) = \cos(3\pi) = -1$

$x(n) = \{1, -1, 1, -1\}$.

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi nk}{N}} \quad , k=0,1\ldots N-1$$

For $N=4$

$$X(k) = \sum_{n=0}^{3} x(n) e^{-\frac{j2\pi nk}{N}} \quad , k=0,1,2,3$$

$k=0$

$$X(0) = \sum_{n=0}^{3} x(n) e^{-\frac{j2\pi n(0)}{4}} = \sum_{n=0}^{3} x(n) e^{0}$$

$$= \sum_{n=0}^{3} x(n) = x(0)+x(1)+x(2)+x(3)$$

$$= 1-1+1-1 = 0$$

$$\boxed{X(0)=0}$$

$k=1$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-\frac{j2\pi n(1)}{4}} = \sum_{n=0}^{3} x(n) e^{-j\pi n/2}$$

$$= x(0) + x(1) e^{-j\pi/2} + x(2) e^{-j\pi} + x(3) e^{-j3\pi/2}$$

$$= 1 - 1(0-j) + 1(-1) - 1(+j) = 1+j-1-j$$

$$\boxed{X(1)=0}$$

$\underline{\underline{k=2}}$

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j4\frac{\pi n}{4}} = \sum_{n=0}^{3} x(n) e^{-jn\pi}$$

$$= x(0) + x(1) e^{-j\pi} + x(2) e^{-j2\pi} + x(3) e^{-j3\pi}$$

$$= 1 - j(\cos\pi - j\sin\pi) + 1(\cos 2\pi - j\sin 2\pi) - 1 (\cos 3\pi - j\sin 3\pi).$$

$$\boxed{X(2) = 3 + j}$$

$\underline{\underline{k=3}}$

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j6\pi n/4} = \sum_{n=0}^{3} x(n) e^{-j3\pi n/2}$$

$$= x(0) + x(1) e^{-j3\pi/2} + x(2) e^{-j3\pi} + x(3) e^{-j9\pi/2}$$

$$= 1 - 1(\cos 3\pi/2 - j\sin 3\pi/2) + 1(\cos 3\pi - j\sin 3\pi) - 1(\cos 9\pi/2 - j\sin 9\pi/2)$$

$$\boxed{X(3) = 0}$$

$$\boxed{X(k) = \{0, 0, 3+j, 0\}}$$

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15. perform the circular convolution of the following sequences $x(n) = \{1, 1, 2, 1\}$, $h(n) = \{1, 2, 3, 4\}$.

using DFT and IDFT method.

soln

we know $X_3(k) = X_1(k) X_2(k)$

$$X_1(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} , \quad k = 0, 1, \dots N-1$$

Given $x_1(n) = \{1, 1, 2, 1\}$ and $N = 4$

$$X_1(0) = \sum_{n=0}^{3} x_1(n) = 1+1+2+1 = 5$$

$$X_1(1) = \sum_{n=0}^{3} x_1(n) e^{-j\pi n/2} = 1 - j - 2 + j = -1$$

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$$X_1(2) = \sum_{n=0}^{3} x_1(n)\, e^{-j\pi n} = 1-1+2-1 = 1$$

$$X_1(3) = \sum_{n=0}^{3} x_1(n)\, e^{-j3\pi n/2} = 1+j-2-j = -1$$

$$X_1(k) = \{5, -1, 1, -1\}$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n)\, e^{-j2\pi nk/N} \quad , k = 0, 1 \dots N-1$$

$$X_2(0) = \sum_{n=0}^{3} x_2(n) = 1+2+3+4 = 10.$$

$$X_2(1) = \sum_{n=0}^{3} x_2(n)\, e^{-j\pi n/2} = 1+2(-j)+3(-1)+4(j)$$

$$= -2+2j$$

$$X_2(2) = \sum_{n=0}^{3} x_2(n)\, e^{-j\pi n} = 1+2(-1)+3(1)+4(-1)$$

$$= -2$$

$$X_2(3) = \sum_{n=0}^{3} x_2(n)\, e^{-j3\pi n/2} = 1+2(j)+3(-1)+4(-j)$$

$$= -2-j2$$

$$X_2(k) = \{10, -2+j2, -2, -2-j2\}.$$

$$X_3(k) = X_1(k) \cdot X_2(k) = \{50, 2-j2, -2, 2+j2\}$$

$$x_3(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_3(k)\, e^{j2\pi nk/N} \quad , \quad n = 0,1 \ldots N-1$$

$$x_3(0) = \frac{1}{4} \sum_{k=0}^{3} X_3(k) = \frac{1}{4}(50+2-j2-2+2+j2)$$

$$= 13.$$

$$x_3(1) = \frac{1}{4}\left[\sum_{k=0}^{3} X_3(k)\, e^{j\pi k/2}\right]$$

$$= \frac{1}{4}\left[50 + (2-j2)j + (-2)(-1) + (2+j2)(-j)\right]$$

$$= 14$$

$$x_3(2) = \frac{1}{4}\left[\sum_{k=0}^{3} X_3(k)\, e^{j\pi k}\right]$$

$$= \frac{1}{4}\left[50 + (2-j2)(-1) + (-2)(1) + (2+j2)(-1)\right]$$

$$= 11$$

$$x_3(3) = \frac{1}{4}\left[\sum_{k=0}^{3} X_3(k)\, e^{j3\pi k/2}\right]$$

$$= \frac{1}{4}\left[50 + (2-j2)(-j) + (-2)(-1) + (2+j2)(j)\right]$$

$$= 12$$

$$\boxed{x_3(n) = \{13, 14, 11, 12\}}$$

6 .Fast Fourier Transform:

The Fast Fourier transform is a highly
efficient procedure for computing the DFT of a finite
series and requires less number of computations by
~~taking advantage of the fact that~~ the ~~calculation~~
than that of direct evaluation of DFT. It reduces
the computations by taking advantage of the
fact that the calculation of the coefficients of
the DFT can be carried out iteratively. Due to
this, FFT computation technique is used in digital
spectral analysis, filter simulation, autocorrelation
and pattern recognition.

The FFT is based on decomposition
and breaking the transform into smaller
transforms and combining them to get the total
transform. FFT reduces the computation time
required to compute a discrete Fourier transform
and improves the performance by a factor 100
or more over direct evaluation of the DFT.

Consider the following DFT where $N = 8$

$$X(k) = \sum_{k=0}^{7} x(n)\, e^{-j2\pi kn/8} \quad , \quad k = 0, 1, \ldots 7 \qquad -(1)$$

Substituting $(k2\pi/8) = k$ in the above equation we get,

$$X(k) = x(0)\, e^{-jk0} + x(1)\, e^{-jk} + x(2)\, e^{-jk2} + x(3)\, e^{-jk3} + x(4)\, e^{-jk4} + x(5)\, e^{-jk5} + x(6)\, e^{-jk6} + x(7)\, e^{-jk7}$$

$$k = 0, 1, \ldots 7 \qquad -(2)$$

From eqn (2) has eight terms in the right hand side in which each term contains multiplication of a real term with complex exponential. Thus, for example $x(1)\, e^{-jk} = x(1)\,(\cos k - j\sin k)$ requires two multiplications and one addition for each value of k where $k = \frac{2\pi k}{8}$, $k = 0, 1, 2 \ldots 7$.

Thus, in eqn (2) each term in the right hand side requires eight complex multiplications and seven additions. The 8 point DFT therefore requires $8 \times 8 = 8^2 = 64$ complex multiplications $8 \times 7 = 8(8-1) = 56$ additions.

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In general, for an N point DFT, N^2 multiplications and $N(N-1)$ additions are required. For $N = 1024$, about 10^8 multiplications and equal number of additions are required which results in computational burden. Further such a ~~large~~ huge number of mathematical operations limit the bandwidth of digital signal processors. Several algorithms have been developed to reduce the computation burden and ease the implementation of DFT. The algorithm developed by Cooley and Tukey in 1965 is the most efficient one and is called Fast Fourier transform (FFT). The application FFT algorithms are discussed below with illustrated examples.

Radix - 2 FFT Algorithms:

For efficient computation of DFT, several algorithms have been developed based on divide and ~~conquer~~ conquer methods. However, the method is applicable for N not being a prime number.

Consider the case when $N = r_1 r_2 r_3 \ldots r_v$ where the (r_j) are prime. If $r_1 = r_2 = r_3 = \ldots = r_j$ then $N = r^v$. In such a case the DFTs are of size r. The number r is called the radix of the FFT algorithm. The most widely used FFT algorithms are radix-2 and radix-4 algorithms and are discussed in the following sections.

For performing radix-2 FFT, the value of N should be such that, $N = 2^m$. Here the decimation can be performed m times, where $m = \log_2 N$.

In direct computation of N-point DFT, the total number of complex addition are $N(N-1)$ and total number of complex multiplications are N^2. In radix-2 FFT, the total number of complex additions are reduced to $N \log_2 N$ and total number of complex multiplications are $\left(\frac{N}{2}\right) \log_2 N$. Comparison of number of computations by DFT and FFT is shown in table.

Table: Comparison of number of computations by DFT and FFT.

Number of points N	Direct computation		Radix-2 FFT	
	Addition $N(N-1)$	multiplication N^2	Addition $N \log_2 N$	Multiplication $\left(\frac{N}{2}\right) \log_2 N$
4	12	16	8	4
8	56	64	24	12
16	240	256	64	32
32	992	1024	160	80
64	4032	4096	384	192

Classical DFT approach does not use the two important properties of twiddle factor namely symmetry and periodicity properties which are given below.

$$W_N^{k+N/2} = -W_N^k$$

$$W_N^{k+N} = W_N^k$$

Radix-2 FFT algorithm exploits these two properties thereby removing redundant mathematical operations. However the results obtained using FFT algorithms is exactly the same as that of DFT. Further, the efficiency of FFT algorithm increases as N is increased. For example, if N = 512, DFT requires nearly 110 times more multiplications than FFT algorithm. The basic principle of FFT algorithm is therefore to decompose DFT into successively smaller DFTs. The manner in which this decomposition is done leads to different FFT algorithms. The ~~is~~ two basic classes of algorithms are :

1. Decimation in Time (DIT)
2. Decimation in Frequency (DIF)

In the algorithm developed by DIT, the sequence x(n) is decomposed into successively smaller subsequences. In DIF algorithm, the sequence of DFT coefficients

The N-point DFT of $x(n)$ can be written as

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} \qquad k = 0, 1, \ldots N-1 \qquad \text{--- (2)}$$

Separating $x(n)$ into even and odd indexed values of $x(n)$, we obtain

$$X(k) = \sum_{\substack{n=0 \\ (\text{even})}}^{N-1} x(n)\, W_N^{nk} + \sum_{\substack{n=0 \\ (\text{odd})}}^{N-1} x(n)\, W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_N^{2nk} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1)\, W_N^{(2n+1)k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_N^{2nk} + W_N^{k} \sum_{n=0}^{\frac{N}{2}-1} x(2n+1)\, W_N^{2nk} \qquad \text{--- (3)}$$

substituting eqn (1) in eqn (3) we have

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x_e(n)\, W_N^{2nk} + W_N^{k} \sum_{n=0}^{\frac{N}{2}-1} x_o(n)\, W_N^{2nk}$$

we can write — (4)

$$W_N^{2} = \left(e^{-j2\pi/N}\right)^2 = e^{-j2\pi/N/2} = W_{N/2}$$

i.e., $W_N^{2} = W_{N/2}$ — (5)

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Substituting eqn (5) in eqn (4) we get

$$X(k) = \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x_e(n) W_{N/2}^{nk}}_{N/2\text{-Point DFT of even indexed sequence}} + W_N^k \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x_o(n) W_{N/2}^{nk}}_{N/2\text{ point DFT of odd indexed sequence}} \quad \text{—— (6)}$$

$$= X_e(k) + W_N^k X_o(k)$$

—— (7)

Each of the sums in eqn (6) is an $\frac{N}{2}$-point DFT, the first sum being the $\frac{N}{2}$-point DFT of the even-indexed sequence and the second being the $\frac{N}{2}$-Point DFT of the odd-indexed sequence. Although the index k ranges from $k = 0, 1, \ldots N-1$, each of the sums is computed only for $k = 0, 1, \ldots \frac{N}{2}-1$, since $X_e(k)$ and $X_o(k)$ are periodic in k with period $\frac{N}{2}$. After the two DFTs are computed, they are combined according to eqn (7) to get the N-Point DFT of $x(k)$. So eqn (7) holds for the values of $k = 0, 1, \ldots \frac{N}{2}-1$.

For $k \geq N/2$

$$W_N^{k+N/2} = -W_N^k \quad \text{—— (8)}$$

Now $X(k)$ for $k \geq N/2$ is given by

$$X(k) = X_e\left(k - \frac{N}{2}\right) - W_N^{k-N/2} X_o\left(k - \frac{N}{2}\right)$$

for $k = \frac{N}{2}, \frac{N}{2}+1, \ldots N-1.$

—— (9)

Let us take $N = 8$:

Then $X_e(k)$ and $X_o(k)$ are 4-point $(N/2)$ DFTs of even-indexed sequence $x_e(n)$ and odd-indexed sequence $x_o(n)$ respectively. where

$x_e(0) = x(0)$; $\quad x_o(0) = x(1)$

$x_e(1) = x(2)$; $\quad x_o(1) = x(3)$

$x_e(2) = x(4)$; $\quad x_o(2) = x(5)$

$x_e(3) = x(6)$; $\quad x_o(3) = x(7)$

From eqn (7) and eqn (9) we have

$$X(k) = X_e(k) + W_8^k X_o(k) \quad \text{for } 0 \leq k \leq 3$$

$$X(k) = X_e(k-4) - W_8^{k-4}\, X_o(k-4) \quad \text{for } 4 \le k \le 7 \qquad \text{—(10)}$$

By substituting different values of k we get

$X(0) = X_e(0) + W_8^0 X_o(0)$; $X(4) = X_e(0) - W_8^0 X_o(0)$

$X(1) = X_e(1) + W_8^1 X_o(1)$; $X(5) = X_e(1) - W_8^1 X_o(1)$

$X(3) = X_e(2) + W_8^2 X_o(2)$; $X(6) = X_e(2) - W_8^2 X_o(2)$

$X(4) = X_e(3) + W_8^3 X_o(3)$; $X(7) = X_e(3) - W_8^3 X_o(3)$

—(11)

From the above set of equations we can find that $X(0)$ and $X(4)$, $X(1)$ and $X(5)$, $X(2)$ and $X(6)$, $X(3)$ and $X(7)$ have same inputs. $X(0)$ is obtained by multiplying $X_o(0)$ with W_8^0 and adding the product to $X_e(0)$. Similarly $X(4)$ is obtained by multiplying $X_o(0)$ with W_8^0 and subtracting the product form $X_e(0)$. This operation can be represented by a butterfly diagram as shown in fig. 1.

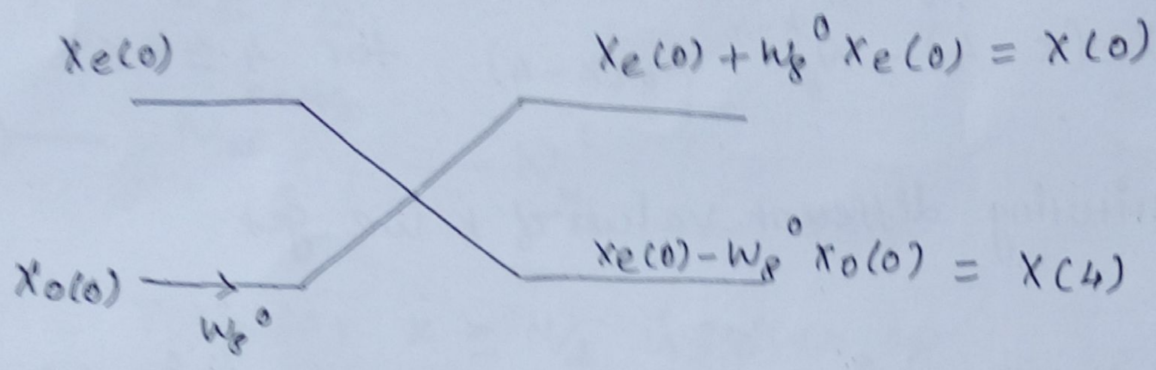

fig. 1. Flow graph of butterfly diagram for eqn(11).

Now the values $X(k)$ for $k = 0, 1, 2, 3, 4, 5, 6, 7$ can be obtained and an 8-point DFT flowgraph can be constructed from two 4-point DFTs as shown in fig (2).

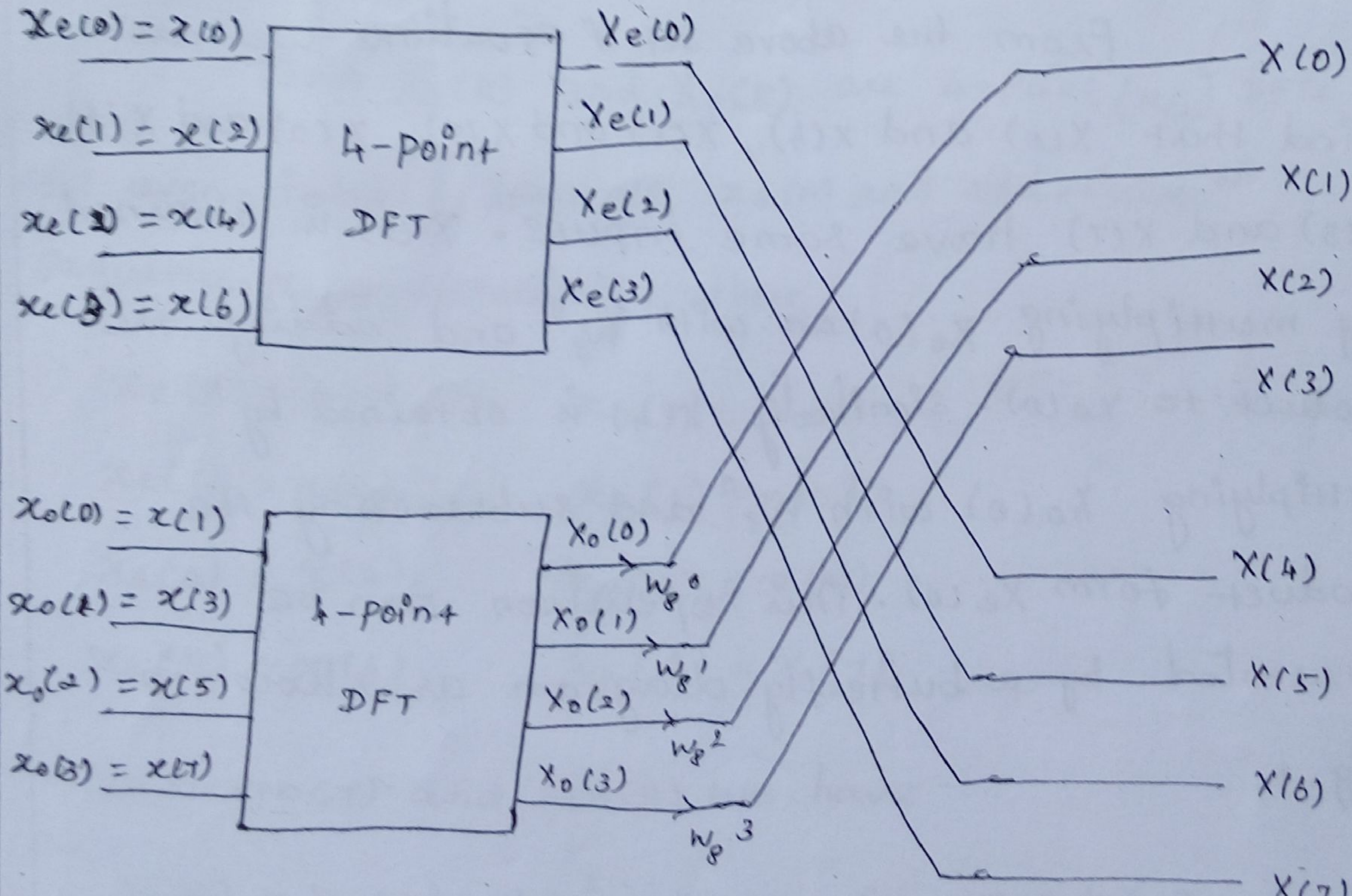

fig. 2. construction of an 8-point DFT from two 4 point DFTs.

From fig (2) we can find that initially the sequence $x(n)$ is shuffled into even indexed sequence $x_e(n)$ and odd indexed sequence $x_o(n)$ and then transformed to give $X_e(k)$ and $X_o(k)$. For $k=0,1,2,3$ the values $X_e(k)$ and $X_o(k)$ are obtained according to eqns (11) and using butterfly structure shown in fig. 1 the 8-point DFT is obtained. The inputs to the butterfly is separated by $N/2$ samples i.e. 4 samples and the power of the twiddle factors associated in this set of butterflies are in natural order.

Now we apply the same approach to decompose each of $N/2$ sample DFT. This can be done by dividing the sequence $x_e(n)$ and $x_o(n)$ into two sequences consisting of even and odd numbers of the sequences. The $N/2$-point DFTs can be expressed as a combination of $\frac{N}{4}$-point DFTs. That is, $X_e(k)$ for $0 \le k \le \frac{N}{2}-1$ can be written as

$$X_e(k) = X_{ee}(k) + W_N^{2k} X_{eo}(k) \quad \text{for } 0 \le k \le \frac{N}{2}-1$$

$$= X_{ee}\left(k-\frac{N}{4}\right) - W_N^{2(k-N/4)} X_{eo}\left(k-\frac{N}{4}\right) \quad \text{for } \frac{N}{4} \le k \le \frac{N}{2}-1$$

where $X_{ee}(k)$ is $\frac{N}{4}$ point DFT of the even members of $x_e(n)$ and $X_{eo}(k)$ is $\frac{N}{4}$ - point of DFT of the odd members of $x_e(n)$.

In the same way

$$X_o(k) = X_{oe}(k) + W_N^{2k} X_{oo}(k) \quad \text{for } 0 \le k \le \frac{N}{2} - 1$$

$$= X_{oe}\left(k - \frac{N}{4}\right) - W_N^{2\left(k-\frac{N}{4}\right)} X_{oo}\left(k - \frac{N}{4}\right)$$

$$\text{for } \frac{N}{4} \le k \le \frac{N}{2} - 1 \qquad \text{— (13)}$$

where $X_{oe}(k)$ is $\frac{N}{4}$ - Point DFT of the even members of $x_o(n)$ and $X_{oo}(k)$ is $\frac{N}{4}$ - Point of DFT of the odd members of $x_o(n)$.

For $N = 8$ the sequence $x_e(n)$ can be divided into even and odd indexed sequences as

$x_{ee}(0) = x_e(0)$; $x_{ee}(1) = x_e(2)$

$x_{eo}(0) = x_e(1)$; $x_{eo}(1) = x_e(3)$

Now from eqn (12) we have

$$X_e(0) = X_{ee}(0) + W_8^0 X_{eo}(0)$$

$$X_e(1) = X_{ee}(1) + W_8^2 X_{eo}(1)$$

$$X_e(2) = X_{ee}(0) - W_8^0 X_{eo}(0)$$

$$X_e(3) = X_{ee}(1) - W_8^2 X_{eo}(1) \qquad (14)$$

where $X_{oe}(k)$ is the 2-point DFT of even members of $x_e(n)$ and $X_{eo}(k)$ is the 2-point DFT of odd members of $x_e(n)$.

Similarly the sequence $x_o(n)$ can be divided into even and odd membered sequence as

$$x_{oe}(0) = x_o(0); \qquad x_{oe}(1) = x_o(2)$$

$$x_{oo}(0) = x_o(1); \qquad x_{oo}(1) = x_o(3)$$

From eqn (13) we can obtain

$$X_o(0) = X_{oe}(0) + W_8^0 X_{oo}(0)$$

$$X_o(1) = X_{oe}(1) + W_8^2 X_{oo}(1)$$

$$X_o(2) = X_{oe}(0) - W_8^0 X_{oo}(0)$$

$$X_o(3) = X_{oe}(1) - W_8^2 X_{oo}(1). \qquad (15)$$

where $X_{oe}(k)$ is the 2-point DFT of the even members of $x_o(n)$.

$X_{oo}(k)$ is the 2-point DFT of the odd members of $x_o(n)$. Fig. 3. shows the resulting flow graph when the four-point DFTs of fig. 2 are evaluated as in eqn (14) and eqn (15).

From fig. 3 we find that the input sequence is again ~~reorded~~ reordered, the input samples ~~of~~ to each butterfly are separated by $\frac{N}{4}$ samples i.e., 2 samples and there are two sets of butterflies. In each set of butterflies the twiddle factor exponents are same and separated by two.

Incomplete

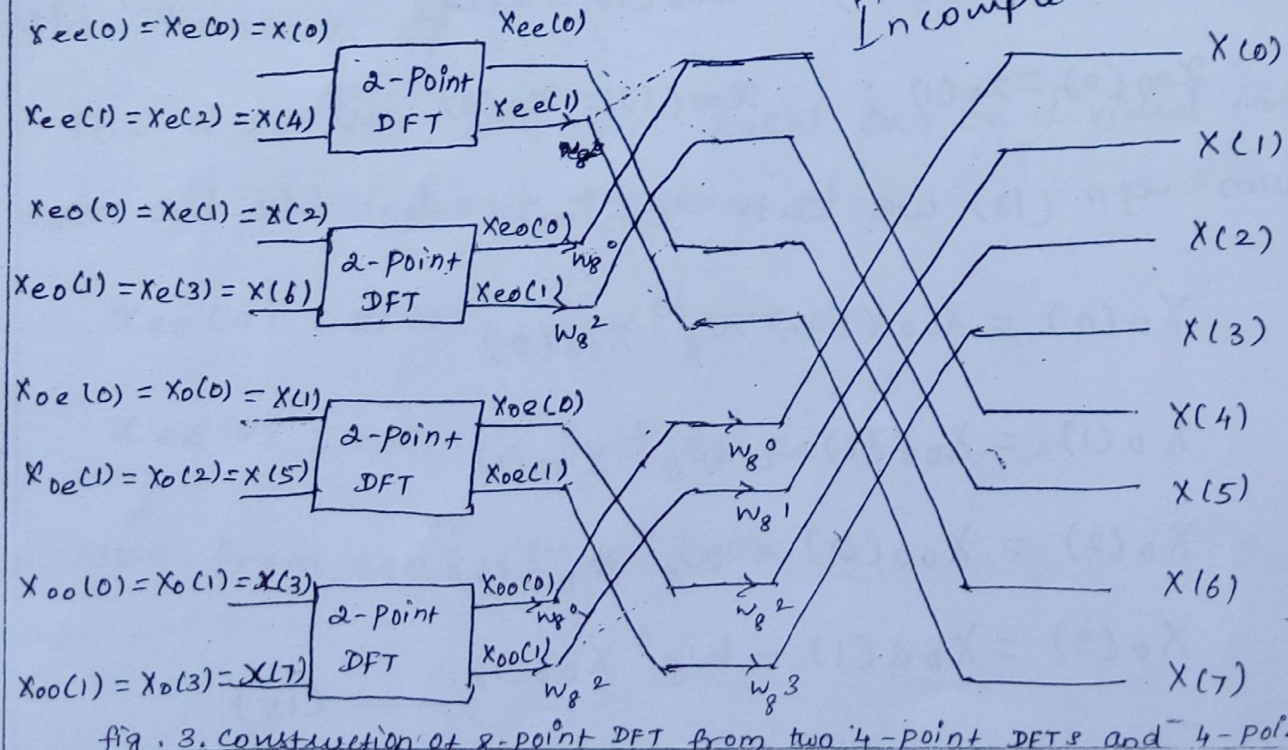

fig. 3. Construction of 8-point DFT from two 4-point DFTs and 4-point DFT from two 2-point DFTS.

For an 8-point DFT the number of stages required is three. So far we have seen the decomposition for stage 3 and stage 2. For stage1 the two point DFT can be easily found by adding and subtracting the input sequences as the twiddle factor associated with first stage is $W_8^0 = 1$. That is the first stage involves no multiplication but addition and subtraction. Now we have

$$X_{ee}(0) = x_{ee}(0) + x_{ee}(1) = x_e(0) + x_e(2) = x(0) + x(4)$$

$$X_{ee}(1) = x_{ee}(0) - x_{ee}(1) = x_e(0) - x_e(2) = x(0) - x(4) \quad \text{---(16)}$$

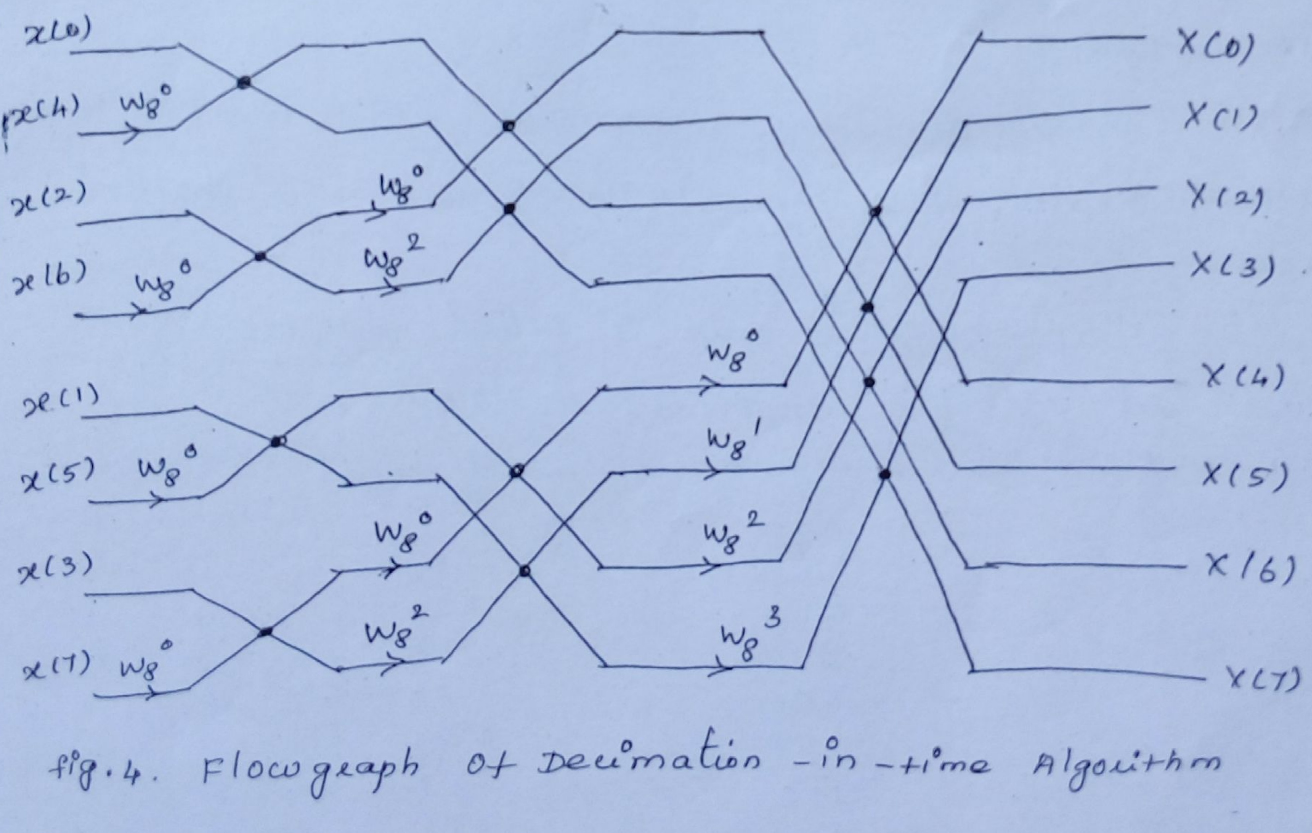

fig.4. Flowgraph of Decimation-in-time Algorithm

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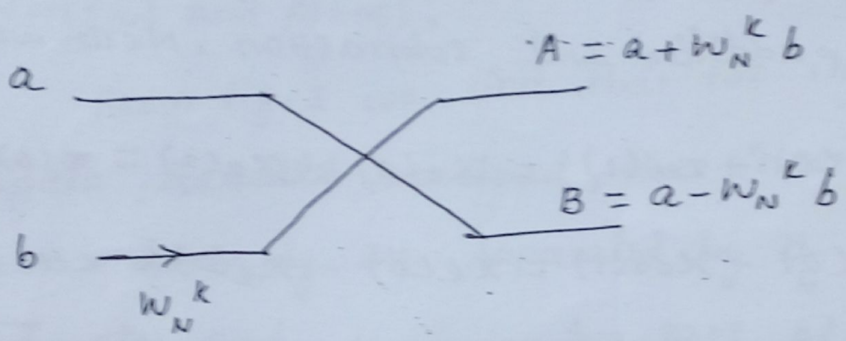
The algorithm has been called decimation in time since at each stage, the input sequence is divided into smaller sequences, i.e., the input sequences are decimated at each stage. From the flow graph several important observations can be made.

The basic flow graph of DIT algorithm is



Bit Reversal:

In DIT algorithm, the output sequence is in a natural order, the input sequence is in a shuffled order.

That shuffled order is called the bit reversal order and can be explained as follows,

input sample index	Binary representation	Bit reversed Binary	Bit reversed sample index
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

Example:

1. Find the DFT of a sequence $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$ using DIT algorithm.

The twiddle factors associated with the flowgraph are

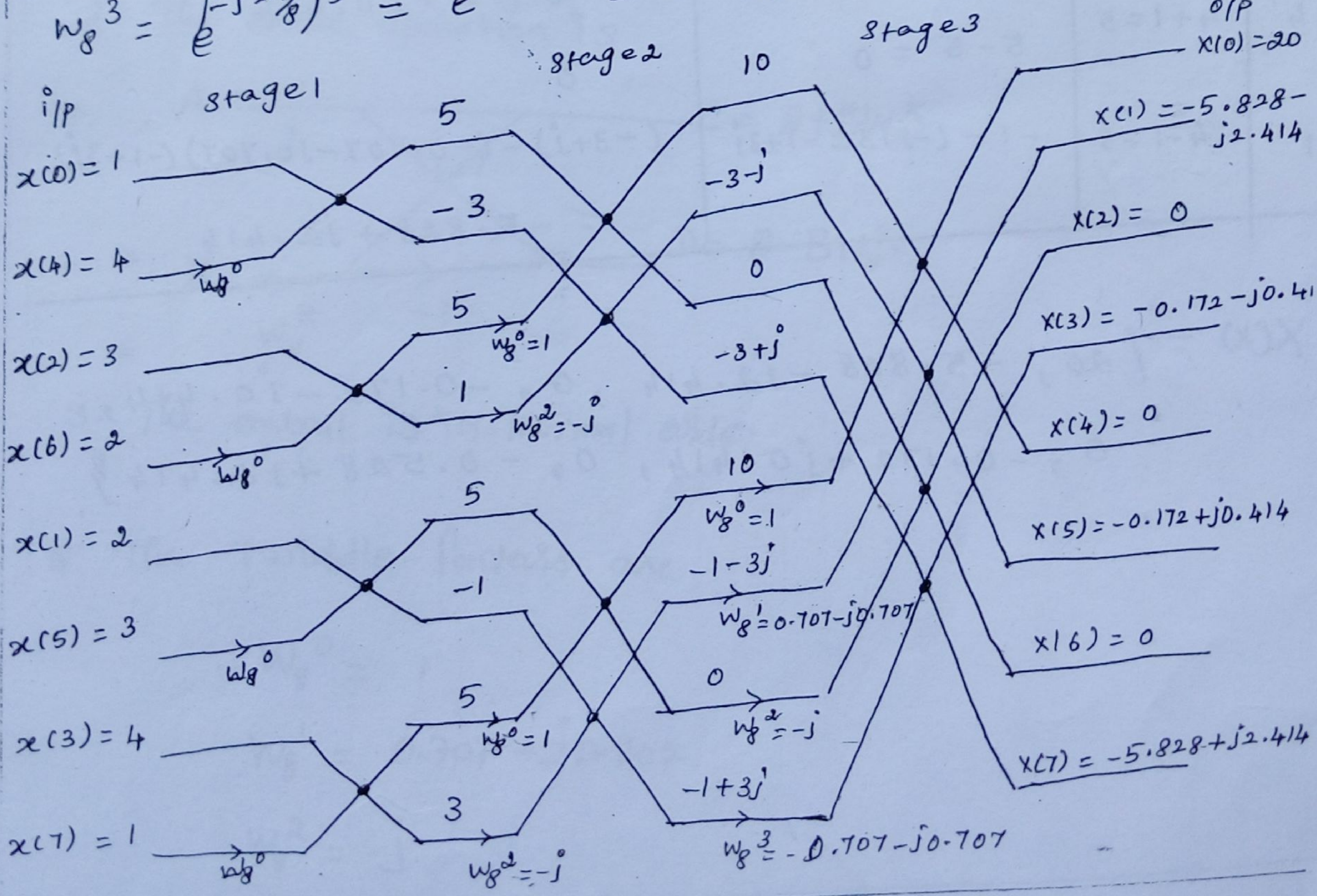
$$W_8^0 = 1, \quad W_8^1 = \left(e^{-j2\pi/8}\right)^1 = e^{-j\pi/4} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4}$$

$$W_8^1 = \cos 45 - j\sin 45 = 0.707 - j0.707$$

$$W_8^2 = \left(e^{-j2\pi/8}\right)^2 = e^{-j\pi/2} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = \cos 90 - j\sin 90$$

$$W_8^2 = -j$$

$$W_8^3 = \left(e^{-j2\pi/8}\right)^3 = e^{-j3\pi/4} = \cos 3\pi/4 - j\sin\frac{3\pi}{4} = -0.707 - j0.707$$



I/p	o/p of stage 1	o/p of stage 2	o/p of stage 3
1	$1+4=5$	$5+5=10$	$10+10=20$
4	$1-4=-3$	$-3+(-j)(1)=-3-j$	$-3-j+(0.707-j0.707)(-1-3j)$ $=-5.828-j2.414$
3	$3+2=5$	$5-5=0$	0
2	$3-2=1$	$-3-(-j)(1)=-3+j$	$(-3+j)+(-0.707-j0.707)(-1+3j)$ $=-0.172-j0.414$
2	$2+3=5$	$5+5=10$	$10-10=0$
3	$2-3=-1$	$-1+(-j)3=-1-3j$	$(-3-j)-(0.707-j0.707)(-1-3j)$ $=-0.172+j0.414$
4	$4+1=5$	$5-5=0$	0
1	$4-1=3$	$-1-(-j)3=-1+3j$	$(-3+j)-(-0.707-j0.707)(-1+3j)$ $=-5.828+j2.414$

$$X(k)=\{20,\ -5.828-j2.414,\ 0,\ -0.172-j0.414,$$
$$0,\ -0.172+j0.414,\ 0,\ -0.528+j2.414\}.$$

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ASSIGNMENT - III

① An 8-point DFT is given by $x[n] = \{2,2,2,2,1,1,1,1\}$. Compute 8point DFT of $x[n]$ by using radix-2 DIT-FFT. Also sketch magnitude and phase spectrum.

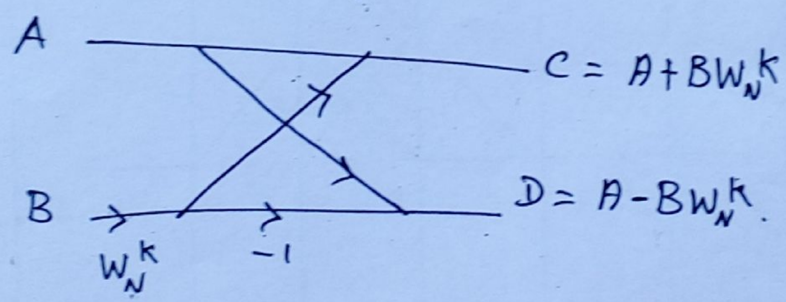
Soln:

Given,

$$x[n] = \{\underset{x(0)}{2}, \underset{x(1)}{2}, \underset{x(2)}{2}, \underset{x(3)}{2}, \underset{x(4)}{1}, \underset{x(5)}{1}, \underset{x(6)}{1}, \underset{x(7)}{1}\}.$$

* DIT-FFT Algorithm.

1. The input is Bit-Reversed.
2. The basic operation is



3. The output is in Normal order.

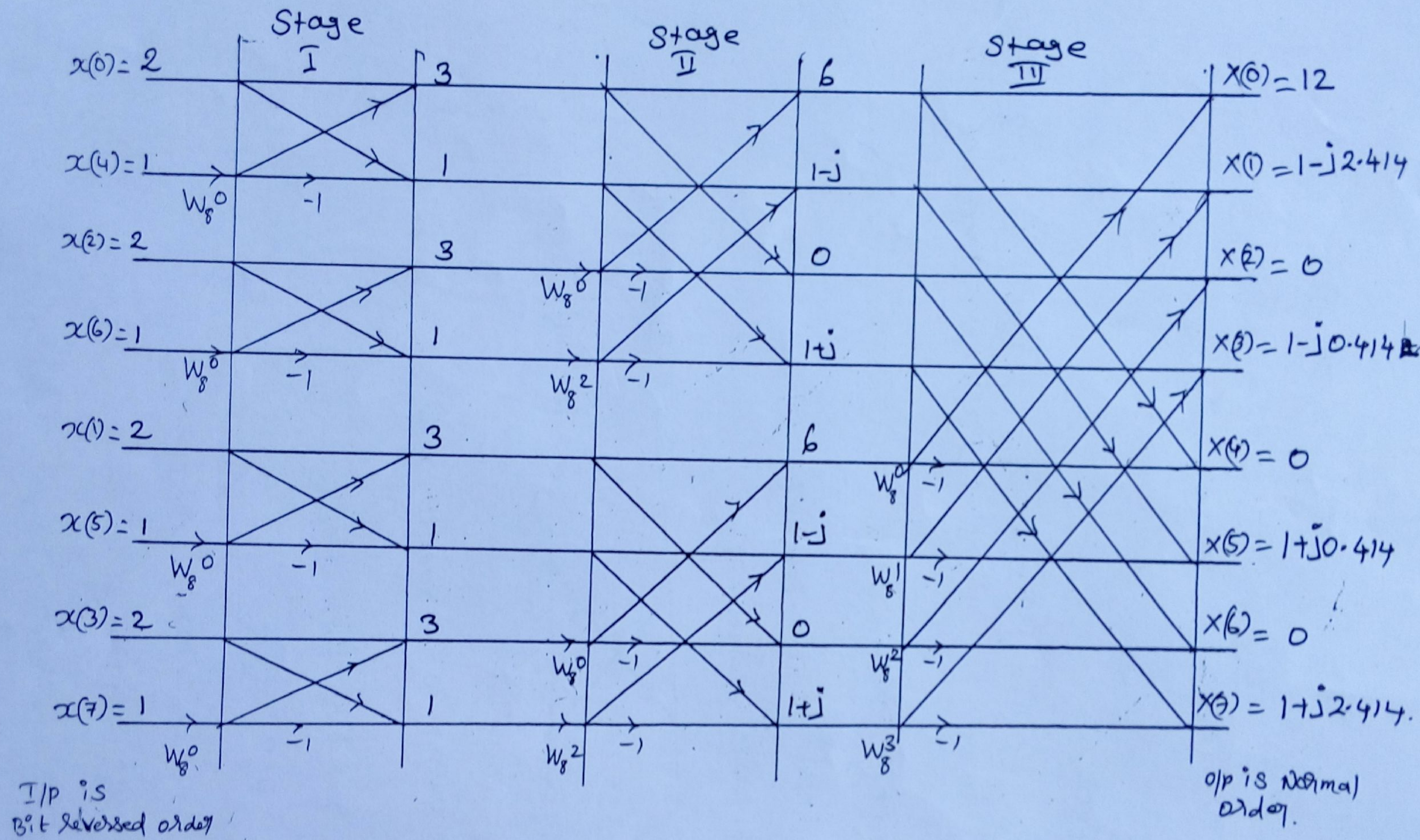
* The Twiddle factors are,

$W_8^0 = 1$

$W_8^1 = 0.707 - j0.707$.

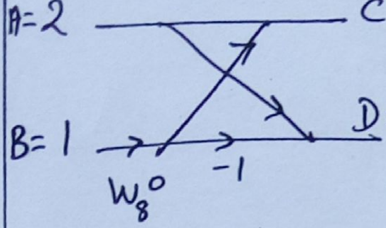
$W_8^2 = -j$

$W_8^3 = -0.707 - j0.707$.



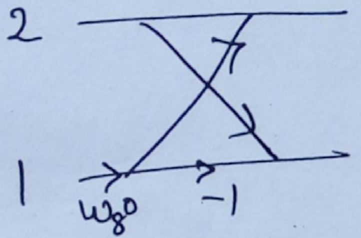
$$\boxed{X(k) = \{12,\ 1-j2.414,\ 0,\ 1-j0.414,\ 0,\ 1+j0.414,\ 0,\ 1+j2.414\}}$$

Stage I



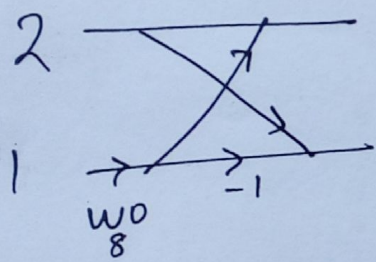
$C = A + BW_N^k = 2 + 1(1) = 3$

$D = A - BW_N^k = 2 - 1 = 1$



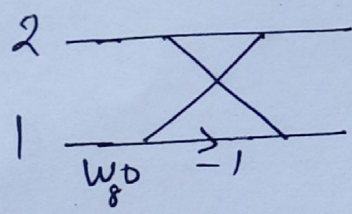
$C = 2 + 1(1) = 3$

$D = 2 - 1(1) = 1$



$C = 2 + 1(1) = 3$

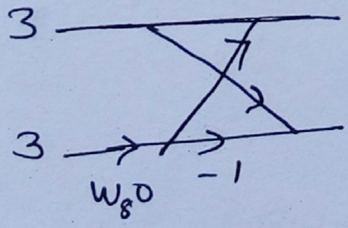
$D = 2 - 1(1) = 1$



$C = 2 + 1(1) = 3$

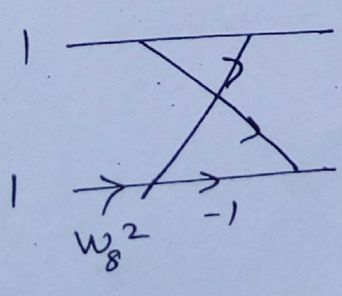
$D = 2 - 1(1) = 1$

Stage II



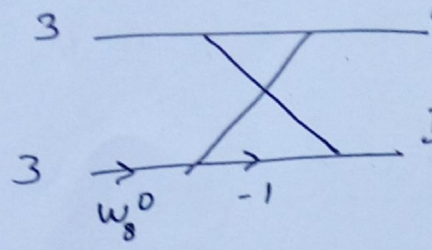
$C = 3 + 3(1) = 6$

$D = 3 - 3(1) = 0$



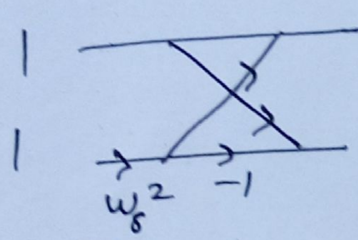
$C = 1 + 1(-j) = 1 - j$

$D = 1 - 1(-j) = 1 + j$



$C = 3 + 3(1) = 6$

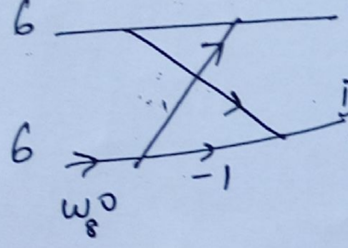
$D = 3 - 3(1) = 0.$



$C = 1 + 1(-j) = 1 - j$

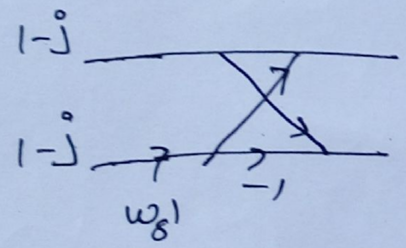
$D = 1 - 1(-j) = 1 + j$

Stage III



$C = 6 + 6(1) = 12$

$D = 6 - 6(1) = 0$



$C = 1 - j2.414.$

$D = 1 + j0.414.$

$$C = (1-j) + (1-j)(0.707 - j0.707)$$

$$= 1 - j + [0.707 - j0.707 - j0.707 - 0.707]$$

$$C = 1 - j2.414$$

$$D = (1-j) - (1-j)(0.707 - j0.707)$$

$$= 1 - j - [0.707 - j0.707 - j0.707 - 0.707]$$

$$= 1 - j - \cancel{0.707} + j0.707 + j0.707 + \cancel{0.707}$$

$$D = 1 + j0.414$$

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DIF Algorithm:

- Decimation in frequency FFT decomposes the DFT by recursively splitting the sequence elements $X(k)$ in the frequency domain into sets of smaller and smaller subsequences.

- To derive the decimation in frequency FFT algorithm for N, a power of 2, the input sequence $x(n)$ is divided into the first half and the last half of the points.

- In this algorithm the input sequence $x(n)$ is partitioned into two sequences each of length $\frac{N}{2}$ samples. The first sequence $x_1(n)$ consists of first $\frac{N}{2}$ samples of $x(n)$ and the second sequence $x_2(n)$ consists of the last $\frac{N}{2}$ samples of $x(n)$ i.e.,

$$x_1(n) = x(n), \quad n = 0, 1, 2, \ldots \frac{N}{2} - 1 \quad \text{— (1)}$$

$$x_2(n) = x\left(n + \frac{N}{2}\right), \quad n = 0, 1, 2, \ldots \frac{N}{2} - 1 \quad \text{— (2)}$$

If $N = 8$ the first sequence $x_1(n)$ has values for $0 \leq n \leq 3$ and $x_2(n)$ has values for $4 \leq n \leq 7$.

The N-point DFT of $x(n)$ can be written as

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{nk} + \sum_{n=\frac{N}{2}}^{N-1} x(n) W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n) W_N^{nk} + \sum_{n=0}^{\frac{N}{2}-1} x_2(n) W_N^{(n+\frac{N}{2})k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n) W_N^{nk} + W_N^{Nk/2} \sum_{n=0}^{\frac{N}{2}-1} x_2(n) W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n) W_N^{nk} + e^{-j\pi k} \sum_{n=0}^{\frac{N}{2}-1} x_2(n) W_N^{nk}$$

when k is even $e^{-j\pi k} = 1$

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} [x_1(n) + x_2(n)] \cdot W_N^{2nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} [x_1(n) + x_2(n)] W_{N/2}^{nk}$$

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} f(n) W_{N/2}^{nk} \quad \text{—— (3.a)}$$

where $f(n) = x_1(n) + x_2(n)$ —— (3.b)

Eqn (3.a) is the $\frac{N}{2}$ -point DFT of the $\frac{N}{2}$ - point sequence f(n) obtained by adding the first-half and the last-half of the input sequence. when k is odd

$$e^{-j\pi k} = -1$$

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} [x_1(n) - x_2(n)] W_N^{(2k+1)n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} [x_1(n) - x_2(n)] W_N^n \, W_{N/2}^{nk}$$

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} g(n) \, W_{N/2}^{nk} \quad \text{——— (4)}$$

where

$$g(n) = (x_1(n) - x_2(n)) W_N^n \quad \text{——— (5)}$$

Eqn (4) is the $\frac{N}{2}$ -point DFT of the sequence g(n) obtained by subtracting the second half of the input sequence from the first half and then multiplying the resulting sequence with W_N^n.

From eqn (3.a) and (4) we find that the even and odd samples of the DFT can be obtained from the $\frac{N}{2}$ -point DFTs of f(n) and g(n) respectively.

The eqn (3.b) and eqn (5) can be represented by a butterfly as shown in fig (1). This is the basic operation of DIF algorithm.

From eqn (3), for N=8, we have

$$X(0) = \sum_{n=0}^{3} [x_1(n) + x_2(n)] = \sum_{n=0}^{3} f(n)$$

$$= f(0) + f(1) + f(2) + f(3) \quad \text{—— (6)}$$

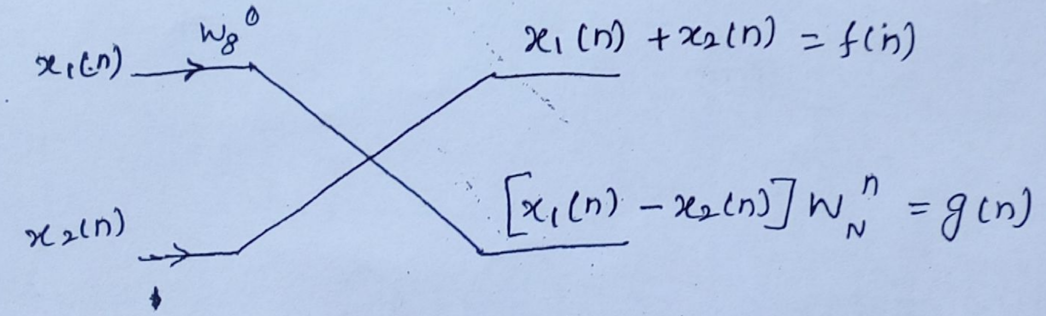

fig.1. flow graph of basic butterfly diagram for DIF algorithm.

$$X(2) = \sum_{n=0}^{3} (x_1(n) + x_2(n))\, W_8^{2n} = \sum_{n=0}^{3} f(n)\, W_8^{2n}$$

$$X(2) = f(0) + f(1)\, W_8^2 - f(2) - f(3)\, W_8^2 \quad \text{—— (7)}$$

$$\left[\begin{array}{l} \because W_8^4 = \left(e^{+j2\pi/8}\right)^4 = -1 \\ W_8^8 = \left(e^{j2\pi/8}\right)^8 = 1 \end{array}\right.$$

$$X(4) = \sum_{n=0}^{3} [x_1(n) + x_2(n)]\, W_8^{4n} = \sum_{n=0}^{3} f(n)\, W_8^{4n} = \sum_{n=0}^{3} f(n)\, (-1)^n$$

$$= f(0) - f(1) + f(2) - f(3) \quad \text{— (8)}$$

$$X(6) = \sum_{n=0}^{3} [x_1(n) + x_2(n)] W_8^{6n} = \sum_{n=0}^{3} f(n) (-W_8^2)^n$$

$$= f(0) - f(1) W_8^2 - f(2) + f(3) W_8^2 \qquad \text{— (9)}$$

From eqn (4) we have

$$X(1) = \sum_{n=0}^{3} [x_1(n) - x_2(n)] W_8^n = \sum_{n=0}^{3} g(n) = g(0) + g(1) + g(2) + g(3) \qquad \text{— (10)}$$

$$X(3) = \sum_{n=0}^{3} [x_1(n) - x_2(n)] W_8^{3n} = \sum_{n=0}^{3} g(n) W_8^{2n}$$

$$= g(0) + g(1) W_8^2 - g(2) - g(3) W_8^2 \qquad \text{— (11)}$$

$$X(5) = \sum_{n=0}^{3} [x_1(n) - x_2(n)] W_8^{5n} = \sum_{n=0}^{3} g(n) W_8^{4n} = \sum_{n=0}^{3} g(n) (-1)^n$$

$$= g(0) - g(1) + g(2) - g(3) \qquad \text{— (12)}$$

$$X(7) = \sum_{n=0}^{3} [x_1(n) - x_2(n)] W_8^{7n} = \sum_{n=0}^{3} g(n) (-W_8^2)^n$$

$$= g(0) - g(1) W_8^2 - g(2) + g(3) W_8^2 \qquad \text{— (13)}$$

We have seen that the even-indexed samples of $X(k)$ can be obtained from the 4-point DFT of the sequence $f(n)$ where

$f(n) = x_1(n) + x_2(n) \qquad n = 0, 1, \ldots, \frac{N}{2} - 1$

i.e.,

$$f(0) = x_1(0) + x_2(0)$$

$$f(1) = x_1(1) + x_2(1)$$

$$f(2) = x_1(2) + x_2(2)$$

$$f(3) = x_1(3) + x_2(3) \qquad \text{——— (14)}$$

The odd-indexed samples of $X(k)$ can be obtained from the 4-point DFT of the sequence $g(n)$ where

$$g(n) = [x_1(n) - x_2(n)] W_8^n$$

i.e.,

$$g(0) = [x_1(0) - x_2(0)] W_8^0$$

$$g(1) = [x_1(1) - x_2(1)] W_8^1$$

$$g(2) = [x_1(2) - x_2(2)] W_8^2$$

$$g(3) = [x_1(3) - x_2(3)] W_8^3 \qquad \text{——— (15)}$$

using the above information and the butterfly structure shown in fig. 1 we can draw the flow graph of 8-point DFT shown in fig (2).

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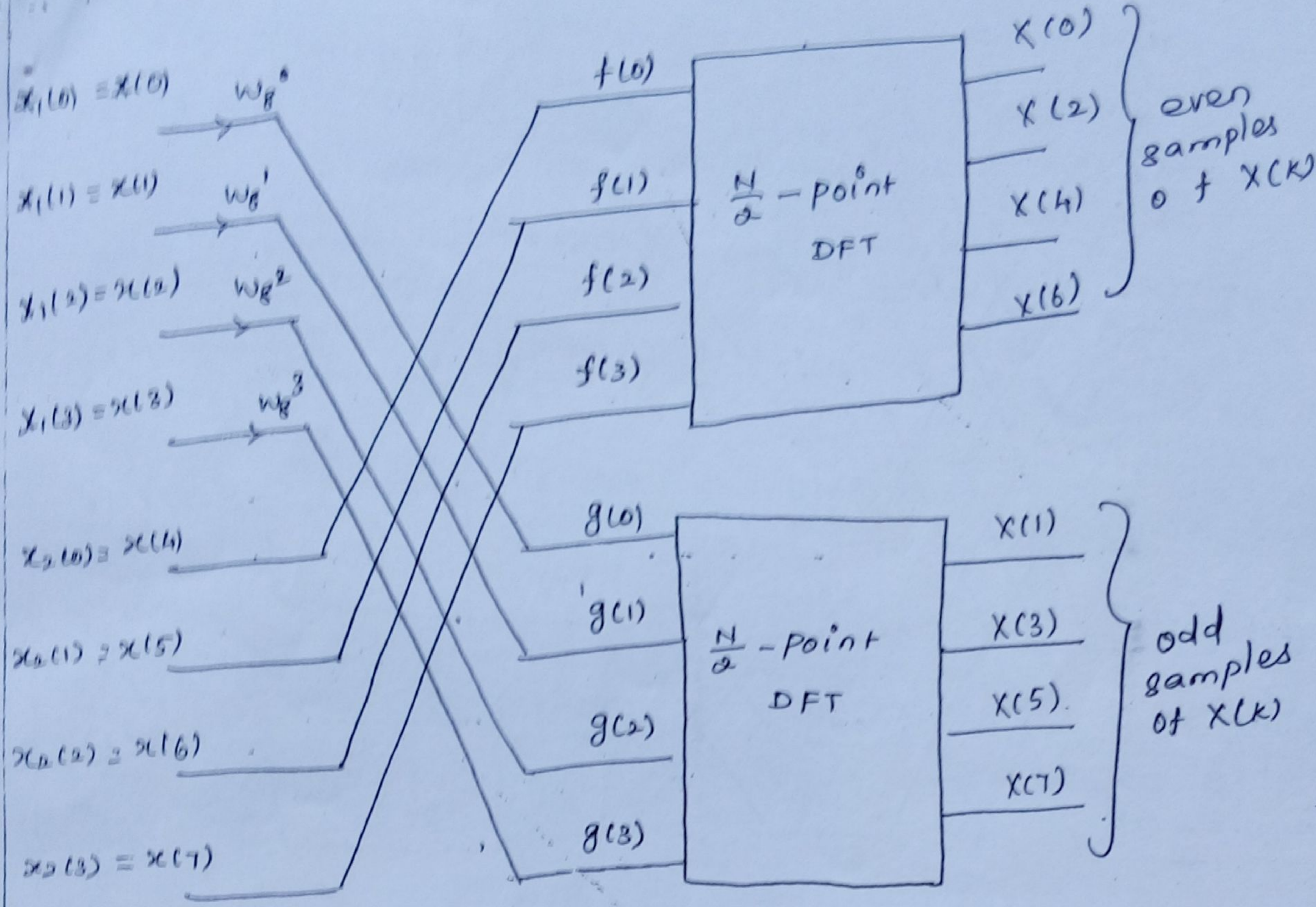


fig(2) Reduction of an 8-point DFT to two 4-point DFTs by decimation in frequency.

Now each $\frac{N}{2}$-point DFT can be computed by combining the first half and the last half of the input points for each of the $\frac{N}{2}$-point DFTs and then computing $\frac{N}{4}$-point DFTs. For the 8-point DFT example the resultant flow graph is shown in fig. 3.

The 2-point DFT can be found by adding and subtracting the input points. The fig(3) can be further reduced as in fig (4).

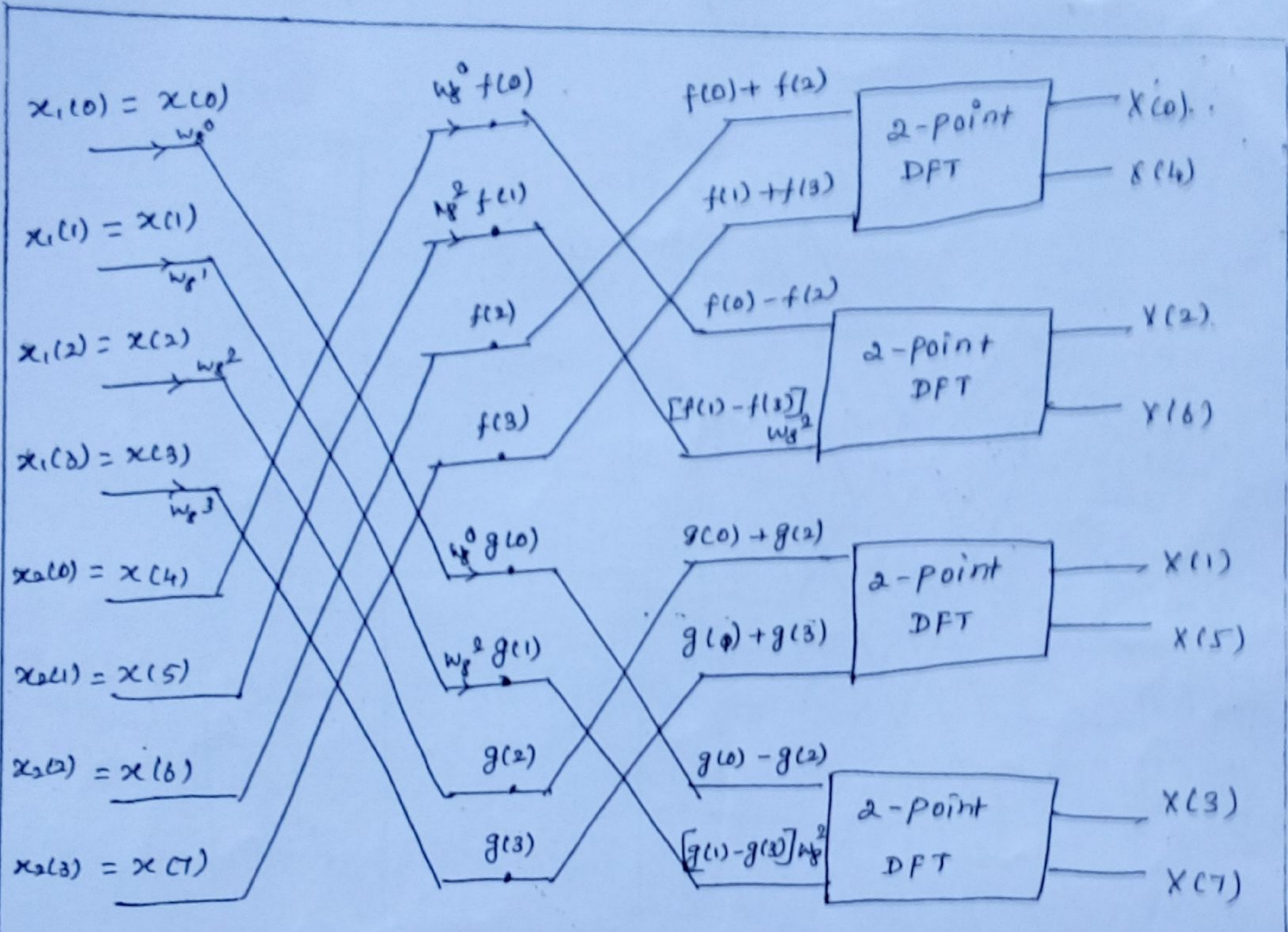
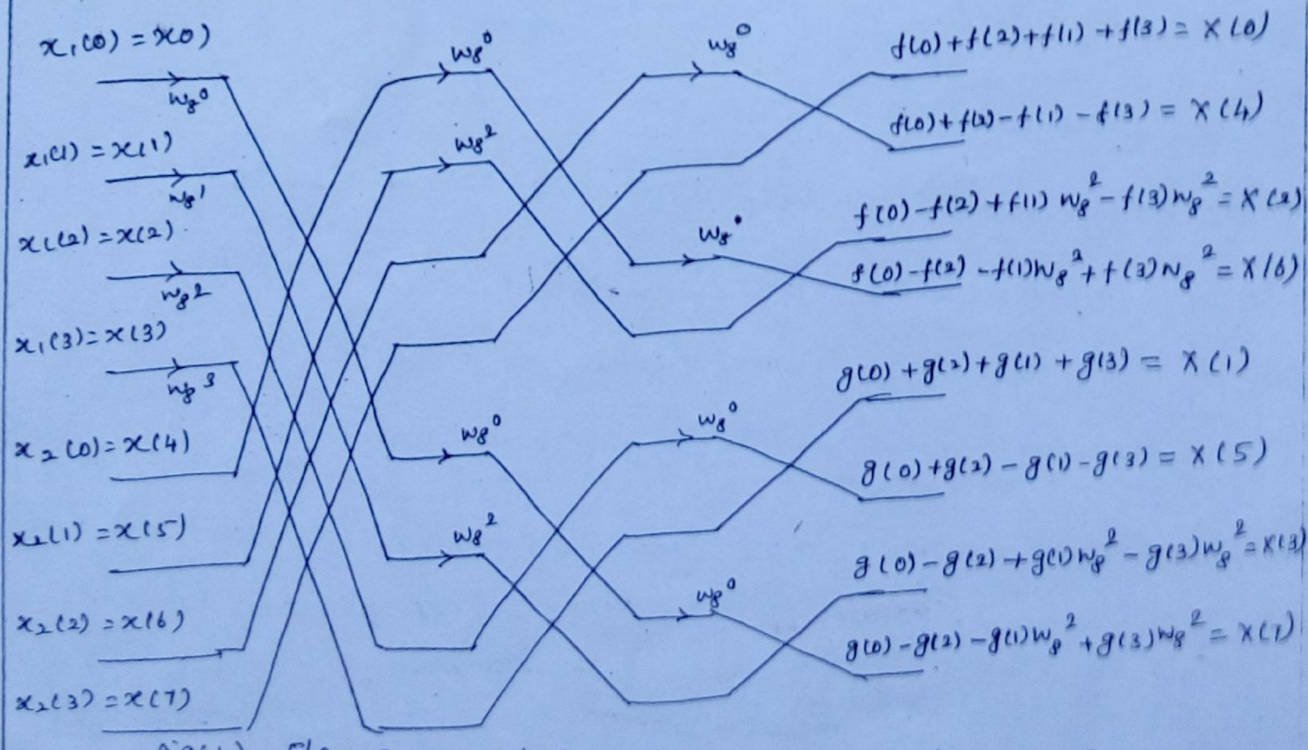


fig. 3 Flow graph of decimation in frequency decomposition of an 8-point DFT into four 2-point DFT computations.



fig(4) Flow graph of 8-point DIF-FFT algorithm.

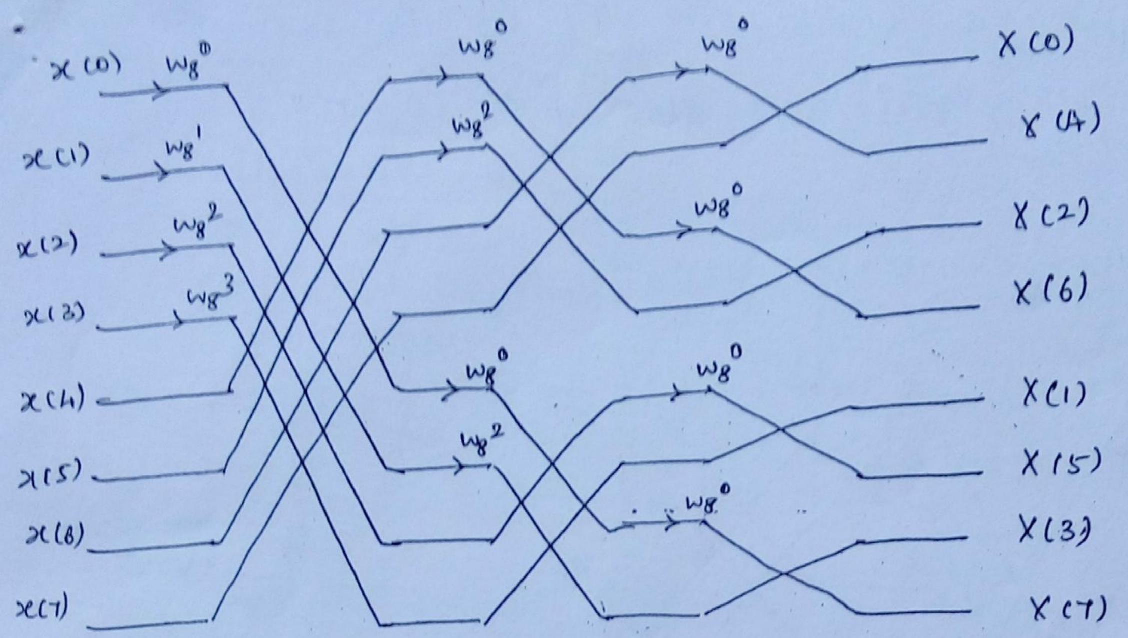


fig. 5. flow graph of complete decimation in frequency decomposition of an 8-point DFT computation.

Differences and Similarities between DIT and DIF Algorithms

Differences:

1. For decimation-in-time (DIT), the input is bit-reversed while the output is in natural order. whereas, for decimation in frequency the input is in natural order while the output is bit reversed order.

2. The DIF butterfly is slightly different from the DIT wherein DIF the complex multiplication takes place after the add-subtract operation.

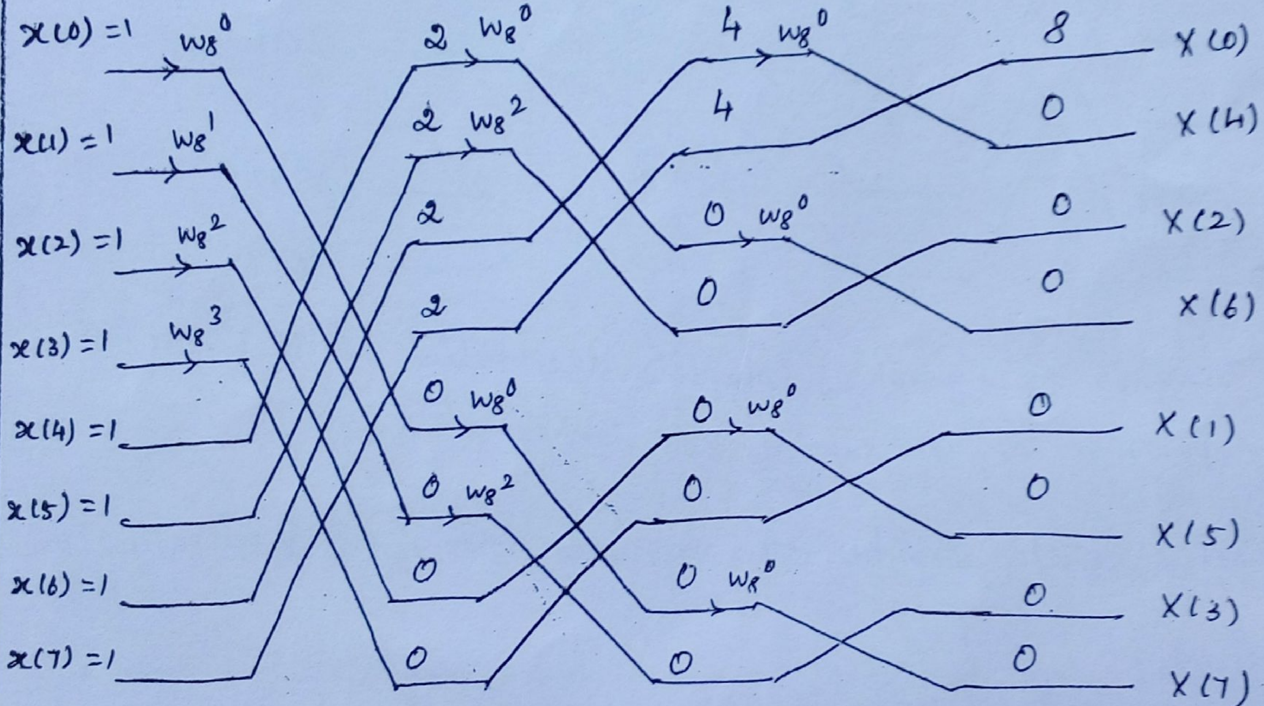
Similarities:

Both algorithms require $N \log_2 N$ operations to compute the DFT. Both algorithms can be done in-place and both need to perform bit reversal at some place during the computation.

1. Compute the eight point DFT of the sequence by using DIF algorithm: $x(n) = \begin{cases} 1 & 0 \le n \le 7 \\ 0 & \text{otherwise.} \end{cases}$

Soln

The given sequence $x(n) = \{1, 1, 1, 1, 1, 1, 1, 1\}$



$W_8^0 = 1, \quad W_8^1 = 0.707 - j0.707, \quad W_8^2 = -j, \quad W_8^3 = -0.707 - j0.707$

I/P	O/P of stage 1	O/P of stage 2	O/P of stage 3
1	$1+1=2$	$2+2=4$	$4+4=8$
1	$1+1=2$	$2+2=4$	$4-4=0$
1	$1+1=2$	$2-2=0$	0
1	$1+1=2$	$2-2=0$	0
1	$1-1=0$	0	0
1	$1-1=0$	0	0
1	$1-1=0$	0	0
1	$1-1=0$	0	0

$X(K) = \{8, 0, 0, 0, 0, 0, 0, 0\}$

IDFT using FFT Algorithm :

FFT algorithms can be used to compute an inverse DFT without any change in the algorithm. The inverse DFT of an N-point sequence $X(k)$, $k=0,1,..N-1$ is defined as

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W^{-nk} \quad \text{--- (1)}$$

where $W = e^{-j2\pi/N}$

Take complex conjugate and multiply by N, we obtain

$$N x^*(n) = \sum_{k=0}^{N-1} X^*(k)\, W^{nk} \quad \text{--- (2)}$$

The right hand side of eqn (2) is DFT of the sequence $X^*(k)$ and may be computed using any FFT algorithm. The desired output sequence $x(n)$ can then be found by complex conjugating the DFT and dividing by N to give:

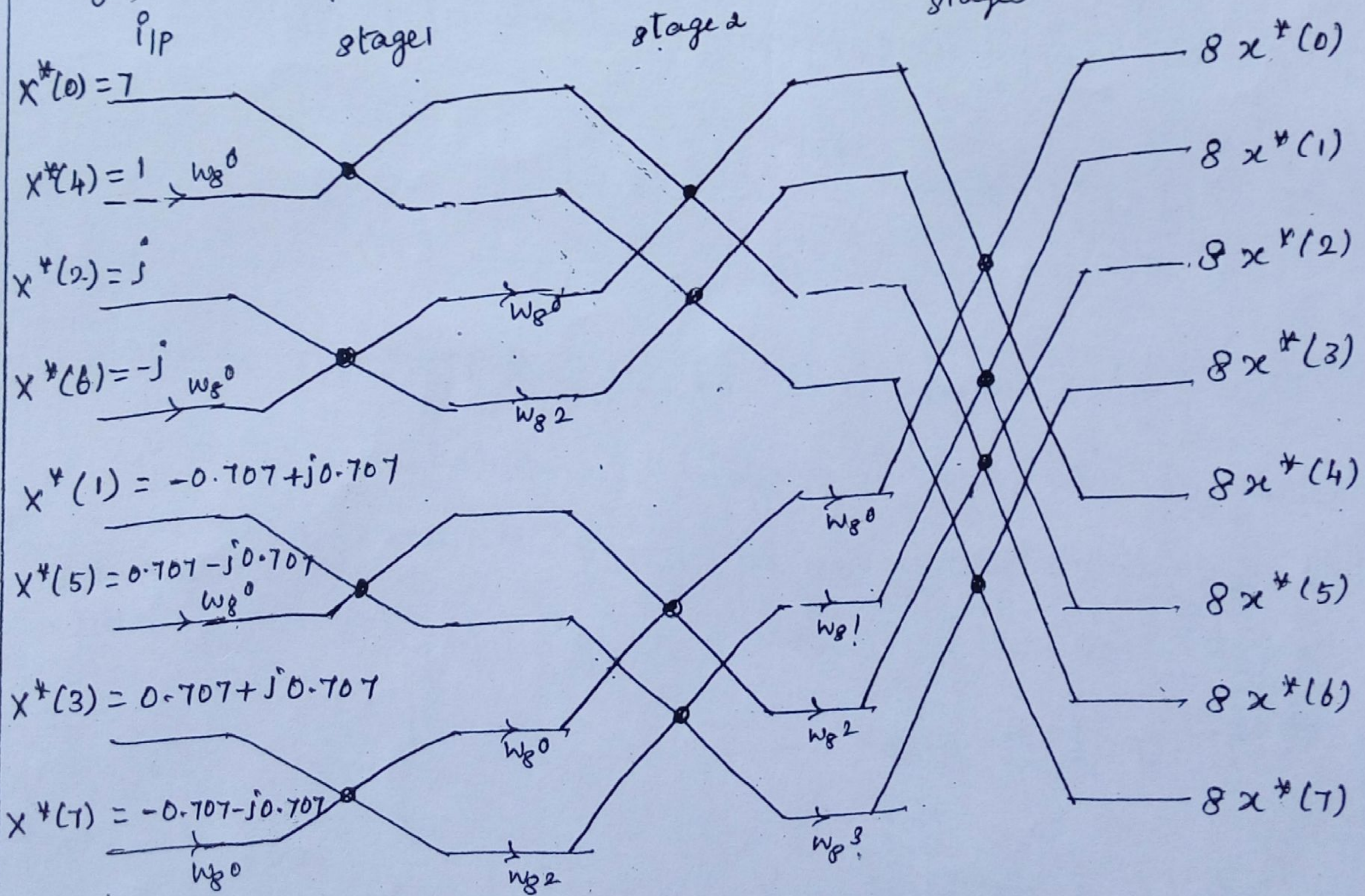
$$x(n) = \frac{1}{N}\left[\sum_{k=0}^{N-1} X^*(k)\, W^{nk}\right]^* \quad \text{--- (3)}$$

1. Compute IDFT of the sequence

$X(k) = \{7, -0.707 - j0.707, -j, 0.707 - j0.707, 1, 0.707 + j0.707, j, -0.707 + j0.707\}$ using DIT algorithm.

<u>Soln</u>

Take complex conjugate of X(k) and apply bit reversal index inputs to flow graph of 8-point DIT algorithm.



input	stage of 1	stage of 2	stage of 3
7	$7+1=8$	$8+0=8$	$8+0=8$
1	$7-1=6$	$6+(-j)2j=8$	$8+0(W^1)=8$
j	$j-j=0$	$8-0=8$	$8+0(W^2)=8$
$-j$	$j-(-j)=2j$	$6-(j)(2j)=4$	$4+(-0.707-j0.707)(-2.828+j2.828)=8$
$-0.707+j0.707$	$(-0.707+j0.707)+(0.707-j0.707)=0$	0	$8-0=8$
$0.707-j0.707$	$(-0.707+j0.707)-(0.707-j0.707)=-1.414+j1.414$	$(-1.414+j1.414)+(-j)(1.414+j1.414)=0$	$8-0(W^1)=8$
$0.707+j0.707$	$(0.707+j0.707)+(-0.707-j0.707)=0$	0	$8-0(W^2)=8$
$-0.707-j0.707$	$0.707+j0.707-(-0.707-j0.707)=1.414+j1.414$	$(-1.414+j1.414)-(-j)(1.414+j1.414)=-2.828+j2.828$	$4-(-0.707-j0.707)(-2.828+j2.828)=0$

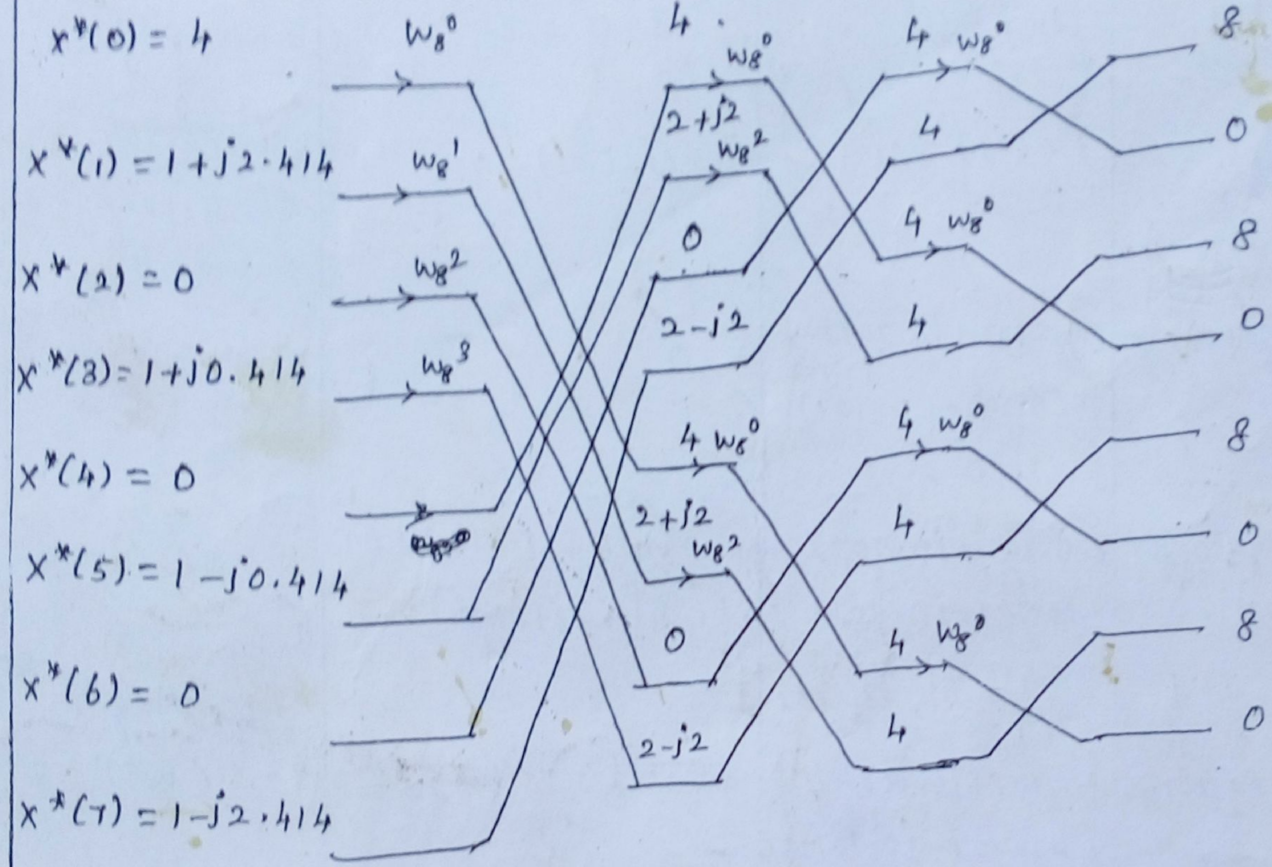
Output :

$$Nx^*(n) = \{8,8,8,8,8,8,8,0\}$$

$$x^*(n) = \{1,1,1,1,1,1,1,0\}$$

$$x(n) = \{1,1,1,1,1,1,1,0\}$$

2. Find the IDFT of the sequence $X(K) = \{4, 1-j2.414, 0, 1-j0.414, 0, 1+j0.414, 0, 1+j2.414\}$ using DIF algorithm.



The output $8x^*(n)$ is in bit reversal order

Therefore

$$x(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$$

T-12

UNIT - III

FIR FILTER DESIGN

①

3 design Methods :

1) Fourier Series Method
2) Window Method.
3) Frequency Sampling Method.

Drawback of ~~Frequency~~ Fourier Series Method:

(X) Gibb's oscillation
(2m) (OR)
Gibb's phenomenon

Window Method:

FIR → Linear phase FIR, Zero phase FIR

Linear phase FIR:

$h(n) = h(N-1-n)$

phase delay $\alpha = \frac{N-1}{2}$

$0 \le n \le N-1$

Zero phase FIR

$h(n) = h(-n)$

$\alpha = 0$

$-\frac{(N-1)}{2} \le n \le \frac{(N-1)}{2}$

Procedure : (for window method)

1. Select Frequency response $H_d(w)$
2. Take inverse fourier transform, we get

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(w)\, e^{jwn}\, dw$$

Impulse response of ideal filter

3. Select Window sequence $w(n)$

4. $h(n) = w(n) \cdot h_d(n)$

5. Take z Transform, we get $H(z)$

Frequency response Table for Linear phase FIR:

Filter type	$H_d(\omega)$
LPF	$H_d(\omega) = \begin{cases} e^{-j\omega\alpha}, & -\omega_c \le \pi \le \omega_c \\ 0, & \text{otherwise} \end{cases}$
HPF	$H_d(\omega) = \begin{cases} e^{-j\omega\alpha}, & -\pi \le \omega \le -\omega_c, \\ & \omega_c \le \omega \le \pi \\ 0, & \text{otherwise} \end{cases}$
BPF	$H_d(\omega) = \begin{cases} e^{-j\omega\alpha}, & -\omega_{c2} \le \omega \le -\omega_{c1}, \\ & \omega_{c1} \le \omega \le \omega_{c2} \\ 0, & \text{otherwise} \end{cases}$
BSF	$H_d(\omega) = \begin{cases} e^{-j\omega\alpha}, & -\pi \le \omega \le -\omega_{c2}, \\ & -\omega_{c1} \le \omega \le \omega_{c1} \\ & \omega_{c2} \le \omega \le \pi \\ 0, & \text{otherwise} \end{cases}$

Window Sequences:

1. Rectangular Window

$$w(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

(or)

$$\begin{cases} 1, & -\frac{(N-1)}{2} \le n \le \frac{(N-1)}{2} \\ 0, & \text{otherwise} \end{cases}$$

2 3

2. Triangular (or) Bartler window

$$w(n) = 1 - \frac{2|n|}{N-1}, \quad -\frac{(N-1)}{2} \leq n \leq \frac{(N-1)}{2}$$

0, otherwise

(or)

$$w(n) = 1 - \frac{2\left|n - \frac{N-1}{2}\right|}{N-1}, \quad 0 \leq n \leq N-1$$

0, otherwise

3. Hamming window:

$$w(n) = 0.54 + 0.46 \cos\left(\frac{2\pi n}{N-1}\right), \quad -\left(\frac{N-1}{2}\right) \leq n \leq \left(\frac{N-1}{2}\right)$$

(or)

$$= 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right), \quad 0 \leq n \leq N-1$$

3, 4 Raised cosine window $w(n) = \alpha + (1-\alpha)\cos\left(\frac{2\pi n}{N-1}\right)$

4) Hanning window:

$$w(n) = 0.5 + 0.5 \cos\left(\frac{2\pi n}{N-1}\right), \quad -\left(\frac{N-1}{2}\right) \leq n \leq \left(\frac{N-1}{2}\right)$$

(or)

$$= 0.5 - 0.5 \cos\left(\frac{2\pi n}{N-1}\right), \quad 0 \leq n \leq N-1$$

5) Blackman window:

$$w(n) = 0.42 + 0.5 \cos\left(\frac{2\pi n}{N-1}\right) + 0.08 \cos\left(\frac{4\pi n}{N-1}\right), \quad -\frac{(N-1)}{2} \leq n \leq \left(\frac{N-1}{2}\right)$$

(or)

$$= 0.42 - 0.5 \cos\left(\frac{2\pi n}{N-1}\right) + 0.08 \cos\left(\frac{4\pi n}{N-1}\right), \quad 0 \leq n \leq N-1$$

6) Kaiser window.

26.7.12

1. Design a linear Low pass FIR filter having nine samples with cut off 1.2 rad/sec use hanning window.

2. Design a filter with $H_d(\omega) = \begin{cases} e^{-j3\omega}, & -\frac{\pi}{4} \le \omega \le \frac{\pi}{4} \\ 0, & \text{Otherwise} \end{cases}$ use hamming window.

3. Design a filter with $H_d(\omega) = \begin{cases} 1, & -\frac{\pi}{4} \le \omega \le \frac{\pi}{4} \\ 0, & \text{Otherwise} \end{cases}$ N = 7. Use rectangular window.

If Window is not mentioned, take rectangular window.

1. Given data :- Linear phase FIR LPF

$N = 9$ $\qquad \alpha = \frac{N-1}{2} = 4$

$\omega_c = 1.2$ rad/sec.

hanning window.

i. $H_d(\omega) = \begin{cases} e^{-j\omega\alpha}, & -\omega_c \le \pi \le \omega_c \\ 0, & \text{Otherwise} \end{cases}$

ii. Take inverse fourier transform

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} H_d(\omega)\, e^{j\omega n}\, d\omega.$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\alpha)}\, d\omega.$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega_c(n-\alpha)}}{j(n-\alpha)} - \frac{e^{-j\omega_c(n-\alpha)}}{j(n-\alpha)}\right]$$

$$= \frac{1}{\pi(n-\alpha)}\left[\frac{e^{j\omega_c(n-\alpha)} - e^{-j(n-\alpha)}}{2j}\right]$$

$$h_d(n) = \frac{1}{\pi(n-\alpha)} \sin \omega_c(n-\alpha) \quad , n \neq \alpha$$

apply L Hospital rule.

$$\lim_{\theta \to 0} \frac{\sin A\theta}{\theta} = A.$$

$$= \lim_{n \to \alpha} \frac{1}{\pi(n-\alpha)} \sin \omega_c (n-\alpha)$$

$$h_d(n) = \frac{\omega_c}{\pi} \quad , n = \alpha.$$

iii. Hanning window is given by

$$w(n) = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right), & 0 \leq n \leq N-1 \\ 0 & , \text{otherwise} \end{cases}$$

iv. $h(n) = w(n) . h_d(n)$

$$h(n) = \frac{1}{\pi(n-\alpha)} \sin \omega_c(n-\alpha) . \left\{0.5 - 0.5\cos\frac{2\pi n}{N-1}\right\}, n \neq \alpha$$

$$h(n) = \frac{\omega_c}{\pi}\left\{0.5 - 0.5\cos\frac{2\pi n}{N-1}\right\}, n = \alpha \qquad 0 \leq n \leq N-1$$

$$0 \leq n \leq 8$$

$$h(0) = \frac{1}{-4\pi} \sin \cancel{\omega_c}^{(1.2)}(-4)\left\{0.5 - 0.5\cos\frac{0}{9-1}\right\}$$

$$= \frac{\sin \cancel{\omega_c}^{(1.2)}(4)}{4\pi}\{0.5 - 0.5\}$$

$$= 0$$

$$h(1) = \frac{1}{-3\pi} \sin(1.2)(3) \left\{ 0.5 - 0.5 \cos \frac{2\pi}{9-1} \right\}$$

$$= -0.0469 \{ 0.5 - 0.5(0.707) \}$$

$$= -0.00686$$

$$h(2) = \frac{1}{-2\pi} \sin(1.2)(-2) \left\{ 0.5 - 0.5 \cos \frac{4\pi}{8} \right\}$$

$$= -0.0704 \{ 0.5 \} = +0.5375$$

$$h(3) = \frac{1}{-\pi} \sin(-1.2) \left\{ 0.5 - 0.5 \cos \frac{6\pi}{8} \right\}$$

$$= 0.258$$

$$h(4) = \frac{1.2}{\pi} \left(0.5 - 0.5 \cos \frac{8\pi}{8} \right) = 0.3819$$

$$h(n) = h(N-1-n)$$

$$h(n) = h(8-n) \qquad h(0) = h(8)$$

$$h(1) = h(7)$$

$$h(2) = h(6)$$

$$h(3) = h(5)$$

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$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$= h(0) + h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3} + h(4) z^{-4} + h(5) z^{-5} + h(6) z^{-6} + h(7) z^{-7} + h(8) z^{-8}$$

$$= 0 - 0.00686 z^{-1} + 0.5375 z^{-2} + 0.258 z^{-3} + 0.3819 z^{-4} + 0.258 z^{-5} + 0.5375 z^{-6} - 0.00686 z^{-7} + 0$$

$$= -0.00686(z^{-1}+z^{-7}) + 0.5375(z^{-2}+z^{-6}) + 0.258(z^{-3}+z^{-5})$$

$$+0.8819z^{-4}$$

2. $H_d(\omega) = \begin{cases} e^{-j3\omega} & ; -\pi/4 \le \omega \le \pi/4 \\ 0 & ; \text{otherwise} \end{cases}$

$\alpha = 3$

$\alpha = \frac{N-1}{2}$

$3 = \frac{N-1}{2}$

$N = 7$ Similar to prob. 1

3. $H_d(\omega) = \begin{cases} 1 & -\pi/4 \le \omega \le \pi/4 \\ 0 & \text{otherwise} \end{cases}$

given: $N = 7$

zero phase FIR LPF

(i) $H_d(\omega) = \begin{cases} 1, & -\pi/4 \le \omega \le \pi/4 \\ 0, & \text{otherwise} \end{cases}$

(ii) Take inverse fourier transform

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} H_d(\omega) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1 \cdot e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi jn}\left[e^{j\omega_c n} - e^{-j\omega_c n}\right]$$

$$= \frac{1}{\pi n}\sin\omega_c n$$

$h_d(n) = \frac{1}{\pi n} \sin w_c n \quad , n \neq 0.$

apply L Hospital rule

$\lim_{\theta \to 0} \frac{\sin A\theta}{\theta} = A$

$= \lim_{n \to 0} \frac{\sin w_c n}{\pi n}$

$h_d(n) = \frac{w_c}{\pi} \quad n = 0.$

(iii) Rectangular window is given by

$$w(n) = \begin{cases} 1 & -\frac{(N-1)}{2} \le n \le \frac{(N-1)}{2} \\ 0 & \text{Otherwise.} \end{cases}$$

(iv) $h(n) = w(n)\, h_d(n)$

$h(n) = \frac{1}{\pi n} \sin w_c n \, , \; n \neq 0$

$h(n) = \frac{w_c}{\pi} \, , \; n = 0 \qquad -\frac{(N-1)}{2} \le n \le \frac{(N-1)}{2}$

$-3 \le n \le 3$

$h(-3) = \frac{1}{-3\pi} \sin \frac{\pi}{4}(-3) = 0.0750$

$h(-2) = \frac{1}{-2\pi} \sin \frac{\pi}{4}(-2) = 0.1591$

$h(-1) = \frac{1}{-\pi} \sin \frac{\pi}{4}(-1) = 0.2251$

$h(0) = \frac{w_c}{\pi} = \frac{\pi}{4\pi} = 0.25$

$h(1) = \frac{1}{\pi} \sin \frac{\pi}{4} = 0.2251$

$h(2) = \frac{1}{2\pi} \sin \frac{2\pi}{4} = 0.1591$

$h(3) = \frac{1}{3\pi} \sin \frac{3\pi}{4} = 0.0750$

$$H(z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{N-1} h(n) z^{-n}$$

$$= h(-3)z^{3} + h(-2)z^{2} + h(-1)z^{1} + h(0)z^{0} + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3}$$

$$= 0.0750(z^{3}+z^{-3}) + 0.1591(z^{-2}+z^{2}) + 0.2251(z^{-1}+z) + 0.25.$$

4. Design a linear phase and zero phase HP FIR filter with cut off freq. 1.2 rad/sec and $N = 9$. use Hamming window.

given: Linear phase

$N = 9$

$w_c = 1.2$ rad/sec

Hamming window

i. $H_d(\omega) = \begin{cases} e^{-j\omega\alpha}, & -\pi \le \omega \le -\omega_c \\ 0 \text{ otherwise} & \omega_c \le \omega \le \pi \end{cases}$

ii. Take inverse fourier transform.

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} H_d(\omega) e^{j\omega n}\, d\omega.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{-j\omega\alpha} e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{-j\omega\alpha} e^{j\omega n}\, d\omega.$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{\omega_c}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_c(n-\alpha)} - e^{-\pi j(n-\alpha)} + e^{j\pi(n-\alpha)} - e^{j\omega_c(n-\alpha)}}{j(n-\alpha)}\right]$$

$$= \frac{1}{\pi(n-\alpha)}\left[\frac{e^{-j\omega_c(n-\alpha)} - e^{j\omega_c(n-\alpha)}}{\cancel{j(n-\alpha)}\, 2j} + \frac{e^{j\pi(n-\alpha)} - e^{-j\pi(n-\alpha)}}{2j}\right]$$

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$$h_d(n) = \frac{1}{\pi(n-\alpha)}\left[\sin(n-\alpha)\pi - \sin \omega_c(n-\alpha)\right] \quad , n \neq \alpha$$

Applying L'Hospital rule

$$\lim_{\theta \to 0} \frac{\sin A\theta}{\theta} = A$$

$$h_d(n) = \frac{\pi}{\pi} - \frac{\omega_c}{\pi} = 1 - \frac{\omega_c}{\pi} \quad , n = \alpha$$

(iii) Hamming window is given by

$$w(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right); & n = 0 \text{ to } N-1 \\ 0 & \text{, otherwise} \end{cases}$$

(iv) $h(n) = h_d(n)w(n)$

$$h(n) = \frac{1}{\pi(n-\alpha)}\left[\sin(n-\alpha)\pi - \sin\omega_c(n-\alpha)\right]\left\{0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)\right\}$$

$n \neq \alpha$

$$h(n) = \left(1 - \frac{\omega_c}{\pi}\right)\left(0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)\right) \quad ; n = \alpha$$

$0 \leq n \leq N-1$

$0 \leq n \leq 8$

$$\alpha = \frac{N-1}{2} = \frac{9-1}{2} = 4$$

$$h(0) = \frac{1}{-4\pi}\left[\sin(-4\pi) - \sin 1.2(-4)\right]\left\{0.54 - 0.46\cos 0\right\}$$

$$= 0.0792 \times 0.08$$

$$= 0.006336$$

$$h(1) = \frac{1}{-3\pi}\left[\sin(-3\pi) - \sin 1.2(-3)\right]\left\{0.54 - 0.46\cos\left(\frac{2\pi}{8}\right)\right\}$$

$$= 0.0469 \times 0.2148 = 0.01007$$

$$h(2) = \frac{1}{-2\pi}\left[\sin(-2\pi) - \sin 1.2(-2)\right]\left\{0.54 - 0.46\cos\left(\frac{4\pi}{8}\right)\right\}$$

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$$= -0.0581$$

$$h(3) = \frac{1}{-\pi}\left[\sin(-\pi) - \sin 1.2(-1)\right]\left\{0.54 - 0.46\cos\frac{6\pi}{8}\right\}$$

$$= -0.2566.$$

$$h(4) = \left(1 - \frac{1.2}{\pi}\right)\left(0.54 - 0.46\cos\frac{8\pi}{8}\right)$$

$$= 0.6181.$$

$$h(n) = h(N-1-n)$$
$$h(n) = h(8-n)$$
$$h(0) = h(8), \quad h(1) = h(7), \quad h(2) = h(6), \quad h(3) = h(5)$$

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} + h(8)z^{-8}$$

$$= 0.0063 + 0.01007z^{-1} - 0.0581z^{-2} - 0.2566z^{-3} + 0.6181z^{-4} - 0.2566z^{-5} - 0.0581z^{-6} + 0.01007z^{-7} + 0.006336z^{-8}$$

$$= (1+z^{-8})0.0063 + (z^{-1}+z^{-7})0.01007 - 0.0581(z^{-2}+z^{-6}) - 0.2566(z^{-3}+z^{-5})$$

Realization of FIR filters:

1. Direct form (Transversal Structure)
2. Cascade form.
3. Linearphase realization
4. Polyphase realization.

1. Realize the following system using direct form.

a) $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-5}$

b) $y(n) = x(n) + 3x(n-1) + 5x(n-3) + 6x(n-5)$

2. Realize the following system using cascade form.

a) $H(z) = (1 + 5z^{-1} + 3z^{-2})(1 + 5z^{-1} + 6z^{-3} + 7z^{-4})$

3. Realize the following using linearphase structure (minimum no. of multiplier).

a) $H(z) = (1 + 3z^{-1} + 5z^{-2} + 6z^{-3} + 5z^{-4} + 3z^{-5} + z^{-6})$.

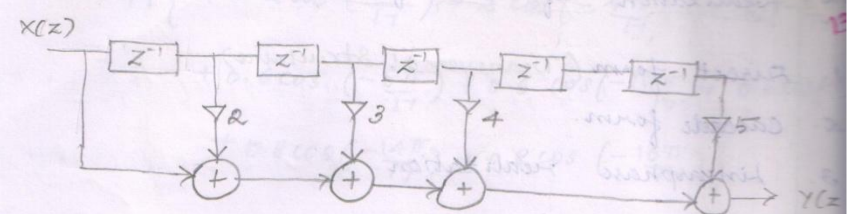
4. Design a linear phase Low pass Filter with cut off freq. $\pi/2$ radian/second and the no. of Samples $N = 9$. using Sampling method.

5. Design a filter with $H_d(w) = \begin{cases} e^{-j2w}, & |w| \le \pi/4 \\ 0 & \text{otherwise} \end{cases}$ using triangular window.

1. a) $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-5}$

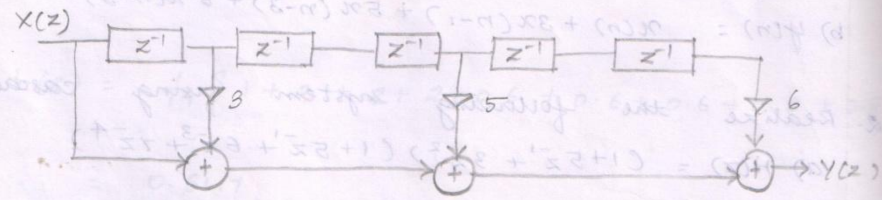
$$\frac{Y(z)}{X(z)} = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-5}$$

$$Y(z) = X(z) + 2z^{-1}X(z) + 3z^{-2}X(z) + 4z^{-3}X(z) + 5z^{-5}X(z)$$



b) $y(n) = x(n) + 3x(n-1) + 5x(n-3) + 6x(n-5)$

$Y(z) = X(z) + 3z^{-1}X(z) + 5z^{-3}X(z) + 6z^{-5}X(z)$



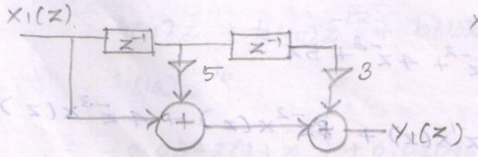
2. $H(z) = (1+5z^{-1}+3z^{-2})(1+5z^{-1}+6z^{-3}+7z^{-4})$

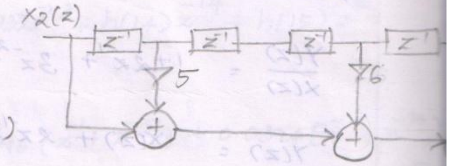
$H(z) = H_1(z) \cdot H_2(z)$

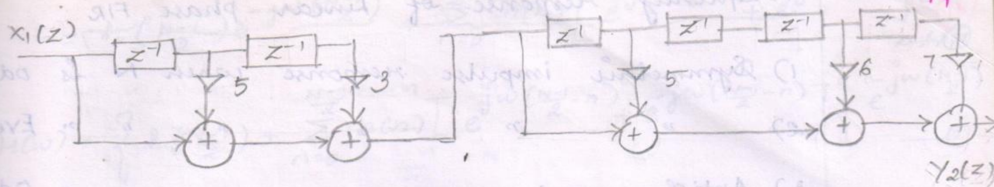
$H_1(z) = 1+5z^{-1}+3z^{-2}$ $\quad H_2(z) = 1+5z^{-1}+6z^{-3}+7z^{-4}$

$\frac{Y_1(z)}{X_1(z)} = 1+5z^{-1}+3z^{-2}$ $\quad \frac{Y_2(z)}{X_2(z)} = 1+5z^{-1}+6z^{-3}+7z^{-4}$

$Y_1(z) = X_1(z) + 5z^{-1}X_1(z) + 3z^{-2}X_1(z)$ $\quad Y_2(z) = X_2(z) + 5z^{-1}X_2(z) + 6z^{-3}X_2(z) + 7z^{-4}X_2(z)$







3. $H(z) = 1+3z^{-1}+5z^{-2}+6z^{-3}+5z^{-4}+3z^{-5}+z^{-6}$

Linear phase condition of FIR filter is given by

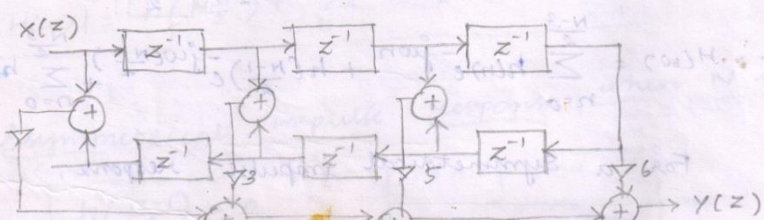
$$h(n) = h(N-1-n)$$

$$h(n) = \{1,3,5,6,5,3,1\}$$

$$N=7$$

$h(n) = h(6-n)$ $\quad h(0)=h(6) \quad h(1)=h(5) \quad h(2)=h(4)$

The above equation satisfies linear phase condition.



$$\frac{Y(z)}{X(z)} = 1+3z^{-1}+5z^{-2}+6z^{-3}+5z^{-4}+3z^{-5}+z^{-6}$$

$$Y(z) = X(z)+3z^{-1}X(z)+5z^{-2}X(z)+6z^{-3}X(z)+5z^{-4}X(z)+3z^{-5}X(z)+z^{-6}X(z).$$

$$=1(X(z)+z^{-6}X(z)) + 3(z^{-1}X(z)+z^{-5}X(z)) + 5(z^{-2}X(z)+z^{-4}X(z)) + 6X(z)z^{-3}.$$

7.8.12

Frequency response of Linear phase FIR:

1) Symmetric impulse response when N is odd
2) " " " " " Even
3) AntiSymmetric " " " " " Odd
4) " " " " " " Even

1) Symmetric Impulse response when N is odd

W.k.t

$$H(\omega) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n} \quad \text{—— ①}$$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n) e^{-j\omega n}$$

Let $n = N-1-n$

In 3rd term, $\frac{N+1}{2} = N-1-n$

$$n = N-1-\left(\frac{N+1}{2}\right) = \boxed{\frac{N-3}{2}}$$

$N-1 = N-1-n$

$\boxed{n=0}$

$$\therefore H(\omega) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n) e^{-j\omega(N-1-n)}$$

For a Symmetrical impulse response,

$$\boxed{h(n) = h(N-1-n)}$$

$$\therefore H(\omega) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega\left(\frac{N-1}{2} - \frac{N-1}{2} + n\right)} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega\left(\frac{N-1}{2} + \frac{N-1}{2} - n\right)}$$

$$= h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[e^{j\omega\left(\frac{N-1}{2}\right)} e^{-j\omega n} e^{-j\omega\left(\frac{N-1}{2}\right)} + e^{-j\omega\left(\frac{N-1}{2}\right)} e^{-j\omega\left(\frac{N-1}{2} - n\right)} \right]$$

$$= h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n)\left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right]e^{-j\omega\left(\frac{N-1}{2}\right)}$$ (16)

$$H(\omega) = \left\{h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n)\left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right]\right\}e^{-j\omega\left(\frac{N-1}{2}\right)} \quad \text{— (2)}$$

in general,

$$\boxed{H(\omega) = |H(\omega)|e^{j\theta(\omega)}} \quad \text{— (3)}$$

$$\boxed{\begin{aligned} |H(\omega)| &= h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n)\, 2\cos\omega\left(\frac{N-1}{2}-n\right) \\ \theta(\omega) &= -\left(\frac{N-1}{2}\right)\omega = -\alpha\omega \end{aligned}}$$

Let $k = \frac{N-1}{2} - n$

$n = \frac{N-1}{2} - k$

when $n = 0$, $k = \frac{N-1}{2}$

$n = \frac{N-3}{2}$, $k = 1$.

$$\boxed{|H(\omega)| = h\left(\frac{N-1}{2}\right) + \sum_{k=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-k\right)\cos\omega k}$$

2) Asymmetrical impulse response when N is odd:

$$\boxed{\begin{aligned} h\left(\frac{N-1}{2}\right) &= 0 \\ h(n) &= -h(N-1-n) \end{aligned}}$$

$$H(\omega) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n} \quad \text{— (1)}$$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n)\,e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n)\,e^{-j\omega n}$$

Let $n = N-1-n$

In 3rd term, $\frac{N+1}{2} = N-1-n$ | $N-1 = N-n-1$

$$\boxed{n = \frac{N-3}{2}} \qquad \boxed{n=0}$$

$$\therefore \; H(\omega) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\tfrac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n) e^{-j\omega(N-1-n)}$$

For antisymmetrical impulse response

$$\boxed{h(n) = -h(N-1-n)}$$

$$h\left(\tfrac{N-1}{2}\right) = 0.$$

$$H(\omega) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} - \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega(N-1-n)}$$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega\left(\frac{N-1}{2} - \frac{N-1}{2} + n\right)} - \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega\left(\frac{N-1}{2} + \frac{N-1}{2} - n\right)}$$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[e^{j\omega\left(\frac{N-1}{2}\right)} e^{-j\omega n} e^{-j\omega\left(\frac{N-1}{2}\right)} - e^{-j\omega\left(\frac{N-1}{2}\right)} e^{-j\omega\left(\frac{N-1}{2}\right)} e^{+j\omega n} \right]$$

$$H(\omega) = \sum_{n=0}^{\frac{N-3}{2}} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} - e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right] e^{-j\omega\left(\frac{N-1}{2}\right)} \quad \text{--- (2)}$$

$$= \left\{ \sum_{n=0}^{\frac{N-3}{2}} h(n)\, 2j \sin\omega\left(\tfrac{N-1}{2} - n\right) \right\} e^{-j\omega\left(\frac{N-1}{2}\right)} = \left\{ \sum_{n=0}^{\frac{N-3}{2}} h(n)\, 2\sin\omega\left(\tfrac{N-1}{2} - n\right) \right\} e^{-j\left(\omega\left(\frac{N-1}{2}\right) - \frac{\pi}{2}\right)}$$

in general

$$H(\omega) = |H(\omega)| e^{j\theta(\omega)} \quad \text{--- (3)}$$

$$j = e^{j\frac{\pi}{2}}$$

$$\boxed{\begin{aligned} |H(\omega)| &= \sum_{n=0}^{\frac{N-3}{2}} h(n)\, 2\sin\omega\left(\tfrac{N-1}{2} - n\right) \\ \theta(\omega) &= -\left(\omega\left(\tfrac{N-1}{2}\right) - \tfrac{\pi}{2}\right) = -\left(\omega\alpha - \tfrac{\pi}{2}\right) \end{aligned}}$$

Let $k = \frac{N-1}{2} - n$

$n = \frac{N-1}{2} - k$

when $n=0$, $k = \frac{N-1}{2}$

$n = \frac{N-3}{2}$, $k=1$

$$\boxed{|H(\omega)| = \sum_{k=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-k\right) \sin\omega k}$$

3) Symmetrical impulse response when N is even:

W.k.t.

$$H(\omega) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n} \quad \text{--- (1)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n) e^{-j\omega n}$$

Let $n = N-1-n$

$N-1-n = \frac{N}{2}$

$\boxed{n = \frac{N}{2}-1}$

$N-1-n = N-1$

$\boxed{n = 0}$

$$H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(N-1-n) e^{-j(N-1-n)\omega}$$

For Symmetrical $h(n)$,

$\boxed{h(n) = h(N-1-n)}$

$$\therefore H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega\left(\frac{N-1}{2} - \frac{N-1}{2} - n\right)} + \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega\left(\frac{N-1}{2} + \frac{N-1}{2} - n\right)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$\boxed{H(\omega) = \left[\sum_{n=0}^{\frac{N}{2}-1} h(n)\, 2\cos\omega\left(\frac{N-1}{2}-n\right) \right] e^{-j\omega\left(\frac{N-1}{2}\right)}} \quad \text{--- (2)}$$

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in general

$$H(\omega) = |H(\omega)| e^{j\theta(\omega)} \quad \text{---} ③$$

From ② and ③

$$|H(\omega)| = \sum_{n=0}^{\frac{N}{2}-1} 2h(n) \cos\omega\left(\frac{N-1}{2} - n\right)$$

$$\theta(\omega) = -\omega\left(\frac{N-1}{2}\right) = -\omega\alpha$$

Let $k = \frac{N}{2} - n$ | $n = 0, \ k = \frac{N}{2}$

$n = \frac{N}{2} - k$ | $n = \frac{N}{2} - 1, \ k = 1.$

$$\boxed{H(\omega) = \left[\sum_{k=1}^{N/2} 2h\left(\frac{N}{2} - k\right) \cos\omega\left(k - \frac{1}{2}\right)\right] e^{-j\omega\alpha}}$$

4) Antisymmetrical impulse response when N is even:

W.K.T

$$H(\omega) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n} \quad \text{---} ①$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n) e^{-j\omega n}$$

Let $n = N-1-n$ | $N-1-n = N-1$

$N-1-n = \frac{N}{2}$ | $\boxed{n = 0}$

$\boxed{n = \frac{N}{2} - 1}$

$$H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(N-1-n) e^{-j(N-1-n)\omega} \quad ④$$

For antisymmetrical $h(n)$,

$$\boxed{h(n) = -h(N-1-n)}$$

$$H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{j\omega\left(\frac{N-1}{2}-\frac{N-1}{2}-n\right)} - \sum_{n=0}^{\frac{N}{2}-1} h(n) e^{-j\omega\left(\frac{N-1}{2}+\frac{N-1}{2}-n\right)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} - e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$H(\omega) = \left[\sum_{n=0}^{\frac{N}{2}-1} h(n)\, 2j \sin\omega\left(\frac{N-1}{2}-n\right) \right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$H(\omega) = \left[\sum_{n=0}^{\frac{N}{2}-1} h(n)\, 2 \sin\omega\left(\frac{N-1}{2}-n\right) e^{j\frac{\pi}{2}} \right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} h(n)\, 2 \sin\omega\left(\frac{N-1}{2}-n\right) e^{-j\left(\omega\left(\frac{N-1}{2}\right)-\frac{\pi}{2}\right)} \quad \text{--- ②}$$

in general

$$H(\omega) = |H(\omega)| e^{j\theta(\omega)} \quad \text{--- ③}$$

from ② & ③

$$|H(\omega)| = \sum_{n=0}^{\frac{N}{2}-1} 2h(n) \sin\omega\left(\frac{N-1}{2}-n\right)$$

$$\theta(\omega) = -\omega\left(\frac{N-1}{2}-\frac{\pi}{2}\right) = -\left(\omega(\alpha)-\frac{\pi}{2}\right)$$

Let $k = \frac{N}{2}-n$, $n = \frac{N}{2}-k$ | $n=0,\ k=\frac{N}{2}$; $n=\frac{N}{2}-1,\ k=1$

$$\boxed{H(\omega) = \left[\sum_{k=1}^{\frac{N}{2}} 2h\left(\frac{N}{2}-k\right) \sin\omega\left(k-\frac{1}{2}\right) \right] e^{-j\left(\omega(\alpha)-\frac{\pi}{2}\right)}}$$

9.8.12 UNIVERSITY PROBLEMS:

1. Design an ideal differentiator with frequency response $H_d(e^{j\omega}) = j\omega$, $-\pi \le \omega \le \pi$ using hamming window with $N = 8$

2. Design a Hilbert transformer whose frequency response is $H_d(e^{j\omega}) = \begin{cases} -j, & 0 \le \omega \le \pi \\ j, & -\pi \le \omega \le 0 \end{cases}$ with $N = 11$ using Blackman window.

1. CONDITION FOR IDEAL DIFFERENTIATOR:

(i) In Ideal differentiator if N is even,

$$h_d(n) = \frac{-\sin \pi(n-\alpha)}{\pi(n-\alpha)} \quad n \neq \alpha$$

if N is odd,

$$h_d(n) = \frac{\cos \pi(n-\alpha)}{\pi(n-\alpha)} \quad n \neq \alpha$$

$$h_d(n) = 0 \quad , \quad n = \alpha.$$

(ii) Freq. response of ideal differentiator is antisymmetric i.e $h_d(n) = -h_d(-n)$ and $h(n) = -h(-n)$

2. CONDITION FOR HILBERT TRANSFORMER:

Impulse response is antisymmetric i.e.

$h_d(n) = -h_d(-n)$

2. Given: $H_d(e^{j\omega}) = \begin{cases} -j, & 0 \le \omega \le \pi \\ j, & -\pi \le \omega \le 0 \end{cases}$

$N = 11.$

Taking inverse fourier transform of $H_d(e^{j\omega})$.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} j e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{0}^{\pi} -j e^{j\omega n} d\omega.$$

$$= \frac{j}{2\pi}\left\{\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{0} - \left[\frac{e^{j\omega n}}{jn}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{2\pi n}\left[e^{0} - e^{-j\pi n} - e^{j\pi n} + e^{0}\right]$$

$$= \frac{1}{2\pi n}\left[2 - (e^{j\pi n} + e^{-j\pi n})\right]$$

$$= \frac{1}{2\pi n}\left[2 - 2\cos\pi n\right]$$

$$\boxed{h_d(n) = \frac{1}{\pi n}(1-\cos n\pi)}, \quad n \neq 0$$

Apply L'Hospital rule

$$h_d(n) = \lim_{n \to 0} \frac{-\sin n\pi}{\pi} = 0$$

$$\boxed{h_d(n) = 0}, \quad n = 0$$

Blackman window (zero phase) is

$$w(n) = 0.42 + 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right),$$

$$-\left(\frac{N-1}{2}\right) \le n \le \left(\frac{N-1}{2}\right)$$

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$$h(n) = h_d(n)\, w(n)$$

$$h(n) = \frac{1}{\pi n}[1-\cos n\pi]\left\{0.42 + 0.5\cos\frac{2\pi n}{N-1} + 0.08\cos\frac{4\pi n}{N-1}\right\}$$

$n \neq 0$

$$h(n) = 0 \quad, n = 0 \quad, -\frac{(N-1)}{2} \le n \le \frac{(N-1)}{2}$$

$$-5 \le n \le 5$$

$$\boxed{h(n) = -h(-n)}$$

$h(0) = 0$

$$h(1) = \frac{1}{\pi}[1-\cos\pi]\left\{0.42 + 0.5\cos\frac{2\pi}{10} + 0.08\cos\frac{4\pi}{10}\right\}$$

$$= \frac{1}{\pi}(2)\{0.42 + 0.4045 + 0.0247\}$$

$$= 0.5406$$

$$h(2) = \frac{1}{2\pi}[1-\cos 2\pi]\left\{0.42 + 0.5\cos\frac{4\pi}{10} + 0.08\cos\frac{8\pi}{10}\right\}$$

$$= \frac{1}{2\pi}(1-1)\left\{0.42 + 0.5\cos\frac{4\pi}{10} + 0.08\cos\frac{8\pi}{10}\right\}$$

$$= 0$$

$$h(3) = \frac{1}{3\pi}[1-\cos 3\pi]\left\{0.42 + 0.5\cos\frac{6\pi}{10} + 0.08\cos\frac{12\pi}{10}\right\}$$

$$= \frac{1}{3\pi}(2)\{0.42 + (-0.1545) - 0.0647\}$$

$$= 0.0425$$

$$h(4) = \frac{1}{4\pi}[1-\cos 4\pi]\left\{0.42 + 0.5\cos\frac{8\pi}{10} + 0.08\cos\frac{16\pi}{10}\right\}$$

$$= \frac{1}{4\pi}(0)$$

$$= 0$$

$$h(5) = \frac{1}{5\pi}[1-\cos 5\pi]\left\{0.42 + 0.5\cos\frac{10\pi}{10} + 0.08\cos\frac{20\pi}{10}\right\}$$

$$= \frac{1}{5\pi}(2)\left\{0.42 - 0.5 + 0.08\right\}$$

$$= 0$$

$h(1) = -h(-1)$ $h(2) = -h(-2)$ $h(3) = -h(-3)$ $h(4) = -h(-4)$ $h(5) = -h(-5)$

$$H(z) = \sum_{n=-\frac{(N-1)}{2}}^{\frac{(N-1)}{2}} h(n) z^{-n}$$

$$= h(-5)z^5 + h(-4)z^{+4} + h(-3)z^{+3} + h(-2)z^2 + h(-1)z^1 + h(0)z^0 +$$
$$h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5}$$

$$= 0.0425(z^{-3} - z^3) + 0.5406(z^{-1} - z^1)$$

1. $H_d(e^{j\omega}) = j\omega$, $-\pi \le \omega \le \pi$

$N = 7$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} j\omega e^{j\omega n} d\omega$$

$$= \frac{j}{2\pi}\left[\frac{\omega e^{j\omega n}}{jn} - \frac{e^{j\omega n}}{(jn)^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi n}\left[\pi(e^{jn\pi} + e^{-jn\pi}) - \left(\frac{e^{jn\pi} - e^{-jn\pi}}{jn}\right)\right]$$

$$= \frac{1}{2\pi n}\left[2\pi\cos n\pi - \frac{2\sin n\pi}{n}\right]$$

$$h_d(n) = \frac{1}{n}\left[\cos n\pi - \frac{\sin n\pi}{n\pi}\right], \quad n \ne 0.$$

$$h_d(n) = 0, \quad n = 0$$

* Design a filter with $H_d(\omega) = \begin{cases} e^{-j2\omega}, & |\omega| \le \pi/4 \\ 0, & \text{otherwise} \end{cases}$ 25

using triangular window.

i. $H_d(\omega) = \begin{cases} e^{-j2\omega}, & -\frac{\pi}{4} \le \omega \le \pi/4 \\ 0, & \text{otherwise} \end{cases}$

ii. Take inverse fourier transform

$$h_d(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} H_d(\omega) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{-j2\omega} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega(n-2)}}{j(n-2)} \right]_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{2\pi j(n-2)} \left\{ e^{j\frac{\pi}{4}(n-2)} - e^{-j\frac{\pi}{4}(n-2)} \right\}$$

$$h_d(n) = \frac{1}{\pi(n-2)} \sin\frac{\pi}{4}(n-2), \quad n \ne 2$$

Applying L'Hospital rule.

$$h_d(n) = \frac{1}{4}, \quad n = 2.$$

iii. Triangular window

$$w(n) = \begin{cases} 1 - \dfrac{2\left|n - \frac{N-1}{2}\right|}{N-1}, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

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iv. $h(n) = h_d(n)\,w(n)$

$$h(n) = \left\{\frac{1}{\pi(n-2)} \sin\frac{\pi}{4}(n-2)\right\}\left[1 - \frac{2\left|n-\frac{N-1}{2}\right|}{N-1}\right], \quad n \neq 2$$

$$h(n) = \frac{1}{4}\left[1 - \frac{2\left|n-\frac{N-1}{2}\right|}{N-1}\right], \quad n = 2 \qquad 0 \le n \le 4$$

$$\alpha = 2 = \frac{N-1}{2}$$

$$N = 5$$

$$h(0) = \left\{\frac{1}{-2\pi} \sin\frac{\pi}{4}(-2)\right\}\left[1 - \frac{2(2)}{4}\right] = 0$$

$$h(1) = \left\{\frac{1}{-\pi} \sin\frac{\pi}{4}(-1)\right\}\left[1 - \frac{2|1-2|}{4}\right]$$

$$= 0.2251 \times 0.5$$

$$= 0.1125$$

$$h(2) = \frac{1}{4}\left\{1 - 2\,\frac{|2-2|}{4}\right\} = \frac{1}{4} = 0.25$$

$$h(n) = h(N-1-n)$$

$$h(n) = h(4-n)$$

$$h(0) = h(4) \qquad h(1) = h(3)$$

$$H(z) = \sum_{n=0}^{\frac{N-1}{2}} h(n)\,z^{-n}$$

$$= h(0)z^{-0} + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4}$$

$$= 0.1125\,(z^{-1} + z^{-3}) + 0.25\,z^{-2}$$

problem continues...

Ideal differentiator.

Hamming window is given by

$$w(n) = 0.54 + 0.46 \cos\left(\frac{2\pi n}{N-1}\right), \left(-\frac{(N-1)}{2} \le n \le \frac{(N-1)}{2}\right)$$

$$h(n) = h_d(n)\,w(n)$$

$$h(n) = \frac{1}{n}\left[\cos n\pi - \frac{\sin n\pi}{n\pi}\right]\left\{0.54 + 0.46 \cos \frac{2\pi n}{N-1}\right\} \quad n \neq 0$$

$$h(n) = 0 \quad ; n = 0 \quad ; -3 \le n \le 3$$

$$h(0) = 0$$

$$h(1) = \frac{1}{1}\left[\cos\pi - \frac{\sin\pi}{\pi}\right]\left\{0.54 + 0.46 \cos\frac{2\pi}{6}\right\}$$

$$= (-1)(0.9383)$$

$$= -0.9383$$

$$h(2) = \frac{1}{2}\left[\cos 2\pi - \frac{\sin 2\pi}{2\pi}\right]\left\{0.54 + 0.46 \cos\frac{4\pi}{6}\right\}$$

$$= \frac{1}{2}(0.31)$$

$$= 0.155$$

$$h(3) = \frac{1}{3}\left[\cos 3\pi - \frac{\sin 3\pi}{3\pi}\right]\left\{0.54 + 0.46 \cos\frac{6\pi}{6}\right\}$$

$$= -\frac{1}{3}(0.08)$$

$$= -0.0266.$$

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$h(n) = -h(-n)$

$h(-1) = -h(1)$

$h(-2) = -h(2)$

$h(-3) = -h(3)$

$$H(z) = \sum_{n=-\frac{(N-1)}{2}}^{\frac{(N-1)}{2}} h(n) z^{-n}$$

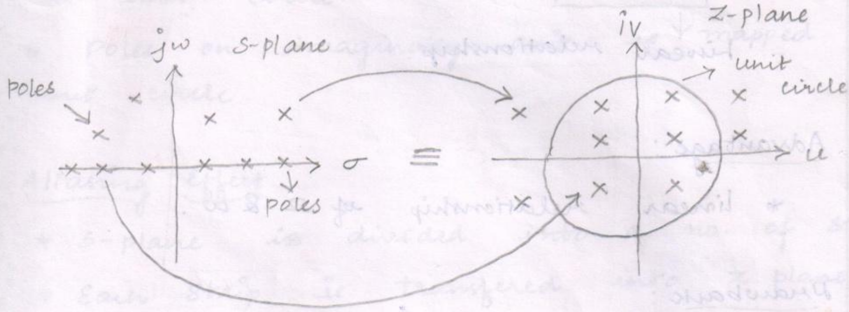
$$= h(-3)z^{3} + h(-2)z^{2} + h(-1)z^{1} + h(0)z^{-0} + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3}$$

$$= 0.0266(z^{3} - z^{-3}) + 0.155(z^{-2} - z^{2}) + 0.9383(z^{+1} - z^{-1})$$

5-7-12 UNIT-II (1)

IIR Filter Design
(INFINITE IMPULSE RESPONSE)

* Analog filter → Transfer function $H(s)$
* Digital filter → Transfer function $H(z)$



No. of poles = No. of zeros
{all the poles should lie in left half of S-plane = Stability of analog Filter}

IIR
- Butterworth IIR filter (Analog) $H(s)$
- Chebyshev IIR filter (Analog) $H(s)$

→ Digital filter $H(z)$
- Impulse Invariant Transformation (IIT) $H(z)$
- Bilinear Transformation (BLT) — Approximation of derivatives.

(1) IIT

$$\frac{1}{S - P_i} \xrightarrow{\text{is transformed to}} \frac{1}{1 - e^{+P_i T} z^{-1}}$$

where $P_i \rightarrow$ poles
$T \rightarrow$ Sampling period

1 Relationship between Ω & ω: 2

$\Omega \rightarrow$ Analog freq
$\omega \rightarrow$ digital freq } rad/sec

$\boxed{\omega = \Omega T}$

$\downarrow$

Linear relationship.

Advantage:

* linear relationship of Ω & ω.

Drawback:

* Aliasing Effect

2 BLT (conformal mapping):

$$s \xrightarrow{\text{is transformed to}} \frac{2}{T}\,\frac{1-z^{-1}}{1+z^{-1}}$$

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

Advantage:

* No aliasing effect

Drawback:

* non linear relationship of Ω & ω.

It will give warping effect.

* This warping can be eliminated by "Prewarping technique".

* Due to non linear relationship of Ω and ω it will introduce distortion in the frequency axis this effect is called as warping.

Mapping: Analog into Digital

* Poles on left half of s plane is mapped inside the unit circle of z plane
* Poles on right half is mapped outside the unit circle.
* Poles on imaginary axis is mapped on the unit circle.

Aliasing Effect:

* s-plane is divided into a no. of strips
* Each strip is transfered into z plane
* Overlapping of strips is called Aliasing effect.

Conformal Mapping:

* Entire left half of s plane is transferred to z plane.

1. Determine digital filter transfer function of $H(s) = \dfrac{2}{s^2+5s+6}$ using (a) IIT (b) BLT. Assume $T = 1$ sec.

(a) IIT:

$$H(s) = \frac{2}{(s+2)(s+3)}$$

Apply partial fraction.

$$= \frac{A}{s+2} + \frac{B}{s+3}$$

(3) 4

$$2 = A(s+3) + B(s+2)$$

put $s = -2$ put $s = -3$

$A = 2$ $B = -2$

$$H(s) = \frac{2}{s+2} + \frac{(-2)}{s+3}$$

$$= 2 \cdot \frac{1}{s-(-2)} - 2 \cdot \frac{1}{s-(-3)}$$

$$H(z) = 2 \cdot \frac{1}{1-e^{+(-2)T}z^{-1}} - 2 \cdot \frac{1}{1-e^{+(-3)T}z^{-1}}$$

$$= \frac{2}{1-e^{-2}z^{-1}} - \frac{2}{1-e^{-3}z^{-1}}$$

$$= \frac{2}{1-7.38z^{-1}} - \frac{2}{1-20.08z^{-1}}$$

$$= 2\frac{(1-20.08z^{-1}-1+7.38z^{-1})}{(1-7.38z^{-1})(1-20.08z^{-1})}$$

$$= \frac{-25.4z^{-1}}{1-7.38z^{-1}-20.08z^{-1}+148.19z^{-2}}$$

$$H(z) = \frac{-25.4z^{-1}}{1-27.46z^{-1}+148.19z^{-2}}$$

$$= \frac{2}{1-0.1353z^{-1}} - \frac{2}{1-0.0497z^{-1}}$$

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$$= \frac{2\,(1-0.0497z^{-1} - 1 + 0.1353z^{-1})}{(1-0.1353z^{-1})(1-0.0497z^{-1})}$$

$$= \frac{0.1712z^{-1}}{1-0.0497z^{-1} - 0.1353z^{-1} + 0.0067z^{-2}}$$

$$H(z) = \frac{0.1712z^{-1}}{1-0.185z^{-1}+0.0067z^{-2}}$$

(b) BLT:

$$s \longrightarrow \frac{2}{T}\,\frac{1-z^{-1}}{1+z^{-1}}$$

$$H(s) = \frac{2}{s^2+5s+6}$$

$$H(z) = \frac{2}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 5\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)+6}$$

$$= \frac{2}{\frac{2(1-z^{-1})^2 + 10(1-z^{-1})(1+z^{-1}) + 6(1+z^{-1})^2}{(1+z^{-1})^2}}$$

$$= \frac{2(1+z^{-1})^2}{4[(1+z^{-2}-2z^{-1})]+[10-10z^{-2}+6+6z^{-2}+12z^{-1}]}$$

$$= \frac{2(1+z^{-1})^2}{20+4z^{-1}}$$

$$H(z) = \frac{(1+z^{-1})^2}{2(5+z^{-1})}$$

H.W. 6

2. $x(n) = \{1, 2, 1, 2, 3, 1, 4, 5\}$

$h(n) = \{2, 1\}$

$x_1(n) = \{1, 2\}$

$x_2(n) = \{1, 2\}$

$x_3(n) = \{3, 1\}$

$x_4(n) = \{4, 5\}$

* Add $N_2 - 1$ zeros to all the sequence

$x_1(n) = \{1, 2, 0\}$ $x_2(n) = \{1, 2, 0\}$ $x_3(n) = \{3, 1, 0\}$

$x_4(n) = \{4, 5, 0\}$ $h(n) = \{2, 1, 0\}$

* $$x_1(n) \circledast h(n) = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$$

$$x_2(n) \circledast h(n) = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$$

$$x_3(n) \circledast h(n) = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$$

$$x_4(n) \circledast h(n) = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \\ 5 \end{bmatrix}$$

$x_1(n) \circledast h(n) = [2 \quad 5 \quad 2]$

$x_2(n) \circledast h(n) = [2 \quad 5 \quad 2]$

$x_3(n) \circledast h(n) = [6 \quad 5 \quad 1]$

$x_4(n) \circledast h(n) = [8 \quad 14 \quad 5]$

$y(n) = \{2, 5, 4, 5, 8, 5, 9, 14, 5\}$

(b) Overlap save method:

$x(n) = \{1, 2, 1, 2, 3, 1, 4, 5\}$

$h(n) = \{2, 1\}$

* $x_1(n) = \{0, 1, 2\}$

$x_2(n) = \{2, 1, 2\}$

$x_3(n) = \{2, 3, 1\}$

$x_4(n) = \{1, 4, 5\}$

$x_5(n) = \{5, 0, 0\}$

* Add $N_2 - 1$ zeroes to $h(n)$. $h(n) = \{2, 1, 0\}$

$$x_1(n) \circledast h(n) = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

$$x_2(n) \circledast h(n) = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix}$$

$$x_3(n) \circledast h(n) = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 5 \end{bmatrix}$$

$$x_4(n) \circledast h(n) = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ 14 \end{bmatrix}$$

$$x_5(n) \circledast h(n) = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 0 \end{bmatrix}$$

$x_1(n) \circledast h(n) = [\cancel{2} \quad 2 \quad 5]$

$x_2(n) \circledast h(n) = [\cancel{6} \quad 4 \quad 5]$

$x_3(n) \circledast h(n) = [\cancel{5} \quad 8 \quad 5]$

$x_4(n) \circledast h(n) = [\cancel{7} \quad 9 \quad 14]$

$x_5(n) \circledast h(n) = [\cancel{10} \quad 5 \quad 0]$

$y(n) = \{2, 5, 4, 5, 8, 5, 9, 14, 5, 0\}$

9.7.12 **Design of Low Pass IIR Butterworth Filter:**

Specifications:

$A_1 \rightarrow$ Gain at P.B } No unit
$A_2 \rightarrow$ Gain at S.B

$k_1 \rightarrow$ Gain at P.B } unit dB
$k_2 \rightarrow$ Gain at S.B

$\Omega_1 \rightarrow$ Analog freq. at P.B
$\Omega_2 \rightarrow$ " " at S.B
$\omega_1 \rightarrow$ Digital freq at P.B
$\omega_2 \rightarrow$ " " at S.B
} unit rad/sec

$f_1 \rightarrow$ P.B freq in Hz

$f_2 \rightarrow$ S.B freq in Hz

$\Omega_1 = 2\pi f_1 \qquad \omega_1 = 2\pi f_1 / F$

$\Omega_2 = 2\pi f_2 \qquad \omega_2 = 2\pi f_2 / F$

F = Sampling freq.

$$A_1 = 10^{+k_1/20}$$

$$A_2 = 10^{+k_2/20}$$

$k_1, k_2 \rightarrow$ always negative value.

Procedure:

1. Choose IIT / BLT
2. Calculate $\frac{\Omega_2}{\Omega_1}$

For IIT, $\frac{\Omega_2}{\Omega_1} = \frac{\omega_2}{\omega_1}$

For BLT, $\frac{\Omega_2}{\Omega_1} = \frac{\tan \omega_2/2}{\tan \omega_1/2}$

3. Determine the order of the filter N

$$N_1 = \frac{\frac{1}{2} \log \left\{ \left(\frac{1}{A_2^2} - 1\right) / \left(\frac{1}{A_1^2} - 1\right) \right\}}{\log \left(\frac{\Omega_2}{\Omega_1}\right)}$$

$$\boxed{N \geq N_1}$$

4. Calculate analog cut off freq. (Ω_c)

For IIT, $\Omega_c = \frac{\omega_1/T}{\left[\frac{1}{A_1^2} - 1\right]^{1/2N}}$ (or) $\frac{\Omega_1}{\left[\frac{1}{A_1^2} - 1\right]^{1/2N}}$

For BLT, $\Omega_c = \frac{\frac{2}{T} \tan \omega_1/2}{\left[\frac{1}{A_1^2} - 1\right]^{1/2N}}$ (or) $\frac{\Omega_1}{\left[\frac{1}{A_1^2} - 1\right]^{1/2N}}$

5. Find H(s) (normalized filter)

If N is even,

$$H(s) = \prod_{k=1}^{N/2} \frac{1}{s^2 + b_k s + 1}$$

{condition for normalized filter : $\Omega_c = 1$ rad/sec }

If N is odd,

$$H(S) = \frac{1}{S+1} \prod_{k=1}^{\frac{N-1}{2}} \frac{1}{S^2 + b_k S + 1}$$

where $b_k = 2\sin\left(\frac{(2k-1)\pi}{2N}\right)$

6) Analog frequency transformation

LPF	$S \to \frac{S}{\Omega_c}$
HPF	$S \to \frac{\Omega_c}{S}$

7) Convert $H(S) \to H(Z)$ using BLT / IIT.

1. Design a butterworth digital LPF IIR filter using BLT to satisfy following specifications

$$0.6 \leq |H(e^{j\omega})| \leq 1.0 \; ; \text{for } 0 \leq \omega \leq 0.35\pi$$

$$|H(e^{j\omega})| \leq 0.1 \; ; \quad 0.7\pi \leq \omega \leq \pi$$

In General: $A_1 \leq |H(e^{j\omega})| \leq 1.0 \; ; \; 0 \leq \omega \leq \omega_1$

$$|H(e^{j\omega})| \leq A_2 \; ; \; \omega_2 \leq \omega \leq \pi$$

Given data:

$A_1 = 0.6 \qquad \omega_1 = 0.35\pi$ rad/Sec

$A_2 = 0.1 \qquad \omega_2 = 0.7\pi$ rad/Sec

IIR Butterworth LPF using BLT.

(i) Given transformation is BLT.

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(ii) $\frac{\Omega_2}{\Omega_1} = \frac{\tan \omega_2/2}{\tan \omega_1/2}$

$= \frac{\tan \frac{0.7\pi}{2}}{\tan \frac{0.35\pi}{2}}$

$= \frac{1.9626}{0.6128}$

$\frac{\Omega_2}{\Omega_1} = 3.2026$

(iii) $N_1 = \frac{1/2 \log\{(\frac{1}{A_2^2}-1)/(\frac{1}{A_1^2}-1)\}}{\log(\Omega_2/\Omega_1)}$

$= \frac{\frac{1}{2}\log\{(\frac{1}{(0.1)^2}-1)/(\frac{1}{(0.6)^2}-1)\}}{\log(3.2026)}$

$= \frac{1/2 \log\{99/1.7778\}}{\log(3.2026)}$

$= \frac{\frac{1}{2}\log 55.6868}{\log 3.2026}$

$N_1 = 1.7267$

$N = 2$

(iv) $\Omega_c = \frac{\frac{2}{T}\tan \omega_1/2}{[1/A_1^2 - 1]^{1/2N}}$

assume $T = 1$ sec

$$\Omega_c = \frac{2 \tan \frac{0.35\pi}{2}}{\left[\frac{1}{(0.6)^2} - 1\right]^{1/4}}$$

$$= \frac{2 \times 0.6128}{1.1547}$$

$$\Omega_c = 1.0614 \text{ rad/sec}$$

(v) N is even

$$H(s) = \prod_{k=1}^{N/2} \frac{1}{s^2 + b_k s + 1}$$

$$H(s) = \frac{1}{s^2 + b_1 s + 1}$$

$$b_1 = 2\sin\left(\frac{2k-1}{2N}\right)$$

$$b_1 = 0.4948$$

$$H(s) = \frac{1}{s^2 + 0.4948 s + 1}$$

(vi) for LPF,

$$s \to s/\Omega_c$$

$$H(s) = \frac{1}{\left(\frac{s}{1.06}\right)^2 + 0.4948\left(\frac{s}{1.06}\right) + 1}$$

$$\boxed{H(s) = \frac{1.1236}{s^2 + 0.5245 s + 1.1236}}$$

(vii) converting $H(s)$ to $H(z)$

$$s \to \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

$$H(z) = \frac{1.1236}{\left(2 \cdot \frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.5245\left(2 \cdot \frac{1-z^{-1}}{1+z^{-1}}\right) + 1.1236}$$

$$= \frac{1.1236\,(1+z^{-1})^2}{4(1+z^{-2}-2z^{-1}) + 1.049(1-z^{-2}) + 1.1236(1+z^{-1})^2}$$

$$\cancel{H(z) = \frac{1.1236(1+z^{-1})^2}{2.951z^{-2} - 8z^{-1} + 6.1126}}$$

2\. Given: $0.707 \le |H(e^{j\omega})| \le 1.0$; $0 \le \omega \le 0.45\pi$

$|H(e^{j\omega})| \le 0.2$; $0.65\pi \le \omega \le \pi$

$A_1 = 0.707$ $\quad \omega_1 = 0.45\pi$ rad/sec

$A_2 = 0.2$ $\quad \omega_2 = 0.65\pi$ rad/sec.

(i) Given transformation is BLT.

(ii) $\frac{\Omega_2}{\Omega_1} = \frac{\tan \omega_2/2}{\tan \omega_1/2}$

$$= \frac{\tan \frac{0.65\pi}{2}}{\tan \frac{0.45\pi}{2}}$$

$$= \frac{1.6318}{\cancel{0.7068}\ 0.8541}$$

$\frac{\Omega_2}{\Omega_1} = \cancel{2.3087}\ 1.9105$

(13)

(iii) $N_1 = \dfrac{\frac{1}{2}\log\left[\left(\frac{1}{A_2^2}-1\right)/\left(\frac{1}{A_1^2}-1\right)\right]}{\log(\Omega_2/\Omega_1)}$

$= \dfrac{\frac{1}{2}\log\left\{\left(\frac{1}{(0.2)^2}-1\right)/\left(\frac{1}{(0.707)^2}-1\right)\right\}}{\log(1.9105)}$

$= \dfrac{\frac{1}{2}\log(24/1.0006)}{\log(1.9105)}$

$= \dfrac{0.6899}{0.2811}$

$N_1 = 2.45$

$N = 3$

(iv) $\Omega_c = \dfrac{\frac{2}{T}\tan \omega_1/2}{\left[\frac{1}{A_1^2}-1\right]^{1/2N}}$

$= \dfrac{2\tan\frac{0.45\pi}{2}}{\left[\frac{1}{(0.707)^2}-1\right]^{1/6}}$

$= \dfrac{2\times 0.8541}{(1.0006)^{1/6}}$

$\Omega_c = 1.708$ rad/sec

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(v) N is odd

$$H(s) = \frac{1}{s+1} \prod_{k=1}^{\frac{N-1}{2}} \left(\frac{1}{s^2 + b_k s + 1} \right)$$

$$= \left(\frac{1}{s^2 + b_k s + 1} \right) \left(\frac{1}{s+1} \right)$$

$$b_1 = 2 \sin \frac{(2k-1)}{2N}$$

$$= 2 \sin(1/6)$$

$$b_1 = 0.3318$$

$$H(s) = \left(\frac{1}{s^2 + 0.3318s + 1} \right) \left(\frac{1}{s+1} \right)$$

(vi) for LPF,

$$s \to s/\Omega_c$$

$$H(s) = \frac{1}{(s/1.708)^2 + 0.3318 \frac{s}{1.708} + 1} \cdot \frac{1}{(s/1.708 + 1)}$$

$$= \frac{2.9173}{(s^2 + \cancel{0.1443}\,0.5667 s + \cancel{1}\,2.9173)} \frac{1.708}{(s + 1.708)}$$

$$H(s) = \frac{4.9827}{(s + 1.708)(s^2 + \cancel{0.1443}\,0.5667 s + 2.91)}$$

(vii) converting $H(s)$ into $H(z)$

$$s \to \frac{2}{T} \left(\frac{1-z^{-1}}{1+z^{-1}} \right)$$

$$H(z) = \frac{4.9827}{\left(\frac{2}{T}\cdot\frac{1-z^{-1}}{1+z^{-1}}+1.708\right)\left(\left(\frac{2}{T}\cdot\frac{1-z^{-1}}{1+z^{-1}}\right)^2+\overset{0.5667}{\cancel{0.1943}}\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)+\right.}$$

$$H(z) = \frac{4.9827\,(1+z^{-1})^3}{[2(1-z^{-1})+1.708(1+z^{-1})]\,[4(1-z^{-1})^2+\overset{1.1334}{\cancel{0.3886}}(1-z^{-2})+(1+z^{-1})^2}$$

11.7.12

3. Determine the order of Butterworth filter

$\alpha_p = 1\,dB$ $\quad \Omega_p = 200$ rad/sec

$\alpha_s = 30\,dB$ $\quad \Omega_s = 600$ rad/sec

4. Determine the order of Butterworth filter that has $\underline{3\,dB}$ attenuation at 500 hz PB and attenuation of $\underline{40\,dB}$ at 1000 hz. SB

5. Design a butterworth LPF satisfying the following specification.

$f_p = 0.1\,Hz$ $\quad \alpha_p = 0.5\,dB$ $\quad F = 1\,Hz$

$f_s = 0.15\,Hz$ $\quad \alpha_s = 15\,dB$

6. For the given specification $\alpha_p = 3\,dB$ $\alpha_s = 15\,dB$ $\Omega_p = 1000$ rad/sec $\Omega_s = 500$ rad/sec. Design HPF butterworth filter (Maximally Flat filter) or (Filter with monotonically decreasing fn. of magnitude response) req. filter is analog

3. $\Omega_1 = \Omega_p = 200$ rad/sec

$\Omega_2 = \Omega_s = 600$ rad/sec

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$k_1 = -1\,dB$

$k_2 = -30\,dB$

$A_1 = 10^{k_1/20}$ $\quad A_2 = 10^{k_2/20}$

$A_1 = 10^{-1/20}$ $\quad A_2 = 10^{-30/20}$

$A_1 = 0.8913$ $\quad A_2 = 0.0316$

$$N_1 = \frac{\frac{1}{2}\log\left\{\left(\frac{1}{A_2^2}-1\right)/\left(\frac{1}{A_1^2}-1\right)\right\}}{\log\{\Omega_2/\Omega_1\}}$$

$$= \frac{\frac{1}{2}\log\left\{\left(\frac{1}{(0.0316)^2}-1\right)/\left(\frac{1}{(0.8913)^2}-1\right)\right\}}{\log\{600/200\}}$$

$$= \frac{\frac{1}{2}\log\{1000.44/0.2587\}}{\log 3}$$

$$N_1 = \frac{\frac{1}{2}\log 3867.18}{\log 3} = \frac{15.0376}{4} = 3.76$$

$N = \cancel{5}\ 4.$

4. Given:

$k_1 = -3\,dB$

$k_2 = -40\,dB$

$f_1 = 500\,Hz$

$f_2 = 1000\,Hz$

$\Omega_1 = 2\pi f_1 = 2\times\cancel{3.14}\pi\times 500 = 1000\pi$ rad/sec

$\Omega_2 = 2\pi f_2 = 2\times\pi\times1000 = 2000\pi$ rad/sec

$A_1 = 10^{k_1/20}$ $\quad\quad A_2 = 10^{k_2/20}$

$= 10^{-3/20}$ $\quad\quad = 10^{-40/20}$

$A_1 = 0.7079$ $\quad\quad A_2 = 0.01$

$$N_1 = \frac{\frac{1}{2}\log\left\{\left(\frac{1}{A_2^2}-1\right)/\left(\frac{1}{A_1^2}-1\right)\right\}}{\log(\Omega_2/\Omega_1)}$$

$$= \frac{\frac{1}{2}\log\left\{\left(\frac{1}{(0.01)^2}-1\right)/\left(\frac{1}{(0.7079)^2}-1\right)\right\}}{\log(2000\pi/1000\pi)}$$

$$= \frac{\frac{1}{2}\log\{9999/0.9954\}}{\log 2}$$

$$= \frac{1}{2}\cdot\frac{4.0019}{0.3010}$$

$N_1 = 6.6476$

$N = 7$.

5. Given :

If transformation is not given, the analog filter has to be designed

$\alpha_p = k_1 = -0.5\,dB$

$\alpha_s = k_2 = -15\,dB$

$$= \frac{0.2\pi}{(0.1219)^{1/14}}$$

$$\Omega_c = 0.7303 \text{ rad/sec}$$

(iii) N is odd.

$$H(s) = \frac{1}{s+1} \prod_{K=1}^{\frac{N-1}{2}} \frac{1}{s^2 + b_K s + 1}$$

$$= \frac{1}{(s+1)} \cdot \frac{1}{(s^2+b_1 s+1)} \cdot \frac{1}{(s^2+b_2 s+1)} \cdot \frac{1}{(s^2+b_3 s+1)}$$

$$b_K = 2\sin\left(\frac{2K-1}{2N}\right)$$

$$b_1 = 2\sin\left(\frac{1}{14}\right) = 0.1427 \qquad b_2 = 2\sin\frac{3}{14} = 0.4253 \qquad b_3 = 2\sin\frac{5}{14} = 0.6992$$

$$H(s) = \frac{1}{(s+1)} \cdot \frac{1}{(s^2+0.1427s+1)} \cdot \frac{1}{(s^2+0.4253s+1)} \cdot \frac{1}{(s^2+0.6992s+1)}$$

(iv) for LPF

$$s \rightarrow s/\Omega_c$$

$$H(s) = \frac{1}{\left(\frac{s}{0.7303}+1\right)} \cdot \frac{1}{\left[\left(\frac{s}{0.7303}\right)^2 + 0.1427\left(\frac{s}{0.7303}\right)+1\right]} \cdot \frac{1}{\left[\left(\frac{s}{0.7303}\right)^2 + 0.4253\left(\frac{s}{0.7303}\right)\right.} \cdot \frac{1}{\left[\left(\frac{s}{0.7303}\right)^2 + 0.6992\left(\frac{s}{0.7303}\right)\right.}$$

$$= \frac{0.7303}{(s+0.7303)} \cdot \frac{(0.7303)^2}{(s^2+0.1042s+0.5333)} \cdot \frac{(0.7303)^2}{(s^2+0.3106s+0.5333)} \cdot \frac{(0.7303)^2}{(s^2+0.5106s+0.5333)}$$

(22)

$$H(s) = \frac{0.1108}{(s+0.7303)(s^2+0.1042s+0.5333)(s^2+0.3106s+0.5333)(s^2+0.5106s+0.5333)}$$

6. Given: HPF butterworth filter

$\Omega_p = 500$ rad/sec

$\Omega_s = 1000$ rad/sec

$\alpha_p = K_1 = -3dB$

$\alpha_s = K_2 = -15dB$.

$A_1 = 10^{K_1/20}$

$= 10^{-3/20}$

$A_1 = 0.7079$

$A_2 = 10^{K_2/20}$

$= 10^{-15/20}$

$A_2 = 0.1778$.

(i) $$N_1 = \frac{\frac{1}{2}\log\left\{\left(\frac{1}{A_2^2}-1\right)/\left(\frac{1}{A_1^2}-1\right)\right\}}{\log(\Omega_2/\Omega_1)}$$

$$= \frac{\frac{1}{2}\log\left\{\left(\frac{1}{(0.1778)^2}-1\right)/\left(\frac{1}{(0.7079)^2}-1\right)\right\}}{\log(1000/500)}$$

$$= \frac{\frac{1}{2}\log\{30.6316/0.9954\}}{\log 2}$$

$N_1 = 2.47$

$N = 3$.

(ii) $$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{A_1^2}-1\right]^{1/2N}}$$

$$= \frac{500}{\left[\frac{1}{(0.7079)^2}-1\right]^{1/6}}$$

$$= \frac{500}{(0.9954)^{1/6}}$$

$$\Omega_c = 500.4 \text{ rad/sec}$$

(iii) N is odd

$$H(s) = \frac{1}{(s+1)} \prod_{k=1}^{\frac{N-1}{2}} \frac{1}{s^2+b_k s+1}$$

$$= \frac{1}{(s+1)} \cdot \frac{1}{(s^2+b_1 s+1)}$$

$$b_1 = 2\sin\left(\frac{2k-1}{2N}\right)$$

$$= 2\sin\left(\frac{1}{6}\right)$$

$$b_1 = 0.3318$$

$$H(s) = \frac{1}{(s+1)} \cdot \frac{1}{(s^2+0.3318s+1)}$$

(iv) for HPF

$$s \rightarrow \Omega_c/s$$

$$H(s) = \frac{1}{\left(\frac{500.4}{s}+1\right)} \cdot \frac{1}{\left[\left(\frac{500.4}{s}\right)^2 + 0.3318\times\frac{500.40}{s}+1\right]}$$

$$H(s) = \frac{s}{s+500.4} \cdot \frac{s^2}{s^2+166.03s+250400.16}$$

7.12 Design of digital Low pass Chebyshev filter :

Steps :

1. Similar to butterworth
2. " " " "
3. Find the order N

$$N_1 = \frac{\cosh^{-1}\left(\sqrt{(1/A_2^2 - 1)/(1/A_1^2 - 1)}\right)}{\cosh^{-1}(\Omega_2/\Omega_1)}$$

$$N > N_1$$

4. Same
5. Determine Analog T.F. H(s) for nonmalized filter when N is even

$$H(s) = \prod_{k=1}^{N/2} \frac{B_k}{s^2+b_k s+c_k}$$

When N is odd

$$H(s) = \frac{B_0}{s+C_0} \prod_{k=1}^{\frac{N-1}{2}} \frac{B_k}{s^2+b_k s+c_k}$$

where

$$b_k = 2 y_N \sin\left(\frac{(2k-1)\pi}{2N}\right)$$

$$c_k = y_N^2 + \cos^2\left(\frac{(2k-1)\pi}{2N}\right)$$

$$c_0 = y_N.$$

$$Y_N = \frac{1}{2}\left[\left(\sqrt{\frac{1}{\varepsilon^2}+1}+\frac{1}{\varepsilon}\right)^{1/N} - \left(\sqrt{\frac{1}{\varepsilon^2}+1}+\frac{1}{\varepsilon}\right)^{-1/N}\right]$$

ε = attenuation constant $= \sqrt{\frac{1}{A_1^2}-1}$

For even values of N, find B_K.

$$H(0) = \frac{1}{\sqrt{1+\varepsilon^2}}$$

For odd values of N, find B_k

$$H(0) = 1.$$

In general, $B_0 = B_1 = B_2 = \dots B_K$.

6 & 7. Same

1. Design chebyshev digital low pass IIR filter using impulse invariant transformation to satisfy the following spec.

passband ripple ≤ 0.9151 dB

stopband attenuation ≥ 12.3958 dB

Passband edge freq. $= 0.25\pi$ rad/sec

Stopband edge freq. $= 0.5\pi$ rad/sec

Given:

$K_1 = -0.9151$ dB

$K_2 = -12.3958$ dB

$\omega_1 = 0.25\pi$ rad/sec

$\omega_2 = 0.5\pi$ rad/sec ,

$A_1 = 10^{K_1/20}$ $\quad A_2 = 10^{K_2/20}$

$= 10^{-0.9151/20}$ $\quad = 10^{-12.3958/20}$

$A_1 = 0.9$, $\quad A_2 = 0.2399$

(i) Given transformation is IIT.

(ii) $\frac{\Omega_2}{\Omega_1} = \frac{\omega_2}{\omega_1} = \frac{0.5\pi}{0.25\pi} = 2$

(iii) $N_1 = \frac{\cosh^{-1}\sqrt{[(1/A_2^2 - 1)/(1/A_1^2 - 1)]}}{\cosh^{-1}(\Omega_2/\Omega_1)}$

$= \frac{\cosh^{-1}\sqrt{(1/(0.2399)^2 - 1)/(1/(0.9)^2 - 1)}}{\cosh^{-1}(2)}$

$= \frac{\cosh^{-1}\sqrt{16.3756/0.2345}}{1.3169}$

$= \frac{2.8126}{1.3169}$

$N_1 = 2.1357$

$N = 3$

(iv) $\Omega_c = \frac{\omega_1/T}{\left[\frac{1}{A_1^2} - 1\right]^{1/2N}}$

$= \frac{0.25\pi}{\left[\frac{1}{(0.9)^2} - 1\right]^{1/6}} = \frac{0.25\pi}{[0.2345]^{1/6}}$

$\Omega_c = 1.0001$ rad/sec.

(v) N is odd

$$H(s) = \frac{B_0}{s+C_0} \prod_{k=1}^{\frac{N-1}{2}} \frac{B_k}{s^2+b_k s+C_k}$$

$$= \frac{B_0}{s+C_0} \cdot \frac{B_1}{s^2+b_1 s+C_1}$$

$$b_k = 2Y_N \sin\left(\frac{(2k-1)\pi}{2N}\right)$$

$$b_1 = 2Y_N \sin\frac{\pi}{6}$$

$$\varepsilon = \sqrt{\frac{1}{A_1^2}-1} = \sqrt{\frac{1}{(0.9)^2}-1} = (0.2345)^{1/2} = 0.4843.$$

$$Y_N = \frac{1}{2}\left[\left(\sqrt{\frac{1}{\varepsilon^2}+1}+\frac{1}{\varepsilon}\right)^{1/N} - \left(\sqrt{\frac{1}{\varepsilon^2}+1}+\frac{1}{\varepsilon}\right)^{-1/N}\right]$$

$$= \frac{1}{2}\left[\left(\sqrt{\frac{1}{(0.4843)^2}+1}+\frac{1}{0.4843}\right)^{1/3} - \left(\sqrt{\frac{1}{(0.4843)^2}+1}+\frac{1}{0.4843}\right)^{-1/3}\right]$$

$$= \frac{1}{2}\left[(2.2942+2.0648)^{1/3} - (2.2942+2.0648)^{-1/3}\right]$$

$$= \frac{1}{2}\left[(4.359)^{1/3} - (4.359)^{-1/3}\right]$$

$$= \frac{1}{2}\left[1.6335 - 0.6122\right]$$

$$Y_N = 0.5106$$

$$b_1 = 2\times 0.5106 \sin\frac{\pi}{6} = 0.5106$$

$$C_k = Y_N^2 + \cos^2\left(\frac{(2k-1)\pi}{2N}\right)$$

$$C_1 = (0.5106)^2 + \cos^2\frac{\pi}{6}$$

$$C_1 = 0.2607 + 0.7499$$

$$C_1 = 1.0106$$

$$C_0 = Y_N = 0.5106.$$

$$H(s) = \frac{B_0}{s+0.5106} \cdot \frac{B_1}{s^2+0.5106s+1.01}$$

for N is odd,

$$H(0) = 1$$

put $s=0$ in $H(s)$

$$H(0) = \frac{B_0}{0.5106} \cdot \frac{B_1}{1.01}$$

$$H(0) = \frac{B_0 B_1}{0.5157} = 1$$

$$B_0 B_1 = 0.5157$$

$$B_0 = B_1$$

$$B_0^2 = 0.5157.$$

$$B_0 = 0.718 = B_1$$

$$\boxed{H(s) = \frac{0.5157}{(s+0.5106)(s^2+0.5106s+1.01)}}$$

vi. $s \rightarrow \frac{s}{\Omega_c} \rightarrow s$ $\left(\because \Omega_c = 1 \text{ rad/sec } \{1.0001\} \right)$

vii. $\frac{1}{s-p_i} \rightarrow \frac{1}{1-e^{p_i T} z^{-1}}$

$$H(s) = \frac{0.515}{(s+0.51)(s^2+0.51s+1.01)}$$

$$= \frac{A}{s+0.51} + \frac{Bs+C}{s^2+0.51s+1.01}$$

$$0.515 = A(s^2+0.51s+1.01) + Bs(s+0.51) + C(s+0.51)$$

put $s = -0.51$

$0.515 = A(1.01)$

$A = 0.5099$

put $s = 0$

$0.515 = A(1.01) + 0.51C$

$C = 1.96\times10^{-5} \approx 0$

$$s = \frac{-0.51 \pm \sqrt{(0.51)^2 - 4\times1.01}}{2}$$

$$= \frac{-0.51 \pm 1.9442j}{2}$$

$$s = -0.255 \pm 0.9721j$$

$$H(s) = \frac{0.515}{(s+0.51)(s+0.255-0.9721j)(s+0.255+0.9721j)}$$

$$0.515 = \frac{A}{s+0.51} + \frac{B}{s+0.255-0.9721j} + \frac{C}{s+0.255+0.9721j}$$

The values of A, B and C can be found by partial fraction method. 30

H(s) is converted into H(z) using IIT.

Realization of IIR filters (Structure of IIR)

1. Direct form - I
2. Direct form - II
3. Cascade form
4. Parallel form

Basic elements are

(i) Multipliers ——▷——
(ii) unit delay element —— $\boxed{z^{-1}}$ ——
(iii) Adder →⊕→

Realize the following system using DF-I, DF-II, cascade & parallel.

$$y(n) - 5y(n-1) + 6y(n-2) = x(n) + 2x(n-1)$$

Realize the following system using cascade & parallel

$$H(z) = \frac{1+5z^{-1}}{1+13z^{-1}+40z^{-2}}$$

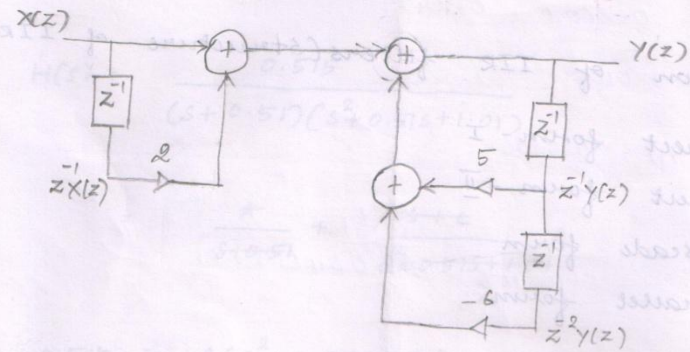
$$y(n) - 5y(n-1) + 6y(n-2) = x(n) + 2x(n-1)$$

DF-I

i. Take z-transform on both sides of gn. equ.

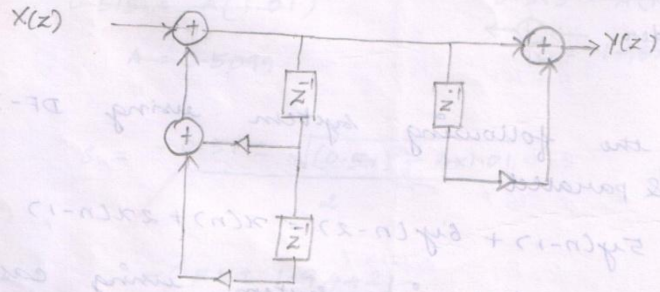
$$Y(z) - 5z^{-1}Y(z) + 6z^{-2}Y(z) = X(z) + 2z^{-1}X(z)$$

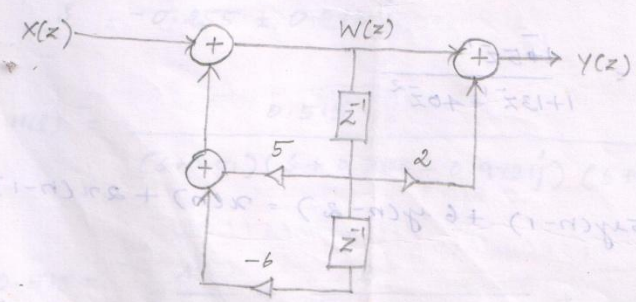
$$Y(z) = X(z) + 2z^{-1}X(z) + 5z^{-1}Y(z) - 6z^{-2}Y(z)$$



We use seperate delay element for i/p & o/p in D

DF - II :

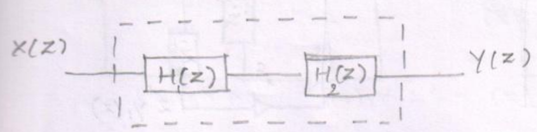




DF - II is better than DF - I because the total no. of delay elements are reduced.

Cascade Form :

$$H(z) = H_1(z) \cdot H_2(z)$$



Cascade form

$$y(n) - 5y(n-1) + 6y(n-2) = x(n) + 2x(n-1)$$

i. Take z transform on both sides.

$$Y(z) - 5z^{-1}Y(z) + 6z^{-2}Y(z) = X(z) + 2z^{-1}X(z).$$

$$Y(z)\{1 - 5z^{-1} + 6z^{-2}\} = X(z)\{1 + 2z^{-1}\}$$

$$\frac{Y(z)}{X(z)} = \frac{1+2z^{-1}}{1-5z^{-1}+6z^{-2}} = H(z)$$

$$H(z) = \frac{1+2z^{-1}}{(1-2z^{-1})(1-3z^{-1})} = \frac{(1+2z^{-1})}{(1-3z^{-1})} \frac{1}{(1-2z^{-1})}$$

$$H(z) = H_1(z) \cdot H_2(z)$$

Realize $H_1(z)$ using DF-II

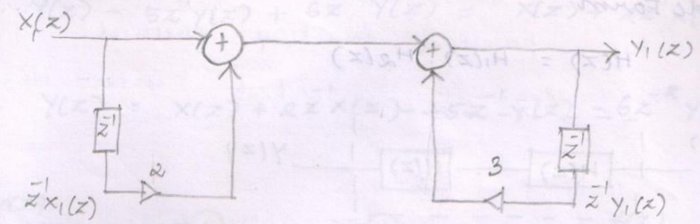
$$H_1(z) = \frac{1+2z^{-1}}{1-3z^{-1}}$$

$$\frac{Y_1(z)}{X_1(z)} = \frac{1+2z^{-1}}{1-3z^{-1}}$$

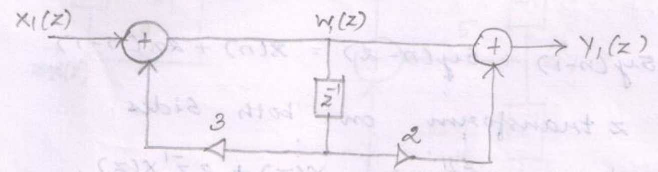
$$Y_1(z) - 3z^{-1}Y_1(z) = X_1(z) + 2z^{-1}X_1(z).$$

$$Y_1(z) = X_1(z) + 2z^{-1}X_1(z) + 3z^{-1}Y_1(z).$$

D.F I :



DF II :



Realize $H_2(z)$:

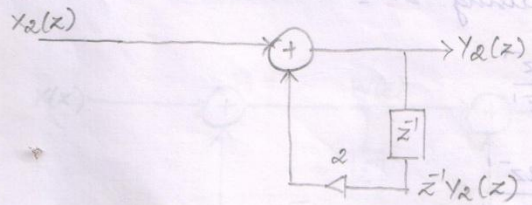
$$H_2(z) = \frac{1}{1-2z^{-1}}$$

$$\frac{Y_2(z)}{X_2(z)} = \frac{1}{1-2z^{-1}}$$

$$Y_2(z) - 2z^{-1}Y_2(z) = X_2(z)$$

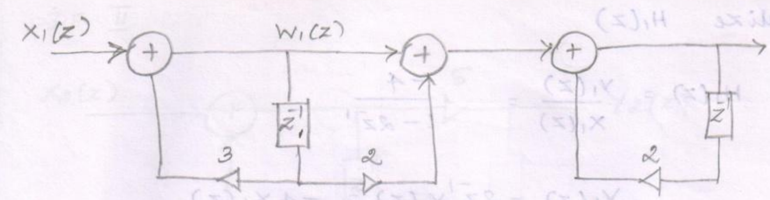
$$Y_2(z) = X_2(z) + 2z^{-1}Y_2(z)$$

DF I :



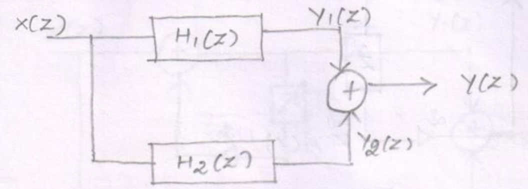
~~The resultant DF I~~

This is applicable to DF II.



Cascade form

-2-12 Parallel form :



$$\boxed{H(z) = H_1(z) + H_2(z)}$$

$$H(z) = \frac{1+2z^{-1}}{1-5z^{-1}+6z^{-2}}$$

$$= \frac{1+2z^{-1}}{(1-2z^{-1})(1-3z^{-1})} = \frac{A}{1-2z^{-1}} + \frac{B}{1-3z^{-1}}$$

$$1+2z^{-1} = A(1-3z^{-1}) + B(1-2z^{-1})$$

put $z^{-1} = \frac{1}{3}$

$1 + 2/3 = 0 + B(1 - 2/3)$

$B = 5$

put $z^{-1} = 1/2$

$1 + 2\cdot\frac{1}{2} = A(1 - \frac{3}{2})$

$A = -4$.

$$H(z) = \frac{-4}{1-2z^{-1}} + \frac{5}{1-3z^{-1}}$$

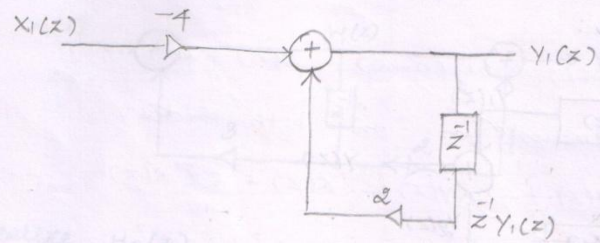
$$H_1(z) = \frac{-4}{1-2z^{-1}} \qquad H_2(z) = \frac{5}{1-3z^{-1}}$$

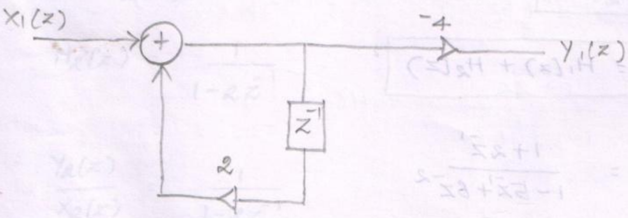
Realize $H_1(z)$

$$H_1(z) = \frac{Y_1(z)}{X_1(z)} = \frac{-4}{1-2z^{-1}}$$

$$Y_1(z) - 2z^{-1}Y_1(z) = -4X_1(z)$$

$$Y_1(z) = -4X_1(z) + 2z^{-1}Y_1(z).$$



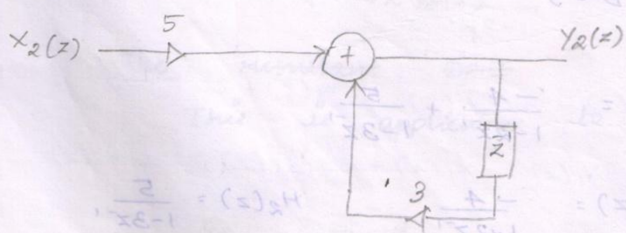


Realize $H_2(z)$

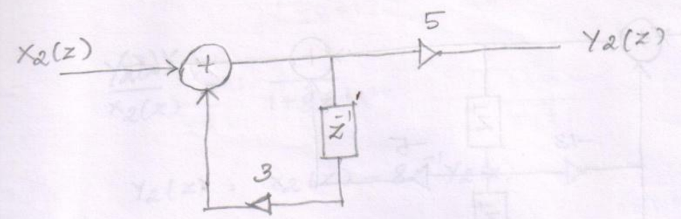
$$H_2(z) = \frac{Y_2(z)}{X_2(z)} = \frac{5}{1-3z^{-1}}$$

$$Y_2(z) - 3z^{-1}Y_2(z) = 5X_2(z)$$

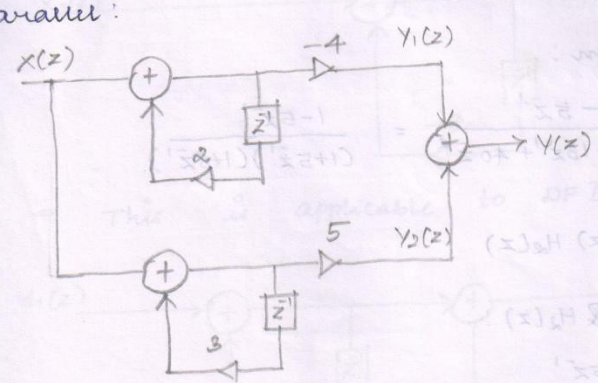
$$Y_2(z) = 5X_2(z) + 3z^{-1}Y_2(z)$$



DF II



Parallel :



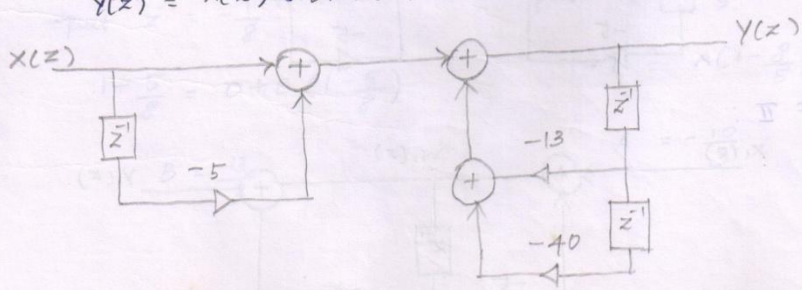
2. Given: $H(z) = \dfrac{1-5z^{-1}}{1+13z^{-1}+40z^{-2}}$

DF I :

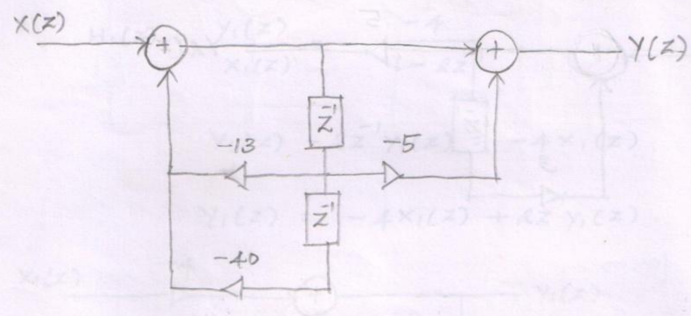
$$\frac{Y(z)}{X(z)} = \frac{1-5z^{-1}}{1+13z^{-1}+40z^{-2}}$$

$$Y(z) + 13z^{-1}Y(z) + 40z^{-2}Y(z) = X(z) - 5z^{-1}X(z)$$

$$Y(z) = X(z) - 5z^{-1}X(z) - 13z^{-1}Y(z) - 40z^{-2}Y(z)$$



DF II :



Cascade form :

$$H(z) = \frac{1-5z^{-1}}{1+13z^{-1}+40z^{-2}} = \frac{1-5z^{-1}}{(1+5z^{-1})(1+8z^{-1})}$$

$$H(z) = H_1(z) \cdot H_2(z)$$

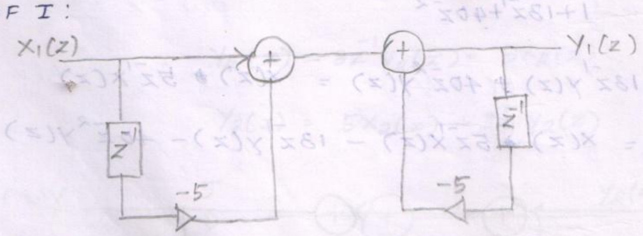
Realize $H_1(z)$ & $H_2(z)$:

$$H_1(z) = \frac{1-5z^{-1}}{1+5z^{-1}}$$

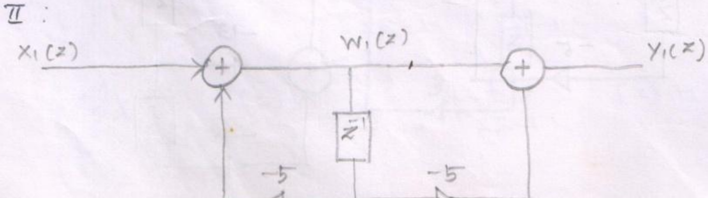
$$\frac{Y_1(z)}{X_1(z)} = \frac{1-5z^{-1}}{1+5z^{-1}}$$

$$Y_1(z) = X_1(z) - 5z^{-1}X_1(z) - 5z^{-1}Y_1(z)$$

DF I :



DF II :

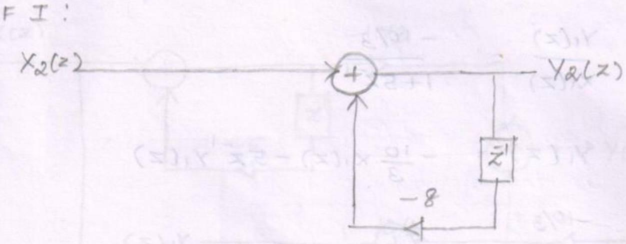


$$H_2(z) = \frac{1}{1+8z^{-1}}$$

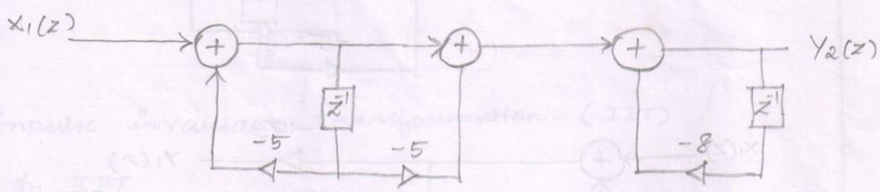
$$\frac{Y_2(z)}{X_2(z)} = \frac{1}{1+8z^{-1}},$$

$$Y_2(z) = X_2(z) - 8z^{-1}Y_2(z)$$

DF I:



This is applicable to DF II:



Cascade form

Parallel form:

$$H(z) = \frac{1-5z^{-1}}{(1+5z^{-1})(1+8z^{-1})} = \frac{A}{1+5z^{-1}} + \frac{B}{1+8z^{-1}},$$

$$1-5z^{-1} = A(1+8z^{-1}) + B(1+5z^{-1})$$

put $z^{-1} = -\frac{1}{8}$ put $z^{-1} = -\frac{1}{5}$

$$1+\frac{5}{8} = 0 + B\left(1-\frac{5}{8}\right) \qquad 1+1 = A\left(1-\frac{8}{5}\right)+0$$

$$B = \frac{13}{3} \qquad A = -\frac{10}{3}$$

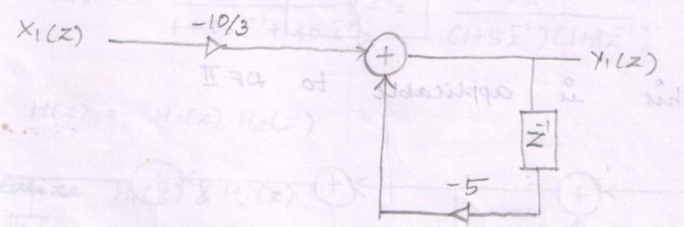
$$H(z) = \frac{-10/3}{1+5z^{-1}} + \frac{13/3}{1+8z^{-1}}$$

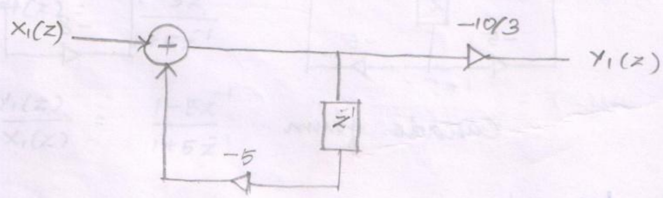
$$H_1(z) = \frac{-10/3}{1+5z^{-1}} \qquad H_2(z) = \frac{13/3}{1+8z^{-1}}$$

Realize $H_1(z)$

$$H_1(z) = \frac{Y_1(z)}{X_1(z)} = \frac{-10/3}{1+5z^{-1}}$$

$$Y_1(z) = -\frac{10}{3}X_1(z) - 5z^{-1}Y_1(z)$$

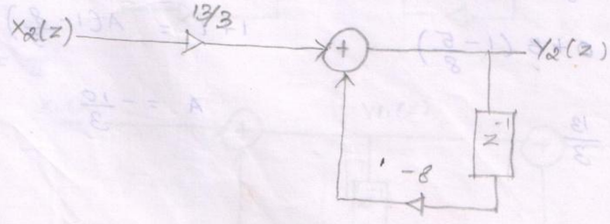


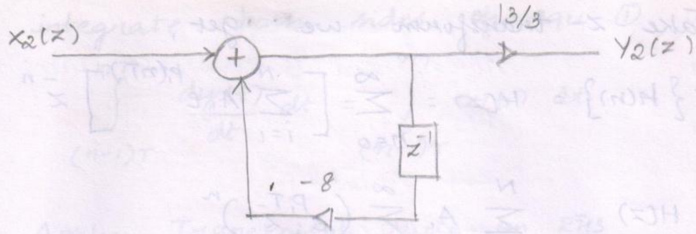


Realize $H_2(z)$

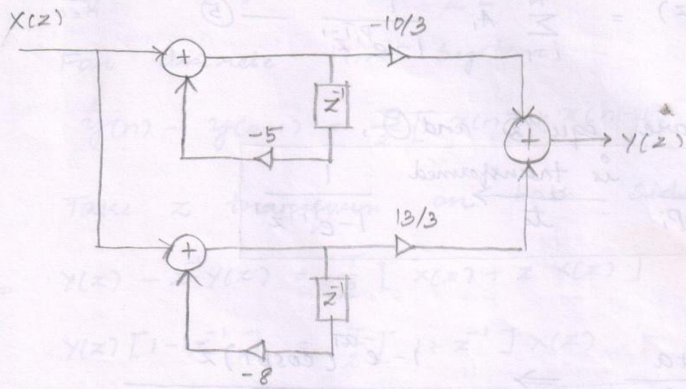
$$H_2(z) = \frac{Y_2(z)}{X_2(z)} = \frac{13/3}{1+8z^{-1}}$$

$$Y_2(z) = \frac{13}{3}X_2(z) - 8z^{-1}Y_2(z)$$





Parallel form:



5-7-12

Impulse invariant transformation: (IIT)

In IIT,

$$h(n) = h(t)|_{t=nT} \quad \text{—— ①}$$

Transfer function of Analog filter is given by

$$H(S) = \sum_{i=1}^{N} \frac{A_i}{S-p_i} \quad \text{—— ②}$$

Taking inverse laplace transform, we get

$$h(t) = \sum_{i=1}^{N} A_i e^{p_i t} u(t) \quad \text{—— ③}$$

From ①,

$$h(n) = \sum_{i=1}^{N} A_i e^{p_i nT} u(nT) \quad \text{—— ④}$$

Take z- transform we get,

$$z\{h(n)\} = H(z) = \sum_{n=0}^{\infty}\left[\sum_{i=1}^{N} A_i e^{P_i(nT)}\right] z^{-n}$$

$$H(z) = \sum_{i=1}^{N} A_i \sum_{n=0}^{\infty}\left(e^{P_i T} z^{-1}\right)^n$$

$$H(z) = \sum_{i=1}^{N} A_i \frac{1}{1-e^{P_i T} z^{-1}} \quad — ⑤ \qquad \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

Compare equ ② and ⑤,

$$\boxed{\frac{1}{S-P_i} \xrightarrow[\text{to}]{\text{is transformed}} \frac{1}{1-e^{P_i T} z^{-1}}}$$

Note:

$$\frac{S+a}{(S+a)^2+b^2} \Rightarrow \frac{1-e^{-aT}(\cos bT) z^{-1}}{1-2e^{-aT}(\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

$$\frac{b}{(S+a)^2+b^2} \Rightarrow \frac{e^{-aT}(\sin bT) z^{-1}}{1-2e^{-aT}(\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

Bilinear transformation (BLT):

* It is a conformal mapping that transforms imaginary axis of S-plane into unit circle in z plane only once

First order differential equation of analog system is given by

$$\frac{d(y(t))}{dt} = x(t) \quad — ①$$

integrate both sides of equ. ①

$$\int_{(n-1)T}^{nT} \frac{dy(t)}{dt}\, dt = \int_{(n-1)T}^{nT} x(t)\, dt$$

Apply Trapezoidal rule on RHS

$$y(nT) - y((n-1)T) = \frac{T}{2}\left[x(nT) + x((n-1)T) \right]$$

For discrete time system,

$$y(n) - y(n-1) = \frac{T}{2}\left[x(n) + x(n-1) \right] \quad —②$$

Take z transform on both sides of equ. ②

$$Y(z) - z^{-1}Y(z) = \frac{T}{2}\left[X(z) + z^{-1}X(z) \right]$$

$$Y(z)\left[1 - z^{-1}\right] = \frac{T}{2}\left[1 + z^{-1} \right] X(z)$$

$$\boxed{X(z) = Y(z)\, \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]} \quad —③$$

Take laplace transform of equ. ①

$$\boxed{SY(S) = X(S)} \quad —④$$

Compare equ. ③ and ④ we get

$$\boxed{S \xrightarrow[\text{to}]{\text{is transformed}} \frac{2}{T}\cdot\frac{1-z^{-1}}{1+z^{-1}}}$$

Approximation of Derivatives:

$$\boxed{S \xrightarrow[\text{to}]{\text{is transformed}} \frac{1-z^{-1}}{T}}$$

UNIT V - DIGITAL SIGNAL PROCESSORS

Features of DSP processors

1. DSP processors should have multiple registers so that data exchange from register to register to fast.

2. It requires multiple operands simultaneously. Hence DSP processors should have multiple operand fetch capacity.

3. DSP processors should have circular buffers to support circular shift operations.

4. It should be able to perform multiply and accumulate operations very fast.

5. It should have multiple pointers to support multiple operands, jumps and shifts.

6. To support the DSP operations fast, the DSP processors should have on chip memory.

7. For real time applications, interrupts and timers are required. Hence DSP processors should have powerful interrupt structure and timers.

Types of Architectures

There are three types of standard architecture for microprocessors.

i) Modified Harvard Architecture

In this architecture data memory can be shared by data as well as programs.

Normally the program memory and data memory addresses are generated by separate address generators. The data address generator for programs can address program memory as well as data memory. This provides flexibility in use of these memories.

ii) The speed of the operation is also increased. The architecture shown in figure is normally on chip. Today's commonly used **Von-Neumann Architecture**

General purpose processors normally have this type of architecture. The architecture shares same memory for program and data.

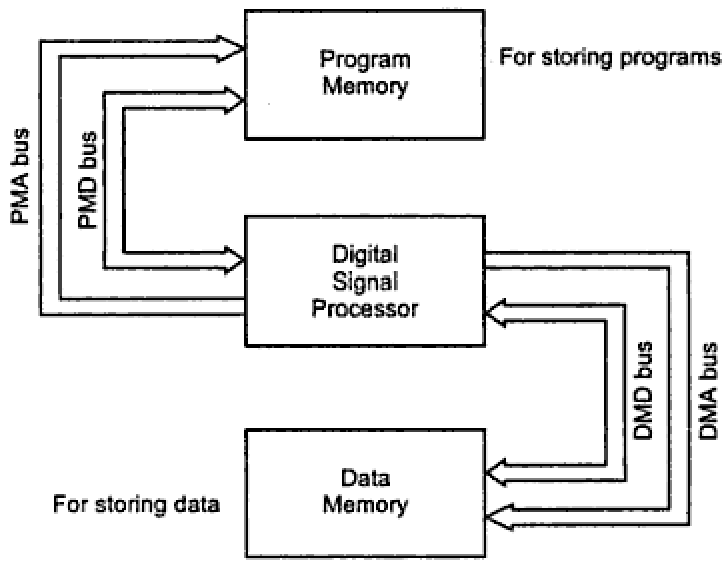
The architecture shares same memory for memory for program and data. The processors perform instruction fetch, decode and execute operations sequentially.

In such architecture the speed can be increased by pipelining. This type architecture contains common interval address and data bus, ALU, accumulator, I/O devices and common memory for program and data.

This type of architecture is not suitable for DSP processors.

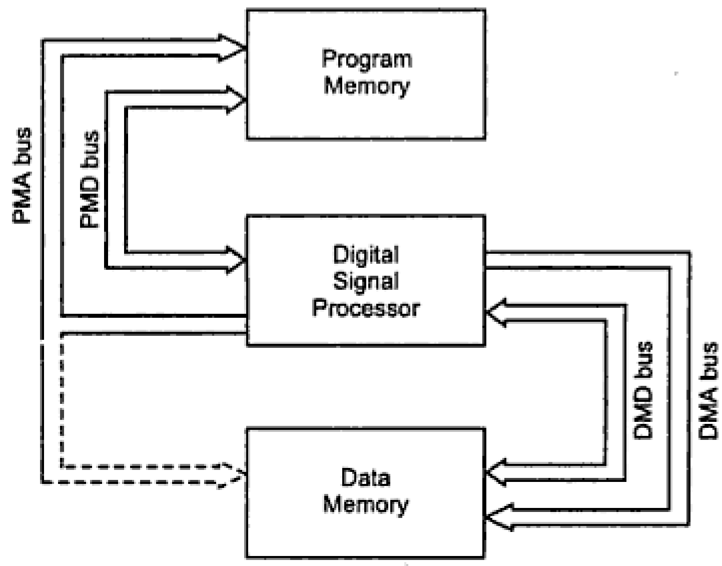
iii) <u>Harvard Architecture</u>

The Harvard architecture has separate memories for program and data. There are also separate address and data buses for program and data. Because of these separate on chip memories and internal buses, the speed of execution in harvard architecture is high.



Harvard Architecture

In the above figure observe that there is Program Memory Address (PMA) bus and Program Memory Data (PMD) bus separate for program memory. Similarly there is separate Data Memory Data (DMD) bus and Data Memory Address (DMA) bus of data memory. This is all on chip.
DSP processors normally have this type of architecture.



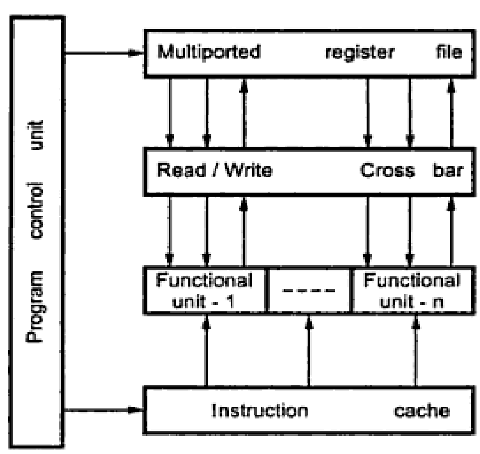
Modified Harvard Architecture

Dedicated MAC unit

Most of the operations in DSP involve array multiplication. The operation such as convolution, correlation requires multiply and accumulates operations. In real time applications the array multiplications and accumulation must be completed before next sample of input comes. This requires fast implementation of multiplication and accumulation.

The dedicated hardware unit called Multiplier Accumulator (MAC). It is one of the computational units in processor. The complete MAC operation is executed in one clock cycle.
The DSP processors have a special instruction called MACD. This means multiply accumulate with data shift.

Multiple ALUs



Some of the DSP processors use Very Long Instruction Word (VLIW) architecture. Such architecture consists of multiply number of ALUs, MAC units, shifters etc.

The above architecture consists of multiported register file. It is used to fetching the operands and storing the results.

The Read/write cross bar provides random access by functional units to the multiported register file. The functional units work concurrently with the load/store operation of data between a RAM and the register file.

The program control unit provides the algorithm that executes independent parallel operations. The performance of VLIW architecture depends upon degree of parallelism.

Normally 8 functional units are preferred. The number is limited by hardware cost of the multiported register file and crossbar switch

Pipelining

Any instruction cycle can split in following micro instructions:

Fetch: In this phase, any instruction is fetched from the memory.

Decode: In this phase, an instruction is decoded.

Read: An operand required for the instruction is fetched from the data memory.

Execute: The operation is executed and results are stored at appropriate place.

Each of the above operations can be separately executed in different functional units. The figure shows how the instruction is executed without pipeline.

Value of T	Fetch	Decode	Read	Execute
1	I 1			
2		I 1		
3			I 1	
4				I 1
5	I 2			
6		I 2		
7			I 2	
8				I 2

In the above figure observe that when I1 is in fetch phase, other units such as decode, read and execute are in idle. Similarly when I1 is in decode phase, other phase are idle. This means each functional unit is busy only for 25% of the total time.

Figure shows the instruction execution with pipe line. Here observe that when I1 is in decode phase, next instruction I2 is fetched. Similarly when I2 goes to decode phase, next instruction I3 is fetched. Thus observe that the functional units are executing four successive instructions at any time. We observe that five instructions are executed in the same time if pipe line is used.

Value of T	Fetch	Decode	Read	Execute
1	I 1			
2	I 2	I 1		
3	I 3	I 2	I 1	
4	I 4	I 3	I 2	I 1
5	I 5	I 4	I 3	I 2
6		I 5	I 4	I 3
7			I 5	I 4
8				I 5

Addressing modes of DSP

Conventional microprocessor has addressing modes such as direct, indirect, immediate etc. The DSP processors have additional modes because of which execution is fast.

1. Short Immediate Addressing

The operand is specified using a short constant. This short constant becomes the part of a single word instruction. In TMS320C5X series of DSP processors 8-bit operand can be specified as one if the operand in single word instructions such as add, subtract, AND, OR,XOR etc.

2. Short Direct Addressing

The lower order address of the operand is specified in the single word instruction. In TMS320CXX DSP processors lower 7 bits of the address are specified as the part of the instruction. Higher 9 bits of the address are stored in the data page pointer. Each such data page consists of 128 words.

3. Memory mapped Addressing

The CPU and I/O registers are accessed as memory location. These registers are mapped in the starting page or final page of the memory space. In TMS320C5X page0 corresponds to CPU and I/O registers.

4. Indirect Addressing

The addresses of operands are stored in the indirect address registers. In TMS320CXX processors such registers are called auxiliary registers. Any of these auxiliary registers can be updated when operands fetched by these registers are being executed. The auxiliary registers are incremented or decremented automatically by the value specified in offset register. In TMS320CXX processors the offset register is called INDEX register.

5. Bit Reversed Addressing Mode

For the computation of FFT, the input data is required in bit reversed format. There is no need to actually reshuffle the data in bit reversed sequence. The serially arranged data in the memory or buffer can be given to the processor in bit reversed mode with the help of bit reversed addressing mode. With this addressing mode an address is incremented / decremented by number represented in bit reversed form.

6. Circular Addressing

With this mode, the data stored in the memory can be read / written in circular fashion. This increases the utility of the memory. The memory organized as a circular buffer. The beginning and ending addresses are continuously monitored. If the address exceeds ending address of the memory, then it is set at the beginning address of the memory.

TMS 302C54X Processors

- This processors series contain all the features of the basic architecture. It has number of additional features for improved speed and performance.
- This series of processors have advanced modified Harvard architecture.
- The TMS 302C54X is upward compatible to earlier fixed point processors such as ‘C2X’, ‘C2XX and ‘C5X processors.
- It is 16 bit fixed-point DSP processor family.

Advantages of C54X Devices

1. Enhanced Harvard architecture, which include one program bus, three data buses and four address buses.
2. CPU has high degree of parallelism and application specific hardware logic.
3. It has highly specialized instruction set for faster algorithms.
4. Modular architecture design for fast development of spinoff devices.
5. It has increased performance and low power consumption.

Features of C54X

A. CPU

1. One program bus, three data buses and four address buses.
2. 40 bit ALU, including 40 bit barrel shifter and two independent 40 bit accumulators.
3. 17 bit x 17 bit parallel multiplier coupled to 40 bit dedicated adder for non pipelined single cycle multiply / accumulate (MAC) operation.
4. Compare, select, store unit (CSSU) for the add /compare selection of viterbi operator.
5. Exponent encoder to compute the exponent of 40 bit accumulator value in single cycle.
6. Two address generators, including eight auxiliary registers and two auxiliary register arithmetic units.
7. Multiple-CPU/ core architecture on some devices.

B. Memory

1. 192 K words x 16 bit bit addressable memory space.
2. Extended program memory in some devices.

C. Instruction Set

1. Single instruction repeat and block repeat operations.
2. Block memory move operations.
3. 32 bit long operand instructions.
4. Instructions with 2 or 3 operand simultaneous reads.
5. Parallel load and parallel store instructions.
6. Conditional store instructions.
7. Fast return from interrupt.

D. On- Chip peripherals

1. Software programmable wait state generator.
2. Programmable bank switching logic.
3. On-chip PLL generator with internal generator with internal oscillator.
4. External bus-off control to disable the external data bus, address bus and control signals.
5. Programmable timer.
6. Bus hold feature for data bus.

ARCHITECTURE OF TMS 320C54X DSP PROCESSOR

Bus Architecture

- There are eight major 16 bit buses (four program/data bus and four address buses).
- Program bus (PB) carries instruction code and immediate operands from program memory.
- Three address buses (CB, DB and EB) interconnect CPU, data address generation logic, program address generation logic, on chip peripherals and data memory.
- Four address buses (PAB, CAB, DAB and EAB) carry the addresses needed for instruction execution.

Internal Memory Organization

- There are three individually selectable spaces: program, data and I/O space.
- There are 26 CPU registers plus peripheral registers that are mapped in data memory space.
- The ‘C54X devices can contain RAM as well as ROM.
- On-chip Rom is part of program memory space, and in some cases part of data memory space.
- There can be DARAM, SARAM, Two way shared RAM on the chip.
- On-chip memory can be protected from being manipulated externally.

CPU

The CPU of the ’54x devices contain:

- A 40-bit arithmetic logic unit (ALU)
- Two 40-bit accumulators
- A barrel shifter
- A 17 × 17-bit multiplier/adder
- A compare, select, and store unit (CSSU)

Arithmetic Logic Unit:

The ’54x devices perform 2s-complement arithmetic using a 40-bit ALU and two 40-bit accumulators (ACCA and ACCB). The ALU also can perform Boolean operations.

The ALU can function as two 16-bit ALUs and perform two 16-bit operations simultaneously when the C16 bit in status register 1 (ST1) is set.

Accumulators:

There are two accumulators A and B. They store the output from the ALU or the multiplier / adder block.

The bits in each accumulator is grouped as follows:

- Guard bits (bits 32–39)
- A high-order word (bits 16–31)
- A low-order word (bits 0–15)

Instructions are provided for storing the guard bits, the high-order and the low-order accumulator words in data memory, and for manipulating 32-bit accumulator words in or out of data memory. Also, any of the accumulators can be used as temporary storage for the other.

Barrel shifter:

- It is a 40 bit input came from a accumulator or data memory(CB,DB)
- Its 40 bit output is connected to ALU or data memory
- It can produces left shift of 0 to 31 bits and right shift of 0 to 16 bits

Multiplier / Adder Unit:

- It performs 17x17 bit 2s compliment multiplication and 40 bit addition in a single instruction cycle.
- The unit also contains fractional control, zero detector, a rounder and overflow/saturation logic.
- The fractional mode selected when FRCT bit =1.
- The fast on-chip multiplier allows the ’54x to perform operations such as convolution, correlation, and filtering efficiently

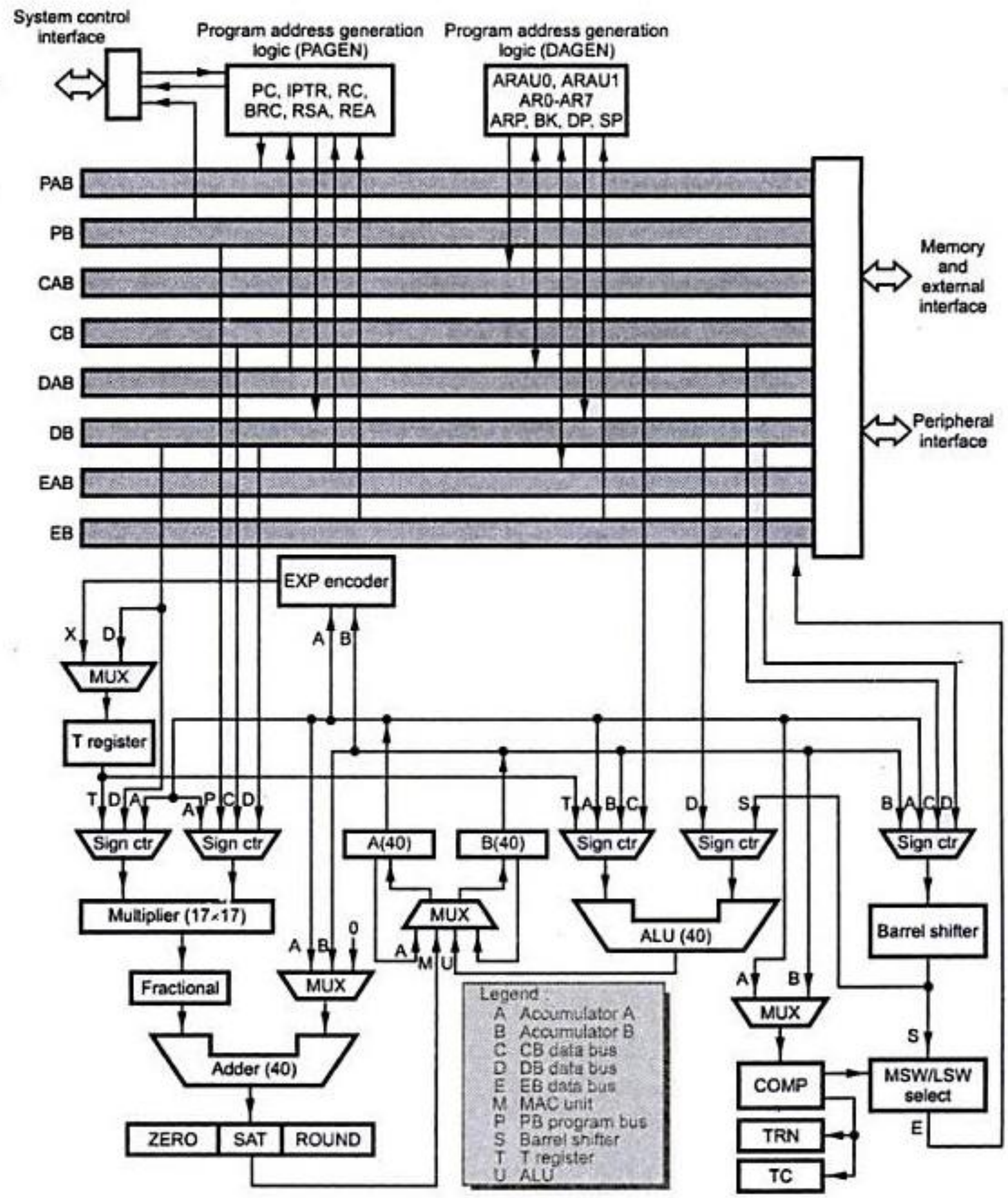
Compare, Select and Store Unit (CSSU):

- The compare, select, and store unit (CSSU) performs maximum comparisons between the accumulator’s high and low words, allows the test/control (TC) flag bit of status register 0 (ST0) and the transition (TRN) register to keep their transition histories, and selects the larger word in the accumulator to be stored in data memory.
- The CSSU also accelerates Viterbi-type butterfly computation with optimized on-chip hardware.

Data Addressing

The C54X processors have seven basic data addressing modes

1. Immediate addressing
2. Absolute addressing
3. Direct addressing
4. Indirect addressing
5. Memory mapped register addressing
6. Stack addressing



Functional block diagram of TMS 320C54X DSP processor

Status Register (ST0, ST1)

- The status registers, ST0 and ST1, contain the status of the various conditions and modes for the ’54x devices.

- ST0 contains the flags (OV, C, and TC) produced by arithmetic operations and bit manipulations ST1 contains the various modes and instructions that the processor operates on and executes.

Auxiliary Registers (AR0-AR7)

- The eight 16-bit auxiliary registers (AR0–AR7) can be accessed by the central airthmetic logic unit (CALU) and modified by the auxiliary register arithmetic units (ARAUs).

- The primary function of the auxiliary registers is generating 16-bit addresses for data space. However, these registers also can act as general-purpose registers or counters.

Temporary Registers (T REG)

- One of the multiplicands for multiply and multiply/accumulate instruction

- It can hold a shift count for instructions with shift operation such as ADD, LD and SUB

- It also holds a dynamic bit address for BITT instruction

Transition Register (TRN)

- This 16 bit register holds the transition decision for the path to new metrics to perform Viterbi algorithm.
- CMPS (Compare select max store) instruction updates the contents of TRN register on the basis of comparison of accumulator high word and acc low word.
-

Stack Point Register (SP)

- The SP is a 16-bit register that contains the address at the top of the system stack. The SP always points to the last element pushed onto the stack.
- The stack is manipulated by interrupts, traps, calls, returns, and the PUSHD, PSHM, POPD, and POPM instructions.

Program Memory Addressing

- The program memory is addressed with program counter (PC) the PC is used to fetch individual instructions.

- Program Counter is loaded by program address generator (PAGEN). PAGEN increments Program counter.

Pipeline Operation

- The C54X DSP has six levels: prefetch, fetch, decode, access, read and execute.
- One to six instructions can be active in a single cycle.

On-chip Peripherals

All the ’54x devices have the same CPU structure; however, they have different on-chip peripherals connected to their CPUs. The on-chip peripheral options provided are:

- General purpose I/O pins
- Software programmable Wait state Generator
- Programmable Bank-Switching Logic
- Hardware timer
- Clock generator
- DMA controller
- Host Port Interface
- Serial ports

General purpose I/O pins:
These pins can be read or written through software control. These pins are BIO and XF.

Software programmable Wait state Generator:
It extends external bus cycles up to seven machine cycles to interface with slower off-chip memory and I/O devices.

The software wait-state generator is incorporated without any external hardware. For off-chip memory access, a number of wait states can be specified for every 32K-word block of program and data memory space, and for one 64K-word block of I/O space within the software wait-state register (SWWSR)

Programmable Bank-Switching Logic:
It can automatically insert one cycle when an access crosses memory bank boundaries inside program memory or data memory space.
One cycle can also be inserted when crossing from program-memory space to data-memory space (’54x) or from one program memory page to another program memory page on selected devices.

Hardware timer:
It provides 16-bit timing circuit with 4-bit prescaler. The timer can be stopped, restarted, reset or disabled by specific status bits.

Clock generator:
The clock can be generated by two options (a) internal oscillator or (b) PLL circuit.

DMA controller:
It transfers data between by two points in the memory map without intervention by the CPU. The data can be moved to and from program data memory, on-chip peripherals or external memory devices.
Some of the features of DMA controller are as follows

- The DMA operates independently of the CPU.
- The DMA has six channels. The DMA can keep track of the contexts of six independent block transfers.
- The DMA has higher priority than the CPU for both internal and external accesses.
- Each channel has independently programmable priorities.

Host Port Interface (HPI):

It is parallel port. It provides an interface to a host processor. The information is exchanged between C54X and host processor through on-chip memory.

Serial ports:

There are four types of serial ports i) Synchnous ii) Buffered
iii) Multichannel buffered and iv) Time division multiplexed

Comparison between DSP processor and General purpose processor

S.No	Parameter	DSP Processors	General purpose processors
1.	Instruction cycle	Instructions are executed in single cycle of the clock i.e True instruction cycle.	Multiple clock cycles are required for execution of one instructions.
2.	Instruction execution	Parallel execution is possible	Execution of instruction is always sequential.
3.	Operand fetch from memory	Multiple operand are fetched simultaneously	Operands are fetched sequentially
4.	Memories	Separate program and data memories	Normally no such separate memories
5.	On-Chip/off-chip memories	Program and data memories are present onchip and extendable offchip.	Normally onchip cache memory is present. Main memory is offchip.
6.	Program flow control	Program sequencer and instruction register takes care of program flow.	Program counter maintains the flow of execution.
7.	Queuing/pipelining	Queuing is implicite through instruction register and instruction cache.	Queue is performed explicitely by queue registers for pipelining
8.	Address generation	Addresses are generated combinely by DAGs and program sequencer	Program counter is incremental sequentially to generate addresses.
9.	On-chip address and data buses.	Separate address and data buses for program memory and data memories and result bus. i.e. PMA, DMA, PMD, DMD and R-bus.	Address and data buses are the two buses on the chip.
10.	Addressing modes	Direct and indirect addressing is supported.	Direct, indirect, register, register indirect, immediate, etc addressing modes are supported.
11.	Suitable for	Array processing operations	General purpose processing.