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PROBABILITY AND COMPLEX FUNCTION MA3303

UNIT I
PROBABILITY AND RANDOM VARIABLES

Probability is a mahematical measurement of uncertainty.

Example :

* In a tossing of a coin one is not sure if a head or tail is obatined.
* If a dice is thrown we may get any of faces 1,2,3,4,5or6 but nobody knows which one wii actually occur.
* A tossing of a coin is a trial.
* Getting a head or trial is an event.
* The set of all possible oucomes of an experiment is called as a sample space and it is denoted by S.

Example:

i) tossing a coin S ={H,T}

ii)Throwing a dice S = {1,2,3,4,5,6}

iii)Tossing 2 coins S = {HH,HT,TH,TT}

iii)Tossing 2 dice S = { (1,1) (1,2) (1,3) (1,4) (1,5) (1,6)
(2,1) (2,2) (2,3) (2,4) (2,5) (2,6)
- -
(6,1) (6,2) (6,3) (6,4) (6,5) (6,6)}

* The Probability of an event A Occuring is denoted as P(A) is defined by

$$P(A)=\frac{N(A)}{N(S)}$$

RANDOM VARIABLE

A random variable is a rule that assigns a numerical value to each possible outcome of an experiment.

Example :

An experiment consist of tosses of two coins consider the random variable X which is the number of heads

Outcomes	:	HH	HT	TH	TT
getting Head X	:	2	1	1	0

$\therefore X = \{0,1,2\}$

DISCRETE RANDOM VARIABLE

A random variable which can assume only a countable number of real values is called a discrete random variable.

PROBABILITY MASS FUNCTION (pmf)

if X is a discrete random variable which takes the values X_1, X_2, X_3 etc. Then $P(X = x_i) = P(x_i)$ is called probabllity mass function.

It is denoted by tabular format

X	x_1	x_2	x_3
P(x)	$P(x_1)$	$P(x_2)$	$P(x_3)$

This called probability distribution $P(x_i)$ must satisfy the following condition

i) $P(x_i) \geq 0$ ii) $\sum_{i=1}^{\infty} P(x_i) = 1$

CONTINUOUS RANDOM VARIABLE

A random variable X is said to be continuous if it takes all possible values between certain limits.

PROBABLITY DENSITY FUNCTION (pdf)

The probability function f(x) of a continuous random variable X is called a probability density function.

The function f(x) satisfies the following conditions

i) $f(x) \geq 0$ ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

CUMULATIVE DISTRIBUION FUNCTION (cdf)

(or)

DISTRIBUTION FUNCTION

If X is a random variable discrete or continuous then $F(x) = P(X \leq x)$ is called the cumulative distribution function .

For discrete Random variable

$$F(x) = P(X \leq x) = \sum P(x)$$

For continuous Random variable

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)dx$$

PROBLEMS

1. If a random variable X takes the values 1,2,3,4 such that 2 P(X=1)=3P(X=2)=P(X=3)=5P(X=4). Find the probability distribution of X

Solution:

Assume P(X=3) = α By the given equation

$P(X=1) = \frac{\alpha}{2}$ $P(X=2) = \frac{\alpha}{3}$ $P(X=4) = \frac{\alpha}{5}$ For a probability distribution (and mass function) $\sum P(x) = 1$

P(1)+P(2)+P(3)+P(4) =1

$$\frac{\alpha}{2} + \frac{\alpha}{3} + \alpha + \frac{\alpha}{5} = 1 \quad \Rightarrow \quad \frac{61}{30}\alpha = 1 \quad \Rightarrow \alpha = \frac{30}{61}$$

$$P(X=1) = \frac{15}{61}\ ; P(X=2) = \frac{10}{61};\ P(X=3) = \frac{30}{61};\ P(X=4) = \frac{6}{61}$$

The probability distribution is given by

X	1	2	3	4
$p(x)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

2. A random variable X has the following probability distribution.

X:	**0**	**1**	**2**	**3**	**4**	**5**	**6**
f(x):	**0**	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

Find (i) the value of k (ii) p(1.5 < X < 4.5 | X > 2) and (iii) the smallest value of λ such that p(X≤λ) > $\frac{1}{2}$.

Solution:

$$\sum P(x) = 1$$

$$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$10k^2 + 9k - 1 = 0 \quad \Rightarrow k = -1, \frac{1}{10}$$

$$k = \frac{1}{10} = 0.1$$

$A = 1.5 < X < 4.5 = \{2,3,4\}$

$B = X > 2 = \{3,4,5,6,7\}$

(ii) $A \cap B = \{3,4\}$

$$p(1.5 < X < 4.5 | X > 2) = p(A | B) = \frac{p(A \cap B)}{p(B)} = \frac{p(3,4)}{p(3,4,5,6,7)}$$

$$= \frac{2k + 3k}{2k + 3k + k^2 + 2k^2 + 7k^2 + k} = \frac{5k}{10k^2 + 6k} = \frac{\frac{5}{10}}{\frac{7}{10}} = \frac{5}{7}$$

(iii)

X	p(X)	F(X)
0	0	0
2	2k = 0.2	0.3
3	2k = 0.2	0.5
4	3k = 0.3	0.8
5	k^2=0.01	0.81
6	$2k^2$ = 0.02	0.83
7	$7k^2$+k = 0.17	1.00

From the table for X = 4,5,6,7 p(X) > $\frac{1}{2}$ and the smallest value is 4 Therefore $\lambda = 4$.

3. Let X be a continuous random variable having the probability density function $f(x) = \begin{cases} \frac{2}{x^3}, & x \geq 1 \\ 0, & otherwise \end{cases}$ **Find the distribution function of x.**

Solution:

$$F(x) = \int_1^x f(x)\, dx = \int_1^x \frac{2}{x^3} dx = \left[-\frac{1}{x^2} \right]_1^x = 1 - \frac{1}{x^2}$$

4.A random variable X has the probability density function *f(x)* given by $f(x)=\begin{cases} cxe^{-x}, & x\ge 0 \\ 0, & otherwise \end{cases}$**. Find the value of c and CDF of X.**

Solution:

$$\int_0^\infty f(x)\,dx = 1$$

$$\int_0^\infty cxe^{-x}\,dx = 1$$

$$c\left[-xe^{-x}-e^{-x}\right]_0^\infty = 1$$

$$c(1) = 1$$

$$c = 1$$

$$F(x)=\int_0^x f(x)\,dx$$

$$=\int_0^x cxe^{-x}\,dx$$

$$=\int_0^x xe^{-x}\,dx$$

$$=\left[-xe^{-x}-e^{-x}\right]_0^x$$

$$=1-xe^{-x}-e^{-x}$$

5.A continuous random variable X has the probability density function *f(x)* given by $f(x)=ce^{-|x|}, -\infty<x<\infty$**. Find the value of c and CDF of X.**

Solution:

$$\int_{-\infty}^\infty f(x)\,dx = 1$$

$$\int_{-\infty}^\infty ce^{-|x|}\,dx = 1$$

$$2\int_0^\infty ce^{-|x|}\,dx = 1$$

$$2\int_0^\infty ce^{-x}\,dx = 1$$

$$2c\left[-e^{-x}\right]_0^\infty = 1$$

$$2c(1) = 1$$

$$c=\frac{1}{2}$$

$Case(i)\ x < 0$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{x} c\,e^{-|x|}\,dx$$

$$= c\int_{-\infty}^{x} e^{x}\,dx$$

$$= c\left[e^{x}\right]_{-\infty}^{x}$$

$$= \frac{1}{2}e^{x}$$

$Case(ii)\ x > 0$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{x} c\,e^{-|x|}\,dx$$

$$= c\int_{-\infty}^{0} e^{x}\,dx + c\int_{0}^{x} e^{-x}\,dx$$

$$= c\left[e^{x}\right]_{-\infty}^{0} + c\left[-e^{-x}\right]_{0}^{x}$$

$$= c - ce^{-x} + c$$

$$= c\left(2 - e^{-x}\right)$$

$$= \frac{1}{2}\left(2 - e^{-x}\right)$$

$$F(x) = \begin{cases} \frac{1}{2}e^{x}, & x > 0 \\ \frac{1}{2}\left(2 - e^{-x}\right), & x < 0 \end{cases}$$

6.If a random variable has the probability density $f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & otherwise \end{cases}$**. Find the probability that it will take on a value between 1 and 3. Also, find the probability that it will take on value greater than 0.5.**

Solution:

$$P(1 < X < 3) = \int_1^3 f(x)\,dx = \int_1^3 2e^{-2x}dx = \left[-e^{-2x}\right]_1^3 = e^{-2} - e^{-6}$$

$$P(X > 0.5) = \int_{0.5}^{\infty} f(x)\,dx = \int_{0.5}^{\infty} 2e^{-2x}dx = \left[-e^{-2x}\right]_{0.5}^{\infty} = e^{-1}$$

7.Is the function defined as follows a density function?

$$f(x) = \begin{cases} 0, & x < 2 \\ \frac{1}{18}(3+2x), & 2 \le x \le 4 \\ 0, & x > 4 \end{cases}$$

Solution:

$$\int_2^4 f(x)\,dx = \int_2^4 \frac{1}{18}(3+2x)\,dx = \left[\frac{(3+2x)^2}{72}\right]_2^4 = 1$$

Hence it is density function.

8.The cumulative distribution function (CDF) of a random variable X is $F(X) = 1-(1+x)e^{-x}$, $x > 0$**. Find the probability density function of X.** **Solution:**

$$\begin{aligned} f(x) &= F'(x) \\ &= 0 - \left[(1+x)\left(-e^{-x}\right) + (1)\left(e^{-x}\right)\right] \\ &= xe^{-x}, \quad x > 0 \end{aligned}$$

9.A continuous random variable X has the distribution function

$$F(x) = \begin{cases} 0 & : \quad x \le 1 \\ k(1-x)^4 & : \quad 1 < x \le 3 \\ 0 & : \quad x > 3 \end{cases}$$

Find *k*, the probability density function *f(x)* and *P(X <2)*.

Solution:

Since it is a distribution function

$$F(\infty) = F(3) = 1$$

$$k(3-1)^4 = 1$$

$$k = \frac{1}{16}$$

The density function is $f(x) = F'(x) = \frac{1}{16}4(1-x)^3 = \frac{(1-x)^3}{4}, \ 1 \le x \le 3$

p(X < 2) = F(2) = $\frac{1}{16}(2-1)^4 = \frac{1}{16}$

10.If the cumulative distribution function of a R.V X is given by $F(x) = \begin{cases} 1-\frac{4}{x^2}, & x > 2 \\ 0, & x \le 2 \end{cases}$ **find (i) P(X < 3) (ii) P(4 < X < 5) (iii) P (X ≥3).**

Solution:

(i) P(X < 3) = F(3) = $1-\frac{4}{3^2} = \frac{5}{9}$

(ii) P(4 < X < 5) = F(5) - F(4) = $\left(1-\frac{4}{5^2}\right) - \left(1-\frac{4}{4^2}\right) = \frac{21}{25} - \frac{3}{4} = \frac{9}{100}$

(iii) P (X ≥3) = 1- F(3) = 1 - $\left(1-\frac{4}{3^2}\right)$ = 1- $\frac{5}{9} = \frac{4}{9}$

11.A continuous random variable X has the distribution function

$$F(x) = \begin{cases} 0 & : & x \le 1 \\ k(1-x)^4 & : & 1 < x \le 3 \\ 0 & : & x > 3 \end{cases}$$

Find *k*, the probability density function *f(x)* and *P(X <2)*.

Solution:

Since it is a distribution function

$$F(\infty) = F(3) = 1$$

$$k(3-1)^4 = 1$$

$$k = \frac{1}{16}$$

The density function is $f(x) = F'(x) = \frac{1}{16}4(1-x)^3 = \frac{(1-x)^3}{4}, \ 1 \le x \le 3$

p(X < 2) = F(2) = $\frac{1}{16}(2-1)^4 = \frac{1}{16}$

12.If the cumulative distribution function of a R.V X is given by $F(x) = \begin{cases} 1-\frac{4}{x^2}, & x > 2 \\ 0, & x \le 2 \end{cases}$ **find (i) P(X < 3) (ii) P(4 < X < 5) (iii) P (X ≥3).**

Solution:

(i) $P(X < 3) = F(3) = 1-\frac{4}{3^2} = \frac{5}{9}$

(ii) $P(4 < X < 5) = F(5) - F(4) = \left(1-\frac{4}{5^2}\right)-\left(1-\frac{4}{4^2}\right) = \frac{21}{25}-\frac{3}{4} = \frac{9}{100}$

(iii) $P (X \geq 3) = 1- F(3) = 1 - \left(1-\frac{4}{3^2}\right) = 1- \frac{5}{9} = \frac{4}{9}$

13.Given the p.d.f of a continuous r.v X as follows:

$f(x) = \begin{cases} 6x(1-x), & 0 \le x \le 1 \\ 0, & elsewhere \end{cases}$ **Find the CDF of X.**

Solution:

$$F(x) = \int_0^x f(x)\ dx = \int_0^x 6x(1\_x)\ dx = \int_0^x 6x\_6x^2\ dx = \left[3x^2 - 2x^3\right]_0^x = 3x^2 - 2x^3$$

14.A continuous random variable X has the probability function $f(x) = k(1+x),\ \ 2 \le x \le 5$. **Find P(X<4).**

Solution:

$$\int_2^4 f(x)dx = 1 \qquad \Rightarrow k\int_2^5 (1+x)dx = 1$$

$$\Rightarrow k\left[\frac{(1+x)^2}{2}\right]_2^5 = 1$$

$$\Rightarrow k\frac{27}{2} = 1$$

$$\Rightarrow k = \frac{2}{27}$$

$$P(X < 4) = \int_2^4 f(x)\ dx = \frac{2}{27}\int_2^4 (1+x)\ dx = \frac{2}{27}\left[\frac{(1+x)^2}{2}\right]_2^4 = \frac{1}{25}(25-9) = \frac{16}{27}$$ **15.Given the p.d.f of a continuous R.V X as follows:**

$f(x) = \begin{cases} 12.5x - 1.25 & 0.1 \le x \le 0.5 \\ 0, & elsewhere \end{cases}$

Find P(0.2 < X < 0.3)

Solution:

$$P(0.2 < X < 0.3) = \int_{0.2}^{0.3} (12.5x - 1.25)\, dx$$

$$= \left[12.5\frac{x^2}{2} - 1.25x \right]_{0.2}^{0.3}$$

$$= 1.25\left[5(0.3)^2 - 0.3 - 5(0.2)^2 + 0.2\right]$$

$$= 0.1875$$

16.If a RV X has the pdf $f(x) = \begin{cases} \frac{1}{4}, & |x| < 2 \\ 0, & otherwise \end{cases}$ **.**

Obtain (i) p(X <1) (ii) p($|X|$>1) (iii) p(2X+3 > 5)

(iv) p ($|X|$ < 0.5 | X < 1

Solution:

(i) p(X < 1) = $\int_{-2}^{1} \frac{1}{4} dx = \frac{1}{4}[x]_{-2}^{1} = \frac{3}{4}$

(ii) p($|X| \le 1$) = $\int_{-1}^{1} \frac{1}{4} dx = \frac{1}{4}[x]_{-1}^{1} = \frac{1}{2}$

Hence p($|X|$>1)= 1 - p($|X| \le 1$) = $\frac{1}{2}$

(iii) p(2X+3 > 5) = p(X > 1) = 1- p(X ≤1) = 1- $\frac{3}{4}$ = $\frac{1}{4}$

(iv) p ($|X|$ < 0.5 | X < 1) = $\dfrac{p(|X| < 0.5 \cap X < 1)}{p(X < 1)}$

$$= \frac{p([-0.5 < X < 0.5] \cap X < 1)}{p(X < 1)}$$

$$= \frac{p([-0.5 < X < 0.5])}{p(X < 1)} = \frac{\int_{-1}^{1} \frac{1}{4} dx}{\frac{3}{4}} = \frac{[x]_{-0.5}^{0.5}}{3} = \frac{1}{3}$$

EXPECTIATION (MEAN)

The average process when applied to the random variable is called expectation . It is denoted by E[X] or mean value of X.

For discrete case : $E[X] = \sum xP(x)$

For continuous case : $E[X] = \int_{-\infty}^{\infty} xf(x)dx$

MOMENT

For discrete case

$$\mu_1' = E[X] = \sum xP(x)$$

$$\mu_2' = E[X^2] = \sum x^2 P(x)$$

$$\mu_r' = E[X^r] = \sum x^r P(x)$$

For continuous case

$$\mu_1' = E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

$$\mu_2' = E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)dx$$

$$\mu_r' = E[X^r] = \int_{-\infty}^{\infty} x^r f(x)dx$$

PROPERTIES OF EXPECTATION

1. E[a] = a
2.E[aX] = aE[X]
3. E[aX+b] = aE[X]+b

VARIANCE

Var(X) = E[X^2] - [E(x)]^2

Standard Deviation = $\sqrt{Var(X)}$

MOMENT GENERAING FUNCTION FUNCTION(mgf)

The mgf of a random variable X is denoted by

$$M_X(t) = E[e^{tx}]$$

NOTE:

$M_X'(0) = \mu_1'$ and $M_X''(0) = \mu_2'$

17.Find the MGF of the RV X, whose pdf is given by $f(x) = \frac{1}{2} e^{-|x|}, \; -\infty < x < \infty$**. Hence its mean and variance.**

Solution:

$$M_X(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{0} e^{tx} e^{x}\; dx + \int_{0}^{\infty} e^{tx} e^{-x}\; dx$$

$$= \int_{-\infty}^{0} e^{(t+1)x}\; dx + \int_{0}^{\infty} e^{-(1-t)x}\; dx$$

$$= \left[\frac{e^{(t+1)x}}{(t+1)}\right]_{-\infty}^{0} + \left[\frac{e^{-(1-t)x}}{-(1-t)}\right]_{0}^{\infty}$$

$$M_X(t) = \frac{1}{2}\left(\frac{1}{1+t} + \frac{1}{1-t}\right) = \frac{1}{1-t^2} = 1 + t^2 + t^4 + \ldots.$$

Mean = E(X) = (coefficient of t) 1! = 0

$E(X^2)$ = (coefficient of t^2)2! =2

Variance = $E(X^2) - E(X)^2 = 2$

18.The p.m.f of a RV X, is given by $p(X = j) = \frac{1}{2^j}, \; j = 1,2,3\ldots$ **Find MGF, mean and variance.**

Solution:

$$M_X(t) = E\left(e^{tX}\right) = \sum e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{1}{2^x}$$

$$= \sum_{x=0}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \left(\frac{e^t}{2}\right)^4 + \ldots.$$

$$= \frac{e^t}{2}\left(1+\left(\frac{e^t}{2}\right)+\left(\frac{e^t}{2}\right)^2+\left(\frac{e^t}{2}\right)^3+\left(\frac{e^t}{2}\right)^4+..\right)$$

$$= \frac{e^t}{2}\frac{1}{1-\frac{e^t}{2}} = \frac{e^t}{2-e^t}$$

Differentiating twice with respect to t

$$M'_X(t) = \frac{\left(2-e^t\right)\left(e^t\right)-e^t\left(-e^t\right)}{\left(2-e^t\right)^2} = \frac{2e^t}{\left(2-e^t\right)^2}$$

$$M''_X(t) = \frac{\left(2-e^t\right)^2\left(2e^t\right)-2e^t 2\left(2-e^t\right)\left(-e^t\right)}{\left(2-e^t\right)^4} = \frac{4e^t+2e^{2t}}{\left(2-e^t\right)^3}$$

put t = 0 above $E(X) = M'_X(0) = 2$

$E\left(X^2\right) = M''_X(0) = 6$

$Variance = E\left(X^2\right) - E(X)^2 = 6-4=2$

19.Find MGF of the RV X, whose pdf is given by $f(x) = \lambda e^{-\lambda x},\ x>0$ **and hence find the first four central moments**.

Solution:

$$M_X(t) = E\left(e^{tX}\right) \quad = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^{\infty} e^{-(\lambda - t)x}\, dx$$

$$= \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)}\right]_0^{\infty}$$

$$= \frac{\lambda}{(\lambda - t)}$$

Expanding in powers of t

$$M_X(t) = \frac{\lambda}{(\lambda - t)} = \frac{1}{1-\left(\frac{t}{\lambda}\right)} = 1 + \left(\frac{t}{\lambda}\right) + \left(\frac{t}{\lambda}\right)^2 + \left(\frac{t}{\lambda}\right)^3 + \ldots$$

Taking the coefficient we get the raw moments about origin

$$E(X) = (coefficient \ of \ t)1! = \frac{1}{\lambda}$$

$$E\left(X^2\right) = \left(coefficient \ of \ t^2\right)2! = \frac{2}{\lambda^2}$$

$$E\left(X^3\right) = \left(coefficient \ of \ t^3\right)3! = \frac{6}{\lambda^3}$$

$$E\left(X^4\right) = \left(coefficient \ of \ t^4\right)4! = \frac{24}{\lambda^4}$$

and the central moments are

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - 2C_1\mu_1'\mu_1' + \mu_1'^2$$

$$= \frac{2}{\lambda^2} - 2\frac{1}{\lambda^2} + \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\mu_3 = \mu_3' - 3C_1\mu_2'\mu_1' + 3C_2\mu_1'\mu_1'^2 - \mu_1'^3$$

$$= \frac{6}{\lambda^3} - 3\frac{2}{\lambda^2}\frac{1}{\lambda} + 3\frac{1}{\lambda}\frac{1}{\lambda^2} - \frac{1}{\lambda^3} = \frac{2}{\lambda^3}$$

$$\mu_4 = \mu_4' - 4C_1\mu_3'\mu_1' + 4C_2\mu_2'\mu_1'^2 - 4C_3\mu_1'^4 + \mu_1'^4$$

$$= \frac{24}{\lambda^4} - 4\frac{6}{\lambda^3}\frac{1}{\lambda} + 6\frac{2}{\lambda^2}\frac{1}{\lambda^2} - 4\frac{1}{\lambda^4} + \frac{1}{\lambda^4} = \frac{9}{\lambda^4}$$

20.If the MGF of a (discrete) RV X is $\frac{1}{5-4e^t}$ find the distribution of X and p (X = 5 or 6).

Solution:

$$M_X(t) = \frac{1}{5-4e^t} = \frac{1}{5\left(1-\frac{4e^t}{5}\right)}$$

$$= \frac{1}{5}\left[1 + \left(\frac{4e^t}{5}\right) + \left(\frac{4e^t}{5}\right)^2 + \left(\frac{4e^t}{5}\right)^3 + \ldots\right]$$

By definition

$$M_X(t) = E\left(e^{tX}\right) = \sum e^{tx} p(x)$$

$$= 1 + e^{t0} p(0) + e^{t} p(1) + e^{t2} p(2) + \ldots$$

On comparison

$$p(0) = \frac{1}{5} \qquad p(1) = \frac{4}{25} \qquad p(2) = \frac{16}{125} \qquad p(3) = \frac{64}{625}$$

In general $p(X = r) = \frac{1}{5}\left(\frac{4}{5}\right)^r, \quad r = 0,1,2,3$

$$p(X = 5 \text{ or } 6) = p(X = 5) + p(X = 6)$$

$$= \frac{1}{5}\left(\frac{4}{5}\right)^5 + \frac{1}{5}\left(\frac{4}{5}\right)^6$$

$$= \frac{1}{5}\left(\frac{4}{5}\right)^5\left(1 + \frac{4}{5}\right)$$

$$= \frac{9}{25}\left(\frac{4}{5}\right)^5$$

21.If X has the probability density function $f(x) = k\,e^{-3x},\ x > 0$

Find (i) k (ii) p(0.5 ≤X≤1) (iii) Mean of X.

Solution:

(i)

$$p(0.5 \le X \le 1) = \int_{0.5}^{1} f(x)\, dx$$

$$= \int_{0.5}^{1} 3e^{-3x}\, dx$$

$$= 3\left[\frac{-e^{-3x}}{3}\right]_{0.5}^{1}$$

$$= -e^{-3} + e^{-1.5}$$

(iii)

$$Mean = E(X) = \int_0^\infty x\, f(x)\, dx$$

$$= \int_0^\infty 3x\, e^{-3x}\, dx$$

$$= 3\left[-x\frac{e^{-3x}}{3} - \frac{e^{-3x}}{9}\right]_0^\infty$$

$$= 3\left(\frac{1}{9}\right) = \frac{1}{3}$$

22.If X has the distribution function

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{3}, & 1 \le x < 4 \\ \frac{1}{2}, & 4 \le x < 6 \\ \frac{5}{6}, & 6 \le x < 10 \\ 1, & x > 10 \end{cases} \qquad \textbf{(1)}$$

Probability distribution of X (2) p(2<X<6) (3) Mean (4) variance Solution:

(1)As there is no x terms in the distribution function given is a discrete random variable. Hence the probability distribution is given by

X	1	4	6	10
$p(X)$	$\frac{1}{3}$	$\frac{1}{2}-\frac{1}{3}$	$\frac{5}{6}-\frac{1}{2}$	$1-\frac{5}{6}$
	$\frac{1}{3}$	$=\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$

(2) p(2<X<6) = p(4) = $\frac{1}{6}$

(3) Mean = E(X) = $\sum x\ p(x) = (1)\left(\frac{1}{3}\right) + (4)\left(\frac{1}{6}\right) + (6)\left(\frac{1}{3}\right) + (10)\left(\frac{1}{6}\right) = \frac{14}{3}$

(4) E(X^2) = $\sum x^2\ p(x) = (1)\left(\frac{1}{3}\right) + (16)\left(\frac{1}{6}\right) + (36)\left(\frac{1}{3}\right) + (100)\left(\frac{1}{6}\right) = \frac{95}{3}$

Variance = E(X^2) – E(X)^2 = $\frac{95}{3}$ - $\frac{196}{9}$ = $\frac{89}{9}$

23.If the MGF of a continuous R.V X is given by $M_X(t) = \frac{3}{3-t}$**.**

Find the mean and variance of X.

Solution:

$$M_X(t) = \frac{3}{3-t} = \frac{1}{1-\frac{t}{3}} = \left(1-\frac{t}{3}\right)^{-1} = 1+\frac{t}{3}+\left(\frac{t}{3}\right)^2+\left(\frac{t}{3}\right)^3+\ldots$$

$$E(X) = (coefficient\ \ of\ \ \ t)\ 1! = \frac{1}{3}\ \ is\ \ the\ \ \ mean$$

$$E(X^2) = \left(coefficient\ of\ t^2\right)2! = \frac{1}{9}2! = \frac{2}{9}$$

$$Variance\ = E(X^2) - E(X)^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

24.If the MGF of a discrete R.V X is given by $M_X(t) = \frac{1}{81}\left(1+2e^t\right)^4$,find the distribution of X.

Solution:

$$M_X(t) = \frac{1}{81}\left(1+2e^t\right)^4 = \frac{1}{81}\left(1+4C_1\left(2e^t\right)+4C_2\left(2e^t\right)^2+4C_3\left(2e^t\right)^3+4C_4\left(2e^t\right)^4\right)$$

$$= \frac{1}{81}+\frac{8}{81}e^t+\frac{24}{81}e^{2t}+\frac{32}{81}e^{3t}+\frac{16}{81}e^{4t}$$

By definition of MGF ,

$$M_X(t) = \sum e^{tx}p(x) = p(0) + p(1)e^t + p(2)e^{2t} + p(3)e^{3t} + p(4)e^{4t}$$

On comparison with above expansion the probability distribution is

X	0	1	2	3	4
$p(x)$	$\frac{1}{81}$	$\frac{8}{81}$	$\frac{24}{81}$	$\frac{32}{81}$	$\frac{16}{81}$

25.Find the MGF of the R.V X whose p.d.f is $f(x) = \begin{cases} \frac{1}{10}, & 0 < x < 10 \\ 0, & elsewhere \end{cases}$ .Hence its mean.

Solution:

$$M_X(t) = \int_0^{10} \frac{1}{10} e^{tx}\, dx$$

$$= \frac{1}{10}\left(\frac{e^{tx}}{t}\right)_0^{10}$$

$$= \frac{1}{10}\left(\frac{e^{10t}-1}{t}\right)$$

$$= \frac{1}{10t}\left(1+10t+\frac{100t^2}{2!}+\frac{1000t^3}{3!}+....-1\right)$$

$$= 1+5t+\frac{1000}{31}t^2+.....$$

$$Mean = coefficient\ of\ \ t = 5$$

Given the probability density function $f(x) = \dfrac{k}{1+x^2}, -\infty < x < \infty$, find k and C.D.F.

Solution:

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{k}{1+x^2}\, dx = 1$$

$$\Rightarrow k\left[\tan^{-1} x\right]_{-\infty}^{\infty} = 1$$

$$\Rightarrow k\left[\left[\tan^{-1}\infty\right]-\left[\tan^{-1} -\infty\right]\right] = 1$$

$$\Rightarrow k\left[\frac{\pi}{2}+\frac{\pi}{2}\right] = 1$$

$$\Rightarrow k = \frac{1}{\pi}$$

$$F(x) = \int_{-\infty}^{x} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{k}{1+x^2}\, dx$$

$$= \frac{1}{\pi}\left[\tan^{-1} x\right]_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left[\left[\tan^{-1}\infty\right]-\left[\tan^{-1} - x\right]\right]$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - \tan^{-1} x\right) = \frac{1}{\pi}\cot^{-1} x$$

26.The density function of a random variable X is given by $f(x) = kx(2-x),\ 0 \le x \le 2$. Find *k*, mean , variance and *r*th moment.

Solution:

$$\int_0^2 f(x)\,dx = 1 \qquad \int_0^2 kx(2-x)\,dx = 1$$

$$k\int_0^2 \left(2x - x^2\right)dx = 1 \qquad k\left[x^2 - \frac{x^3}{3}\right]_0^2 = 1$$

$$k\left(4 - \frac{8}{3}\right) = 1 \qquad k = \frac{3}{4}$$

$$\mu_r' = \int_0^2 x^r \frac{3}{4}x(2-x)\,dx$$

$$= \frac{3}{4}\int_0^2 \left(2x^{r+1} - x^{r+2}\right)dx$$

put $r = 1,2$

$$= \frac{3}{4}\left[2\frac{x^{r+2}}{r+2} - \frac{x^{r+3}}{r+3}\right]_0^2 \qquad = \frac{3}{4}\left[\frac{2^{r+3}}{r+2} - \frac{2^{r+3}}{r+3}\right]$$

$$= \frac{3}{4}2^{r+3}\left[\frac{1}{r+2} - \frac{1}{r+3}\right] \qquad = 6\left(2^r\right)\frac{1}{(r+2)(r+3)}$$

$$\mu_{1'} = \frac{12}{(3)(4)} = 1 \qquad \mu_2' = \frac{24}{(4)(5)} = \frac{6}{5}$$

Mean = 1 and variance = $\mu_{2'} - \mu_{1'^2} = \frac{6}{5} - 1 = \frac{1}{5}$

27.The monthly demand for Allwyn watches is known to have the following probability distribution.

Demand:	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Probability:	**0.0**	**0.3*k***	**0.19**	**0.24**	**k^2**	**0.1**	**0.07**	**0.04**

Determine the expected demand for watches. Also, compute the variance. Solution:

$$\sum P(x) = 1$$

$$(0.08) + (0.3k) + (0.19) + (0.24) + (k^2) + (0.1) + (0.07)+)(0.04) = 1$$

$$k^2 + 0.3k - 0.28 = 0 \quad \Rightarrow k = 0.4$$

$$E(X) = \sum x\,P(x) = (1)(0.18) + (2)(0.12) + (3)(0.19) +$$

$$(4)(0.24) + (5)(0.16) + (6)(0.1) + (7)(0.07) + (8)(0.04)$$

$$= 4.02 \;\; is \;\; the \;\; mean$$

$$E(X^2) = \sum x^2 \; P(x) \; = (1)(0.18) + (4)(0.12) + (9)(0.19) +$$
$$(16)(0.24) + (25)(0.16) + (36)(0.1) + (49)(0.07) + (64)(0.04)$$
$$= 19.7$$
$$Variance \; = E(X^2) - E(X)^2 = 19.07 - 4.02^2 = 3.54$$

28.The number of hardware failures of a computer system in a week of operations has the following probability mass function:

No of failures:	**0**	**1**	**2**	**3**	**4**	**5**	**6**
Probability :	**0.18**	**0.28**	**0.25**	**0.18**	**0.06**	**0.04**	**0.01**

Find the mean of the number of failures in a week.

Solution:

$$E(X) = \sum x \; P(x) \; = (0)(0.18) + (1)(0.28) + (2)(0.25) + (3)(0.18) +$$
$$(4)(0.06) + (5)(0.04) + (6)(0.01)$$
$$= \; 1.92$$

STANDARD DISTRIBUTION

DIRCRETE DISTRIBUTION	**CONTINUOUS DISTRIBUTION**
*Binomial	*Uniform
*Poisson	*Exponetial
*Geometric	*Normal

BINOMIAL DISTRIBUTION

A random variable X is said to follow binomial distribution if

$$P(x) = P(X = x) = {}_nC_x p^x q^{n-x}$$

Probability mass function

X = 0,1,2…..n

p = probability for success

q = probability for failure

$$p+q = 1 \quad \Rightarrow \quad q = 1-p$$

where n,p are parameters of binomial distribution.

MOMENT GENERATING FUNCTION

$$Mgf \ M_x(t) = E(e^{tx}) = \sum e^{tx} \ p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} {}_n c_x p^x \ q^{n-x} = \sum_{x=1}^{\infty} {}_n c_x (pe^t)^x \ q^{n-x}$$

$$= q^n + {}_n c_1 (pe^t)^1 q^{n-1} + {}_n c_2 (pe^t)^2 q^{n-2} + ... + (pe^t)^n$$

$$= (q + \ pe^t)^n$$

$$M_X(t) \ = \ (q + \ pe^t)^n$$

MEAN AND VARIANCE

Mean = np

Variance = npq

29. The mean of a binomial distribution is 20 and SD is 4. Find the parameter of the disbution.

Soltion:

Mean = 20

$$np \quad = \ 20 \ ------(1)$$

S.D = 4

$$\sqrt{npq} \ = \ 4$$

$$npq \quad = \ 16--------(2)$$

Subs (1) in (2)

$$(2) \Rightarrow \ npq = \ 16$$

$$20q \ = \ 16$$

$$q \quad = \frac{4}{5}$$

$$p + q = 1$$

$$q = 1 - p$$

$$q = 1 - \frac{4}{5}$$

$$p = \frac{1}{5}$$

$$(1) \Rightarrow \quad np = 20$$

$$n\frac{1}{5} = 20$$

$$n = 100$$

$\therefore$ Parameter are (n, p) = $(100, \frac{1}{5})$

24. Four coins are tossed simultaneously. What is the probability of getting i) two heads ii) atleast 2 heads iii) atmost 2 heads.

Solution:

X denotes the number of heads

n = 4 X=0,1,2,3,4

$$p = \frac{1}{2}$$

$$p + q = 1 \Rightarrow q = \frac{1}{2}$$

Binomial distributon is

$$P(X = x) = {}_nC_x p^x q^{n-x}$$

$$P(X = x) = {}_4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}$$

i)two head

$$P(X = 2) = {}_4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2}$$

$$= {}_4c_2 \left(\frac{1}{2}\right)^4$$

$$P(X = 2) = \frac{3}{8}$$

ii)atleast 2 heads

$$P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4)$$

$$= {}_4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{4-2} + {}_4C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{4-3} + {}_4C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{4-4}$$

$$=\left(\frac{1}{2}\right)^4\left[{}_4c_2 + {}_4c_3 + {}_4c_4\right]$$

$$=\left(\frac{1}{2}\right)^4\left[6+4+1\right]$$

$$P(X \geq 2) = \frac{11}{16}$$

iii)atmost 2

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= {}_4C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{4-0} + {}_4C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{4-1} + {}_4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{4-2}$$

$$=\left(\frac{1}{2}\right)^4\left[{}_4c_0 + {}_4c_1 + {}_4c_2\right]$$

$$=\left(\frac{1}{2}\right)^4\left[1+4+6\right]$$

$$= \frac{11}{16}$$

If 10% of the screw produced by an automatic machine are defective. Find the probability that out of 20 screw selected at random there are i)exactly 2 defective ii)atmost 3 defective iii)atleast 2 defective iv)between one and three defective (inclusive)

Solution:

X denotes the number of heads

n = 20 X=0,1,2,3,…20

$$p = 10\% = \frac{1}{10}$$

$$p + q = 1 \Rightarrow q = \frac{9}{10}$$

Binomial distributon is

$$P(X = x) = {}_{n}C_{x} p^{x} q^{n-x}$$

$$P(X = x) = {}_{20}C_{x} \left(\frac{1}{10}\right)^{x} \left(\frac{9}{10}\right)^{20-x}$$

i)exactly 2 defective

$$P(X = 2) = {}_{20}C_{2} \left(\frac{1}{10}\right)^{2} \left(\frac{9}{10}\right)^{20-2}$$

$$= {}_{20}c_{2} \left(\frac{1}{10}\right)^{2} \left(\frac{9}{10}\right)^{18}$$

$$P(X = 2) = 0.285$$

ii)atmost 3 defective

$$P(X \le 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

$$= {}_{20}C_{0} \left(\frac{1}{10}\right)^{0} \left(\frac{9}{10}\right)^{20} + {}_{20}C_{1} \left(\frac{1}{10}\right)^{1} \left(\frac{9}{10}\right)^{19}$$

$$+ {}_{20}C_{2} \left(\frac{1}{10}\right)^{2} \left(\frac{9}{10}\right)^{18} + {}_{20}C_{3} \left(\frac{1}{10}\right)^{3} \left(\frac{9}{10}\right)^{17}$$

$$= \frac{9^{17}}{10^{20}} \left[{}_{20}c_{0} 9^{3} + {}_{20}c_{1} 9^{2} + {}_{20}c_{2} 9^{1} + {}_{20}c_{3} \right]$$

$$= \frac{9^{17}}{10^{20}} [5199]$$

$$P(X \le 3) = 0.8670$$

iii)atleast 2 heads

$$P(X \ge 2) = 1 - P(X < 2)$$

$$= 1 - [P(X = 0) + P(X = 1)]$$

$$= 1 - \left[{}_{20}C_{0} \left(\frac{1}{10}\right)^{0} \left(\frac{9}{10}\right)^{20} + {}_{20}C_{1} \left(\frac{1}{10}\right)^{1} \left(\frac{9}{10}\right)^{19} \right]$$

$$= 1 - \left[\frac{9^{19}}{10^{20}} \left({}_{20}C_{0} + {}_{20}C_{1} \right) \right]$$

$$=1-0.3917 \quad = 0.6083$$

$$P(X \geq 2) = 0.6083$$

iv) between one and three defective (inclusive)

$$P(1 \leq X \leq 3) \quad = \quad P(X = 1) + P(X = 2) + P(X = 3)$$

$$= {}_{20}C_1\left(\frac{1}{10}\right)^1\left(\frac{9}{10}\right)^{19} + {}_{20}C_2\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^{18}$$

$$+ {}_{20}C_3\left(\frac{1}{10}\right)^3\left(\frac{9}{10}\right)^{17}$$

$$= \frac{9^{17}}{10^{20}}\left[{}_{20}c_1 9^2 + {}_{20}c_2 9^1 + {}_{20}c_3\right]$$

$$= \frac{9^{17}}{10^{20}}[4470]$$

The recurrence relation for the moments of the Binomial distribution.

The k^{th} order central moment is given by

$$\mu_k = E\left[(X-\overline{X})^k\right] = E\left[(X-np)^k\right]$$

$$= \sum_{x=0}^{n} (x-np)^k \, p(x)$$

$$= \sum_{x=0}^{n} (x-np)^k \, nc_x p^x q^{n-x}$$

$$\mu_k = \sum_{x=0}^{n} nc_x \left[(x-np)^k p^x q^{n-x}\right] \text{-------------} (1)$$

Differentiating (1) w.r.to p,we have

$$\frac{d\mu_k}{dp} = \sum_{x=0}^{n} nc_x \left[k(x-np)^{k-1}(-n)p^x q^{n-x} + (x-np)^k \left(x.p^{x-1}q^{n-x} + p^x(n-x)q^{n-x-1}(-1) \right) \right]$$

$$= -nk\sum_{x=0}^{n} nc_x (x-np)^{k-1} p^x q^{n-x} + \sum_{x=0}^{n} nc_x (x-np)^k p^{x-1} q^{n-x-1} \left[xq - (n-x)p \right]$$

$$= -nk\sum_{x=0}^{n} nc_x (x-np)^{k-1} p^x q^{n-x} + \sum_{x=0}^{n} nc_x (x-np)^k \frac{p^x}{p} \frac{q^{n-x}}{q} \left[x(p+q) - np \right]$$

$$= -nk\mu_{k-1} + \sum_{x=0}^{n} nc_x (x-np)^k \frac{p^x}{p} \frac{q^{n-x}}{q} \left[x - np \right]$$

$$= -nk\mu_{k-1} + \frac{1}{pq} \sum_{x=0}^{n} nc_x (x-np)^{k+1} p^x q^{n-x}$$

$$= -nk\mu_{k-1} + \frac{1}{pq}\mu_{k+1} \qquad , \; by\,(1)$$

$$\Rightarrow \mu_{k+1} = pq\left[\frac{d\mu_k}{dp} + nk\mu_{k-1} \right]$$

This is the recurrence relation for the moments of the Binomial distribution

POISSON DISTRIBUTION

A Random variable X is said to follow poisson distribution if its probability if its probability mass function is given by

$$P(x) = P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad , \qquad x = 0,1,2....n$$

λ - parameter of the distribution

NOTE:

- The number of trials is infinitely large . ie, $n \to \infty$

- P is the probability of success in each trial. It is very small
- $np = \lambda$

29. Prove that poisson distribution is the limiting case of Binomial distribution.

(or)

Poisson distribution is a limiting case of Binomial distribution under the following conditions

(i) **n , the no.of trials is indefinitely large , i.e,** $n \to \infty$

(ii) **p, the constant probability of success in each trial is very small ,i.e** $p \to 0$

(iii) $np = \lambda \, is \inf inite \; or \; p = \frac{\lambda}{n} \, and \; q = 1 - \frac{\lambda}{n}, \lambda is \; positive \; real$

Soln: If Xis binomial r.v with parameter n & p ,then

$$p(X = x) = nc_x p^x q^{n-x}, \; x = 0,1,2,....n$$

$$= \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x}$$

$$= \frac{n(n-1)(n-2)....(n-(x-1))(n-x)!}{(n-x)!x!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n(n-1)(n-2)....(n-(x-1))}{x!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{n^x \, x!} n.n\left(1-\frac{1}{n}\right) n\left(1-\frac{2}{n}\right).....n\left(1-\frac{x-1}{n}\right) \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right).....\left(1-\frac{x-1}{n}\right)\left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

Taking limit as $n \to \infty$ on both sides

$$\lim_{n\to\infty} p(X = x) = \lim_{n\to\infty} \frac{\lambda^x}{x!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right).....\left(1-\frac{x-1}{n}\right)\left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right).....\left(1-\frac{x-1}{n}\right) \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!}(1.1....1).\left(e^{-\lambda}\right).1 = \frac{e^{-\lambda}\lambda^x}{x!} \quad , x = 0,1,2,.....$$

$\therefore \; p(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad , x = 0,1,2,.....$ and it is poisson distbn.

Hence the proof.

30. Find the recurrence relation for the central moments of the poisson distbn. and hence find the first three central moments .

Soln: The k^{th} order central moment μ_k is given by

$$\mu_k = E(X-\overline{X})^k = E(X-\lambda)^k$$

$$= \sum_{x=0}^{\infty}(x-\lambda)^k p(x)$$

$$\therefore \quad \mu_k = \sum_{x=0}^{\infty}(x-\lambda)^k \frac{e^{-\lambda}.\lambda^x}{x!} \text{ ----------- }(1)$$

Diff. (1) w.r.to λ we have

$$\frac{d\mu_k}{d\lambda} = \sum_{x=0}^{\infty}\left[k(x-\lambda)^{k-1}(-1)\frac{e^{-\lambda}.\lambda^x}{x!} + \frac{(x-\lambda)^{k-1}}{x!}\left[e^{-\lambda}.x\lambda^{x-1} + (-e^{-\lambda}).\lambda^x\right]\right]$$

$$= -k\sum_{x=0}^{\infty}(x-\lambda)^{k-1}\frac{e^{-\lambda}.\lambda^x}{x!} + \sum_{x=0}^{\infty}(x-\lambda)^k \frac{e^{-\lambda}.\lambda^{x-1}}{x!}(x-\lambda)$$

$$= -k\mu_{k-1} + \sum_{x=0}^{\infty}(x-\lambda)^{k+1}\frac{e^{-\lambda}.\lambda^x}{x!\lambda} \qquad by(1)$$

$$= -k\mu_{k-1} + \frac{1}{\lambda}\mu_{k+1}$$

$$\therefore \mu_{k+1} = \lambda\left[\frac{d\mu_k}{d\lambda} + k\mu_{k-1}\right] \text{ --------------- }(2)$$

which is the recurrence formula for the central moments of the poisson distbn.

since $\mu_0 = 1 \; and \; \mu_1 = 0$

put k= 1 in (2)

$$\mu_{1+1} = \lambda\left[\frac{d}{d\lambda}\mu_1 + 1\mu_{1-1}\right]$$

$$= \lambda\left[\frac{d}{d\lambda}(0) + 1\mu_0\right]$$

$$\mu_2 = \lambda$$

put k= 2 in (2)

$$\mu_{2+1} = \lambda\left[\frac{d}{d\lambda}\mu_2 + 2\mu_{2-1}\right]$$

$$= \lambda\left[\frac{d}{d\lambda}(\lambda) + 2\mu_1\right]$$

$$\mu_3 = \lambda(1+0)$$

$$= \lambda$$

31.Prove that the sum of two independent poisson variates is a poisson variate, while the difference is not a poisson variate.

Soln: Let X_1 and X_2 be independent r.v.s that follow poisson distbn. with

Parameters λ_1 and λ_2 respectively.

Let $X = X_1 + X_2$

$$p(X = n) = p(X_1 + X_2 = n)$$

$$= \sum_{r=0}^{n} p[X_1 = r].p[X_2 = n-r] \quad \sin ce\, X_1\ \&\ X_2 are\, independent$$

$$= \sum_{r=0}^{n} \frac{e^{-\lambda_1}.\lambda_1^{\ r}}{r!}.\frac{e^{-\lambda_2}.\lambda_2^{\ n-r}}{(n-r)!}$$

$$= e^{-\lambda_1} e^{-\lambda_2} \sum_{r=0}^{n} \frac{.\lambda_1^{\ r}}{r!}.\frac{1}{n!}\frac{n\,!.}{(n-r)!}\lambda_2^{\ n-r}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{r=0}^{n}.\frac{n\,!.}{r!\,(n-r)!}\lambda_1^{\ r}\lambda_2^{\ n-r}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{r=0}^{n}.nc_r\lambda_1^{\ r}\lambda_2^{\ n-r} \quad = \frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^n$$

This is poisson with parameter $(\lambda_1+\lambda_2)$

(ii) Difference is not poisson

Let $X = X_1 - X_2$

$$E(X) = E[X_1 - X_2]$$

$$= E(X_1) - E(X_2)$$

$$= \lambda_1 - \lambda_2$$

$$E(X^2) = E\left[(X_1 - X_2)^2\right]$$

$$= E\left[X_1^{\ 2} + X_1^{\ 2} - 2X_1X_2\right]$$

$$= E\left[X_1^{\ 2}\right] + E\left[X_2^{\ 2}\right] - 2E[X_1]E[X_2]$$

$$= (\lambda_1^{\ 2} + \lambda_1) + (\lambda_2^{\ 2} + \lambda_2) - 2(\lambda_1\lambda_2)$$

$$= (\lambda_1 - \lambda_2)^2 + (\lambda_1 + \lambda_2)$$

$$\neq (\lambda_1 - \lambda_2)^2 + (\lambda_1 - \lambda_2)$$

It is not poisson.

32. If X and Y are two independent poisson variates , show that the conditional distbn. of X, given the value of X+Y is Binomial.

Soln: Let X and Y follow poisson with parameters λ_1 and λ_2 respectively.

$$p[X = r/X+Y=n] = \frac{p[X = r \, and \, X+Y=n]}{p[X+Y=n]}$$

$$= \frac{p[X=r].p[X+Y=n]}{p[X+Y=n]} \quad by \ independent$$

$$= \frac{\frac{e^{-\lambda_1}.\lambda_1^{\ r}}{r!}.\frac{e^{-\lambda_2}.\lambda_2^{\ n-r}}{(n-r)!}}{\frac{e^{-(\lambda_1+\lambda_2)}.(\lambda_1+\lambda_2)^n}{n!}}$$

$$= \frac{n!}{r!(n-r)!}\frac{e^{-\lambda_1}.\lambda_1^{\ r}.e^{-\lambda_2}.\lambda_2^{\ n-r}}{e^{-\lambda 1}e^{-\lambda_2}..(\lambda_1+\lambda_2)^n}$$

$$= n c_r \frac{\lambda_1^{\ r}.\lambda_2^{\ n-r}}{(\lambda_1+\lambda_2)^r.(\lambda_1+\lambda_2)^{n-r}}$$

$$= n c_r \left[\frac{\lambda_1}{\lambda_1+\lambda_2}\right]^r \left[\frac{\lambda_2}{\lambda_1+\lambda_2}\right]^{n-r} = n c_r \ p^r \ q^{n-r}$$

where $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$ and $q = \frac{\lambda_2}{\lambda_1+\lambda_2}$

This is binomial distbn.

33. It is known that the probability of an item produced by a certain machine will be defective is 0.05. If the produced items are sent to the market in packets of 20, find the no. of packets containing atleast ,exactly,atmost 2 defectives in a consignment of 1000 packets using poisson.

Soln: Give n = 20 , p = 0.05 , N = 1000

Mean $\lambda = n\,p = 1$

Let X denote the no. of defectives.

$$p[X = x] = \frac{e^{-\lambda}.\lambda^x}{x!} = \frac{e^{-1}.1^x}{x!} = \frac{e^{-1}.}{x!} \quad x = 0,1,2,...$$

$$p[x \geq 2] = 1 - p[x < 2]$$

$$= 1 - [p(x=0) + p(x=1)]$$

$$= 1 - \left[\frac{e^{-1}}{0!} + \frac{e^{-1}}{1!}\right] = 1 - 2\,e^{-1} = 0.2642$$

Therefore , out of 1000 packets , the no. of packets containing atleast 2 defectives

$$= N.p[x \geq 2] = 1000 * 0.2642 \cong 264 \ packets$$

(ii) $p[x=2]=\frac{e^{-1}}{2!}=0.18395$

Out of 1000 packets,=N*p[x=2]=184 packets

(iii)

$$p[x\le 2]=p[x=0]+p[x=1]+p[x=2]$$

$$=\frac{e^{-1}}{0!}+\frac{e^{-1}}{1!}+\frac{e^{-1}}{2!}=0.91975$$

For 1000 packets = 1000*0.91975=920 packets approximately.

34. The atoms of radio active element are randomly disintegrating. If every gram of this element , on average, emits 3.9 alpha particles per second, what is the probability during the next second the no. of alpha particles emitted from 1 gram is
(i) atmost 6 (ii) atleast 2 (iii) atleast 3 and atmost 6 ?

Soln: Given $\lambda=3.9$

Let X denote the no. of alpha particles emitted

$$(i)\ p(x\le 6)=p(x=0)+p(x=1)+p(x=2)+......+p(x=6)$$

$$=\frac{e^{-3.9}(3.9)^0}{0!}+\frac{e^{-3.9}(3.9)^1}{1!}+\frac{e^{-3.9}(3.9)^2}{2!}+......+\frac{e^{-3.9}(3.9)^6}{6!}$$

$$=0.898$$

$$(ii)\quad p(x\ge 2)=1-p(x<2)$$

$$=1-[p(x=0)+p(x=1)]$$

$$=1-\left[\frac{e^{-3.9}(3.9)^0}{0!}+\frac{e^{-3.9}(3.9)^1}{1!}\right]$$

$$=0.901$$

$$(iii)\quad p(3\le x\le 6)=p(x=3)+p(x=4)+p(x=5)+p(x=6)$$

$$=\frac{e^{-3.9}(3.9)^3}{3!}+\frac{e^{-3.9}(3.9)^4}{4!}+\frac{e^{-3.9}(3.9)^5}{5!}+\frac{e^{-3.9}(3.9)^6}{6!}$$

$$=0.645$$

35.The no. of monthly breakdowns of a computer is a r.v. having poisson distbn with mean 1.8. Find the probability that this computer will function for a month with only one breakdown.

Soln: $p(X=x)=\dfrac{e^{-\lambda}\lambda^{x}}{x!}$, $given\ \lambda=1.8$

$$p(x=1)=\frac{e^{-1.8}(1.8)^{1}}{1!}=0.2975$$

36.A discrete r.v X has mgf $M_x(t)=e^{2(e^t-1)}$ .Find E(x), var(x) , and p(x=0).

Soln: Given $M_x(t)=e^{2(e^t-1)}$

We know that mgf of poisson is $M_x(t)=e^{\lambda(e^t-1)}$

Therefore $\lambda=2$

In poisson E(x) = var(x) = λ

$\therefore\ Mean\ E(x)=\text{var}(x)=2$

$$p(X=x)=\frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$\therefore\ p(X=0)=\frac{e^{-\lambda}\lambda^{0}}{0!}=e^{-\lambda}=e^{-2}=0.1353$$

GEOMETRIC DISTRIBUTION

A discrete random variable X is said to follow geometric distribution if its probability mass function is given by

$P(X=x)=q^{x-1}p\ ,\ x=1,2,3......$

Where X denotes the number of trials needed to obtain the first success.

P(X= 1) = probability of success in first trial

P(X= 2) = fail in first and success in second trial

P(X= 3) = fail in 1st and 2nd trial and success in third trial

MGF of geometric distribution.

The pmf of geometric distribution is given by

$$Mgf\ M_x(t)=E(e^{tx})=\sum e^{tx}\ p(x)$$

$$=\sum_{x=1}^{\infty}e^{tx}\,p\,q^{x-1}=\sum_{x=1}^{\infty}e^{tx}\,p\ q^{x}\ q^{-1}$$

$$=\frac{p}{q}\sum_{x=1}^{\infty}\left(e^{t}\right)^{x}q^{x}=\frac{p}{q}\sum_{x=1}^{\infty}\left(qe^{t}\right)^{x}$$

$$=\frac{p}{q}\left[qe^{t}+\left(qe^{t}\right)^{2}+\left(qe^{t}\right)^{3}+\ldots\right]$$

$$= \frac{p}{q}\, qe^t \left[1 + qe^t + \left(qe^t\right)^2 + \ldots\right]$$

$$= pe^t \left(1 - qe^t\right)^{-1} 1 - qe^t$$

$$\therefore\ M_x(t) = 1 - qe^t$$

Mean and Variance of Geometric distribution .

The pmf of Geometric distbn is given by

$$p(X = x) = p\, q^{x-1} \quad , x = 1,2,3, \ldots..$$

$$Mean\ E(x) = \sum x\, p(x)$$

$$= \sum_{x=1}^{\infty} x\, p\, q^{x-1} = p \sum_{x=1}^{\infty} x\ q^{x-1}$$

$$= p\left[1 q^{1-1} + 2q^{2-1} + 3q^{3-1} + \ldots..\right]$$

$$= p\left[1 + 2q \quad + 3q^2 + \ldots..\right] = p[1-q]^2$$

$$= p\, p^{-2} = p^{-1} = \frac{1}{p}$$

Mean $= \frac{1}{p}$

$$E\left[x^2\right] = \sum x^2 p(x)$$

$$= \sum_{x=1}^{\infty} x^2\ p\, q^{x-1}$$

$$= \sum_{x=1}^{\infty} \left[x(x+1) - x\right] p\, q^{x-1}$$

$$= \sum_{x=1}^{\infty} x(x+1)\, p\, q^{x-1} - \sum_{x=1}^{\infty} x\, p\, q^{x-1}$$

$$= 1(1+1)\, p\, q^{1-1} + 2(2+1)\, pq^{2-1} + 3(3+1)\, pq^{3-1} + \ldots.. - \frac{1}{p}$$

$$= 2p + 2(3)\, pq^1 + 3(4)\, pq^2 + \ldots.. - \frac{1}{p}$$

$$= 2p\left[1 + 3q \quad + 6q^2 + \ldots..\right] - \frac{1}{p}$$

$$= 2p\,[1-q]^{-3} - \frac{1}{p} = 2\, p\, p^{-3} - \frac{1}{p}$$

$$= \frac{2}{p^2} - \frac{1}{p}$$

Variance $= E(x^2) - [E(x)]^2$

$$= \frac{2}{p^2} - \frac{1}{p} - \left(\frac{1}{p}\right)^2 = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2} = \frac{q}{p^2}$$

Variance $= \frac{q}{p^2}$

37. Establish the memoryless property of geometric distbn.

Soln: If X is a discrete r.v. following a geometric distbn.

$$\therefore\ p(X = x) = pq^{x-1} \quad , x = 1,2,.....$$

$$p(x > k) = \sum_{x=k+1}^{\infty} pq^{x-1}$$

$$= p\left[q^k + q^{k+1} + q^{k+2} + ...\right]$$

$$= p\,q^k\left[1 + q + q^2 + ...\right] = p\,q^k (1-q)^{-1}$$

$$= p\,q^k\,p^{-1} = q^k$$

Now

$$p[x > m + n/x > m] = \frac{p[x > m + n \ and \ x > m]}{p[\,x > m]}$$

$$= \frac{p[x > m + n\ ]}{p[\,x > m]} = \frac{q^{m+n}}{q^m} = q^n = p[x > n]$$

$$\therefore\ p[x > m + n/x > m] = p[x > n]$$

38. If X_1, X_2 be independent r.v. each having geometric distbn $pq^k, k = 0,1,2,....$

Show that th conditional distribution of X_1 given $X_1 + X_2$ is Uniform distbn.

Soln:

$$p[X_1 = r / X_1 + X_2 = n] = \frac{p[X_1 = r \; and \;\; X_1 + X_2 = n]}{p[\, X_1 + X_2 = n]}$$

$$= \frac{p[X_1 = r \;\; and \;\; X_2 = n - r]}{\sum_{s=0}^{n} p[X_1 = s \;\; and \;\; X_2 = n - s]}$$

$$= \frac{p[X_1 = r].p[X_2 = n - r]}{\sum_{s=0}^{n} p[X_1 = s].p[X_2 = n - s]} \qquad by \;\; independent$$

$$= \frac{q^r \; p \, q^{n-r} \; p}{\sum_{s=0}^{n} q^s \; p \, q^{n-s} \; p} = \frac{q^n}{\sum_{s=0}^{n} q^n} = \frac{1}{n+1}, \;\; r = 0,1,2,.....n$$

$$(i.e) \;\; p[X_1 = r / X_1 + X_2 = n] = \frac{1}{n+1} \;\; , \; this \; is \; uniform \; distbn$$

39. Suppose that a trainee soldier shoots a target in an independent fashion. If the probability That the target is shot on any one shot is 0.7.

(i) What is the probability that the target would be hit in 10 th attempt?

(ii) What is the probability that it takes him less than 4 shots?

(iii)What is the probability that it takes him an even no. of shots?

(iv) What is the average no. of shots needed to hit the target?

Soln: Let X denote the no. of shots needed to hit the target and X follows geometric distribution with pmf $p[X = x] = p\,q^{x-1} \;,\; x = 1,2,....$

Given p=0.7 , and q=1-p=0.3

(i) $p[x = 10] = (0.7)(0.3)^{10-1} = 0.0000138$

(ii)

$$p[x < 4] = p(x=1) + p(x=2) + p(x=3)$$
$$= (0.7)(0.3)^{1-1} + (0.7)(0.3)^{2-1} + (0.7)(0.3)^{3-1}$$
$$= 0.973$$

(iii)

$$p[x \, is \, an \, even \, number] = p(x=2) + p(x=4) + ...$$
$$= (0.7)(0.3)^{2-1} + (0.7)(0.3)^{4-1} + ...$$
$$= (0.7)(0.3)[1 + (0.3)^2 + (0.3)^4 ....]$$

$$= 0.21\left[1 + \left((0.3)^2\right) + \left((0.3)^2\right)^2 + ...\right]$$
$$= 0.21\left[1 - (0.3)^2\right]^{-1} = (0.21)\,(0.91)^{-1}$$
$$= \frac{0.21}{0.91} = 0.231$$

(iv) Average no. of shots $= E(X) = \frac{1}{p} = \frac{1}{0.7} = 1.4286$

CONTINUOUS DISTRIBUTION

UNIFORM DISTRIBUTION

A random variable X is said to have uniform distribution if its probability density function is given by $f(x) = \begin{cases} \frac{1}{b-a} & , \ a < x < b \\ 0 & , \ otherwise \end{cases}$

Where a and b are parameters.

40.Find the MGF of triangular distribution whose density function is given by $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & elsehwere \end{cases}$**.Hence its mean and variance.**

Solution:

$$M_X(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$
$$= \int_0^1 e^{tx}\, x\, dx + \int_1^2 e^{tx}(2-x)\, dx$$
$$= \left[x\frac{e^{tx}}{t} - \frac{e^{tx}}{t^2}\right]_0^1 + \left[(2-x)\frac{e^{tx}}{t} - (-1)\frac{e^{tx}}{t^2}\right]_1^2$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}$$

$$M_X(t) = \frac{e^{2t} - 2e^t + 1}{t^2}$$

expanding the above in powers of t, we get

$$M_X(t)=\frac{e^{2t}-2e^{t}+1}{t^2}=\frac{1}{t^2}\left[\left(1+2t+\frac{4t^2}{2!}+\frac{8t^3}{3!}+\frac{16t^4}{4!}+\dots\right)-2\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+.\right)-1\right]$$

$$=\frac{1}{t^2}\left(\frac{2t^2}{2!}+\frac{6t^3}{3!}+\frac{14t^4}{4!}+\dots\right)$$

$$=1+t+\frac{7t^2}{12}+\frac{t^3}{4}+\dots.$$

Mean = E(X) = (coefficient of t) 1! = 1

$E(X^2)$ = (coefficient of t^2)2! = $\frac{7}{6}$

Variance = $E(X^2) - E(X)^2 = \frac{1}{6}$

41.Show that for the uniform distribution $f(x)=\frac{1}{2a}, -a<x<a$ **,the mgf about origin is** $\frac{\sinh at}{at}$

Soln: Given $f(x)=\frac{1}{2a}, -a<x<a$

MGF $M_x(t)=E\left[e^{tx}\right]$

$$=\int_{-\infty}^{\infty} e^{tx} f(x)dx=\int_{-a}^{a} e^{tx}\frac{1}{2a}dx$$

$$=\frac{1}{2a}\int_{-a}^{a} e^{tx}dx=\frac{1}{2a}\left[\frac{e^{tx}}{t}\right]_{-a}^{a}$$

$$=\frac{1}{2at}\left[e^{at}-e^{-at}\right]=\frac{1}{2at}2\sinh at=\frac{\sinh at}{at}$$

$$M_x(t)=\frac{\sinh at}{at}$$

42.The number of personel computer (pc) sold daily at a computer world is uniformly distributed with a minimum of 2000 pc and a maximum of 5000 pc. Find

(1) The probability that daily sales will fall between 2500 and 3000 pc

(2)What is the probability that the computer world will sell atleast 4000 pc's?

(3) What is the probability that the computer world will sell exactly 2500 pc's?

Soln: Let X~U(a , b) , then the pdf is given by

$$f(x) = \frac{1}{b-a} \quad , a < x < b$$

$$= \frac{1}{5000-2000} , \quad 2000 < x < 5000$$

$$= \frac{1}{3000} \quad , \quad 2000 < x < 5000$$

(1)

$$p[2500 < x < 3000] = \int_{2500}^{3000} f(x)\,dx$$

$$= \int_{2500}^{3000} \frac{1}{3000} dx = \frac{1}{3000} [x]_{2500}^{3000}$$

$$= \frac{1}{3000}[3000 - 2500] = 0.166$$

(2)

$$p[x \geq 4000] = \int_{4000}^{5000} f(x)\,dx$$

$$= \int_{4000}^{5000} \frac{1}{3000} dx = \frac{1}{3000} [x]_{4000}^{5000}$$

$$= \frac{1}{3000}[5000 - 4000] = 0.333$$

(3) $p[x = 2500] = 0$, (i.e) it is particular point ,the value is zero.

43. Starting at 5.00 am every half an hour there is a flight from San Fransisco airport to Losangles .Suppose that none of three planes is completely sold out and that they always have room for passengers . A person who wants to fly to Losangles arrive at a random time between 8.45 am and 9.45 am. Find the probability that she waits (a)Atmost 10 min (b) atleast 15 min

Soln: Let X be the uniform r.v. over the interval (0,60)

Then the pdf is given by

$$f(x) = \frac{1}{b-a} \quad , a < x < b$$

$$= \frac{1}{60} \quad , \quad 0 < x < 60$$

(a) The passengers will have to wait less than 10 min. if she arrives at the airport

$$= p(5 < x < 15) + p(35 < x < 45)$$

$$= \int_5^{15} \frac{1}{60} dx + \int_{35}^{45} \frac{1}{60} dx$$

$$= \frac{1}{60}[x]_5^{15} + \frac{1}{60}[x]_{35}^{45}$$

$$= \frac{1}{3}$$

(b)The probability that she has to wait atleast 15 min.

$$= p(15 < x < 30) + p(45 < x < 60)$$

$$= \int_{15}^{30} \frac{1}{60} dx + \int_{45}^{60} \frac{1}{60} dx$$

$$= \frac{1}{60}[x]_{15}^{30} + \frac{1}{60}[x]_{45}^{60}$$

$$= \frac{1}{2}$$

EXPONENTIAL DISTRIBUTION

A continuous random variable X is said to follow an exponential with parameter $\lambda > 0$ if its probability function is given by

$$f(x) = \begin{cases} \lambda e -^{\lambda x} & , x > 0 \\ 0 & , otherwise \end{cases}$$

44. Define exponential density function and find mean and variance of the same.

Soln: The density function of exponential distribution is given by

1. $f(x) = \lambda e^{-\lambda x}, x \geq 0$

$$Mean = E[x] = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_0^{\infty} x\lambda e^{-\lambda x}\, dx = \lambda \int_0^{\infty} x e^{-\lambda x}\, dx$$

2.
$$= \lambda \left[\frac{-xe^{-\lambda x}}{\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \right]_0^{\infty}$$

3.
$$= \lambda \left[(0-0) - \left(0 - \frac{1}{\lambda^2} \right) \right] = \lambda \left(\frac{1}{\lambda^2} \right) = \frac{1}{\lambda}$$

$$Mean = \frac{1}{\lambda}$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_0^{\infty} x^2 \lambda e^{-\lambda x}\, dx = \lambda \int_0^{\infty} x^2 e^{-\lambda x}\, dx$$

$$= \lambda \left[\frac{-x^2 e^{-\lambda x}}{\lambda} - \frac{2xe^{-\lambda x}}{\lambda^2} - \frac{2e^{-\lambda x}}{\lambda^3} \right]_0^{\infty}$$

$$= \lambda \left[(0-0-0) - \left(0 - 0 - \frac{2}{\lambda^3} \right) \right] = \lambda \left(\frac{2}{\lambda^3} \right) = \frac{2}{\lambda^2}$$

$$Variance = E(x^2) - [E(x)]^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda} \right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

45.Suppose that the duration 'X' in minutes of long distance calls from your home, follows exponential law with p.d.f $f(x) = \frac{1}{5} e^{-\frac{x}{5}},\ x > 0$**. Find p(X > 5), p(3≤X≤6),mean and variance.**

Solution:

(i) $p(X > 5) = \int_5^{\infty} f(x)\, dx = \int_5^{\infty} \frac{1}{5} e^{-\frac{x}{5}} dx = \left[-e^{-\frac{x}{5}} \right]_5^{\infty} = e^{-1}$ (ii)

$$p(3 < X < 6) = \int_3^6 f(x)\,dx = \int_3^6 \frac{1}{5}\, e^{-\frac{x}{5}} dx = \left[-e^{-\frac{x}{5}}\right]_3^6 = -e^{-1.2} + e^{-0.5} \qquad \text{(iii)}$$

$$E(X) = \int_0^\infty x f(x)\,dx = \int_0^\infty \frac{1}{5}\, x e^{-\frac{x}{5}} dx$$

$$= \frac{1}{5}\left[-xe^{-\frac{x}{5}} 5 - e^{-\frac{x}{5}} 25\right]_0^\infty \qquad \text{(iv)}$$

$$= \frac{1}{5}(0 + 25) = 5$$

$$E(X^2) = \int_0^\infty x^2 f(x)\,dx = \int_0^\infty \frac{1}{5}\, x^2 e^{-\frac{x}{5}} dx$$

$$= \frac{1}{5}\left[-x^2 e^{-\frac{x}{5}} 5 - 2xe^{-\frac{x}{5}} 25 + 2e^{-\frac{x}{5}} 125\right]_0^\infty$$

Variance =

$$= \frac{1}{5}(0 + 250) = 50$$

$$E\left(X^2\right) - E(X)^2 = 50 - 25 = 25$$

46.Establish the memoryless property of exponential distbn.

Soln: If X is exponentially distributed , then

$$p[x > s + t/x > s] = p[x > t] \;\; for\, any\, s, t > 0$$

The pdf of exponential distbn is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \ge 0 \\ 0 & , \; otherwise \end{cases}$$

$$p(x > k) = \int_k^\infty f(x)\,dx$$

$$= \int_k^\infty \lambda e^{-\lambda x}\,dx = \lambda\left[\frac{e^{-\lambda x}}{-\lambda}\right]_k^\infty$$

$$= -\left[0 - e^{-\lambda k}\right] = e^{-\lambda x} \text{ --------}(1)$$

$$p[x > s+t/x > s] = \frac{p[x > s+t \ and \ x > s]}{p[x > s]}$$

$$= \frac{p[x > s+t\ ]}{p[x > s]} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = p[x > t]$$

$$\therefore \quad p[x > s+t/x > s] = p[x > t] \ \ for\, any\, s, t > 0$$

47. The time (in hours) required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{2}$.

(a)What is the probability that the repair time exceeds 2 hrs ?

(b)What is the conditional probability that a repair takes atleast 11 hrs given that Its direction exceeds 8 hrs ?

Soln: If X represents the time to repair the machine, the density function

Of X is given by

$$f(x) = \lambda e^{-\lambda x} \qquad , x \geq 0$$

$$= \frac{1}{2} e^{-x/2} \qquad x \geq 0$$

(a)

$$p(x > 2) = \int_2^\infty f(x)\, dx = \int_2^\infty \lambda e^{-\lambda x}\, dx$$

$$= \int_2^\infty \frac{1}{2} e^{\frac{-x}{2}}\, dx = \frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}\right]_2^\infty$$

$$= -\left[0 - e^{-1}\right] = 0.3679$$

$$p[x \geq 11/x > 8] = p[x > 3]$$

$$= \int_3^\infty f(x)\, dx = \int_3^\infty \lambda e^{-\lambda x}\, dx$$

$$= \int_3^\infty \frac{1}{2} e^{\frac{-x}{2}}\, dx = \frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}\right]_3^\infty$$

$$= -\left[0 - e^{-\frac{3}{2}}\right] = e^{-\frac{3}{2}} = 0.2231$$

48. The time (in hours) required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{2}$.

(a)What is the probability that the repair time exceeds 2 hrs ?

(b)What is the conditional probability that a repair takes atleast 11 hrs given that Its direction exceeds 8 hrs ?

Soln: If X represents the time to repair the machine, the density function

Of X is given by

$$f(x) = \lambda e^{-\lambda x} \qquad , x \geq 0$$

$$= \frac{1}{2} e^{-x/2} \qquad x \geq 0$$

(a)

$$p(x > 2) = \int_2^\infty f(x)\, dx = \int_2^\infty \lambda e^{-\lambda x}\, dx$$

$$= \int_2^\infty \frac{1}{2} e^{\frac{-x}{2}}\, dx = \frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}\right]_2^\infty$$

$$= -\left[0 - e^{-1}\right] = 0.3679$$

$$p[x \geq 11/x > 8] = p[x > 3]$$

$$= \int_3^\infty f(x)\, dx = \int_3^\infty \lambda e^{-\lambda x}\, dx$$

$$= \int_3^\infty \frac{1}{2} e^{\frac{-x}{2}}\, dx = \frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}\right]_3^\infty$$

$$= -\left[0 - e^{-\frac{3}{2}}\right] = e^{-\frac{3}{2}} = 0.2231$$

GAMMA DISTRIBUTION

A continuous random variable X having the following density function is said to follow gamma distribution with parameters λ and k

$$f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma_k} \quad , \ k, \lambda > 0 \ , \ 0 < x < \infty$$

49.In a certain city the daily consumption of electric power in millions of kilowatt hrs can be treated as central gamma distbn with $\lambda = \frac{1}{2}, k = 3$. If the power plant has a daily capacity of 12 million kilowatt hours . What is the probability that the power supply will be inadequate on any given day.

Soln: Let X be the daily consumption of electric power
Then the density function of X is given by

$$f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma_k}$$

$$= \frac{\left(\frac{1}{2}\right)^3 x^{3-1} e^{-\frac{x}{2}}}{\Gamma_3} = \frac{\left(\frac{1}{8}\right) x^2 e^{-\frac{x}{2}}}{2!}$$

$$= \frac{x^2 e^{-\frac{x}{2}}}{16}$$

p[the power supply is inadequate]=p[x>12]

$$= \int_{12}^{\infty} f(x)\,dx = \int_{12}^{\infty} \frac{x^2 e^{-\frac{x}{2}}}{16} dx$$

$$= \frac{1}{16} \int_{3}^{\infty} x^2 e^{-\frac{x}{2}} dx$$

$$= \frac{1}{16} \left[-2x^2 e^{-\frac{x}{2}} - 8xe^{-\frac{x}{2}} - 16e^{-\frac{x}{2}} \right]_{12}^{\infty}$$

$$= 0.0625$$

50. The daily concemption of milk in a city in excess of 20,000 litres is approximately distributed as an Gamma distbn with parameter $\lambda = \frac{1}{10000}, k = 2$. The city has a daily stock of 30,000 litres. What is the probability that the stock is insufficient on a particular day.

Soln: Let X be the daily consumption ,so ,the r.v. Y=X-20000.

Then

$$f_Y(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{\Gamma_k}$$

$$= \frac{\left(\frac{1}{10000}\right)^2 y^{2-1} e^{-\frac{y}{10000}}}{\Gamma_2} = \frac{y e^{-\frac{y}{10000}}}{(10000)^2\, 1!}$$

$$= \frac{y e^{-\frac{y}{10000}}}{(10000)^2}$$

p[insufficient stock]=p[X>30000]

=p[Y>10000]

$$p[Y > 10000] = \int_{10000}^{\infty} f(y)\,dy = \int_{10000}^{\infty} \frac{y e^{-\frac{y}{10000}}}{(10000)^2}\,dy$$

$$= \frac{1}{(10000)^2} \int_{3}^{\infty} y e^{-\frac{y}{10000}}\,dy$$

$$= 2e^{-1} \quad , \left[By\ substitution\ method\ , put\ t = \frac{y}{10000} \right]$$

$$= 0.7357$$

NORMAL DISTRIBUTION

A continuous random variable X with parameters μ and σ^2 is said to follow a normal distribution if its probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \text{ , - } \infty < x < \infty \text{ , } \sigma > 0$$

X is called normal random variable.

Note:

1. 1.Since the normal distribution is used frequently in statistics , a special notation is used for it.

The notation X ~N(μ , σ^2) means X is normally distributed with mean μ and variance σ^2.

2. The graph of the normal distribution is called the normal curve . It is a bell shaped curve symmetric about x = μ .
3. In the Binomial distribution with parameters n and p , when n is very large and p is nearly 1/2 the binomial approaches normal. We shall now prove that the parameters μ and σ^2 are the mean and variance of the normal distribution.

Standard Normal Distribution

Let X ~N(μ , σ^2) . Since X is a continuous random variable ,

$$p(x_1 < X < x_2) = \int_{x_1}^{x_2} f(x)dx = \int_{x_1}^{x_2} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} , \; -\infty < x < \infty$$

The area $\int_0^z \phi(z)dz$ is computed for different values of z and the area table is constructed , which can be used to compute the probabilities

$$\therefore \; p(z_1 < Z < z_2) = \int_0^{z_2} \phi(z)dz - \int_0^{z_1} \phi(z)dz$$

$$P(X < x_1) = P(Z < z_1) = \int_{-\infty}^{z_1} \phi(z)\,dz$$

$$= 0.5 + \int_0^{z_1} \phi(z)\,dz \quad \text{if } z_1 > 0$$

$$= 0.5 - \int_0^{z_1} \phi(z)\,dz \quad \text{if } z_1 < 0$$

because area left of z=0 is o.5 and the area right of z= 0 is 0.5 .

Area Property

$P(\mu - \sigma < X < \mu + \sigma) = P(-1<Z<1) = 0.6826$

$P(\mu - 2\sigma < X < \mu + 2\sigma) = P(-2<Z<2) = 0.9545$

$P(\mu - 3\sigma < X < \mu + 3\sigma) = P(-3<Z<3) = 0.9973$

Therefore , outside the interval ($\mu - 3\sigma$, $\mu + 3\sigma$) the area is 0.0027. This property forms the basis for the entire large sample theory.

Normal Distribution Problems

51. X is a normal variate with mean 30 and SD 5. Find the probabilities that (i) $26 \leq X \leq 40$ (ii) $X \geq 5$ (iii) $|X - 30| > 5$

Solution : Given $\mu=30$,$\sigma=5$

We know that , $Z = \frac{X-\mu}{\sigma}$

(i) When X= 26 , $Z = \frac{26-30}{5} = -0.8$

When X = 40 , $Z = \frac{40-30}{5} = 2$

$\therefore$ $P(26 \leq X \leq 40) = P(-0.8 \leq Z \leq 2)$

$= P(-0.8 \leq Z \leq 0) + P(0 \leq Z \leq 2)$

$= P(0 \leq Z \leq 0.8) + P(0 \leq Z \leq 2)$

$= 0.2881 + 0.4772$

$= 0.7653$

(ii) When X= 45 , $Z = \frac{45-30}{5} = 3$

$P(X \geq 45) = P(Z \geq 3)$

$= 0.5 - P(0 \le Z \le 3)$
$= 0.5 - 0.4987$
$= 0.0013$

(iii) $P(|X-30| \le 5) = P(25 \le X \le 35)$

When X = 25 , $Z = \frac{25-30}{5} = -1$
When X = 35 , $Z = \frac{35-30}{5} = 1$

$= P(-1 \le Z \le 1)$
$= 2\, P(0 \le Z \le 1)$
$= 2 \text{ X } 0.3413$
$= 0.6826$

$\therefore P(|X-30| > 5) = 1 - P(|X-30| \le 5)$
$= 1 - 0.6826$
$= 0.3174$

52. The savings bank account of a customer showed an average balance of Rs.1500 and a standard deviation of Rs.50. Assuming that the account balance are normally distributed.

1. What percentage of account is over Rs. 200?
2. What percentage of account is between Rs. 120 and Rs. 170?
3. What percentage of account is less than Rs.75?

Solution:

Given $\mu = 150$, $\sigma = 50$

1. $P(X \ge 200)$

We know that , $Z = \frac{X-\mu}{\sigma}$

When X= 200 , $Z = \frac{200-150}{50} = \frac{50}{50} = 1$

$P(X \ge 200) = P(Z \ge 1)$
$= 0.5 - P(0 \le Z \le 1)$
$= 0.5 - 0.3413$

$=0.1587$

Percentage of account is over Rs.200 is 15.87%

2. $P(120 < X < 170)$

When X= 120 , $Z = \frac{120-150}{50} = -0.6$

When X = 170 , $Z = \frac{170-150}{50} = 0.4$

$\therefore \quad P(120 < X < 170) = P(-0.6 < Z < 0.4)$

$= P(-0.6 < Z < 0) + P(0 < Z < 0.4)$

$= P(0 < Z < 0.6) + P(0 < Z < 0.4)$

$= 0.2257 + 0.1554$

$= 0.3811$

Percentage of account is between Rs.120 and Rs. 170 is 38.11%

3. $P(X < 75)$

When X= 75 , $Z = \frac{75-150}{50} = -1.5$

$P(X < 75) = P(Z < -1.5)$

$= 0.5 - P(0 < Z < 1.5)$

$= 0.5 - 0.4332$

$= 0.0668$

Percentage of account is less than Rs.75 is 6.68%

UNIT – II

TWO DIMENSIONAL RANDOM VARIABLES

DEFINITON:

Let S be the sample space of a random experiment. Let X & Y be two random variable defined on S then the pair (X,Y) is called two dimensional random variable.

TWO DIMENSIONAL DISCRETE RANDOM VARIABLE

If the possible values of (X,Y) are countable then (X,Y) is called a two dimensional discrete random variable.

It can be represented as (x_i, y_j) where i = 1,2,3….n

j = 1,2,3…n

TWO DIMENSIONAL CONTINUOUS RANDOM VARIABLE

If (X,Y) can takes all the values in certain interval then (X,Y) is called two dimensional continuous random variable.

JOINT PROBABILITY MASS FUNCTION

If (X,Y) is a two dimensional discrete random variable such that $P(x_i, y_j) = P(X = x_i, Y = y_j) = P_{ij}$ is called the joint probability function or joint probability mass function of (X,Y) provided the following conditions are satisfied.

$$i)\ P_{ij} \geq 0\ ,\ \forall i \text{ and } j$$
$$ii) \sum_j \sum_j P_{ij} = 1$$

X \ Y	Y_1	Y_2	Y_3	Y_4
X_1	P_{11}	P_{12}	P_{13}	P_{14}
X_2	P_{21}	P_{22}	P_{23}	P_{24}

X_3	P_{31}	P_{32}	P_{33}	P_{34}
X_4	P_{41}	P_{42}	P_{43}	P_{44}

MARGINAL PROBABILITY MASS FUNCTION OF X

$$\begin{aligned}P_X(x_i) &= P(X = x_i)\\ &= P[X = x_i, Y = y_1] + P[X = x_i, Y = y_2] + ... + P[X = x_i, Y = y_m]\\ &= p_{i1} + p_{i2} + ... + p_{im}\\ &= \sum_{j=1}^{\infty} p_{ij} = \sum_{j=1}^{\infty} p(x_i, y_j) = p_{i\bullet}\end{aligned}$$

MARGINAL PROBABILITY MASS FUNCTION OF Y

$$\begin{aligned}P_Y(y_j) &= P(Y = y_j)\\ &= P[X = x_1, Y = y_j] + P[X = x_2, Y = y_j] + ... + P[X = x_n, Y = y_j]\\ &= p_{1j} + p_{2j} + ... + p_{nj}\\ &= \sum_{j=1}^{n} p_{ij} = \sum_{j=1}^{n} p(x_i, y_j) = p_{\bullet j}\end{aligned}$$

CONDITIONAL PROBABILITY OF X GIVEN Y

$$P\{X = x_i / Y = y_j\} = \frac{P\{X = x_i \,\&\, Y = y_j\}}{P(Y = y_j)}$$

$$P(x / y) = \frac{P(x, y)}{P(y)}$$

CONDITIONAL PROBABILITY OF Y GIVEN X

$$P\{Y = y_j / X = x_i\} = \frac{P\{X = x_i \,\&\, Y = y_j\}}{P(X = x_i)}$$

$$P(y / x) = \frac{P(x, y)}{P(x)}$$

INDEPENDENT RANDOM VARIABLES

Two R.V's X and Y are said to be independent if
$P(x, y) = P(x) \bullet P(y)$

TWO DIMENSIONAL CONTINUOUS RANDOM VARIABLE

JOINT PROBABILITY DENSITY FUNCTION

A joint probability density of the two dimensional random variable (X,Y) is denoted by f(x,y) and it satisfies the following conditions.

$$i)\ f(x, y) \geq 0$$

$$ii)\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = 1$$

MARGINAL DENSITY FUNCTION OF X & Y

$$f(x) = \int_{-\infty}^{\infty} f(x, y)dy$$

$$f(y) = \int_{-\infty}^{\infty} f(x, y)dx$$

CONDITIONAL DENSITY OF X GIVEN Y

$$f(x/y) = \frac{f(x, y)}{f(y)}$$

CONDITIONAL DENSITY OF Y GIVEN X

$$f(y/x) = \frac{f(x, y)}{f(x)}$$

INDEPENDENT RANDOM VARIABLES

Two R.V's X and Y are said to be independent if
$f(x, y) = f(x) \bullet f(y)$

CUMULATIVE DISTRIBUTION FUNCTION

$$F(x, y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(x, y)dxdy$$

Properties of joint distribution of (X, Y) are

$$(i)\ F[-\infty, y] = 0 = F[x, -\infty] \text{ and } F[-\infty, \infty] = 1$$
$$(ii)\ P[a < X < b, Y \le y] = F(b, y) - F(a, y)$$
$$(iii)\ P[X \le x, c < Y < d] = F[x, d] - F[x, c]$$
$$(iv)\ P[a < X < b, c < Y < d] = F[b, d] - F[a, d] - F[b, c] + F[a, c]$$
$$(v)\ \text{At points of continuity of } f(x, y),\ \frac{\partial^2 F}{\partial x \partial y} = f(x, y)$$

PROBLEMS

1.Three balls are drawn at random without replacement from a box containing 2 white, 3 red and 4 black balls. If X denotes the number of white balls drawn and Y denotes the number of red balls drawn, find the joint probability distribution of (X, Y).

<u>Solution</u>:

As there are only 2 white balls in the box, X can take the values 0, 1 and 2 and Y can take the values 0, 1, 2 and 3 since there are only 3 red balls.

$$P[X=0, Y=0] = P[\text{drawing 3 balls none of which is white or red}]$$

$$= P[\text{all three balls drawn are black}] = \frac{4c_3}{9c_3} = \frac{1}{21}$$

$$P[X=0, Y=1] = P[\text{drawing 3 balls 1 red and 2 black}] = \frac{3c_1 \times 4c_2}{9c_3} = \frac{3}{14}$$

$$P[X=0, Y=2] = P[\text{drawing 3 balls 2 red and 1 black}] = \frac{3c_2 \times 4c_1}{9c_3} = \frac{1}{7}$$

$$P[X=0, Y=3] = P[\text{drawing 3 red balls}] = \frac{3c_3}{9c_3} = \frac{1}{84}$$

$$P[X=1, Y=0] = P[\text{drawing 1 white 2 black}] = \frac{2c_1 \times 4c_2}{9c_3} = \frac{1}{7}$$

$$P[X=1, Y=1] = P[\text{drawing 1 white 1 red and 1 black}] = \frac{2c_1 \times 3c_1 \times 4c_1}{9c_3} = \frac{2}{7}$$

$$P[X=1\,,\,Y=2] = P[drawing\,1\,white\,2\,red\,] = \frac{2c_1 \times 3c_2}{9c_3} = \frac{1}{14}$$

$P[X=1\,,\,Y=3] = 0$ [since only 3 balls are drawn]

$$P[X=2\,,\,Y=0] = P[drawing\,2\,white\,1\,black\,] = \frac{2c_2 \times 4c_1}{9c_3} = \frac{1}{21}$$

$$P[X=2\,,\,Y=1] = P[drawing\,2\,white\ and\,1\,red\,balls\,] = \frac{2c_2 \times 3c_1}{9c_3} = \frac{1}{28}$$

$P[X=2\,,\,Y=2] = 0$ [since only 3 balls are drawn]

$P[X=2\,,\,Y=3] = 0$ [since only 3 balls are drawn]

The joint probability distribution of (X,Y) may be represented in the form

of a table as given below

X \ Y	0	1	2	3
0	$\frac{1}{21}$	$\frac{3}{14}$	$\frac{1}{7}$	$\frac{1}{84}$
1	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{14}$	0
2	$\frac{1}{21}$	$\frac{1}{28}$	0	0

2. The joint probability mass function of (X,Y) is given by $p(x,y) = k(2x+3y)$, $x=0,1,2$, $y=1,2,3$. Find all the marginal and conditional probability distributions. Also find the probability distribution of $X+Y$.

<u>Solution</u>:

The joint probability distribution of (X,Y) is given below

X \ Y	1	2	3
0	3k	6k	9k
1	5k	8k	11k
2	7k	10k	13k

Since $p(x, y)$ is a probability mass function, we have

$$\sum\sum p(x, y) = 1$$

$$3k + 6k + 9k + 5k + 8k + 11k + 7k + 10k + 13k = 1$$

$$72k = 1$$

$$k = \frac{1}{72}$$

Marginal probability distribution of X

$$P[X = 0] = 3k + 6k + 9k = 18k = \frac{18}{72} = \frac{1}{4}$$

$$P[X = 1] = 5k + 8k + 11k = 24k = \frac{24}{72} = \frac{1}{3}$$

$$P[X = 0] = 7k + 10k + 13k = 30k = \frac{30}{72} = \frac{5}{12}$$

Marginal probability distribution of Y

$$P[Y = 1] = 3k + 5k + 7k = 15k = \frac{15}{72} = \frac{5}{24}$$

$$P[Y = 2] = 6k + 8k + 10k = 24k = \frac{24}{72} = \frac{1}{3}$$

$$P[Y = 3] = 9k + 11k + 13k = 33k = \frac{33}{72} = \frac{11}{24}$$

Conditional distribution of X given $Y=1$

$$P[X=0/Y=1]=\frac{P[X=0\ ,\ Y=1]}{P[Y=1]}=\frac{3k}{15k}=\frac{3}{15}=\frac{1}{5}$$

$$P[X=1/Y=1]=\frac{P[X=1\ ,\ Y=1]}{P[Y=1]}=\frac{5k}{15k}=\frac{5}{15}=\frac{1}{3}$$

$$P[X=2/Y=1]=\frac{P[X=2\ ,\ Y=1]}{P[Y=1]}=\frac{7k}{15k}=\frac{7}{15}$$

Conditional distribution of X given $Y=2$

$$P[X=0/Y=2]=\frac{P[X=0\ ,\ Y=2]}{P[Y=2]}=\frac{6k}{24k}=\frac{6}{24}=\frac{1}{4}$$

$$P[X=1/Y=2]=\frac{P[X=1\ ,\ Y=2]}{P[Y=2]}=\frac{8k}{24k}=\frac{8}{24}=\frac{1}{3}$$

$$P[X=2/Y=2]=\frac{P[X=2\ ,\ Y=2]}{P[Y=2]}=\frac{10k}{24k}=\frac{5}{12}$$

Conditional distribution of X given $Y=3$

$$P[X=0/Y=3]=\frac{P[X=0\ ,\ Y=3]}{P[Y=3]}=\frac{9k}{33k}=\frac{9}{33}=\frac{3}{11}$$

$$P[X=1/Y=3]=\frac{P[X=1\ ,\ Y=3]}{P[Y=3]}=\frac{11k}{33k}=\frac{11}{33}=\frac{1}{3}$$

$$P[X=2/Y=3]=\frac{P[X=2\ ,\ Y=3]}{P[Y=3]}=\frac{13k}{33k}=\frac{13}{33}$$

Conditional distribution of Y given $X=0$

$$P[Y=1/X=0]=\frac{P[X=0\ ,\ Y=1]}{P[X=0]}=\frac{3k}{18k}=\frac{3}{18}=\frac{1}{6}$$

$$P[Y=2/X=0]=\frac{P[X=0\ ,\ Y=2]}{P[X=0]}=\frac{6k}{18k}=\frac{6}{18}=\frac{1}{3}$$

$$P[Y=3/X=0]=\frac{P[X=0\ ,\ Y=3]}{P[X=0]}=\frac{9k}{18k}=\frac{9}{18}=\frac{1}{2}$$

Conditional distribution of Y given $X=1$

$$P[Y=1/X=1]=\frac{P[X=1\ ,\ Y=1]}{P[X=1]}=\frac{5k}{24k}=\frac{5}{24}$$

$$P[Y=2/X=1]=\frac{P[X=1\ ,\ Y=2]}{P[X=1]}=\frac{8k}{24k}=\frac{8}{24}=\frac{1}{3}$$

$$P[Y=3/X=1]=\frac{P[X=1\ ,\ Y=3]}{P[X=1]}=\frac{11k}{24k}=\frac{11}{24}$$

Conditional distribution of Y given $X=2$

$$P[Y=1/X=2]=\frac{P[X=2\ ,\ Y=1]}{P[X=2]}=\frac{7k}{30k}=\frac{7}{30}$$

$$P[Y=2/X=2]=\frac{P[X=2\ ,\ Y=2]}{P[X=2]}=\frac{10k}{30k}=\frac{10}{30}=\frac{1}{3}$$

$$P[Y=3/X=2]=\frac{P[X=2\ ,\ Y=3]}{P[X=2]}=\frac{13k}{30k}=\frac{13}{30}$$

Probability distribution of $(X+Y)$

$X+Y$	p
1	$p_{01}=3k=\frac{3}{72}$
2	$p_{02}+p_{11}\ =6k+5k\ =11k\ =\frac{11}{72}$
3	$p_{03}+p_{12}+p_{21}\ =9k+8k+7k=24k=\frac{24}{72}$
4	$p_{13}+p_{22}\ =11k+10k\ =21k=\frac{21}{72}$
5	$p_{23}=13k\ =\frac{13}{72}$

3. Let X and Y be integer valued random variables with $P[X=m,Y=n]=q^2\ p^{m+n-2}\ ,\ n,m=1,2,\ldots\ldots\ldots$ and $p+q=1$. Are $X\ and\ Y$ independent?

The marginal pmf of X is

$$p(x)=\sum_{n=1}^{\infty}q^2\,p^{\,m+n-2}=\sum_{n=1}^{\infty}q^2\,p^{m-1}\,p^{n-1}=q^2\,p^{\,m-1}\sum_{n=1}^{\infty}p^{n-1}$$
$$=q^2\,p^{\,m-1}\left[1+p+p^2+p^3+.........\right]=q^2\,p^{\,m-1}(1-p)^{-1}$$
$$=q^2\,p^{m-1}\,q^{-1}=q\,p^{\,m-1}$$

The marginal pmf of Y is

$$p(y)=\sum_{m=1}^{\infty}q^2\,p^{\,m+n-2}=\sum_{m=1}^{\infty}q^2\,p^{m-1}\,p^{n-1}=q^2\,p^{\,n-1}\sum_{m=1}^{\infty}p^{m-1}$$
$$=q^2\,p^{\,n-1}\left[1+p+p^2+p^3+.........\right]=q^2\,p^{\,n-1}(1-p)^{-1}$$
$$=q^2\,p^{n-1}\,q^{-1}=q\,p^{\,n-1}$$

$$p(x)\,p(y)=q\,p^{m-1}.q\,p^{n-1}=q^2\,p^{\,m+n-2}=P[X=m\ \ Y=n]$$

Therefore $X\ and\ Y$ are independent random variables.

4. The joint distribution of (X,Y) where X and Y are discrete is given in the

following table

X \ Y	0	1	2
0	**0.1**	**0.04**	**0.06**
1	**0.2**	**0.08**	**0.12**
2	**0.2**	**0.08**	**0.12**

Verify whether X and Y are independent.

<u>Solution</u>:

Marginal distribution of X is

$P[X=0] = 0.1+0.04+0.06 = 0.2$

$P[X=1] = 0.2+0.08+0.12 = 0.4$

$P[X=2] = 0.2+0.08+0.12 = 0.4$

Marginal distribution of Y is

$P[Y=0] = 0.1+0.2+0.2 = 0.5$

$P[Y=1] = 0.04+0.08+0.08 = 0.2$

$P[Y=2] = 0.06+0.12+0.12 = 0.3$

X **and** Y **are independent if** $P[X=i]\times P[Y=j] = P[X=i\ ,\ Y=j]$ **for all** i **and** j

(ie) We have to show that

$P[X=0]\times P[Y=0] = 0.2\times 0.5 = 0.1$ **------------------(1)**

$P[X=0, Y=0]\ 0.1$ **-------------------------------(2)**

From (1) and (2), we have

$P[X=0]\times P[Y=0] = P[X=0\ ,\ Y=0]$

$P[X=0]\times P[Y=1] = 0.2\times 0.2 = 0.04 = P[X=0\ ,\ Y=1]$

$P[X=0]\times P[Y=2] = 0.2\times 0.3 = 0.06 = P[X=0\ ,\ Y=2]$

$P[X=1]\times P[Y=0] = 0.4\times 0.5 = 0.2 = P[X=1\ ,\ Y=0]$

$P[X=1]\times P[Y=1] = 0.4\times 0.2 = 0.08 = P[X=1\ ,\ Y=1]$

$P[X=1]\times P[Y=2] = 0.4\times 0.3 = 0.12 = P[X=1\ ,\ Y=2]$

$P[X=2]\times P[Y=0] = 0.4\times 0.5 = 0.2 = P[X=2\ ,\ Y=0]$

$P[X=2]\times P[Y=1] = 0.4\times 0.2 = 0.08 = P[X=2\ ,\ Y=1]$

$P[X=2]\times P[Y=2] = 0.4\times 0.3 = 0.12 = P[X=2\ ,\ Y=2]$

Therefore for all i **and** j**,** $P[X=i]\times P[Y=j] = P[X=i\ ,\ Y=j]$

Hence, the random variables X and Y are independent.

5. The joint probability density function of the random variable (X,Y) is given by $f(x,y)=k\,x\,y\,e^{-(x^2+y^2)}, x>0, y>0$. Find the value of k.

Solution:

Given $f(x,y)$ is the joint pdf , we have

$$\iint f(x,y)\,dx\,dy=1$$ **put** $x^2=t$

$$\int_0^\infty\int_0^\infty k\,x\,y\,e^{-(x^2+y^2)}\,dx\,dy=1$$ $2x\,dx=dt$

$$k\int_0^\infty\int_0^\infty x\,y\,e^{-x^2}\,e^{-y^2}\,dx\,dy=1$$ $x\,dx=\frac{dt}{2}$

$$k\int_0^\infty y\,e^{-y^2}\left[\int_0^\infty x\,e^{-x^2}\,dx\right]dy=1$$

when $x=0, t=0$ *and when* $x=\infty, t=\infty$

$$k\int_0^\infty y\,e^{-y^2}\left[\int_0^\infty e^{-t}\,\frac{dt}{2}\right]dy=1$$

$$\frac{k}{2}\int_0^\infty y\,e^{-y^2}\left(-e^{-t}\right)_0^\infty dy=1$$

$$\frac{k}{2}\int_0^\infty y\,e^{-y^2}(0+1)\,dy=1$$ **put** $y^2=t$

$$\frac{k}{2}\int_0^\infty e^{-t}\,\frac{dt}{2}=1$$ $2y\,dy=dt$

$$\frac{k}{4}\left(e^{-t}\right)_0^\infty=1$$ $y\,dy=\frac{dt}{2}$

$$\frac{k}{4}(0+1)=1$$

when $y=0, t=0$ *and when* $y=\infty, t=\infty$

$$\frac{k}{4}=1$$

Therefore, the value of k is $k=4$.

6. The joint pdf of the random variable (X,Y) is

$f(x,y)=\begin{cases} k(x+y) & ,0<x<2\,;0<y<2 \\ 0 & ,otherwise \end{cases}$ **. Find the value of k .**

<u>Solution</u>:

Given $f(x,y)$ is the joint pdf , we have

$$\iint f(x,y)\,dx\,dy=1$$

$$\int_0^2\int_0^2 k(x+y)\,dx\,dy=1$$

$$k\int_0^2\left[\left(\frac{x^2}{2}\right)_0^2 + y(x)_0^2\right]dy = 1$$

$$k\int_0^2 [(2-0)+y(2-0)]dy=1$$

$$k\int_0^2 (2+2y)\,dy=1$$

$$k\left[2(y)_0^2+2\left(\frac{y^2}{2}\right)_0^2\right]=1$$

$$k[2(2-0)+(4-0)]=1$$

$$8k=1$$

$$k=\frac{1}{8}$$

7. The joint pdf of the random variable (X,Y) is

$f(x,y)=\begin{cases} c\,x\,y & ,0<x<2\,;0<y<2 \\ 0 & ,otherwise \end{cases}$ **. Find the value of c .**

<u>Solution</u>:

Given $f(x,y)$ is the joint pdf , we have

$$\iint f(x,y)\,dx\,dy=1$$

$$\int_0^2\int_0^2 c\,x\,y\,dx\,dy=1$$

$$c\int_0^2\int_0^2 x\,y\,dx\,dy=1$$

$$c\int_0^2 y\left(\frac{x^2}{2}\right)_0^2 dy=1$$

$$c\int_0^2 y(2-0)\,dy=1$$

$$2c\left[\frac{y^2}{2}\right]_0^2=1$$

$$c[4-0]=1 \Rightarrow 4c=1 \Rightarrow \quad c=\frac{1}{4}$$

Therefore the value of c is $c=\frac{1}{4}$.

8. If two random variables $X \text{ and } Y$ have probability density function

$f(x,y)=k(2x+y)$ **for** $0\le x\le 2 \text{ and } 0\le y\le 3$**. Evaluate** k**.**

<u>Solution</u>:

Given $f(x,y)$ **is the joint pdf , we have**

$$\iint f(x,y)\,dx\,dy=1$$

$$\int_0^3\int_0^2 k(2x+y)\,dx\,dy=1$$

$$k\int_0^3\left[\;2\left(\frac{x^2}{2}\right)_0^2+\;y(x)_0^2\right]dy\;=1$$

$$k\int_0^3 (4+2y)\,dy=1$$

$$k\left[4(y)_0^3+2\left(\frac{y^2}{2}\right)_0^3\right]=1$$

$$k[12+9]=1 \qquad \Rightarrow 21k=1 \qquad \Rightarrow k=\frac{1}{21}$$

9. If the function $f(x,y)=c(1-x)(1-y), 0<x<1,\ 0<y<1$ **is to be a density function, find the value of c.**

<u>Solution</u>:

Given $f(x,y)$ is the joint pdf , we have

$$\iint f(x,y)\,dx\,dy=1$$

$$\int_0^1\int_0^1 c(1-x)(1-y)\,dx\,dy=1$$

$$c\int_0^1\int_0^1 (1-x-y+xy)\,dx\,dy=1$$

$$c\int_0^1\left[(x)_0^1-\left(\frac{x^2}{2}\right)_0^1-y(x)_0^1+y\left(\frac{x^2}{2}\right)_0^1\right]dy=1$$

$$c\int_0^1\left[1-\frac{1}{2}-y+\frac{y}{2}\right]dy=1$$

$$c\int_0^1\left(\frac{1}{2}-\frac{y}{2}\right)dy=1$$

$$c\left[\frac{1}{2}(y)_0^1-\frac{1}{2}\left(\frac{y^2}{2}\right)_0^1\right]=1 \qquad \Rightarrow c\left[\frac{1}{2}-\frac{1}{4}\right]=1 \qquad \Rightarrow \frac{c}{4}=1 \qquad \Rightarrow c=4$$

Therefore the value of c is $c=4$

10. The joint pdf of a two dimensional random variable (X,Y) **is given by** $f(x,y)=xy^2+\frac{x^2}{8}, 0\le x\le 2; 0\le y\le 1$**. Compute (1)** $P[X>1]$ **(2)**

$P\left[Y<\frac{1}{2}\right]$ **(3)** $P\left[X>1 \big/ Y<\frac{1}{2}\right]$ **(4)** $P\left[Y<\frac{1}{2} \big/ X>1\right]$ **(5)** $P[X<Y]$ **and (6)** $P[X+Y\le 1]$

Solution:

(1) $P[X>1]=\int_0^1\int_1^2\left(x\,y^2+\frac{x^2}{8}\right)dx\,dy=\int_0^1\left[y^2\left(\frac{x^2}{2}\right)_1^2+\frac{1}{8}\left(\frac{x^3}{3}\right)_1^2\right]dy$

$$=\int_0^1\left[\frac{y^2}{2}(4-1)+\frac{1}{24}(8-1)\right]dy=\int_0^1\left[\frac{3}{2}y^2+\frac{7}{24}\right]dy$$

$$=\frac{3}{2}\left(\frac{y^3}{3}\right)_0^1+\frac{7}{24}(y)_0^1=\frac{1}{2}(1-0)+\frac{7}{24}(1-0)=\frac{1}{2}+\frac{7}{24}=\frac{19}{24}$$

(2) $P\left[Y<\frac{1}{2}\right]=\int_0^{\frac{1}{2}}\int_0^2\left(x\,y^2+\frac{x^2}{8}\right)dx\,dy=\int_0^{\frac{1}{2}}\left[y^2\left(\frac{x^2}{2}\right)_0^2+\frac{1}{8}\left(\frac{x^3}{3}\right)_0^2\right]dy$

$$=\int_0^{\frac{1}{2}}\left[\frac{y^2}{2}(4-0)+\frac{1}{24}(8-0)\right]dy=\int_0^{\frac{1}{2}}\left[2\,y^2+\frac{1}{3}\right]dy$$

$$=2\left(\frac{y^3}{3}\right)_0^{\frac{1}{2}}+\frac{1}{3}(y)_0^{\frac{1}{2}}=\frac{2}{3}\left(\frac{1}{8}-0\right)+\frac{1}{3}\left(\frac{1}{2}-0\right)=\frac{2}{24}+\frac{1}{6}=\frac{1}{4}$$

(3) $P\left[X>1 \big/ Y<\frac{1}{2}\right]=\dfrac{P\left[X>1,Y<\frac{1}{2}\right]}{P\left[Y<\frac{1}{2}\right]}$

$$P\left[X>1,Y<\frac{1}{2}\right]=\int_0^{\frac{1}{2}}\int_1^2\left(x\,y^2+\frac{x^2}{8}\right)dx\,dy=\int_0^{\frac{1}{2}}\left[y^2\left(\frac{x^2}{2}\right)_1^2+\frac{1}{8}\left(\frac{x^3}{3}\right)_1^2\right]dy$$

$$=\int_0^{\frac{1}{2}}\left[\frac{y^2}{2}(4-1)+\frac{1}{24}(8-1)\right]dy=\int_0^{\frac{1}{2}}\left[\frac{3}{2}y^2+\frac{7}{24}\right]dy$$

$$= \frac{3}{2}\left(\frac{y^3}{3}\right)_0^{\frac{1}{2}} + \frac{7}{24}(y)_0^{\frac{1}{2}} = \frac{1}{2}\left(\frac{1}{8} - 0\right) + \frac{7}{24}\left(\frac{1}{2} - 0\right) = \frac{1}{16} + \frac{7}{48} = \frac{5}{24}$$

$$P\left[X > 1 \middle/ Y < \frac{1}{2}\right] = \frac{\frac{5}{24}}{\frac{1}{4}} = \frac{5}{6}$$

(4) $$P\left[Y < \frac{1}{2} \middle/ X > 1\right] = \frac{P\left[X > 1, Y < \frac{1}{2}\right]}{P[X > 1]} = \frac{\frac{5}{24}}{\frac{19}{24}} = \frac{5}{19}$$

(5) $$P[X < Y] = \int_0^1 \int_0^y \left(x y^2 + \frac{x^2}{8}\right) dx\, dy = \int_0^1 \left[y^2 \left(\frac{x^2}{2}\right)_0^y + \frac{1}{8}\left(\frac{x^3}{3}\right)_0^y \right] dy$$

$$= \int_0^1 \left[\frac{y^2}{2}\left(y^2 - 0\right) + \frac{1}{24}\left(y^3 - 0\right) \right] dy = \int_0^1 \left[\frac{y^4}{2} + \frac{y^3}{24} \right] dy$$

$$= \frac{1}{2}\left(\frac{y^5}{5}\right)_0^1 + \frac{1}{24}\left(\frac{y^4}{4}\right)_0^1 = \frac{1}{10}(1 - 0) + \frac{1}{96}(1 - 0) = \frac{1}{10} + \frac{1}{96} = \frac{53}{480}$$

(6) $$P[X + Y \le 1] = \int_0^1 \int_0^{1-y} \left(x y^2 + \frac{x^2}{8}\right) dx\, dy = \int_0^1 \left[y^2 \left(\frac{x^2}{2}\right)_0^{1-y} + \frac{1}{8}\left(\frac{x^3}{3}\right)_0^{1-y} \right] dy$$

$$= \int_0^1 \left[\frac{y^2}{2}\left((1-y)^2 - 0\right) + \frac{1}{24}\left((1-y)^3 - 0\right) \right] dy$$

$$= \frac{1}{2}\int_0^1 y^2 (1-y)^2\, dy + \frac{1}{24}\int_0^1 (1-y)^3\, dy$$

$$= \frac{1}{2}\int_0^1 y^2 \left(1 - 2y + y^2\right) dy + \frac{1}{24}\int_0^1 (1-y)^3\, dy$$

$$=\frac{1}{2}\int_0^1\left(y^2-2y^3+y^4\right)dy+\frac{1}{24}\int_0^1(1-y)^3\,dy$$

$$=\frac{1}{2}\left[\left(\frac{y^3}{3}\right)_0^1-2\left(\frac{y^4}{4}\right)_0^1+\left(\frac{y^5}{5}\right)_0^1\right]+\frac{1}{24}\left[\frac{(1-y)^4}{-4}\right]_0^1$$

$$=\frac{1}{2}\left[\frac{1}{3}(1-0)+\frac{1}{2}(1-0)+\frac{1}{5}(1-0)\right]-\frac{1}{96}(0-1)$$

$$=\frac{1}{2}\left[\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right]+\frac{1}{96}=\frac{1}{60}+\frac{1}{96}=\frac{13}{480}$$

11. Find the marginal density functions of $X\ and\ Y$ if
$f(x,y)=\frac{2}{5}(2x+5y), 0\le x\le 1, 0\le y\le 1$ **.**

<u>Solution:</u>

Marginal density of X is

$$f_X(x)=\int f(x,y)dy$$

$$=\frac{2}{5}\int_0^1(2x+5y)dy=\frac{2}{5}\left[2x(y)_0^1+5\left(\frac{y^2}{2}\right)_0^1\right]$$

$$=\frac{2}{5}\left[2x+\frac{5}{2}\right]=\frac{4}{5}x+1 \qquad ,\quad 0\le x\le 1$$

Marginal density of Y is

$$f_Y(y)=\int f(x,y)dx$$

$$=\frac{2}{5}\int_0^1(2x+5y)dx=\frac{2}{5}\left[2\left(\frac{x^2}{2}\right)_0^1+5y(x)_0^1\right]$$

$$=\frac{2}{5}[1+5y]=\frac{2}{5}+2y \qquad ,\quad 0\le y\le 1$$

12. The joint pdf of a two dimensional random variable (X,Y) is given by $f(x,y)=k\left(x^3y+xy^3\right)$,$0\le x\le 2;0\le y\le 2$. Find the value of k and marginal and conditional density functions.

<u>Solution</u>:

Given $f(x,y)$ is the joint pdf, we have

$$\iint f(x,y)\,dx\,dy = 1$$

$$k\int_0^2\int_0^2\left(x^3y+xy^3\right)dx\,dy = 1$$

$$k\int_0^2\left[y\left(\frac{x^4}{4}\right)_0^2+y^3\left(\frac{x^2}{2}\right)_0^2\right]dy = 1$$

$$k\int_0^2\left(4y+2y^3\right)dy=1$$

$$k\left[4\left(\frac{y^2}{2}\right)_0^2+2\left(\frac{y^4}{4}\right)_0^2\right]=1$$

$$k[8+8]=1 \qquad \Rightarrow \quad 16k=1$$

$$k=\frac{1}{16}$$

Therefore, $f(x,y)=\dfrac{1}{16}\left(x^3\,y+x\,y^3\right);0\le x\le 2,0\le y\le 2$

Marginal density of X is

$$f_X(x)=\int f(x,y)\,dy=\int_0^2\frac{1}{16}\left(x^3y+x\,y^3\right)dy$$

$$=\frac{1}{16}\left[x^3\left(\frac{y^2}{2}\right)_0^2+x\left(\frac{y^4}{4}\right)_0^2\right]=\frac{1}{16}\left[\frac{x^3}{2}(4-0)+\frac{x}{4}(16-0)\right]$$

$$=\frac{1}{16}\left[2x^3+4x\right]=\frac{x^3+2x}{8}\ ,\ 0\le x\le 2$$

Marginal density of Y is

$$f_Y(y) = \int f(x,y)\,dx = \int_0^2 \frac{1}{16}\left(x^3 y + x y^3\right)dx$$

$$= \frac{1}{16}\left[y\left(\frac{x^4}{4}\right)_0^2 + y^3\left(\frac{x^2}{2}\right)_0^2\right] = \frac{1}{16}\left[\frac{y}{4}(16-0) + \frac{y^3}{2}(4-0)\right]$$

$$= \frac{1}{16}\left[4y + 2y^3\right] = \frac{y^3+2y}{8}\ ,\ 0 \le y \le 2$$

Conditional density of X given Y is

$$f_{X/Y}(x/y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{1}{16}\left(x^3 y + xy^3\right)}{\frac{y^3+2y}{8}} = \frac{8}{16}\cdot\frac{y\left(x^3+xy^2\right)}{y\left(y^2+2\right)}$$

$$f_{X/Y}(x/y) = \frac{\left(x^3+xy^2\right)}{2\left(y^2+2\right)}\ ,\ 0 \le x \le 2$$

Conditional density of Y given X is

$$f_{Y/X}(y/x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{1}{16}\left(x^3 y + xy^3\right)}{\frac{x^3+2x}{8}} = \frac{8}{16}\cdot\frac{x\left(x^2 y+y^3\right)}{x\left(x^2+2\right)}$$

$$f_{Y/X}(y/x) = \frac{\left(x^2 y + y^3\right)}{2\left(x^2+2\right)}\ ;0 \le y \le 2.$$

13.If X and Y have joint pdf $f(x,y) = \begin{cases} x+y & ;0<x<1\ ,0<y<1 \\ 0 & ;otherwise \end{cases}$. Check whether X and Y are independent.

$$f_X(x) = \int f(x,y)\,dy$$

$$= \int_0^1 (x+y)\,dy\ = x(y)_0^1 + \left(\frac{y^2}{2}\right)_0^1 = x + \frac{1}{2}, 0<x<1$$

$$f_Y(y) = \int f(x,y)\,dx$$

$$=\int_0^1 (x+y)dx = \left(\frac{x^2}{2}\right)_0^1 + y(x)_0^1 = y+\frac{1}{2}, 0<y<1$$

$$f_X(x).f_Y(y)=\left(x+\frac{1}{2}\right)\left(y+\frac{1}{2}\right)=xy+\frac{x}{2}+\frac{y}{2}+\frac{1}{4}\neq x+y\neq f(x,y)$$

Therefore, X *and* Y are not independent variables.

14. Given the joint pdf of (X,Y) $f(x,y)=\begin{cases} e^{-(x+y)} & ,x>0, y>0 \\ 0 & ,elsewhere \end{cases}$**. Find the marginal densities of** X **and** Y**. Are** X **and** Y **independent?**

Solution:

Marginal density of X **is**

$$f_X(x)=\int f(x,y)\,dy$$

$$=\int_0^\infty e^{-x}\,e^{-y}\,dy = e^{-x}\int_0^\infty e^{-y}\,dy = e^{-x}\left(-e^{-y}\right)_0^\infty$$

$$=-e^{-x}(0-1)=e^{-x} \quad ,x>0$$

Marginal density of Y **is**

$$f_Y(y)=\int f(x,y)\,dx$$

$$=\int_0^\infty e^{-x}\,e^{-y}\,dx = e^{-y}\int_0^\infty e^{-x}\,dx = e^{-y}\left(-e^{-x}\right)_0^\infty$$

$$=-e^{-y}(0-1)=e^{-y} \quad ,y>0$$

$$f_X(x).f_Y(y)=e^{-x}.e^{-y} = e^{-(x+y)}=f_{XY}(x,y)$$

Therefore X and Y are independent.

15. Given the joint pdf of (X,Y) **as** $f(x,y)=\begin{cases} 8xy\ ; & 0<x<y<1 \\ 0 & ,otherwise \end{cases}$**. Find the marginal and conditional probability density functions of** X **and** Y **Are** X **and** Y **are independent?**

Solution: **Marginal density of X is**

$$f_X(x) = \int f(x,y)\,dy = \int_x^1 8\,x\,y\,dy = 8\,x\int_x^1 y\,dy = 8\,x\left(\frac{y^2}{2}\right)_x^1$$

$$= 4\,x\left(1-x^2\right),\ 0<x<1$$

Marginal density of Y is

$$f_Y(y) = \int f(x,y)\,dx = \int_0^y 8\,x\,y\,dx = 8\,y\int_0^y x\,dx = 8\,y\left(\frac{x^2}{2}\right)_0^y$$

$$= 4\,y\left(y^2-0\right) = 4\,y^3\ ,\ 0<y<1$$

$$f_X(x).f_Y(y) = 4x\left(1-x^2\right).4\,y^3 \neq 8\,xy \neq f_{XY}(x,y)$$

Therefore X and Y are not independent.

Conditional density of X given Y is

$$f_{X/Y}(x/y) = \frac{f(x,y)}{f_Y(y)} = \frac{8\,x\,y}{4\,y^3} = \frac{2\,x}{y^2}\ ,\ 0<x<y$$

Conditional density of Y given X is

$$f_{Y/X}(y/x) = \frac{f(x,y)}{f_X(x)} = \frac{8\,x\,y}{4\,x\left(1-x^2\right)} = \frac{2\,y}{1-x^2}\ ,\ x<y<1$$

16. Given $f_{XY}(x,y) = \begin{cases} cx(x-y)\ ;0<x<2,-x<y<x \\ 0 \qquad\qquad ;otherwise \end{cases}$ **.(1) Evaluate c, find (2)** $f_X(x)$ **(3)** $f_{Y/X}(y/x)$ **and (4)** $f_Y(y)$**.**

Solution:(1) Given $f(x,y)$ **is the joint pdf, we have**

$$\iint f(x,y)\,dx\,dy = 1$$

$$c\int_0^2\int_{-x}^x \left(x^2-xy\right)dy\,dx = 1$$

$$c\int_0^2\left[x^2\,(y)_{-x}^{x}-x\left(\frac{y^2}{2}\right)_{-x}^{x}\right]dx=1$$

$$c\int_0^2\left[x^2\,(x-(-x))-\frac{x}{2}\left(x^2-x^2\right)\right]dx=1$$

$$c\int_0^2\left(2x^3-0\right)dx=1 \qquad \Rightarrow 2c\int_0^2 x^3\,dx=1 \qquad \Rightarrow 2c\left[\frac{x^4}{4}\right]_0^2=1$$

$$\frac{c}{2}[16-0]=1 \qquad \Rightarrow 8c=1 \qquad \Rightarrow c=\frac{1}{8}$$

Therefore , $f(x,y)=\frac{1}{8}\left(x^2-xy\right);0<x<2,-x<y<x$

(2) $f_X(x)=\int_{-\infty}^{\infty} f(x,y)dy=\frac{1}{8}\int_{-x}^{x}\left(x^2-xy\right)dy=\frac{1}{8}\left[x^2\,(y)_{-x}^{x}-x\left(\frac{y^2}{2}\right)_{-x}^{x}\right]$

$$=\frac{1}{8}\left[x^2\,(x-(-x))-\frac{x}{2}\left(x^2-x^2\right)\right]=\frac{1}{8}\left[x^2\,(2x)-0\right]=\frac{x^3}{4},0<x<2.$$

(3) $f_{Y/X}(y/x)=\dfrac{f(x,y)}{f_X(x)}=\dfrac{\frac{1}{8}\left(x^2-xy\right)}{\frac{x^3}{4}}=\dfrac{4}{8}\,\dfrac{x(x-y)}{x^3}=\dfrac{x-y}{2x^2},-x<y<x$

(4) $f_Y(y)=\int f(x,y)\,dx$

$$=\begin{cases}\dfrac{1}{8}\displaystyle\int_{-y}^{2}\left(x^2-xy\right)dx & in\ -2\le y\le 0\\ \dfrac{1}{8}\displaystyle\int_{y}^{2}\left(x^2-xy\right)dx & in\ 0\le y\le 2\end{cases}$$

$$=\begin{cases}\dfrac{1}{8}\left[\left(\dfrac{x^3}{3}\right)_{-y}^{2}-y\left(\dfrac{x^2}{2}\right)_{-y}^{2}\right] & in\ \ -2\le y\le 0\\ \dfrac{1}{8}\left[\left(\dfrac{x^3}{3}\right)_{y}^{2}-y\left(\dfrac{x^2}{2}\right)_{y}^{2}\right] & in\ \ 0\le y\le 2\end{cases}$$

$$=\begin{cases} \frac{1}{8}\left[\frac{1}{3}\left(8+y^3\right)-\frac{y}{2}\left(4-y^2\right)\right] & in \quad -2\le y\le 0 \\ \frac{1}{8}\left[\frac{1}{3}\left(8-y^3\right)-\frac{y}{2}\left(4-y^2\right)\right] & in \quad 0\le y\le 2 \end{cases}$$

$$=\begin{cases} \frac{1}{3}-\frac{y}{4}+\frac{5}{48}y^3 & in \quad -2\le y\le 0 \\ \frac{1}{3}-\frac{y}{4}+\frac{1}{48}y^3 & in \quad 0\le y\le 2 \end{cases}$$

17. Suppose the pdf $f(x,y)$ of (X,Y) is given by

$f(x,y)=\begin{cases} \frac{6}{5}\left(x+y^2\right) ;0\le x\le 1,\ 0\le y\le 1 \\ 0 \qquad ;otherwise \end{cases}$**. Obtain the marginal pdf of X, the conditional pdf of Y given $X=0.8$ and then $E[Y/X=0.8]$.**

<u>**Solution**</u>**: Marginal density of X is**

$$f_X(x)=\int f(x,y)\,dy=\int_0^1 \frac{6}{5}\left(x+y^2\right)dy$$

$$=\frac{6}{5}\left[x(y)_0^1+\left(\frac{y^3}{3}\right)_0^1\right]=\frac{6}{5}\left[x+\frac{1}{3}\right],\ 0\le x\le 1$$

Conditional density of Y given $X=0.8$ is

$$f_{Y/X}(y/x)=\frac{f(x,y)}{f_X(x)}=\frac{\frac{6}{5}\left(x+y^2\right)}{\frac{6}{5}\left(x+\frac{1}{3}\right)}=\frac{x+y^2}{x+\frac{1}{3}}$$

$$f_{Y/X=0.8}=\frac{0.8+y^2}{0.8+\frac{1}{3}}=\frac{3\left[0.8+y^2\right]}{3.4}$$

$$E[Y/X=x]=\int y\,f_{Y/X}(y/x)dy$$

$$=\int_0^1 y\left(\frac{x+y^2}{x+\frac{1}{3}}\right)dy=\frac{1}{x+\frac{1}{3}}\int_0^1\left(xy+y^3\right)dy=\frac{3}{3x+1}\left[x\left(\frac{y^2}{2}\right)_0^1+\left(\frac{y^4}{4}\right)_0^1\right]$$

$$=\frac{3}{3x+1}\left[\frac{x}{2}+\frac{1}{4}\right]=\frac{3(2x+1)}{4(3x+1)}$$

$$E[Y/X=0.8]=\frac{3[2(0.8)+1]}{4[3(0.8)+1]}=\frac{7.8}{13.6}=0.5735$$

18.If X and Y are random variables having the joint density function $f(x,y)=\frac{1}{8}(6-x-y),0<x<2,2<y<4$, find $P[X+Y<3]$.

<u>Solution</u>:

$$P[X+Y<3]=\iint f(x,y)\,dx\,dy$$

$$=\frac{1}{8}\int_2^3\int_0^{3-y}(6-x-y)\,dx\,dy=\frac{1}{8}\int_2^3\left[(6-y)(x)_0^{3-y}-\left(\frac{x^2}{2}\right)_0^{3-y}\right]dy$$

$$=\frac{1}{8}\int_2^3\left[(6-y)(3-y)-\frac{1}{2}(3-y)^2\right]dy=\frac{1}{8}\int_2^3\left[18-9y+y^2-\frac{1}{2}(3-y)^2\right]dy$$

$$=\frac{1}{8}\left[18(y)_2^3-9\left(\frac{y^2}{2}\right)_2^3+\left(\frac{y^3}{3}\right)_2^3-\frac{1}{2}\left[\frac{(3-y)^3}{-3}\right]_2^3\right]$$

$$=\frac{1}{8}\left[18(3-2)-\frac{9}{2}(9-4)+\frac{1}{3}(27-8)+\frac{1}{6}(0-1)\right]$$

$$=\frac{1}{8}\left[18-\frac{45}{2}+\frac{19}{3}-\frac{1}{6}\right]=\frac{5}{24}.$$

19.Let X and Y be continuous random variable with joint pdf $f_{XY}(x,y)=\frac{3}{2}\left(x^2+y^2\right),0<x<1\ ,\ 0<y<1$. Find $f_{X/Y}(x/y)$.

<u>Solution</u>:

$$f_Y(y)=\int f(x,y)\,dx$$

$$= \frac{3}{2}\int_0^1 \left(x^2 + y^2\right)dx = \frac{3}{2}\left[\left(\frac{x^3}{3}\right)_0^1 + y^2\left(x\right)_0^1\right] = \frac{3}{2}\left[\frac{1}{3} + y^2\right]$$

$$= \frac{3}{2}y^2 + \frac{1}{2}$$

$$f_{X/Y}(x/y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{3}{2}\left(x^2+y^2\right)}{\frac{3}{2}\left(y^2+\frac{1}{3}\right)} = \frac{x^2+y^2}{y^2+\frac{1}{3}}.$$

COVARIANCE , CORRELATION AND REGRESSION

COVARIANCE

If X and Y are random variables , then co-variance between them is defined as

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

NOTE:

If X & Y are independent then $Cov(X,Y) = 0$

CORRELATION CO-EFFICIENT

Let X and Y be given random variables. The Karl Pearson's co-efficient of correlation is denoted by $r_{XY}\ or\ r(X,Y)$ and defined as

$$r_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

where $Cov(X,Y) = E(XY) - E(X)E(Y)$

$$\sigma_X{}^2 = Var(X) \quad and \quad \sigma_Y{}^2 = Var(Y)$$

NOTE:

- $-1 \le r_{xy} \le 1$

- If X & Y are independent then $r_{xy} = 0$
- The correlation co-efficient is also denoted by ρ_{xy}
- If $r = 0$ then X and Y are called uncorrelated.
- If $r \neq 0$ then X and Y are called correlated.

20. Let X and Y be two discrete random variable with joint pmf $P[X=x, Y=y] = \begin{cases} \frac{x+2y}{18}, & x=1,2;\ y=1,2 \\ 0 & ,otherwise \end{cases}$. Find the marginal pmf of X and $E[X]$.

<u>Solution</u>:

The joint pmf of (X, Y) is given by

X \ Y	**1**	**2**
1	$\frac{3}{18}$	$\frac{4}{18}$
2	$\frac{5}{18}$	$\frac{6}{18}$

Marginal pmf of X is

$$P[X=1] = \frac{3}{18} + \frac{5}{18} = \frac{8}{18} = \frac{4}{9}$$

$$P[X=2] = \frac{4}{18} + \frac{6}{18} = \frac{10}{18} = \frac{5}{9}$$

$$E[X] = \sum x\, p(x) = (1)\left(\frac{4}{9}\right) + (2)\left(\frac{5}{9}\right) = \frac{4}{9} + \frac{10}{9} = \frac{14}{9}.$$

21. If the joint pdf of (X, Y) is given by $f(x,y) = 2-x-y;\ 0 \le x < y \le 1$, find $E[X]$.

<u>Solution</u>:

$$E[X]=\iint x\,f(x,y)\,dx\,dy$$

$$=\int_0^1\int_0^y x[2-x-y]\,dx\,dy$$

$$=\int_0^1\int_0^y (2x-x^2-xy)\,dx\,dy=\int_0^1\left[2\left(\frac{x^2}{2}\right)_0^y-\left(\frac{x^3}{3}\right)_0^y-y\left(\frac{x^2}{2}\right)_0^y\right]dy$$

$$=\int_0^1\left(y^2-\frac{y^3}{3}-\frac{y^3}{2}\right)dy=\int_0^1\left(y^2-\frac{5}{6}y^3\right)dy=\left(\frac{y^3}{3}\right)_0^1-\frac{5}{6}\left(\frac{y^4}{4}\right)_0^1$$

$$=\frac{1}{3}-\frac{5}{24}=\frac{3}{24}=\frac{1}{8}.$$

22.Let $X \text{ and } Y$ be random variable with joint density function $f_{XY}(x,y)=\begin{cases}4\,x\,y\ ;\ 0\le x\le 1\ ,\ 0\le y\le 1\\ 0 \quad ,otherwise\end{cases}$. Find $E[XY]$.

<u>Solution</u>:

$$E[XY]=\iint xy\,f(x,y)\,dx\,dy$$

$$=\int_0^1\int_0^1 xy(4xy)\,dx\,dy=4\int_0^1\int_0^1 x^2\,y^2\,dx\,dy=4\int_0^1 y^2\left(\frac{x^3}{3}\right)_0^1 dy$$

$$=\frac{4}{3}\int_0^1 y^2\,dy=\frac{4}{3}\left(\frac{y^3}{3}\right)_0^1=\frac{4}{3}\left(\frac{1}{3}\right)=\frac{4}{9}.$$

23.Let $X \text{ and } Y$ be any two random variables and a,b be constants. Prove that $Cov(aX,bY)=ab\,\mathrm{cov}(X,Y)$.

<u>Solution</u>:

$$Cov(X,Y)=E[XY]-E[X]E[Y]$$

$$Cov(aX,bY)=E[aX\ bY]-E[aX]E[bY]$$

$$=ab\,E[XY]-a\,E[X]\,b\,E[Y]=ab[E(XY)-E(X)E(Y)]$$

$$=a\,b\,Cov(X,Y).$$

24. If $Y=-2X+3$, find $Cov(X,Y)$.

<u>Solution</u>:

$$Cov(X,Y)=E[XY]-E[X]E[Y]$$

$$=E[X(-2X+3)-E[X]E[-2X+3]]$$

$$=E[-2X^2+3X]-E[X](-2E[X]+3)$$

$$=-2E[X^2]+3E[X]+2(E[X])^2-3E[X]$$

$$=-2\left[E[X^2]-(E[X])^2\right]=-2Var\,X.$$

25. Find $Corr(X,Y)$ for the following discrete bivariate distribution

Y \ X	5	15
10	0.2	0.4
20	0.3	0.1

<u>Solution</u>:

Correlation co-efficient $=\dfrac{E[XY]-E[X]E[Y]}{\sigma_X\,\sigma_Y}$

Marginal distribution of X is

$P[X=5]=0.2+0.3=0.5$

$P[X=15]=0.4+0.1=0.5$

Marginal distribution of Y is

$P[Y=10]=0.2+0.4=0.6$

$P[Y=20]=0.3+0.1=0.4$

$$E[X] = \sum x\,p(x)$$

$$= 5\times0.5 + 15\times0.5 = 2.5 + 7.5 = 10$$

$$E[Y] = \sum y\,p(y)$$

$$= 10\times0.6 + 20\times0.4 = 6 + 8 = 14$$

$$E[X^2] = \sum x^2\,p(x)$$

$$= (5)^2\times0.5 + (15)^2\times0.5 = 25\times0.5 + 225\times0.5 = 125$$

$$E[Y^2] = \sum y^2\,p(y)$$

$$= (10)^2\times0.6 + (20)^2\times0.4 = 100\times0.6 + 400\times0.4 = 220$$

$$\sigma_X{}^2 = E[X^2] - (E[X])^2$$

$$= 125 - (10)^2 = 125 - 100 = 25$$

$$\sigma_X = 5$$

$$\sigma_Y{}^2 = E[Y^2] - (E[Y])^2$$

$$= 220 - (14)^2 = 220 - 196 = 24$$

$$\sigma_Y = 4.89$$

$$E[XY] = \sum xy\,p(x,y)$$

$$= 5\times10\times0.2 + 15\times10\times0.4 + 5\times20\times0.3 + 15\times20\times0.1 = 130$$

Correlation co-efficient $= \dfrac{E[XY] - E[X]E[Y]}{\sigma_X\,\sigma_Y}$

$$Corr(X,Y) = \frac{130 - (10)(14)}{5\times4.89} = \frac{-10}{24.45} = -0.4089\,.$$

26. Find the correlation co-efficient for the following data

X	**10**	**14**	**18**	**22**	**26**	**30**

Y	18	12	24	6	30	36

Solution: **Here** $n = 6$

X	Y	XY	X^2	Y^2
10	18	180	100	324
14	12	168	196	144
18	24	432	324	576
22	6	132	484	36
26	30	780	676	900
30	36	1080	900	1296
120	126	2772	2680	3276

$$\overline{X} = \frac{\sum X}{n} = \frac{120}{6} = 20$$

$$\overline{Y} = \frac{\sum Y}{n} = \frac{126}{6} = 21$$

$$\sigma_X = \sqrt{\frac{1}{n}\sum X^2 - \left(\overline{X}\right)^2} = \sqrt{\frac{2680}{6} - (20)^2} = 6.83$$

$$\sigma_Y = \sqrt{\frac{1}{n}\sum Y^2 - \left(\overline{Y}\right)^2} = \sqrt{\frac{3276}{6} - (21)^2} = 10.25$$

$$Correlation\ co-efficient = \rho_{XY} = \frac{\frac{1}{n}\sum XY - \left(\overline{X}\right)\left(\overline{Y}\right)}{\sigma_X\,\sigma_Y}$$

$$= \frac{\frac{2772}{6} - (20)(21)}{(6.083)\,(10.25)} = 0.59 = 0.6.$$

27. If X_1 **has mean** 4 **and variance** 9 **while** X_2 **has mean** -2 **and variance** 5 **and the two are independent, find** $Var(2X_1 + X_2 - 5)$**.**

Solution:

Given $E[X_1]=4 \quad ,Var[X_1]=9$

$$E[X_2] = -2 \quad , Var[X_2] = 5$$

$$Var(2X_1 + X_2 - 5) = 4 Var X_1 + Var X_2$$

$$= 4(9) + 5 = 36 + 5 = 41.$$

28. If the independent random variables X and Y have the variances 36 and 16 respectively, find the correlation co-efficient between $X+Y$ and $X-Y$.

<u>Solution</u>: Let $U = X + Y$ **and** $V = X - Y$

Given $Var\,X = \sigma_X^{\,2} = 36 \;\Rightarrow\; \sigma_X = 6$

$Var\,Y = \sigma_Y^{\,2} = 16 \;\Rightarrow\; \sigma_Y = 4$

Correlation co-efficient $= \rho_{UV} = \dfrac{E[UV] - E[U]E[V]}{\sigma_U \, \sigma_V}$

$$E[U] = E[X+Y] = E[X] + E[Y]$$

$$E[V] = E[X-Y] = E[X] - E[Y]$$

$$E[UV] = E[(X+Y)(X-Y)] = E[X^2 - Y^2] = E[X^2] - E[Y^2]$$

$$E[U^2] = E[(X+Y)^2] = E[X^2 + 2XY + Y^2]$$

$$= E[X^2] + 2E[XY] + E[Y^2]$$

$= E[X^2] + 2E[X]E[Y] + E[Y^2]$ [since X and Y are independent]

$$E[V^2] = E[(X-Y)^2] = E[X^2 - 2XY + Y^2]$$

$$= E[X^2] - 2E[XY] + E[Y^2]$$

$= E[X^2] - 2E[X]E[Y] + E[Y^2]$ [since X and Y are independent]

$$E[U]E[V] = (E[X] + E[Y])\,(E[X] - E[Y]) = (E[X])^2 - (E[Y])^2$$

$$\sigma_U^{\,2} = E[U^2] - (E[U])^2$$

$$= \left(E\left[X^2\right] + 2E[X]E[Y] + E\left[Y^2\right]\right) - \left(E[X] + E[Y]\right)^2$$

$$= E\left[X^2\right] + 2E[X]E[Y] + E\left[Y^2\right] - \left(E[X]\right)^2 - 2E[X]E[Y] - E\left[Y^2\right]$$

$$= \left[E\left[X^2\right] - \left(E[X]\right)^2\right] + \left[E\left[Y^2\right] - \left(E[Y]\right)^2\right]$$

$$= {\sigma_X}^2 + {\sigma_Y}^2 = 36 + 16 = 52$$

$$\sigma_U = \sqrt{52}$$

$${\sigma_V}^2 = E\left[V^2\right] - \left(E[V]\right)^2$$

$$= \left(E\left[X^2\right] - 2E[X]E[Y] + E\left[Y^2\right]\right) - \left(E[X] - E[Y]\right)^2$$

$$= E\left[X^2\right] - 2E[X]E[Y] + E\left[Y^2\right] - \left(E[X]\right)^2 + 2E[X]E[Y] - E\left[Y^2\right]$$

$$= \left[E\left[X^2\right] - \left(E[X]\right)^2\right] + \left[E\left[Y^2\right] - \left(E[Y]\right)^2\right]$$

$$= {\sigma_X}^2 + {\sigma_Y}^2 = 36 + 16 = 52$$

$$\sigma_V = \sqrt{52}$$

$$\rho_{UV} = \frac{\left[E\left[X^2\right] - E\left[Y^2\right]\right] - \left[\left(E[X]\right)^2 - \left(E[Y]\right)^2\right]}{\sqrt{52}.\sqrt{52}}$$

$$= \frac{\left[E\left[X^2\right] - \left(E[X]\right)^2\right] - \left[E\left[Y^2\right] - \left(E[Y]\right)^2\right]}{52} = \frac{{\sigma_X}^2 - {\sigma_Y}^2}{52} = \frac{36 - 16}{52}$$

$$= \frac{20}{52} = \frac{5}{13}.$$

29. Find the correlation between X and Y if the joint probability density of X and Y is $f(x, y) = \begin{cases} 2 & for\ x>0\ ,\ y>0\ ,\ x+y<1 \\ 0 & ;elsewhere \end{cases}$.

<u>Solution</u>:

Correlation co-efficient $= \dfrac{E[XY] - E[X]E[Y]}{\sigma_X\ \sigma_Y}$

$$E[X] = \iint x\, f(x, y)\, dx\, dy$$

$$= \int_0^1 \int_0^{1-y} x(2)\,dx\,dy = 2\int_0^1 \left(\frac{x^2}{2}\right)_0^{1-y} dy = \int_0^1 (1-y)^2\, dy$$

$$= \left[\frac{(1-y)^3}{-3}\right]_0^1 = -\frac{1}{3}(0-1) = \frac{1}{3}.$$

$$E[Y] = \iint y\, f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^{1-y} y(2)\,dx\,dy = 2\int_0^1 y\,(x)_0^{1-y}\, dy = 2\int_0^1 y(1-y)\,dy$$

$$= 2\int_0^1 \left(y - y^2\right)dy = 2\left[\left(\frac{y^2}{2}\right)_0^1 - \left(\frac{y^3}{3}\right)_0^1\right]$$

$$= 2\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{3}$$

$$E\left[X^2\right] = \iint x^2\, f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^{1-y} x^2(2)\,dx\,dy = 2\int_0^1 \left(\frac{x^3}{3}\right)_0^{1-y} dy = \frac{2}{3}\int_0^1 (1-y)^3\, dy$$

$$= \frac{2}{3}\left[\frac{(1-y)^4}{-4}\right]_0^1 = -\frac{1}{6}(0-1) = \frac{1}{6}$$

$$E\left[Y^2\right] = \iint y^2\, f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^{1-y} y^2(2)\,dx\,dy = 2\int_0^1 y^2\,(x)_0^{1-y}\, dy = 2\int_0^1 y^2\,(1-y)\,dy$$

$$= 2\int_0^1 \left(y^2 - y^3\right)dy = 2\left[\left(\frac{y^3}{3}\right)_0^1 - \left(\frac{y^4}{4}\right)_0^1\right]$$

$$= 2\left[\frac{1}{3} - \frac{1}{4}\right] = \frac{1}{6}$$

$$\sigma_X{}^2 = E\left[X^2\right] - (E[X])^2$$

$$= \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

$$\sigma_X = \frac{1}{\sqrt{18}}$$

$$\sigma_Y^{\,2} = E[Y^2] - (E[Y])^2$$

$$= \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

$$\sigma_Y = \frac{1}{\sqrt{18}}$$

$$E[XY] = \iint xy\, f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^{1-Y} xy\,(2)\,dx\,dy = 2\int_0^1 y\left(\frac{x^2}{2}\right)_0^{1-y} dy = \int_0^1 y(1-y)^2\,dy$$

$$= \int_0^1 y(1-2y+y^2)\,dy = \int_0^1 (y-2y^2+y^3)\,dy$$

$$= \left(\frac{y^2}{2}\right)_0^1 - 2\left(\frac{y^3}{3}\right)_0^1 + \left(\frac{y^4}{4}\right)_0^1 = \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \frac{1}{12}$$

$$Corr(X,Y) = \frac{\frac{1}{12} - \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)}{\frac{1}{\sqrt{18}}\cdot\frac{1}{\sqrt{18}}} = \frac{\frac{1}{12} - \frac{1}{9}}{\frac{1}{18}} = \frac{\frac{-1}{36}}{\frac{1}{18}} = -\frac{1}{2}$$

30. If $f(x,y) = \begin{cases} 2-x-y\ ,\ 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 \qquad\qquad ;elsewhere \end{cases}$ **is the joint pdf of the random variables** X **and** Y**, find the correlation co-efficient of** X **and** Y**.**

<u>Solution</u>:

Correlation co-efficient $= \dfrac{E[XY] - E[X]E[Y]}{\sigma_X\,\sigma_Y}$

$$E[X] = \iint x\, f(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 x(2-x-y)\,dx\,dy = \int_0^1\int_0^1 (2x - x^2 - xy)\,dx\,dy$$

$$=\int_0^1\left[2\left(\frac{x^2}{2}\right)_0^1-\left(\frac{x^3}{3}\right)_0^1-y\left(\frac{x^2}{2}\right)_0^1\right]dy=\int_0^1\left[1-\frac{1}{3}-\frac{y}{2}\right]dy=\int_0^1\left(\frac{2}{3}-\frac{y}{2}\right)dy$$

$$=\frac{2}{3}(y)_0^1-\frac{1}{2}\left(\frac{y^2}{2}\right)_0^1=\frac{2}{3}-\frac{1}{4}=\frac{5}{12}$$

$$E[Y]=\iint y\,f(x,y)\,dx\,dy$$

$$=\int_0^1\int_0^1 y(2-x-y)\,dx\,dy=\int_0^1\int_0^1\left(2y-xy-y^2\right)dx\,dy$$

$$=\int_0^1\left[2y\,(x)_0^1-y\left(\frac{x^2}{2}\right)_0^1-y^2\,(x)_0^1\right]dy=\int_0^1\left[2y-\frac{y}{2}-y^2\right]dy=\int_0^1\left(\frac{3}{2}y-y^2\right)dy$$

$$=\frac{3}{2}\left(\frac{y^2}{2}\right)_0^1-\left(\frac{y^3}{3}\right)_0^1=\frac{3}{4}-\frac{1}{3}=\frac{5}{12}$$

$$E\left[X^2\right]=\iint x^2\,f(x,y)\,dx\,dy$$

$$=\int_0^1\int_0^1 x^2(2-x-y)\,dx\,dy=\int_0^1\int_0^1\left(2x^2-x^3-yx^2\right)dx\,dy$$

$$=\int_0^1\left[2\left(\frac{x^3}{3}\right)_0^1-\left(\frac{x^4}{4}\right)_0^1-y\left(\frac{x^3}{3}\right)_0^1\right]dy=\int_0^1\left[\frac{2}{3}-\frac{1}{4}-\frac{y}{3}\right]dy=\int_0^1\left(\frac{5}{12}-\frac{y}{3}\right)dy$$

$$=\frac{5}{12}(y)_0^1-\frac{1}{3}\left(\frac{y^2}{2}\right)_0^1=\frac{5}{12}-\frac{1}{6}=\frac{1}{4}$$

$$E\left[Y^2\right]=\iint y^2\,f(x,y)\,dx\,dy$$

$$=\int_0^1\int_0^1 y^2(2-x-y)\,dx\,dy=\int_0^1\int_0^1\left(2y^2-xy^2-y^3\right)dx\,dy$$

$$=\int_0^1\left[2y^2\,(x)_0^1-y^2\left(\frac{x^2}{2}\right)_0^1-y^3\,(x)_0^1\right]dy=\int_0^1\left[2y^2-\frac{y^2}{2}-y^3\right]dy$$

$$= \int_0^1 \left(\frac{3}{2} y^2 - y^3 \right) dy = \frac{3}{2} \left(\frac{y^3}{3} \right)_0^1 - \left(\frac{y^4}{4} \right)_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\sigma_X{}^2 = E[X^2] - (E[X])^2$$

$$= \frac{1}{4} - \left(\frac{5}{12} \right)^2 = \frac{1}{4} - \frac{25}{144} = \frac{11}{144}$$

$$\sigma_X = \frac{\sqrt{11}}{12}$$

$$\sigma_Y{}^2 = E[Y^2] - (E[Y])^2$$

$$= \frac{1}{4} - \left(\frac{5}{12} \right)^2 = \frac{1}{4} - \frac{25}{144} = \frac{11}{144}$$

$$\sigma_Y = \frac{\sqrt{11}}{12}$$

$$E[XY] = \iint xy\, f(x, y) dx\, dy$$

$$= \int_0^1 \int_0^1 xy\ (2 - x - y) dx\, dy = \int_0^1 \int_0^1 (2xy - x^2 y - xy^2) dx\, dy$$

$$= \int_0^1 \left[2y \left(\frac{x^2}{2} \right)_0^1 - y \left(\frac{x^3}{3} \right)_0^1 - y^2 \left(\frac{x^2}{2} \right)_0^1 \right] dy = \int_0^1 \left[y - \frac{y}{3} - \frac{y^2}{2} \right] dy$$

$$= \int_0^1 \left(\frac{2}{3} y - \frac{1}{2} y^2 \right) dy = \frac{2}{3} \left(\frac{y^2}{2} \right)_0^1 - \frac{1}{2} \left(\frac{y^3}{3} \right)_0^1 = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$$

$$Corr(X, Y) = \frac{\frac{1}{6} - \left(\frac{5}{12} \right)\left(\frac{5}{12} \right)}{\frac{\sqrt{11}}{12} . \frac{\sqrt{11}}{12}} = \frac{\frac{1}{6} - \frac{25}{144}}{\frac{11}{144}} = \frac{\frac{-1}{144}}{\frac{11}{144}} = -\frac{1}{11}$$

REGRESSION

Regression is a mathematical measurement of average relationship between two or more variables.

- The lines of Regression of y on x is given y by

$$y-\bar{y}=r\frac{\sigma_Y}{\sigma_X}(x-\bar{x})$$

- The lines of Regression of x on y is given y by

$$x-\bar{x}=r\frac{\sigma_X}{\sigma_Y}(y-\bar{y})$$

REGRESSION COEFFICIENTS

- Regression coefficient of y on x is $b_{yx}=r\frac{\sigma_Y}{\sigma_X}$
- Regression coefficient of x on y is $b_{xy}=r\frac{\sigma_X}{\sigma_Y}$

Correlation coefficient $r=\pm\sqrt{b_{yx}b_{xy}}$

31. From the following data, find (1) the two regression equations (2) the co-efficient of correlation between the marks in Mathematics and Statistics. (3) the most likely marks in Statistics when marks in Mathematics are 30.

Marks in Mathematics	**25**	**28**	**35**	**32**	**31**	**36**	**29**	**38**	**34**	**32**
Marks in Statistics	**43**	**46**	**49**	**41**	**36**	**32**	**31**	**30**	**33**	**39**

<u>Solution</u>: Here $n=10$

x	y	xy	x^2	y^2
25	43	1075	625	1849
28	46	1288	784	2116
35	49	1715	1225	2401
32	41	1312	1024	1681
31	36	1116	961	1296

36	32	1152	1296	1024
29	31	899	841	961
38	30	1140	1444	900
34	33	1122	1156	1089
32	39	1248	1024	1521
320	380	12067	10380	14838

$$\bar{x} = \frac{\sum x}{n} = \frac{320}{10} = 32$$

$$\bar{y} = \frac{\sum y}{n} = \frac{380}{10} = 38$$

$$\sigma_x = \sqrt{\frac{1}{n}\sum x^2 - (\bar{x})^2} = \sqrt{\frac{10380}{10} - (32)^2} = 3.74$$

$$\sigma_y = \sqrt{\frac{1}{n}\sum y^2 - (\bar{y})^2} = \sqrt{\frac{14838}{10} - (38)^2} = 6.31$$

$$Correlation\ co-efficient = r_{XY} = \frac{\frac{1}{n}\sum xy - (\bar{x})(\bar{y})}{\sigma_x \sigma_y}$$

$$= \frac{\frac{12067}{10} - (32)(38)}{(3.74)(6.31)} = -0.39 = -0.4$$

The line of regression of y on x is

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$y - 38 = -0.4 \times \frac{6.31}{3.74}(x - 32)$$

$$y - 38 = -0.67(x - 32)$$

$$y - 38 = -0.67x + 21.44$$

$$y = -0.67x + 21.44 + 38$$

$$y = -0.67x + 59.44$$

The line of regression of x on y is

$$x - \overline{x} = r\frac{\sigma_x}{\sigma_y}\left(y - \overline{y}\right)$$

$$x - 32 = -\,0.4\times\frac{3.74}{6.31}\left(y - 38\right)$$

$$x - 32 = -\,0.24\left(y - 38\right)$$

$$x - 32 = -\,0.24\,y + 9.12$$

$$x = -\,0.24\,y + 9.12 + 32$$

$$x = -\,0.24y + 41.12$$

When marks in Mathematics are 30 (ie) when $x = 30$, we have

$$y = -\,0.67(30) + 59.44 = -\,20.1 + 59.44 = 39.34$$

Therefore marks in Statistics = 39.34

32. If $y = 2x - 3$ and $y = 5x + 7$ are the two regression lines, find the mean values of x and y. Find the correlation co-efficient between x and y. Find an estimate of x when $y = 1$.

<u>Solution</u>: Given $y = 2x - 3$ -----------------(1)

$y = 5x + 7$ ----------------- (2)

Since both the lines of regression passes through the mean values $\overline{x}$ and $\overline{y}$, the point $(\overline{x}, \overline{y})$ must satisfy the two given regression lines.

$\overline{y} = 2\overline{x} - 3$ -------------(3)

$\overline{y} = 5\overline{x} + 7$ ------------- (4)

Subtracting the equations (3) and (4), we have

$$3\overline{x} = -10 \quad \Rightarrow \quad \overline{x} = \frac{-10}{3}$$

$$\bar{y} = 2\left(\frac{-10}{3}\right) - 3 = \frac{-29}{3}$$

Therefore mean values are $\bar{x} = \frac{-10}{3}$ and $\bar{y} = \frac{-29}{3}$.

Let us suppose that equation (1) is the line of regression of y on x and equation

(2) is the equation of the line of regression of x on y, we have

$$(1) \Rightarrow y = 2x - 3$$
$$b_{yx} = 2$$

$$(2) \Rightarrow 5x = y - 7$$

$$x = \frac{1}{5}y - \frac{7}{5}$$

$$b_{xy} = \frac{1}{5}$$

$$r = \sqrt{b_{xy} \times b_{yx}} = \sqrt{\frac{1}{5} \times 2} = \pm 0.63$$

Since both the regression co-efficients are positive, r must be positive.

Correlation co-efficient = $r = 0.63$

Substituting $y = 1$ in (2), we have

$$5x = 1 - 7 = -6$$

$$x = -\frac{6}{5}.$$

33. Find the acute angle between the two lines of regression.

<u>Solution</u>:

The equations of the regression lines are

$$y - \bar{y} = r\frac{\sigma y}{\sigma x}(x - \bar{x}) \text{ ----------- } (1)$$

$$x-\bar{x} = r\frac{\sigma x}{\sigma y}(y-\bar{y}) \text{ -----------}(2)$$

Slope of line (1) is $m_1 = r\frac{\sigma y}{\sigma x}$

Slope of line (2) is $m_2 = \frac{\sigma y}{r\sigma x}$

If θ is the acute angle between the two lines, then

$$\tan\theta = \frac{|m_1 - m_2|}{1+m_1 m_2}$$

$$= \frac{\left| r\frac{\sigma y}{\sigma x} - \frac{\sigma y}{r\sigma x}\right|}{1+r\frac{\sigma y}{\sigma x}.\frac{\sigma y}{r\sigma x}} = \frac{\left|\frac{(r^2-1)}{r}\frac{\sigma y}{\sigma x}\right|}{1+\frac{\sigma y^2}{\sigma x^2}} = \frac{\left|-\frac{(1-r^2)}{r}\frac{\sigma y}{\sigma x}\right|}{\frac{\sigma x^2+\sigma y^2}{\sigma x^2}}$$

$$= \frac{(1-r^2)\sigma x\sigma y}{|r|\,(\sigma x^2+\sigma y^2)} \quad .$$

34. If $X \text{ and } Y$ are random variables such that $Y = aX+b$ where a and b are realconstants, show that the correlation co-efficient $r(X,Y)$ between them has magnitude one.

Solution:

Correlation co-efficient $r(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

$$= E[X(aX+b) - E[X]E[aX+b]]$$

$$= E[aX^2+bX] - E[X](aE[X]+b)$$

$$= aE[X^2] + bE[X] - a(E[X])^2 - bE[X]$$

$$= a\left[E[X^2] - (E[X])^2\right] = a Var\, X = a\sigma_X{}^2 .$$

$$\sigma_Y{}^2 = E\left[Y^2\right] - (E[Y])^2$$

$$= E\left[(aX+b)^2\right] - (E[aX+b])^2 = E\left[a^2 X^2 + 2ab\,X + b^2\right] - (a\,E[X]+b)^2$$

$$= a^2 E\left[X^2\right] + 2ab\,E[X] + b^2 - a^2 (E[X])^2 - 2ab\,E[X] - b^2$$

$$= a^2 \left[E\left[X^2\right] - (E[X])^2\right] = a^2\, Var\, X = a^2 \sigma_X{}^2$$

Therefore $\sigma_Y = a\sigma_X$

and $r(X,Y) = \dfrac{a\sigma_X{}^2}{\sigma_X . a\sigma_X} = 1$

Therefore, the correlation co-efficient $r(X,Y)$ between them has magnitude one.

TRANSFORMAION OF RANDOM VARIABLES

Let X,Y be a continuous random variables with joint density function $f(x,y)$ and u & v be transformation such that

$$u = g(x,y) \qquad\qquad v = h(x,y)$$

Then the joint probability density function of u,v is $f(u,v) = |J| f(x,y)$.

Where J is the jacobian of X and Y with respect to u and v.

35. If the joint pdf of (X,Y) is given by $f_{XY}(x,y) = e^{-(x+y)}\,;x \geq 0\,,\ y \geq 0$, find the pdf of $U = \dfrac{X+Y}{2}$.

<u>**Solution**</u>: Given $f_{XY}(x,y) = e^{-(x+y)}\,;x \geq 0\,,\ y \geq 0$

Introduce the auxiliary random variable $V = Y$

$$U = \frac{X+Y}{2} \qquad\qquad V = Y$$

$$u = \frac{x+y}{2} \qquad\qquad v = y$$

$2u = x + y \qquad\qquad y = v$

$2u = x + v$
$x = 2u - v$

$\frac{\partial x}{\partial u} = 2 \qquad\qquad \frac{\partial y}{\partial u} = 0$

$\frac{\partial x}{\partial v} = -1 \qquad\qquad \frac{\partial y}{\partial v} = 1$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2 - 0 = 2$$

$|J| = |2| = 2$

Therefore, the joint density function of UV is given by

$$f_{UV}(u,v) = |J|\; f_{XY}(x,y) \qquad x \geq 0 \;,\; y \geq 0$$

$$= 2e^{-(x+y)} \qquad 2u - v \geq 0 \,,\; v \geq 0$$

$$= 2e^{-(2u - v + v)} \qquad 2u \geq v \quad ,\, v \geq 0$$

$$= 2e^{-2u} \qquad 0 \leq v \leq 2u$$

The pdf of U is given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u,v)\,dv = \int_0^{2u} 2e^{-2u}\,dv = 2e^{-2u}\int_0^{2u} dv = 2e^{-2u}\,[v]_0^{2u}$$

$$= 2e^{-2u}(2u - 0) = 4u\,e^{-2u} \quad ,u \geq 0 .$$

36. If the joint pdf of (X,Y) is given by $f_{XY}(x,y) = x + y \,;\, 0 \leq x, y \leq 1$, find the pdf of $U = XY$.

<u>Solution</u>: Given $f_{XY}(x,y) = x + y \,;\, 0 \leq x, y \leq 1$

Consider the auxiliary random variable $V = Y$

$$U = XY \qquad V = Y$$

$$u = xy \qquad v = y$$

$$u = xv \qquad y = v$$

$$x = \frac{u}{v}$$

$$\frac{\partial x}{\partial u} = \frac{1}{v} \qquad \frac{\partial y}{\partial u} = 0$$

$$\frac{\partial x}{\partial v} = -\frac{u}{v^2} \qquad \frac{\partial y}{\partial v} = 1$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v} - 0 = \frac{1}{v}$$

$$|J| = \left|\frac{1}{v}\right| = \frac{1}{v}$$

Therefore, the joint density function of UV is given by

$$f_{UV}(u,v) = |J|\ f_{XY}(x,y) \qquad 0 \le x \le 1 \ ,\ 0 \le y \le 1$$

$$= \frac{1}{v}(x+y) \qquad 0 \le \frac{u}{v} \le 1 \ ,\ 0 \le v \le 1$$

$$= \frac{1}{v}\left(\frac{u}{v} + v\right) \qquad 0 \le u \le v \ ,\ 0 \le v \le 1$$

$$= \frac{u}{v^2} + 1 \qquad 0 \le u \le v \le 1$$

The pdf of U is given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u,v)\,dv = \int_u^1 \left(\frac{u}{v^2} + 1\right) dv = u\int_u^1 v^{-2}\,dv + \int_u^1 dv$$

$$= u\left[\frac{v^{-1}}{-1}\right]_u^1 + [v]_u^1 = -u\left(1 - \frac{1}{u}\right) + 1 - u = -u + 1 + 1 + -u$$

$$= 2 - 2u = 2(2 - u)\,,\; 0 \le u \le 1\,.$$

37. If the joint pdf of (X, Y) is given by $f_{XY}(x, y) = e^{-(x+y)}\;; x \ge 0\,,\; y \ge 0$, find the pdf of $U = \dfrac{X+Y}{2}$.

<u>Solution</u>: Given $f_{XY}(x, y) = e^{-(x+y)}\;; x \ge 0\,,\; y \ge 0$

Introduce the auxiliary random variable $V = Y$

$$U = \frac{X+Y}{2} \qquad\qquad V = Y$$

$$u = \frac{x+y}{2} \qquad\qquad v = y$$

$$2u = x + y \qquad\qquad y = v$$

$$2u = x + v$$
$$x = 2u - v$$

$$\frac{\partial x}{\partial u} = 2 \qquad\qquad \frac{\partial y}{\partial u} = 0$$

$$\frac{\partial x}{\partial v} = -1 \qquad\qquad \frac{\partial y}{\partial v} = 1$$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2 - 0 = 2$$

$$|J| = |2| = 2$$

Therefore, the joint density function of UV is given by

$$f_{UV}(u, v) = |J|\; f_{XY}(x, y) \qquad x \ge 0\;,\; y \ge 0$$

$$= 2e^{-(x+y)} \qquad 2u - v \ge 0\,,\; v \ge 0$$

$$= 2e^{-(2u - v + v)} \qquad 2u \ge v \qquad , v \ge 0$$

$$= 2e^{-2u} \qquad 0 \le v \le 2u$$

The pdf of U is given by

$$f_U(u)=\int_{-\infty}^{\infty} f_{UV}(u,v)\,dv = \int_0^{2u} 2e^{-2u}\,dv = 2e^{-2u}\int_0^{2u} dv = 2e^{-2u}\left[v\right]_0^{2u}$$

$$= 2e^{-2u}(2u-0) = 4u\,e^{-2u} \quad ,u\geq 0 \text{.}$$

UNIT – IV Analytic Functions

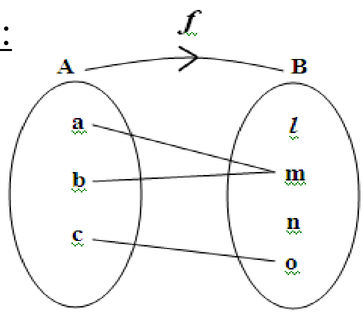
Function

Let A and B be any two sets. A relation f from A to B is called a function if for every $x \in A$ there is a unique $y \in B$ such that $(x, y) \in f$.

A function is also called a map or a mapping or transformation. This map is denoted by f : A $\rightarrow$ B. The set A is said to be domain of f and B is said to be the co-domain of f.

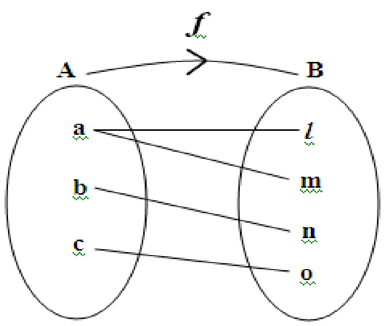
<u>Example:</u>

1)



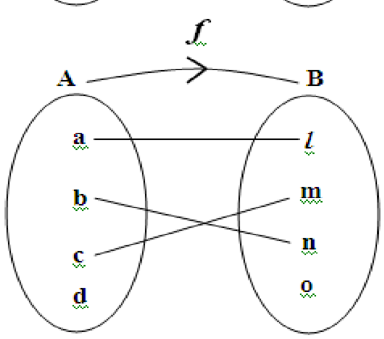
This is a function.

2)



This is not a function, since a$\in$ A is associated to two elements in B.

3)



This is not a function, since d$\in$ A is not associated to any element in B.

If $(x, y) \in f$ then we say that y is the image of x under f and we write $f(x)$ = y.

The set of all image points is called range. In Example (1), Range f = {m, o}

In general, Range f = { $y \in B / y = f(x)$ *for some* $x \in A$ }

Functions of a Complex Variable

Let S be a set of complex numbers. If there is a rule of which assigns to each $z \in S$, a complex number ω then 'f ' is said to be a function defined on S and the value of 'f ' at z is denoted by f (z) so that $\omega = f(z)$. The set S is called the domain of 'f ' and the set of values of 'f ' is its range.

If for each z in S there is only one f (z) of 'f '. Then 'f ' is called a single valued function. Otherwise it is called a multiple valued function.

$$f(z) = z^2 \text{ is a single valued function.}$$

$$f(z) = z^{1/2} \text{ is a two valued function.}$$

Note: Since z = x + iy, f (z) will be in the form u + iv where u and v are functions of two real variables x and y. We usually write $\omega = f(z) = u(x, y) + i\,v(x, y)$.

Limit of a function

Let f (z) be defined in some neighbourhood of z_0 except possibly for the point z_0 itself. Then f (z) is said to tend to a limit ω_0 as z approaches z_0 (along any path) if given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$| f(z) - \omega_0 | < \varepsilon \text{ whenever } 0 < | z - z_0 | < \delta$$

This fact is symbolically written as

$$\lim_{z \to z_0} f(z) = \omega_0.$$

Note: The limit when it exists is always unique, in whatever manner z approaches z_0.

Example:

1) $\displaystyle\lim_{z \to z_0} \bar{z} = \overline{z_0}$

2) $\displaystyle\lim_{z \to 0} \frac{\mathrm{Re}(z)}{|z|} = \lim_{z \to 0} \frac{x}{(x^2 + y^2)^{1/2}}$

Let x approach 0 *along any path* $y = m\,x$, *then*

$$\lim_{z \to 0} \frac{\mathrm{Re}(z)}{|z|} = \lim_{x \to 0} \frac{x}{(x^2 + m^2x^2)^{1/2}} = \lim_{x \to 0} \frac{1}{(1+m^2)^{1/2}} = \frac{1}{(1+m^2)^{1/2}}$$

This limit is different for different values of m. (i.e.) the limit is different for different paths. Hence the limit does not exist.

Continuous function

A function is continuous at a point z_0 if the following conditions are satisfied.

i) $\lim_{z \to z_0} f(z)$ *exists* ii) $f(z_0)$ *exists* iii) $\lim_{z \to z_0} f(z) = f(z_0)$

Statement (iii) says that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon \quad \textit{whenever} \quad |z - z_0| < \delta$$

Example: 1) Polynomial function $P(z) = a_0 + a_1 z + a_2 z^2 + \dots\dots + a_n z^n \ (a_n \neq 0)$

2) e^z, sinz, cosz, tanz, etc.

3) constant function is always continuous.

Bounded function

A function f(z) is said to be bounded in a region R, if there exists a non-negative real number M such that $|f(z)| \leq M \quad \textit{for all } z \in R.$

Derivative

The derivative of f at z_0 is defined by the equation

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \textit{provided the limit exists.}$$

The function f(z) is said to be *differentiable at a point z_0*, when its derivative at z_0 exists.

Note:

1) *If* $\Delta z = z - z_0$, *the above definition becomes*

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad ------ (1)$$

which is another form of definition for derivatives of f at z_0.

2) *If* $\Delta\omega = f(z_0 + \Delta z) - f(z_0)$ *and* $\frac{d\omega}{dz}$ *for* $f'(z_0)$, *then equation* (1) *becomes*

$$\frac{d\omega}{dz} = \lim_{\Delta z \to 0} \frac{\Delta\omega}{\Delta z}$$

which is also another form of definition for derivatives of f at z_0.

Analytic function

A function f(z) is said to be analytic at a point z_0 if its derivative exists not only at z_0 but also at each point z in some neighbourhood of z_0.

Example: 1) $f(z) = z^2$ is analytic everywhere.

2) $f(z) = e^z$ is analytic everywhere.

A function is said to be analytic in a region R if it is analytic at each point in R.

Note: The term holomorphic (or) regular is also used for analyticity.

Entire function

An entire function is a function which is analytic at each point of the entire finite plane.

Example: 1) Every polynomial is an entire function.

2) $f(z) = e^z$ is an entire function.

Harmonic function

A function $u(x,y)$ or $v(x,y)$ is called harmonic function, if first and second order partial derivatives of u or v are continuous and satisfy Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (or) \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$(i.e.)\ u_{xx} + u_{yy} = 0 \quad (or) \quad v_{xx} + v_{yy} = 0$$

Definition

Two families of curves $u(x,y) = c_1$, $v(x,y) = c_2$ are said to form an orthogonal system, if they intersect at right angles at each of their points of intersection.
[(i.e.) product of their slopes = –1]

Necessary condition for f(z) to be analytic

Cauchy-Riemann equations in Cartesian coordinates (CR equations)

The necessary condition for a complex function $f(z) = u(x,y) + iv(x,y)$ to be *analytic are* $u_x = v_y$ *and* $u_y = -v_x$.

Sufficient condition for f(z) to be analytic

The function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D if

i) $u(x, y)$ *and* $v(x, y)$ *are differentiable in D and* $u_x = v_y$ *and* $u_y = -v_x$

ii) The partial derivatives u_x, u_y, v_x, v_y *are all continuous in D.*

Cauchy-Riemann equations in Polar coordinates

The CR equations in polar coordinates are

$$u_r = \frac{1}{r} v_\theta \quad and \quad \frac{1}{r} u_\theta = -v_r$$

Problems

1. *Test whether the function* $f(z) = x^2 + i\,y^2$ *is an analytic or not.*

Sol. $f(z) = x^2 + i\,y^2$

(*i.e.*) $u + i\,v = x^2 + i\,y^2$

$\Rightarrow u = x^2, \quad v = y^2$

$\Rightarrow u_x = 2x, \quad v_x = 0$

$u_y = 0, \quad v_y = 2y$

$\Rightarrow u_x \neq v_y, \; u_y = -v_x$

(*i.e.*) *CR equations are not satisfied.*

$\therefore f(z)$ *is not analytic.*

$$\boxed{\begin{array}{l} \textit{CR equations are} \\ u_x = v_y, \; u_y = -v_x \end{array}}$$

2. *Show that the function* $f(z) = \bar{z}$ *is nowhere differentiable (or) analytic.*

Sol. $f(z) = \bar{z}$

(*i.e.*) $u + i\,v = x - i\,y$

$\Rightarrow u = x, \quad v = -y$

$\Rightarrow u_x = 1, \quad v_x = 0$

$u_y = 0, \quad v_y = -1$

$\Rightarrow u_x \neq v_y, \; u_y = -v_x$

(*i.e.*) *CR equations are not satisfied.*

$\therefore f(z)$ *is nowhere analytic.*

$$\boxed{\begin{array}{l} \textit{CR equations are} \\ u_x = v_y, \; u_y = -v_x \end{array}}$$

3. *Show that* $|z|^2$ *is continuous everywhere but nowhere differentiable except at the origin.*

Sol. *Let* $z = x + i\,y$

$|z|^2 = z\bar{z} = (x + i\,y)(x - i\,y) = x^2 + y^2$

Since $x^2 + y^2$ *is a polynomial in* x *and* y, *it is continuous for all* x *and* y.

$\therefore |z|^2$ *is continuous everywhere.*

Now, $f(z) = |z|^2 = x^2 + y^2$

(*i.e.*) $u + i\,v = x^2 + y^2$

$\Rightarrow u = x^2 + y^2, \quad v = 0$

$\Rightarrow u_x = 2x, \quad v_x = 0, \; u_y = 2y, \quad v_y = 0$

$\Rightarrow u_x \neq v_y, \; u_y \neq -v_x$

(*i.e.*) *CR equations are not satisfied when* $x \neq 0,\ y \neq 0.$

But CR equations are satisfied at $x = 0,\ y = 0.$

$\therefore |z|^2$ *is nowhere differentiable except at the origin.*

4. *Find the analytic region of* $f(z) = (x-y)^2 + 2i(x+y)$

Sol. $f(z) = (x-y)^2 + 2i(x+y)$

(*i.e.*) $u + iv = (x-y)^2 + i(2x+2y)$

$\Rightarrow u = (x-y)^2,\quad v = 2x + 2y$

$\Rightarrow u_x = 2(x-y),\quad v_x = 2$

$u_y = -2(x-y),\quad v_y = 2$

If $x - y = 1$, *then we have* $u_x = v_y,\ u_y = -v_x$

Hence the analytic region is $x - y = 1.$

5. *For what values of* a, b *and* c *the function* $f(z) = (x-2ay) + i(bx - cy)$ *is analytic?*

Sol. $f(z) = (x-2ay) + i(bx-cy)$

(*i.e.*) $u + iv = (x-2ay) + i(bx-cy)$

$\Rightarrow u = (x-2ay),\quad v = bx - cy$

$\Rightarrow u_x = 1,\ u_y = -2a,\ v_x = b,\ v_y = -c$

The condition for analytic is $u_x = v_y,\ u_y = -v_x$

(*i.e.*) $1 = -c,\ -2a = -b$

$\Rightarrow c = -1,\ 2a = b$

6. *If* $u + iv$ *is analytic, show that* $v - iu$ *is also analytic.*

Sol. Since $u + iv$ *is analytic, CR equations are satisfied.*

(*i.e.*) $u_x = v_y,\ u_y = -v_x \text{ ------ } (1)$

Let $U + iV = v - iu$

$\Rightarrow U = v,\quad V = -u$

$\Rightarrow U_x = v_x,\quad V_x = -u_x = -v_y \quad [u\sin g\ (1)]$

$U_y = v_y,\quad V_y = -u_y = v_x \quad [u\sin g\ (1)]$

$\Rightarrow U_x = V_y,\ U_y = -V_x$

$\therefore v - iu$ *is analytic.*

7. Prove that $f(z)=(z-1)(2z+1)$ is analytic and find its derivative.

Sol. $f(z)=(z-1)(2z+1)=2z^2-z-1$

$$(i.e.)\ u+iv=2(x+iy)^2-(x+iy)-1$$
$$=2(x^2-y^2+i\,2xy)-x-iy-1$$
$$\Rightarrow u=2x^2-2y^2-x-1,\quad v=4xy-y$$
$$\Rightarrow u_x=4x-1,\quad v_x=4y$$
$$u_y=-4y,\quad v_y=4x-1$$
$$\Rightarrow u_x=v_y,\ u_y=-v_x$$

(i.e.) CR equations are satisfied.

Further u_x, u_y, v_x, v_y are all continuous.

Hence $f(z)$ is analytic.

$$Now,\ f'(z)=u_x+iv_x$$
$$=(4x-1)+i\,4y$$
$$=4(x+iy)-1$$
$$=4z-1$$

8. Prove that $f(z)=z^n$ is analytic and find $f'(z)$.

Sol. $f(z)=z^n$

$$=(re^{i\theta})^n$$
$$=r^n e^{in\theta}$$

$$\boxed{\begin{aligned} z &= x+iy \\ &= r\cos\theta + i\,r\sin\theta \\ &= r(\cos\theta + i\sin\theta) \\ z &= re^{i\theta} \end{aligned}}$$

$$(i.e.)\ u+iv=r^n(\cos n\theta+i\sin n\theta)$$
$$\Rightarrow u=r^n\cos n\theta,\qquad v=r^n\sin n\theta$$
$$u_r=n\,r^{n-1}\cos n\theta \qquad v_r=n\,r^{n-1}\sin n\theta$$
$$u_\theta=-n\,r^n\sin n\theta \qquad v_\theta=n\,r^n\cos n\theta$$
$$\frac{1}{r}v_\theta=r^{-1}(n\,r^n\cos n\theta)=n\,r^{n-1}\cos n\theta=u_r$$
$$\frac{1}{r}u_\theta=r^{-1}(-n\,r^n\sin n\theta)=-n\,r^{n-1}\sin n\theta=-v_r$$

Hence CR equations are satisfied.

Further $u_r, u_\theta, v_r, v_\theta$ are all continuous.

$\therefore f(z)$ is analytic.

$$Now,\ f'(z) = e^{-i\theta}(u_r + i\,v_r)$$
$$= e^{-i\theta}(n\,r^{n-1}\cos n\theta + i\,n\,r^{n-1}\sin n\theta)$$
$$= n\,r^{n-1}\,e^{-i\theta}(\cos n\theta + i\sin n\theta)$$
$$= n\,r^{n-1}\,e^{-i\theta}\,e^{i\,n\,\theta}$$
$$= n\,r^{n-1}\,e^{i\,(n-1)\,\theta}$$
$$= n\,(\,r\,e^{i\,\theta})^{n-1}$$
$$= n\,z^{n-1}$$

Note: *A function* $\omega = f(z)$ *ceases(not) to be analytic if* $\dfrac{dz}{d\omega} = 0$

9. *Find the point where the function* $\tan z$ *is not analytic.*

Sol. *Let* $\omega = \tan z$

$$\frac{d\omega}{dz} = \sec^2 z = \frac{1}{\cos^2 z}$$

Clearly ω *is not analytic when* $\dfrac{dz}{d\omega} = 0$

$$(i.e.)\ \cos^2 z = 0$$
$$\cos z = 0 = \cos\,(2n+1)\frac{\pi}{2}$$
$$(i.e.)\ z = (2n+1)\frac{\pi}{2},\ \ n = 0,\ \pm 1,\ \pm 2, \ldots\ldots$$

10. *For what values of* z *the function* ω *defined by the equation* $z = e^{-v}(\cos u + i\sin u)$ *is not analytic.*

Sol. *Given* $z = e^{-v}(\cos u + i\sin u)$

$$= e^{-v}e^{i\,u} = e^{i\,(u+iv)} = e^{i\,\omega}$$
$$\Rightarrow \log z = i\,\omega$$
$$\frac{1}{z} = i\,\frac{d\omega}{dz} \ \ \Rightarrow \frac{d\omega}{dz} = \frac{1}{z\,i}$$

Clearly ω *is not analytic when* $\dfrac{dz}{d\omega} = 0$

$$(i.e.)\ z\,i = 0$$
$$(i.e.)\ \ z = 0$$

Home Work

1. Prove that the following function is analytic.
   $(i)\ z^3 \quad (ii)\ e^{-z} \quad (iii)\ \sin z \quad (iv)\ \cosh z$
2. Prove that $(i)\ f(z)=e^y e^{ix} \quad (ii)\ f(z)=e^{x-iy}$ is nowhere analytic.
3. Prove that $f(z)=\log r+i\theta$ (or) $\log z$ is analytic and find $f'(z)$.
4. Prove that $g(z)=\sqrt{r}\,e^{i\theta/2}$ (or) $\sqrt{z}$ is analytic and find $g'(z)$.

Problems

11. *If a function $\omega=f(z)$ is analytic, show that it is independent of $\bar{z}$.*
    (or) Show that every analytic function $\omega=f(z)$ can be expressed as a function of z alone.

Sol. Since $f(z)$ is analytic, CR equations are satisfied.

$$(i.e.)\ u_x=v_y,\ u_y=-v_x \ ------\ (1)$$

Let $\omega=f(z)$

If it is independent of $\bar{z}$, we must have $\dfrac{\partial\omega}{\partial\bar{z}}=0$

We have $z=x+iy,\quad \bar{z}=x-iy$

$$\Rightarrow x=\frac{z+\bar{z}}{2},\quad y=\frac{z-\bar{z}}{2i}$$

$$\Rightarrow \frac{\partial x}{\partial\bar{z}}=\frac{1}{2},\quad \frac{\partial y}{\partial\bar{z}}=-\frac{1}{2i}$$

$$\omega=u+iv$$

$$\frac{\partial\omega}{\partial\bar{z}}=\frac{\partial u}{\partial\bar{z}}+i\frac{\partial v}{\partial\bar{z}}$$

$$=\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial\bar{z}}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial\bar{z}}\right)+i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial\bar{z}}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial\bar{z}}\right)$$

$$=\left[\frac{\partial u}{\partial x}\left(\frac{1}{2}\right)+\frac{\partial u}{\partial y}\left(\frac{-1}{2i}\right)\right]+i\left[\frac{\partial v}{\partial x}\left(\frac{1}{2}\right)+\frac{\partial v}{\partial y}\left(\frac{-1}{2i}\right)\right]$$

$$=\frac{1}{2}\frac{\partial u}{\partial x}-\frac{1}{2i}\frac{\partial u}{\partial y}-\frac{1}{2i}\frac{\partial v}{\partial x}-\frac{1}{2}\frac{\partial v}{\partial y}$$

$$=\frac{1}{2}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)-\frac{1}{2i}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$

$$\frac{\partial\omega}{\partial\bar{z}}=\frac{1}{2}\left(\frac{\partial v}{\partial y}-\frac{\partial v}{\partial y}\right)-\frac{1}{2i}\left(-\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}\right) \quad [using\ (1)]$$

(i.e.) $\frac{\partial\omega}{\partial\bar{z}}=0$

$\Rightarrow \omega$ *is independent of* $\bar{z}$ (*or*) ω *is a function of* z *alone.*

12. *Show that* $u+iv=\dfrac{x-iy}{x-iy+a}$ $(a\neq 0)$ *is not an analytic function of* z, *whereas* $u-iv$ *is such a function at all points where* $z\neq -a$.

Sol. $u+iv=\dfrac{\bar{z}}{\bar{z}+a}=$ *a function of* $\bar{z}$

Since a function of $\bar{z}$ *cannot be analytic,* $u+iv$ *is not analytic.*

$u-iv=$ *conjugate of* $u+iv=\dfrac{z}{z+a}=f(z)$ (*say*)

$\Rightarrow f(z)$ *is a function of* z *alone and* $f'(z)=\dfrac{a}{(z+a)^2}$ *exists everywhere except at* $z=-a$.

$\therefore u-iv$ *is analytic except at* $z=-a$.

13. *Show that an analytic function with*

(*i*) *constant real part is a constant*

(*ii*) *constant modulus is a constant.*

Sol. *Let* $f(z)=u+iv$ *be an analytic function.*

Since $f(z)$ *is analytic, CR equations are satisfied.*

(*i.e.*) $u_x=v_y,\ u_y=-v_x$ ------ (1)

i) *Given* $u=c$

$\Rightarrow u_x=0,\ u_y=0$

$\Rightarrow v_y=0,\ -v_x=0$ [*using* (1)]

(*i.e.*) $v_x=0,\ v_y=0$

$\Rightarrow v=c'$ (*say*)

$\therefore f(z)=c+ic'=constant.$

$ii)\quad Given\ |f(z)| = c$

$$\Rightarrow |u + i\,v| = c$$

$$\Rightarrow \sqrt{u^2 + v^2} = c$$

$$\Rightarrow u^2 + v^2 = c^2 \;-----\; (2)$$

$Diff.\ (2)\ p.w.r.t.\ x\ and\ y,\ we\ get$

$$2u\,u_x + 2v\,v_x = 0$$

$$and\ 2u\,u_y + 2v\,v_y = 0$$

$$u\,u_x + v\,v_x = 0$$

$$and\ u(-v_x) + v(u_x) = 0\ [u\sin g\ (1)]$$

$$(i.e.)\ u\,u_x + v\,v_x = 0 \;----\; (3)$$

$$and\ v\,u_x - u\,v_x = 0 \;----\; (4)$$

$Solving\ (3)\ \&\ (4),\ we\ get$

$$u_x = 0,\ v_x = 0$$

$$\Rightarrow v_y = 0,\ u_y = 0\ [u\sin g\ (1)]$$

$$(i.e.)\ u_x = u_y = 0\ and\ v_x = v_y = 0$$

$$\Rightarrow u\ and\ v\ are\ cons\tan t$$

$$\therefore f(z) = cons\tan t.$$

14. $If\ a\ function\ is\ analytic\ in\ a\ domain\ D,\ show\ that\ f\ is\ a\ cons\tan t$ $function\ when\ f\ is\ real\ valued\ for\ all\ z\ in\ D.$

$Sol.\ Since\ f(z)\ is\ analytic,\ CR\ equations\ are\ satisfied.$

$$(i.e.)\ u_x = v_y,\ u_y = -v_x \;-----\; (1)$$

$Given\ f\ is\ real\ valued\ for\ all\ z.\ (i.e.)\ f(z) = u(x, y)$

$$\therefore v(x, y) = 0$$

$$\Rightarrow v_x = 0,\ v_y = 0$$

$$\Rightarrow u_y = 0,\ u_x = 0\quad [u\sin g\ (1)]$$

$$(i.e.)\ u_x = u_y = 0\ and\ v_x = v_y = 0$$

$$\Rightarrow u\ and\ v\ are\ cons\tan t$$

$$\therefore f(z) = cons\tan t.$$

15. Prove that if $f'(z)=0$ everywhere in a domain D, then $f(z)$ is constant throughout D.

Sol. $f(z)=u+iv$

$f'(z)=u_x+iv_x \quad (or) \quad f'(z)=v_y-iu_y$

Since $f'(z)=0$, we have

$0=u_x+iv_x \quad (or) \quad 0=v_y-iu_y$

$\Rightarrow u_x=0,\ v_x=0,\ v_y=0,\ u_y=0$

$(i.e.)\ u_x=u_y=0$ and $v_x=v_y=0$

$\Rightarrow u$ and v are constant

$\therefore f(z)=$ constant.

16. If $f(z)$ and $\overline{f(z)}$ are analytic in a region, show that $f(z)$ is a constant in that region.

Sol. Assume $f(z)=u+iv$

then $\overline{f(z)}=u-iv=u+i(-v)$

Since $f(z)$ is analytic, CR equations are satisfied.

$(i.e.)\ u_x=v_y,\ u_y=-v_x \ ------\ (1)$

Since $\overline{f(z)}$ is analytic, CR equations are satisfied.

$(i.e.)\ u_x=-v_y,\ u_y=v_x \ ------\ (2)$

$(1)+(2) \Rightarrow \ 2u_x=0,\ 2u_y=0$

$\Rightarrow \ u_x=0,\ u_y=0$

$\Rightarrow u$ is constant.

$(1)-(2) \Rightarrow \ 2v_y=0,\ -2v_x=0$

$\Rightarrow \ v_x=0,\ v_y=0$

$\Rightarrow v$ is constant.

$\therefore f(z)=u+iv=$ constant.

17. If ϕ and ψ are functions of x and y satisfying Laplace's equation namely $\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}=0,\ \dfrac{\partial^2\psi}{\partial x^2}+\dfrac{\partial^2\psi}{\partial y^2}=0$. If $u=\dfrac{\partial\phi}{\partial y}-\dfrac{\partial\psi}{\partial x},\ v=\dfrac{\partial\phi}{\partial x}+\dfrac{\partial\psi}{\partial y}$, Show that $u+iv$ is analytic.

Sol. Given $u = \dfrac{\partial \phi}{\partial y} - \dfrac{\partial \psi}{\partial x}$

$$\frac{\partial u}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} \quad ---- (1)$$

$$v = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}$$

$$\frac{\partial v}{\partial y} = \frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y^2}$$

$$= \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} \quad \left[\because \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0\right]$$

$$= \frac{\partial u}{\partial x} \quad [\textit{from } (1)]$$

(*i.e.*) $u_x = v_y$

Similarly, $\dfrac{\partial u}{\partial y} = \dfrac{\partial^2 \phi}{\partial y^2} - \dfrac{\partial^2 \psi}{\partial x \partial y} \quad ---- (2)$

$$\frac{\partial v}{\partial x} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y}$$

$$= -\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x \partial y} \quad \left[\because \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0\right]$$

$$= -\left[\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x \partial y}\right]$$

$$= -\frac{\partial u}{\partial y} \quad [\textit{from } (2)]$$

(*i.e.*) $u_y = -v_x$

(*i.e.*) *CR equations are satisfied and also* u_x, u_y, v_x, v_y *are continuous.*

Hence $u + iv$ *is analytic.*

18. *Show that* $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} = 4\dfrac{\partial^2}{\partial z \partial \bar{z}}$ (*or*) *S.T.* $\nabla^2 = 4\dfrac{\partial^2}{\partial z \partial \bar{z}}$

Sol. *We have* $z = x + iy, \quad \bar{z} = x - iy$

$$\Rightarrow x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial}{\partial y}\left(\frac{1}{2i}\right)$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right)$$

$$lll^{ly}\quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)$$

$$\therefore\ \frac{\partial}{\partial z}.\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right).\ \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)$$

$$4\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

19. *If* $f(z)$ *is analytic function in any domain, prove that*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2$$

Sol. We have $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} = 4\dfrac{\partial^2}{\partial z \partial \bar{z}}$ (Pr*ove*)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\frac{\partial^2}{\partial z \partial \bar{z}}\,|f(z)|^2$$

$$= 4\frac{\partial^2}{\partial z \partial \bar{z}}\,[f(z)\ \overline{f(z)}\ ]$$

$$= 4\frac{\partial}{\partial z}[f(z)]\ \frac{\partial}{\partial \bar{z}}[f(\bar{z})]$$

$$= 4 f'(z)\ f'(\bar{z})$$

$$= 4 f'(z)\ \overline{f'(z)}$$

$$= 4\,|f'(z)|^2$$

20. *If* $f(z)$ *is analytic function of* z *such that* $f'(z) \neq 0$, *prove that*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0$$

Sol. We have $\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}=4\dfrac{\partial^2}{\partial z\,\partial \bar{z}}$ (Prove)

$$\begin{aligned}
\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)| &= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\log|f'(z)| \\
&= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[\frac{1}{2}\log|f'(z)|^2\right] \\
&= 2\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[\log\{f'(z)\,\overline{f'(z)}\}\right] \\
&= 2\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[\log f'(z)+\log f'(\bar{z})\right] \\
&= 2\frac{\partial}{\partial z}\left[0+\frac{f''(\bar{z})}{f'(\bar{z})}\right] \\
&= 2(0) \\
&= 0
\end{aligned}$$

21. *If* $f(z)$ *is an analytic function of* z, *prove that*

$$\nabla^2|f(z)|^p=p^2\,|f(z)|^{p-2}\,|f'(z)|^2$$

Sol. We have $\nabla^2=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}=4\dfrac{\partial^2}{\partial z\,\partial \bar{z}}$ (Prove)

$$\begin{aligned}
\nabla^2|f(z)|^p &= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}|f(z)|^p \\
&= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[\,|f(z)|^2\,\right]^{\frac{p}{2}} \\
&= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[f(z)\,\overline{f(z)}\,\right]^{\frac{p}{2}} \\
&= 4\left[\frac{\partial}{\partial z}\,f^{\frac{p}{2}}(z)\right]\left[\frac{\partial}{\partial \bar{z}}\,f^{\frac{p}{2}}(\bar{z})\right] \\
&= 4\left[\frac{p}{2}\,f^{\frac{p}{2}-1}(z)\,f'(z)\right]\left[\frac{p}{2}\,f^{\frac{p}{2}-1}(\bar{z})\,f'(\bar{z})\right] \\
&= p^2\left[f^{\frac{p}{2}-1}(z)\,f^{\frac{p}{2}-1}(\bar{z})\right]\left[f'(z)\,f'(\bar{z})\right]
\end{aligned}$$

$$\nabla^2 |f(z)|^p = p^2 \left[f(z)\, f(\bar{z})\right]^{\frac{p}{2}-1} |f'(z)|^2$$
$$= p^2 \left[\,|f(z)|^2\,\right]^{\frac{p-2}{2}} |f'(z)|^2$$
$$= p^2 |f(z)|^{p-2}\ |f'(z)|^2$$

22. *If* $f(z)$ *is an analytic function of* z, *prove that*

$$\left\{\frac{\partial}{\partial x} |f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y} |f(z)|\right\}^2 = |f'(z)|^2$$

Sol. Since $f(z)$ *is analytic, CR equations are satisfied.*

(i.e.) $u_x = v_y\,,\ u_y = -v_x$ ------ (1)

We have $f(z) = u + i\,v$ *and* $f'(z) = u_x + i\,v_x$

$\Rightarrow\ |f(z)| = (u^2+v^2)^{1/2}$ *and* $|f'(z)| = (u_x{}^2 + v_x{}^2)^{1/2}$

$$\frac{\partial}{\partial x} |f(z)| = \frac{1}{2}(u^2+v^2)^{-1/2}(2u\,u_x + 2v\,v_x)$$
$$= (u^2+v^2)^{-1/2}(u\,u_x + v\,v_x) \text{ ------ } (2)$$
$$\frac{\partial}{\partial y} |f(z)| = \frac{1}{2}(u^2+v^2)^{-1/2}(2u\,u_y + 2v\,v_y)$$
$$= (u^2+v^2)^{-1/2}(-u\,v_x + v\,u_x) \text{ ------ } (3)\quad [u\sin g\ (1)]$$

Squaring and adding (2) & (3), *we get*

$$\left\{\frac{\partial}{\partial x} |f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y} |f(z)|\right\}^2 = (u^2+v^2)^{-1}\ [(u\,u_x + v\,v_x)^2 + (-u\,v_x + v\,u_x)^2]$$
$$= (u^2+v^2)^{-1}\ [u^2\,u_x{}^2 + v^2\,v_x{}^2 + 2\,u\,v\,u_x\,v_x$$
$$+ u^2\,v_x{}^2 + v^2\,u_x{}^2 - 2\,u\,v\,u_x\,v_x]$$
$$= (u^2+v^2)^{-1}\ [u^2\,(u_x{}^2 + v_x{}^2) + v^2\,(u_x{}^2 + v_x{}^2)]$$
$$= (u^2+v^2)^{-1}\,(u^2+v^2)\,(u_x{}^2 + v_x{}^2)$$
$$= u_x{}^2 + v_x{}^2$$
$$= |f'(z)|^2$$

Properties of Analytic function

1. Prove that the real and imaginary parts of an analytic function are harmonic functions.

Proof. Let $f(z) = u + iv$ be any analytic function.

Since $f(z)$ *is analytic, CR equations are satisfied.*

$$(i.e.)\ u_x = v_y,\ u_y = -v_x \text{ ------ } (1)$$

Diff. (1) *w.r.t.* x *and* y, *we get*

$$u_{xx} = v_{xy},\ u_{xy} = -v_{xx} \text{ ------ } (2)$$

$$u_{xy} = v_{yy},\ u_{yy} = -v_{xy} \text{ ------ } (3)$$

Adding (2) *and* (3), *we get*

$$u_{xx} + u_{yy} = v_{xy} - v_{xy} = 0 \quad \text{and} \quad v_{xx} + v_{yy} = -u_{xy} + u_{xy} = 0$$

Hence u and v are harmonic functions.

2. If $f(z) = u(x,y) + i\,v(x,y)$ is an analytic function then the curves of the families $u(x,y) = c_1$ and $v(x,y) = c_2$ cut orthogonally where c_1 and c_2 are arbitrary real constants.

Proof. *Since* $f(z)$ *is analytic, CR equations are satisfied.*

$$(i.e.)\ u_x = v_y,\ u_y = -v_x \text{ ------ } (1)$$

Given $u(x, y) = c_1$

$$du = 0$$

$$\text{But } du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$0 = u_x\,dx + u_y\,dy$$

$$-u_x\,dx = u_y\,dy$$

$$\frac{dy}{dx} = -\frac{u_x}{u_y}$$

(*i.e.*) *Slope of the tangent to the u curve is* $-\dfrac{u_x}{u_y} = m_1$ (*say*)

Similarly $v(x, y) = c_2$ *gives*

Slope of the tangent to the u curve is $-\dfrac{v_x}{v_y} = m_2$ (*say*)

$$\text{Product of the slopes} = m_1 m_2$$
$$= \left(-\frac{u_x}{u_y}\right)\left(-\frac{v_x}{v_y}\right)$$
$$= \left(\frac{v_y}{-v_x}\right)\left(\frac{v_x}{v_y}\right) \quad [u\sin g \ (1)]$$
$$= -1$$
$$\therefore \text{The two curves cut orthogonally.}$$

Harmonic Conjugate

If u and v are harmonic functions such that u + *i* v is analytic, then each is called the conjugate harmonic function of the other.

Note: Milne-Thompson formula for the analytic function is

$$f'(z) = u_x(z,0) - i\,u_y(z,0)$$
$$\text{and} \quad f'(z) = v_y(z,0) + i\,v_x(z,0)$$

Problems

1. *Show that $u(x,y) = y^3 - 3x^2 y$ is a harmonic function and find an analytic function $f(z)$ and also find its harmonic conjugate.*

Sol. $u(x,y) = y^3 - 3x^2 y$

$$u_x = -6xy \qquad u_y = 3y^2 - 3x^2$$
$$u_{xx} = -6y \qquad u_{yy} = 6y$$
$$\therefore u_{xx} + u_{yy} = -6y + 6y = 0$$
$$\Rightarrow u \text{ is harmonic.}$$

<u>*To find analytic function*</u>

$$u_x(z,0) = -6(z)(0) = 0$$
$$u_y(z,0) = 3(0)^2 - 3(z)^2 = -3z^2$$

By Milne-Thompson formula, we have

$$f'(z) = u_x(z,0) - i\,u_y(z,0)$$
$$= 0 - i(-3z^2)$$
$$= i3z^2$$

Integrating, we get

$$f(z) = 3i\int z^2\,dz = 3i\left(\frac{z^3}{3}\right) + c = i z^3 + c \quad \rightarrow \text{ Analytic function}$$

$$f(z) = i\,z^3 + c$$
$$= i\,(x + i\,y)^3 + c$$
$$= i\,[x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3] + c$$
$$= i\,[x^3 + i\,3x^2 y - 3\,x\,y^2 - i\,y^3] + c$$
$$= (y^3 - 3\,x^2 y) + i\,(x^3 - 3\,x\,y^2) + c$$
$$\therefore\ v(x, y) = x^3 - 3\,x\,y^2 + c \quad \rightarrow \text{ Harmonic conjugate}$$

2. *Find the harmonic conjugate* $v(x, y)$ *of the function* $u(x, y) = x\,y$ *such that* $v(1,1) = 0$.

Sol. $u(x, y) = x\,y$

$$u_x = y \qquad u_y = x$$
$$u_{xx} = 0 \qquad u_{yy} = 0$$
$$\therefore\ u_{xx} + u_{yy} = 0$$
$\Rightarrow$ *u is harmonic.*

To find analytic function

$$u_x(z,0) = 0, \quad u_y(z,0) = z$$

By Milne-Thompson formula, we have

$$f'(z) = u_x(z,0) - i\,u_y(z,0)$$
$$= 0 - i\,(z)$$
$$= -\,i\,z$$

Integrating, we get

$$f(z) = -i\int z\,dz = -i\left(\frac{z^2}{2}\right) + c \quad \rightarrow \text{ Analytic function}$$
$$= \frac{-i}{2}(x + i\,y)^2 + c$$
$$= \frac{-i}{2}\,[x^2 + i\,2\,x\,y - y^2] + c$$
$$= (xy) + i\left(\frac{y^2 - x^2}{2}\right) + c$$
$$\therefore\ v(x, y) = \frac{y^2 - x^2}{2} + c \quad \rightarrow \text{ Harmonic conjugate}$$

$$Given\ v(1,1) = 0$$

$$(i.e.)\quad 0 = \frac{1-1}{2} + c$$

$$c = 0$$

$$\therefore v(x,y) = \frac{y^2 - x^2}{2}$$

3. *Show that* $v = e^x(x\cos y - y\sin y)$ *is harmonic function. Hence find the analytic function* $f(z) = u + iv$.

Sol. Given $v = e^x(x\cos y - y\sin y)$

$$v_x = e^x(\cos y) + (x\cos y - y\sin y)(e^x)$$

$$= e^x(\cos y + x\cos y - y\sin y)$$

$$v_{xx} = e^x(\cos y) + (\cos y + x\cos y - y\sin y)(e^x)$$

$$= e^x(2\cos y + x\cos y - y\sin y)$$

$$v_y = e^x[-x\sin y - (y\cos y + \sin y.1)]$$

$$v_{yy} = e^x[-x\cos y - (-y\sin y + \cos y.1) - \cos y]$$

$$= e^x(-2\cos y - x\cos y + y\sin y)$$

$$\therefore v_{xx} + v_{yy} = 0$$

$\Rightarrow v$ *is harmonic function.*

<u>To find analytic function</u>

$$v_x(z,0) = e^z(\cos 0 + z\cos 0 - 0) = e^z(1+z)$$

$$v_y(z,0) = e^z(0+0+0) = 0$$

By Milne-Thompson formula, we have

$$f'(z) = v_y(z,0) + i\,v_x(z,0)$$

$$= 0 + i\,e^z(1+z)$$

Integrating, we get

$$f(z) = i\int (1+z)\ e^z\,dz$$

$$= i\left[(1+z)(e^z) - (1)(e^z)\right] + c$$

$$= i[e^z + z\,e^z - e^z] + c$$

$$f(z) = i\,z\,e^z + c$$

Note: In two dimensional steady state flow problems in thermodynamics, hydrodynamics and electronics, we represent the complex potential function as

$$F(z) = \phi(x, y) + i\psi(x, y)$$

$\phi(x, y)$ *is called velocity potential function.*

$\psi(x, y)$ *is called stream function (or) lines of force (or) heat flow lines.*

4. *If* $\psi = x\,y(x^2 - y^2)$ *represents stream function. Find its velocity potential.*

Sol. *Given* $\psi = x\,y(x^2 - y^2)$

$$\psi_x = x\,y(2x) + (x^2 - y^2).y = 3x^2 y - y^3$$

$$\psi_{xx} = 6\,x\,y$$

$$\psi_y = x\,y(-2y) + (x^2 - y^2).x = x^3 - 3\,x\,y^2$$

$$\psi_{yy} = -6\,x\,y$$

$$\therefore \psi_{xx} + \psi_{yy} = 0$$

$\Rightarrow \psi$ *is harmonic function.*

To find velocity potential

$$\psi_x(z,0) = 3(z^2)(0) - 0 = 0$$

$$\psi_y(z,0) = z^3 - 3(z)(0) = z^3$$

By Milne-Thompson formula, we have

$$f'(z) = \psi_y(z,0) + i\,\psi_x(z,0)$$

$$= z^3 + i\,0$$

Integrating, we get

$$f(z) = \int z^3\,dz = \frac{z^4}{4} + c$$

$$\boxed{(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4}$$

$$(i.e.)\ \ f(z) = \frac{1}{4}(x + i\,y)^4 + c$$

$$= \frac{1}{4}[x^4 + 4\,x^3(iy) + 6\,x^2(iy)^2 + 4\,x(iy)^3 + (iy)^4] + c$$

$$= \frac{1}{4}[x^4 + i\,4\,x^3 y - 6\,x^2 y^2 - i\,4\,x\,y^3 + y^4] + c$$

$$= (x^3 y - x\,y^3) + i\,\frac{1}{4}(x^4 - 6\,x^2 y^2 + y^4) + c$$

$$\therefore \textit{Velocity potential is } \phi = \frac{1}{4}(x^4 - 6\,x^2 y^2 + y^4) + c$$

5. *Find the equation of the orthogonal trajectories of the family of curves given by* $3x^2y + 2x^2 - y^3 - 2y^2 = a$, *where 'a' is an arbitrary constant.*

Sol. *Given* $3x^2y + 2x^2 - y^3 - 2y^2 = a$ (i.e.) $u(x, y) = a$

$$\therefore\ u(x, y) = 3x^2y + 2x^2 - y^3 - 2y^2$$

$$u_x = 6xy + 4x \qquad u_y = 3x^2 - 3y^2 - 4y$$

$$u_{xx} = 6y + 4 \qquad u_{yy} = -6y - 4$$

$$\therefore u_{xx} + u_{yy} = 0$$

$$\Rightarrow u \text{ is harmonic.}$$

Now, $u_x(z,0) = 6(z)(0) + 4z = 4z$

$$u_y(z,0) = 3(z)^2 - 0 - 0 = 3z^2$$

By Milne-Thompson formula, we have

$$f'(z) = u_x(z,0) - i\,u_y(z,0)$$

$$= 4z - i(3z^2)$$

Integrating, we get

$$f(z) = 4\int z\,dz - 3i\int z^2\,dz$$

$$= 4\left(\frac{z^2}{2}\right) - 3i\left(\frac{z^3}{3}\right) + c$$

$$f(z) = 2z^2 - iz^3 + c$$

$$\boxed{(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3}$$

$$= 2(x + iy)^2 - i(x + iy)^3 + c$$

$$= 2(x^2 + 2ixy - y^2) - i[x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3] + c$$

$$= 2x^2 - 2y^2 + 4ixy - ix^3 + 3x^2y + i3xy^2 - y^3 + c$$

$$= (2x^2 - 2y^2 - y^3 + 3x^2y) + i(4xy - x^3 + 3xy^2) + c$$

$$\therefore v(x, y) = 4xy - x^3 + 3xy^2 + c$$

Hence the orthogonal trajectories to the given curve is

$$4xy - x^3 + 3xy^2 = b$$

6. *Find the analytic function for the folowing* :

$$(i)\quad u = \frac{1}{2}\log(x^2 + y^2) \qquad (ii)\quad u = \frac{\sin 2x}{\cos 2x + \cosh 2y}$$

Sol. (i) Given $u(x,y) = \frac{1}{2}\log(x^2 + y^2)$

$$u_x = \frac{1}{2}\,\frac{2x}{x^2 + y^2} = \frac{x}{x^2 + y^2}$$

$$u_y = \frac{1}{2}\,\frac{2y}{x^2 + y^2} = \frac{y}{x^2 + y^2}$$

$$Now,\quad u_x(z,0) = \frac{z}{z^2 + 0} = \frac{1}{z}$$

$$u_y(z,0) = \frac{0}{z^2 + 0} = 0$$

By Milne-Thompson formula, we have

$$f'(z) = u_x(z,0) - i\,u_y(z,0)$$

$$= \frac{1}{z} - i(0)$$

Integrating, we get

$$f(z) = \int \frac{1}{z}\,dz$$

$$(i.e.)\ f(z) = \log z + c$$

(ii) Given $u(x,y) = \dfrac{\sin 2x}{\cos 2x + \cosh 2y}$

$$u_x = \frac{(\cos 2x + \cosh 2y)(2\cos 2x) - \sin 2x(-2\sin 2x + 0)}{(\cos 2x + \cosh 2y)^2}$$

$$u_x(z,0) = \frac{(\cos 2z + \cosh 0)(2\cos 2z) - \sin 2z(-2\sin 2z)}{(\cos 2z + \cosh 0)^2}$$

cosh 0 = 1
sinh 0 = 0

$$= \frac{(\cos 2z + 1)(2\cos 2z) + 2\sin^2 2z}{(\cos 2z + 1)^2}$$

$$= \frac{2\cos^2 2z + 2\cos 2z + 2\sin^2 2z}{(\cos 2z + 1)^2}$$

$$= \frac{2(\cos^2 2z + \sin^2 2z) + 2\cos 2z}{(\cos 2z + 1)^2}$$

$$= \frac{2(1 + \cos 2z)}{(1 + \cos 2z)^2} = \frac{2}{1 + \cos 2z}$$

$$u_y = \frac{(\cos 2x + \cosh 2y)(0) - \sin 2x(0 + 2\sinh 2y)}{(\cos 2x + \cosh 2y)^2}$$

$$u_y(z,0) = \frac{0 - \sin 2z(2\sinh 0)}{(\cos 2z + \cosh 0)^2}$$

$$= \frac{0}{(\cos 2z + 1)^2} = 0$$

By Milne-Thompson formula, we have

$$f'(z) = u_x(z,0) - i\,u_y(z,0)$$

$$= \frac{2}{1 + \cos 2z} - i(0)$$

$$= \frac{2}{2\cos^2 z} = \sec^2 z$$

Integrating, we get

$$f(z) = \int \sec^2 z \; dz$$

$$(i.e.)\; f(z) = \tan z + c$$

7. *Construct an analytic function* $u + iv$ *given that* $2u + v = e^x(\cos y - \sin y)$

Sol. *Let* $f(z) = u + iv \quad ----(1)$

$$2f(z) = 2u + i2v$$

$$2i\,f(z) = 2iu - 2v \;----(2)$$

$$(1) + (2) \Rightarrow (1 + 2i)\,f(z) = (u - 2v) + i(2u + v)$$

$$F(z) = U + iV$$

where $F(z) = (1 + 2i)\,f(z),\; U = u - 2v,\; V = 2u + v$

Given $V = 2u + v = e^x(\cos y - \sin y)$

$$V_x = e^x(\cos y - \sin y)$$

$$V_x(z,0) = e^z(\cos 0 - \sin 0) = e^z(1 - 0) = e^z$$

$$V_y = e^x(-\sin y - \cos y)$$

$$V_y(z,0) = e^z(-\sin 0 - \cos 0) = e^z(0 - 1) = -e^z$$

By Milne-Thompson formula, we have

$$F'(z) = V_y(z,0) + i\,V_x(z,0)$$

$$= -e^z + i(e^z)$$

$$= (-1 + i)\,e^z$$

Integrating, we get

$$F(z) = (-1+i)\int e^z \, dz$$

$$(i.e.)\ (1+2i)\, f(z) = (-1+i)\ e^z + c$$

$$f(z) = \frac{-1+i}{1+2i}\ e^z + c$$

$$f(z) = \left(\frac{1+3i}{5}\right) e^z + c$$

$$\boxed{\begin{aligned} \frac{-1+i}{1+2i} &= \frac{(-1+i)(1-2i)}{(1+2i)(1-2i)} \\ &= \frac{-1+2i+i-2i^2}{1-4i^2} \\ &= \frac{-1+3i+2}{1+4} = \frac{1+3i}{5} \end{aligned}}$$

8. *Construct an analytic function* $u+iv$ *given that* $3u+2v = y^2 - x^2 + 16xy$

Sol. Let $f(z) = u + iv$

$$3f(z) = 3u + i3v \quad ---- (1)$$

Also $2f(z) = 2u + i2v$

$$-2i\, f(z) = -2iu + 2v \quad ---- (2)$$

$$(1)+(2) \Rightarrow (3-2i)\, f(z) = (3u+2v) + i(-2u+3v)$$

$$F(z) = U + iV$$

where $F(z) = (3-2i)\, f(z)$, $U = 3u + 2v$, $V = -2u + 3v$

Given $U = 3u + 2v = y^2 - x^2 + 16xy$

$$U_x = -2x + 16y$$

$$U_x(z,0) = -2(z) + 0 = -2z$$

$$U_y = 2y + 16x$$

$$U_y(z,0) = 0 + 16z = 16z$$

By Milne-Thompson formula, we have

$$F'(z) = U_x(z,0) - iU_y(z,0)$$

$$= -2z - i(16z)$$

$$= (-2-16i)\, z$$

Integrating, we get

$$F(z) = (-2-16i)\int z \, dz$$

$$(i.e.)\ (3-2i)\, f(z) = (-2-16i)\left(\frac{z^2}{2}\right) + c$$

$$f(z) = \frac{-1-8i}{3-2i}\ z^2 + c$$

$$f(z) = (1-2i)\ z^2 + c$$

$$\boxed{\begin{aligned} \frac{-1-8i}{3-2i} &= \frac{(-1-8i)(3+2i)}{(3-2i)(3+2i)} \\ &= \frac{-3-2i-24i-16i^2}{9-4i^2} \\ &= \frac{-3-26i+16}{9+4} = \frac{13-26i}{13} \\ &= 1-2i \end{aligned}}$$

Home Work

Find the analytic function $f(z) = u + i\,v$ for the following

1. $u(x, y) = e^{-x}[(x^2 - y^2)\cos y + 2xy\sin y]$
2. $u(x, y) = e^{-x}(x\sin y - y\cos y)$
3. $v(x, y) = x^4 - 6x^2y^2 + y^4$
4. $v(x, y) = e^{-x}(x\cos y + y\sin y)$
5. $u - v = e^x(\cos y - \sin y)$
6. $2u + 3v = e^x(\cos y - \sin y)$

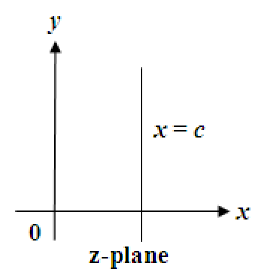
Mapping (or) Transformation

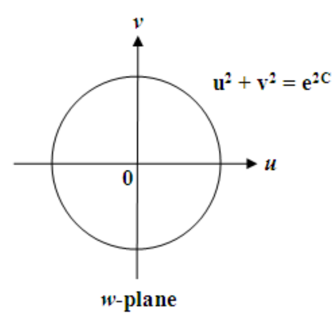
In the complex domain, the function $\omega = f(z)$ $(i.e.)\ u + iv = f(x + iy)$ $----$ (1) involves four variables x, y, u,v. Hence we need a 4 dimensional region to plot (1) in the Cartesian form. It is not possible to have 4 dimensional graph sheets, we make use of 2 complex planes, one for the variable $z = x + iy$ and the other for the variable $\omega = u + iv$. If the point z describes some curve Γ in the z-plane, the point ω will move along a corresponding curve Γ' in the ω-plane. We then, say that a curve Γ in the z-plane is mapped into the corresponding curve Γ' in the ω-plane by the function $\omega = f(z)$ which defines a mapping or transformation of z-plane into the ω-plane.

Example: *Let* $\omega = e^z$

$$(i.e.)\ u + iv = e^{x+iy}$$
$$= e^x e^{iy}$$
$$= e^x(\cos y + i\sin y)$$
$$\Rightarrow u = e^x\cos y,\ v = e^x\sin y$$
$$\Rightarrow u^2 + v^2 = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x}$$
$$When\ x = c,\ we\ have\ u^2 + v^2 = e^{2c}$$

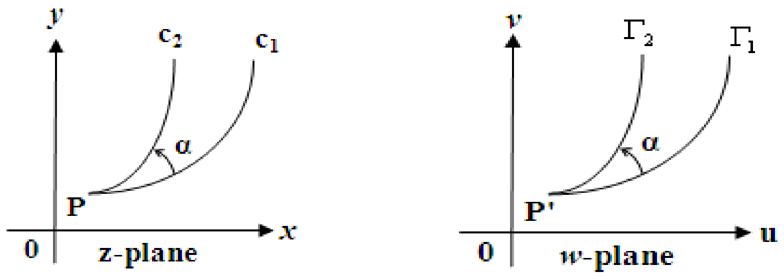
$\therefore$ Straight line parallel to y-axis (i.e.) $x = c$ in z-plane are transformed to a circle $u^2 + v^2 = e^{2c}$ in the ω-plane whose centre is origin and radius is e^c.



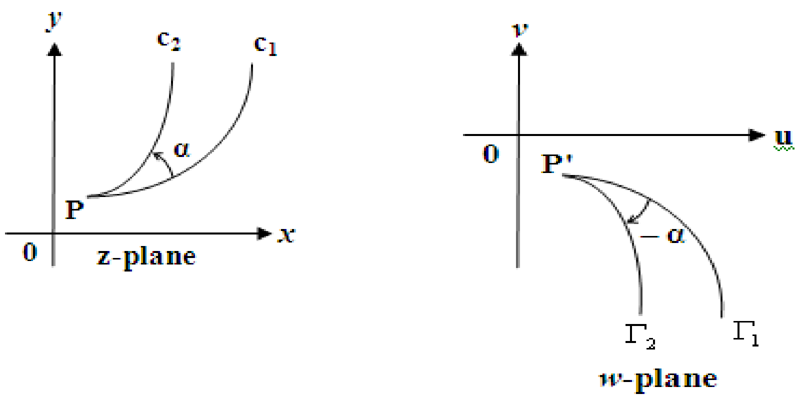


Conformal and Isogonal

A transformation that preserves angles between every pair of curves through a point, both in magnitude and sense, is said to be *conformal* at that point.



A transformation under which angles between every pair of curves through a point are preserved in magnitude, but altered in sense is said to be *isogonal* at that point.



Example:

1) The mapping $\omega = e^z$ is conformal throughout.
2) The mapping $\omega = \bar{z}$, which is a reflection in the real axis is isogonal.

Note:

1) If $f(z)$ is analytic and $f'(z_0) \neq 0$, then the mapping $\omega = f(z)$ is conformal at $z = z_0$.
2) A point z_0 at which the mapping $\omega = f(z)$ is not conformal (or not analytic) (i.e.) if $\frac{d\omega}{dz} = 0$ (or) $f'(z) \neq 0$ is called a critical point of the mapping. The critical points of $z = f^{-1}(\omega)$ are given by $\frac{dz}{d\omega} = 0$. Hence the critical point of the transformation $\omega = f(z)$ are given by $\frac{d\omega}{dz} = 0$ and $\frac{dz}{d\omega} = 0$.

Example:

1. The critical point of $\omega = z^2$ is

$$\frac{d\omega}{dz} = 2z$$

$$0 = 2z$$

$$(i.e.)\ z = 0$$

2. The critical point of $\omega = \frac{1}{z}$ is

$$\frac{d\omega}{dz} = -\frac{1}{z^2} \to \infty \ \ as\ \ z \to 0.$$

$$(i.e.)\ \ \omega = \frac{1}{z}\ \ is\ \ not\ \ analytic\ \ at\ \ z = 0.$$

$$\therefore The\ critical\ po\text{int}\ is\ z = 0.$$

3. Find the critical points of the transformation $\omega^2 = (z-\alpha)(z-\beta)$

Sol. Given $\omega^2 = (z-\alpha)(z-\beta)$

$$\omega = \sqrt{(z-\alpha)(z-\beta)}$$

$$\frac{d\omega}{dz} = \frac{(z-\alpha).1 + (z-\beta).1}{2\sqrt{(z-\alpha)(z-\beta)}}$$

$$0 = \frac{2z-\alpha-\beta}{2\sqrt{(z-\alpha)(z-\beta)}}$$

$$0 = 2z-\alpha-\beta$$

$$z = \frac{\alpha+\beta}{2}$$

$$Also\ \ \omega^2 = (z-\alpha)(z-\beta)\ \ is\ \ not\ \ analytic\ \ at\ \ z = \alpha,\ \beta.$$

$$\therefore The\ critical\ \ po\text{int}\ \ is\ \ z = \alpha,\ \beta,\ \frac{\alpha+\beta}{2}.$$

Some standard transformation

1) The transformation $\omega = c + z$ is known as translation.
2) The transformation $\omega = c\,z$ is known as magnification and rotation.

Problems

1. What is the region of the w-plane into which the rectangular region in the z-plane bounded by the lines $x = 0$, $y = 0$, $x = 1$ and $y = 2$ is mapped under the transformation $w = z + (2 - i)$.

Sol. $\omega = z + (2 - i)$

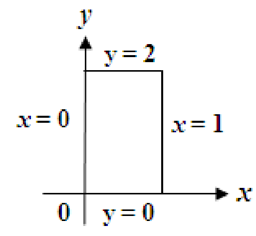
$$u + i\,v = (x + i\,y) + (2 - i)$$
$$= (x + 2) + i\,(y - 1)$$
$$\Rightarrow u = x + 2,\ \ v = y - 1$$

When $x = 0$, *we have* $u = 2$

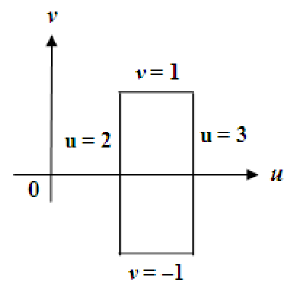
When $x = 1$, *we have* $u = 3$

When $y = 0$, *we have* $v = -1$

When $y = 2$, *we have* $v = 1$



z-plane



w-plane

Hence the lines $x = 0$, $y = 0$, $x = 1$ and $y = 2$ are mapped into the lines $u = 2$, $v = -1$, $u = 3$ and $v = 1$ respectively, which form a rectangle in the w-plane.

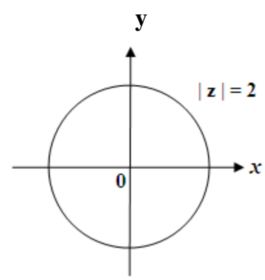
2. Find the image of the circle $|z| = 2$ by the transformation $w = z + 3 + 2i$

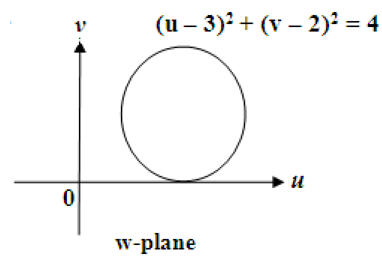
Sol. $\omega = z + 3 + 2i$

$$u + i\,v = (x + i\,y) + 3 + 2i$$
$$= (x + 3) + i\,(y + 2)$$
$$\Rightarrow u = x + 3,\ \ v = y + 2$$

Given $|z| = 2$

$$|x + i\,y| = 2$$
$$\sqrt{x^2 + y^2} = 2$$
$$x^2 + y^2 = 4$$
$$(i.e.)\ (u - 3)^2 + (v - 2)^2 = 4$$





Hence the circle $|z| = 2$ is mapped into $(u - 3)^2 + (v - 2)^2 = 4$ in the w-plane, which is also a circle with centre at (3, 2) and radius 2.

3. Determine the region D' of the w-plane into which the triangular region D enclosed by the lines $x = 0$, $y = 0$, $x + y = 1$ is transformed under the transformation $w = 2z$.

Sol. $\omega = 2z$

$u + iv = 2(x + iy)$

$\Rightarrow u = 2x, \ v = 2y$

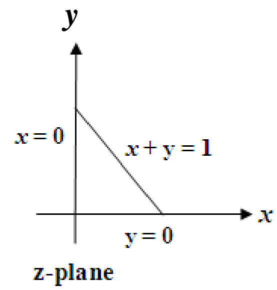
When $x = 0$, *we have* $u = 0$

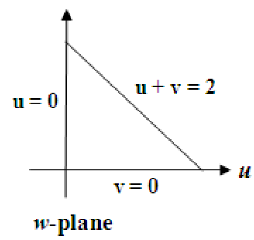
When $y = 0$, *we have* $v = 0$

When $x + y = 1$, *we have* $\dfrac{u}{2} + \dfrac{v}{2} = 1$

$(i.e.) \ u + v = 2$

Hence the lines $x = 0$, $y = 0$, $x + y = 1$ are mapped into the lines $u = 0$, $v = 0$, $u + v = 2$ in the w-plane.





4. Find the image of the strip $0 < x < 1$ under the transformation $w = i\,z$.

Sol. $\omega = iz$

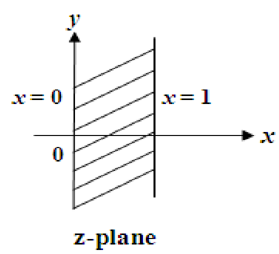
$u + iv = i(x + iy)$

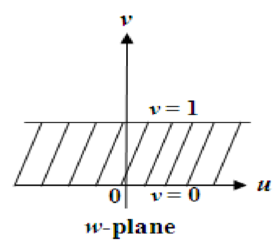
$= ix - y$

$\Rightarrow u = -y, \ v = x$

When $0 < x < 1$, *we have* $0 < v < 1$

$\therefore$ The image of the strip $0 < x < 1$ under the transformation $w = i\,z$ is $0 < v < 1$.





5. Under the transformation $w = i\,z + i$, show that the half plane $x > 0$ maps onto the half plane $v > 1$.

Sol. $\omega = iz + i$

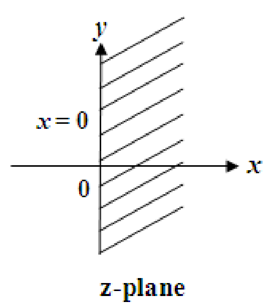
$u + iv = i(x + iy) + i$

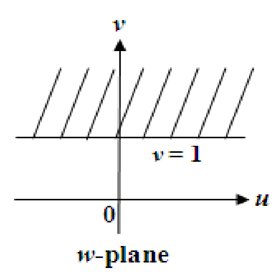
$= ix - y + i$

$\Rightarrow u = -y, \ v = x + 1$

When $x > 0$, *we have* $v - 1 > 0$

$(i.e.) \ v > 1$





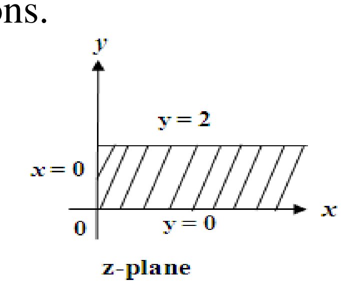
6. Find the image of the semi-infinite strip $x > 0, \ 0 < y < 2$ under the transformation $w = i\,z + 1$. Sketch the graph of the regions.

Sol. $\omega = iz + 1$

$u + iv = i(x + iy) + 1$

$= ix - y + 1$

$\Rightarrow u = 1 - y, \ v = x$



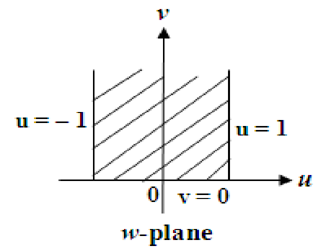
$$When \;\; x > 0, \; we \; have \;\; v > 0.$$

$$When \;\; 0 < y < 2, \; we \; have \; 0 < 1 - u < 2$$

$$0 > u - 1 > -2$$

$$1 > u > -1$$

$$-1 < u < 1$$



7. *Find the image of* $| z - 2i | = 2$ *under the transformation* $w = \dfrac{1}{z}.$

Sol. $\quad | z - 2i | = 2$

$$w = \frac{1}{z} \quad \Rightarrow z = \frac{1}{w}$$

$$(i.e.) \left| \frac{1}{w} - 2i \right| = 2$$

$$|1 - 2i\,w| = 2\,|w|$$

$$|1 - 2i\,(u + iv)| = 2 \;\; |u + iv|$$

$$|\,(1 + 2v) - i\,2u\,| = 2 \;\; |u + iv|$$

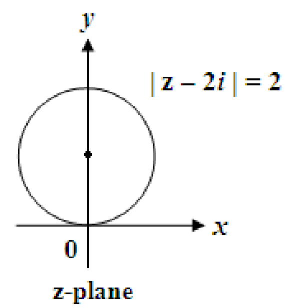
$$\sqrt{(1 + 2v)^2 + (-2u)^2} \; = 2\sqrt{u^2 + v^2}$$

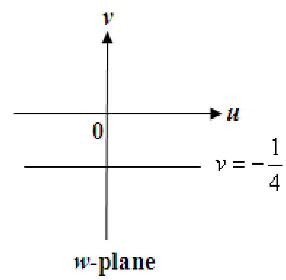
$$(1 + 2v)^2 + 4u^2 = 4(u^2 + v^2)$$

$$1 + 4v + 4v^2 + 4u^2 = 4u^2 + 4v^2$$

$$1 + 4v = 0$$

$$(i.e.) \quad v = -\frac{1}{4}$$





8. *Under the transformation* $w = \dfrac{1}{z}$, *find the image of* $(i)\, | z + i | < 1 \;\; (ii) | z - 1 | < 1$

Sol. $(i) \quad | z + i | < 1 \quad \Rightarrow \left| \dfrac{1}{w} + i \right| < 1$

$$w = \frac{1}{z} \quad \Rightarrow z = \frac{1}{w}$$

$$|1 + i\,w\,| < |\,w\,|$$

$$|1 + i\,(u + iv)| < |u + iv|$$

$$|\,(1 - v) + i \;\, u \;|< |u + iv|$$

$$\sqrt{(1 - v)^2 + (u)^2} \; < \; \sqrt{u^2 + v^2}$$

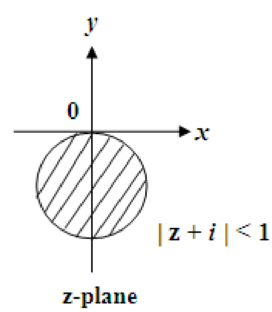
$$(1 - v)^2 + u^2 < u^2 + v^2$$

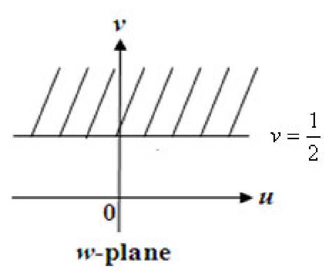
$$1 - 2v + v^2 + u^2 < u^2 + v^2$$

$$1 - 2v < \; 0$$

$$1 < 2v \;\; (or) \;\; 2v > 1$$

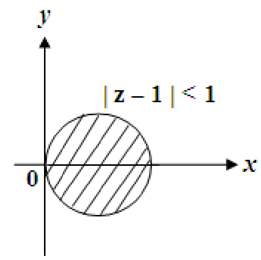
$$(i.e.) \quad v > \frac{1}{2}$$





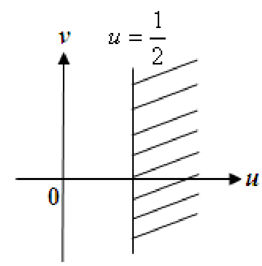
$Sol.\ (ii)\quad |z-1|<1 \quad \Rightarrow \left|\dfrac{1}{w}-1\right|<1$

$$\boxed{w=\frac{1}{z} \quad \Rightarrow z=\frac{1}{w}}$$



z-plane

$$|1-w|<|w|$$
$$|1-(u+iv)|<|u+iv|$$
$$|(1-u)-i\,v|<|u+iv|$$
$$\sqrt{(1-u)^2+(-v)^2}<\sqrt{u^2+v^2}$$
$$(1-u)^2+v^2<u^2+v^2$$
$$1-2u+u^2+v^2<u^2+v^2$$
$$1-2u<0$$
$$1<2u\ (or)\ 2u>1$$
$$(i.e.)\ u>\frac{1}{2}$$

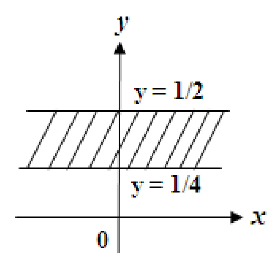


w-plane

9. $Find\ the\ image\ of\ the\ infinite\ strip\ \ (i)\ \dfrac{1}{4}<y<\dfrac{1}{2}\quad (ii)\ 0<y<\dfrac{1}{2e},\ e>0$

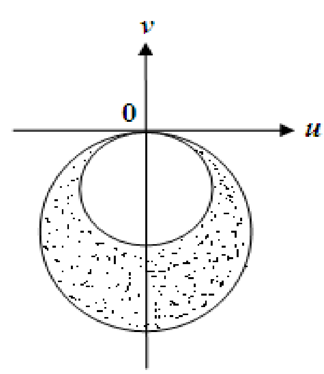
$under\ the\ transformation\ w=\dfrac{1}{z}.$

$Sol.$ $\quad w=\dfrac{1}{z} \Rightarrow z=\dfrac{1}{w}$



z-plane

$$x+i\,y=\frac{1}{u+i\,v}$$
$$=\frac{u-i\,v}{(u+i\,v)(u-i\,v)}$$
$$=\frac{u-i\,v}{u^2+v^2}$$
$$\Rightarrow x=\frac{u}{u^2+v^2},\quad y=\frac{-v}{u^2+v^2}$$



w-plane

$(i)\ Given\ \ strip\ \ is\ \ \dfrac{1}{4}<y<\dfrac{1}{2}$

$When\ \ y=\dfrac{1}{4},\ we\ \ get\ \ \dfrac{1}{4}=\dfrac{-v}{u^2+v^2}$

$$(i.e.)\ u^2+v^2=-4v$$
$$u^2+v^2+4v=0$$
$$u^2+(v+2)^2=4$$

$which\ is\ a\ circle\ with\ centre\ at\ (0,-2)\ and\ radius\ 2\ in\ the\ w\ plane.$

$$When \quad y=\frac{1}{2}, \ we \ get \quad \frac{1}{2}=\frac{-v}{u^2+v^2}$$

$$(i.e.) \ u^2+v^2=-2v$$

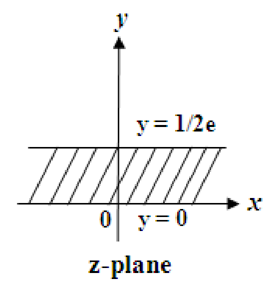
$$u^2+v^2+2v=0$$

$$u^2+(v+1)^2=1$$

which is a circle with centre at $(0,-1)$ *and radius* 1 *in the w plane.*

Hence the infinite strip $\frac{1}{4}<y<\frac{1}{2}$ is mapped into the region common to the circles $u^2+(v+1)^2=1$ and $u^2+(v+2)^2=4$ in the w-plane.

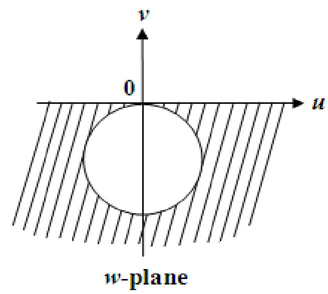
(ii) *Given strip is* $0<y<\frac{1}{2e}, \ e>0$



$$When \quad y>0, \ we \ have \ \frac{-v}{u^2+v^2}>0$$

$$-v>0$$

$$(i.e.) \ v<0$$



$$When \quad y=\frac{1}{2e}, \ we \ get \quad \frac{1}{2e}=\frac{-v}{u^2+v^2}$$

$$(i.e.) \ u^2+v^2=-2ev$$

$$u^2+v^2+2ev=0$$

$$u^2+(v+e)^2=e^2$$

which is a circle with centre at $(0,-e)$ *and radius* e *in the w plane.*

Hence the infinite strip $0<y<\frac{1}{2e}$ is mapped into the region outside the circle $u^2+(v+e)^2=e^2$ in the lower half plane.

10. *Find the image of the rectangular region* $a\le x\le b, \ c\le y\le d$ *under the transformation* $w=e^z$.

Sol. *Let* $\omega=e^z$

$$(i.e.) \ u+iv=e^{x+iy}=e^x e^{iy}=e^x(\cos y+i\sin y)$$

$$\Rightarrow u=e^x\cos y, \ v=e^x\sin y$$

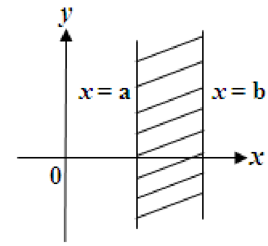
$$\Rightarrow \ u^2+v^2=e^{2x}(\cos^2 y+\sin^2 y)$$

$$u^2+v^2=e^{2x} \ ----(1)$$

$$and \ \frac{v}{u}=\frac{e^x\sin y}{e^x\cos y} \ \Rightarrow \ \frac{v}{u}=\tan y \ ----(2)$$

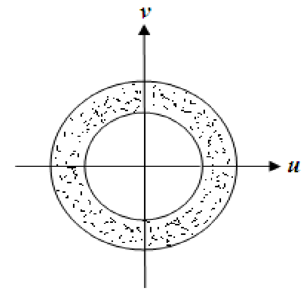
(i) *Given* $a \le x \le b$

When $x = a$, *equation* (1) *becomes* $u^2 + v^2 = e^{2a}$, *which is a circle with centre at* $(0,0)$ *and radius* e^a *in the* $w-$ *plane.*



z-plane

When $x = b$, *equation* (1) *becomes* $u^2 + v^2 = e^{2b}$, *which is a circle with centre at* $(0,0)$ *and radius* e^b *in the* $w-$ *plane.*

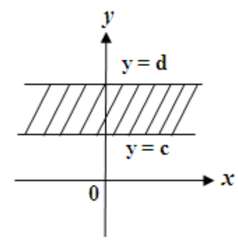


w-plane

Hence the rectangular region $a \le x \le b$ *is mapped into the region common to the circles* $u^2 + v^2 = e^{2b}$ *and* $u^2 + v^2 = e^{2a}$ *in the* $w-$ *plane.*

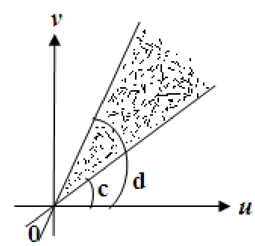
(ii) *Given* $c \le y \le d$

When $y = c$, *equation* (2) *becomes* $v = u \tan c$, *which is a straight line through the origin in the* $w-$ *plane.*



z-plane

When $y = d$, *equation* (2) *becomes* $v = u \tan d$, *which is a straight line through the origin in the* $w-$ *plane.*



w-plane

$\therefore$ *The image of the strip* $c \le y \le d$ *is mapped into the region between the lines* $v = u \tan c$ *and* $v = u \tan d$ *in the* $w-$ *plane.*

11. *Prove that the map* $w = \dfrac{1}{z}$ *maps the totality of circles and lines as circles or lines.*

Sol.

$$w = \frac{1}{z} \quad \Rightarrow \quad z = \frac{1}{w}$$

$$x + i\,y = \frac{1}{u + i\,v} = \frac{u - i\,v}{(u + i\,v)(u - i\,v)} = \frac{u - i\,v}{u^2 + v^2}$$

$$\Rightarrow x = \frac{u}{u^2 + v^2}, \quad y = \frac{-v}{u^2 + v^2}$$

$i)$ *The circle with centre at the origin in the $z-$plane are of the form*

$$x^2 + y^2 = k$$

$$(i.e.)\quad \frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} = k$$

$$(i.e.)\quad \frac{u^2+v^2}{(u^2+v^2)^2} = k$$

$$(i.e.)\quad \frac{1}{u^2+v^2} = k$$

$$\Rightarrow\ u^2+v^2 = \frac{1}{k}$$

$\therefore$ *The circle $x^2 + y^2 = k$ in the $z-$plane transforms into a circle*

$u^2+v^2 = \dfrac{1}{k}$ *in the $w-$plane.*

From this the unit circle with centre at the origin in the $z-$plane transforms into the unit circle in the $w-$plane with centre at the origin.

$ii)$ *We know that the general equation of circle in the $z-$plane is*

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad ------ \quad (1)$$

This transforms into

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \frac{2gu}{u^2+v^2} - \frac{2fv}{u^2+v^2} + c = 0$$

$$(i.e.)\quad \frac{u^2+v^2}{(u^2+v^2)^2} + \frac{2gu}{u^2+v^2} - \frac{2fv}{u^2+v^2} + c = 0$$

$$(i.e.)\quad \frac{1}{u^2+v^2} + \frac{2gu}{u^2+v^2} - \frac{2fv}{u^2+v^2} + c = 0$$

$$\Rightarrow\ c(u^2+v^2) + 2gu - 2fv + 1 = 0 \quad ------ \quad (2)$$

which is also the equation of the circle in $w-$plane.

Hence under the transformation $w = \dfrac{1}{z}$, a circle in $z-$plane transforms to another circle in $w-$plane.

When the circle passes through the origin, we have $c = 0$ in equation (1).

When $c = 0$, equation (2) *gives a straight line $2gu - 2fv + 1 = 0$.*

$iii)$ *A line in the* $z-$ *plane of the form* $l\,x+m\,y+n=0$ *transforms into*

$$\frac{l\,u}{u^2+v^2}-\frac{m\,v}{u^2+v^2}+n=0$$

$$\Rightarrow \; n(u^2+v^2)+l\,u-m\,v=0$$

which is a circle passing through the origin in $w-$ *plane.*

If $n=0$, *the straight line in the* $z-$ *plane* $l\,x+m\,y=0$ *passes through the origin. Its image in the* $w-$ *plane is* $l\,u-m\,v=0$, *which is also a straight line passing through the origin .*

Home Work

1. Find the image of the circle $|z| = 2$ under the transformation $w = 3z$.
2. Find the image of the line $x = k$ under the transformation $w=\frac{1}{z}$.
3. Find the image of a square whose vertices are $z = 1 + i,\ 3 + i,\ 1 + 3i,\ 3 + 3i$ under the transformation $w=\frac{1}{z}$.
4. Find the image of $|z - 1| > ½$ under the transformation $w = i\,z$.
5. Under the transformation $w=\frac{1}{z}$, find the image of
   (i) $|z + 1| = 1$ (ii) $|z - 1| = 1$ (iii) $|z - 3i| = 3$ (iv) $|z - 2i| < 2$.

Bilinear Transformation

The transformation T defined by

$$\omega = T(z) = \frac{az+b}{cz+d} \quad ------ (1)$$

where a, b, c, d are complex constants and ad – bc ≠ 0 is called a bilinear transformation (or) linear fractional transformation (or) Mobius transformation. The constant ad – bc is called the determinant of the transformation. The transformation (1) is said to be normalized if ad – bc = 1.

The inverse transformation T^{-1} is defined by

$$z = T^{-1}(\omega) = \frac{-d\omega + b}{c\omega - a}$$

The determinant of this transformation is (– d)(–a) – bc = ad – bc which is the same as that of (1).

Fixed point (or) Invariant point

The points which coincide with their transformation are called fixed point of the transformation. In otherwords fixed points of the transformation $w = f(z)$ are obtained by the equation $z = f(z)$.

<u>Note:</u> The bilinear transformation which transforms z_1, z_2, z_3 into w_1, w_2, w_3 is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Problems

1. *Find the fixed point of the transformation* $w = \dfrac{3z-4}{z-1}$

Sol. *Given* $w = \dfrac{3z-4}{z-1}$

put $w = z$, *then*

$$z = \frac{3z-4}{z-1}$$

$$z^2 - z = 3z - 4$$

$$z^2 - 4z + 4 = 0$$

$$(z-2)^2 = 0$$

$$z = 2$$

$\Rightarrow z = 2$ *is the only fixed point.*

2. Find the bilinear transformation that maps 2, i and –2 of the z-plane onto 1, i and –1 of the w-plane.

Sol. $Given\ z_1 = 2,\ z_2 = i,\ z_3 = -2\ ;\ w_1 = 1,\ w_2 = -i,\ w_3 = -1$

The bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$(i.e.)\quad \frac{(w-1)(i+1)}{(w+1)(i-1)} = \frac{(z-2)(i+2)}{(z+2)(i-2)}$$

$$\frac{(w-1)}{(w+1)}(-i) = \frac{(z-2)}{(z+2)}\left(\frac{3+4i}{-5}\right)$$

$$\frac{(w-1)}{(w+1)} = \frac{(z-2)}{(z+2)}\left(\frac{3+4i}{5i}\right)$$

$$\frac{(w-1)}{(w+1)} = \frac{(z-2)}{(z+2)}\left(\frac{4-3i}{5}\right)$$

$$\frac{(w-1)+(w+1)}{(w-1)-(w+1)} = \frac{(z-2)(4-3i)+5(z+2)}{(z-2)(4-3i)-5(z+2)}$$

$$\frac{2w}{-2} = \frac{4z-3zi-8+6i+5z+10}{4z-3zi-8+6i-5z-10}$$

$$-w = \frac{9z-3zi+6i+2}{-z-3zi+6i-18}$$

$$-w = \frac{3z(3-i)+2i(3-i)}{-zi(3-i)-6(3-i)}$$

$$-w = \frac{3z+2i}{-zi-6}$$

$$(i.e.)\quad w = \frac{3z+2i}{zi+6}$$

$$\frac{i+1}{i-1} = \frac{(i+1)(i+1)}{(i-1)(i+1)} = \frac{i^2+2i+1}{i^2-1} = \frac{-1+2i+1}{-1-1} = \frac{2i}{-2} = -i$$

$$\frac{i+2}{i-2} = \frac{(i+2)(i+2)}{(i-2)(i+2)} = \frac{i^2+4i+4}{i^2-4} = \frac{-1+4i+4}{-1-4} = \frac{3+4i}{-5}$$

$$\frac{1}{i} = -i$$

$$\therefore \frac{3+4i}{5i} = -i\left(\frac{3+4i}{5}\right) = \frac{4-3i}{5}$$

Componendo & dividendo

If $\frac{a}{b} = \frac{c}{d}$ *then we have*

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}$$

3. Find the bilinear transformation that maps ∞, i and 0 of the z-plane onto 0, i and ∞ of the w-plane.

Sol. $Given\ z_1 = \infty,\ z_2 = i,\ z_3 = 0\ ;\ w_1 = 0,\ w_2 = i,\ w_3 = \infty$

The bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$(i.e.)\quad \frac{(w-w_1)\left(\dfrac{w_2}{w_3}-1\right)}{(w_2-w_1)\left(\dfrac{w}{w_3}-1\right)}=\frac{\left(\dfrac{z}{z_1}-1\right)(z_2-z_3)}{\left(\dfrac{z_2}{z_1}-1\right)(z-z_3)}$$

$$\frac{(w-0)\left(\dfrac{i}{\infty}-1\right)}{(i-0)\left(\dfrac{w}{\infty}-1\right)}=\frac{\left(\dfrac{z}{\infty}-1\right)(i-0)}{\left(\dfrac{i}{\infty}-1\right)(z-0)}$$

$$\frac{(w)\,(-1)}{(i)\,(-1)}=\frac{(-1)(i)}{(-1)\,(z)}$$

$$w=\frac{i^2}{z}\quad\Rightarrow\quad w=-\frac{1}{z}$$

4. Find the Mobius transformation which sends the points z = 0, 1, ∞ into the points w = –5, –1, 3 respectively. What are the invariant points in this transformation?

Sol. $\quad Given\ \ z_1=0,\ z_2=1,\ z_3=\infty\ \ ;\ \ w_1=-5,\ w_2=-1,\ w_3=3$

The bilinear (or) Mobius transformation is

$$\frac{(w-w_1)\,(w_2-w_3)}{(w-w_3)\,(w_2-w_1)}=\frac{(z-z_1)\,(z_2-z_3)}{(z-z_3)\,(z_2-z_1)}$$

$$(i.e.)\quad \frac{(w-w_1)\,(w_2-w_3)}{(w-w_3)\,(w_2-w_1)}=\frac{\left(\dfrac{z_2}{z_3}-1\right)(z-z_1)}{\left(\dfrac{z}{z_3}-1\right)(z_2-z_1)}$$

$$\frac{(w+5)\,(-1-3)}{(w-3)\,(-1+5)}=\frac{\left(\dfrac{1}{\infty}-1\right)(z-0)}{\left(\dfrac{z}{\infty}-1\right)(1-0)}$$

$$\frac{(w+5)\,(-4)}{(w-3)\,(4)}=\frac{(-1)(z)}{(-1)\,(1)}$$

$$\frac{-\,(w+5)}{(w-3)}=z$$

$$-w-5=wz-3z$$

$$wz+w=3z-5\ \Rightarrow\ w=\frac{3z-5}{z+1}$$

To find invariant point, put $w = z$

$$then \quad z = \frac{3z-5}{z+1}$$

$$z^2 + z = 3z - 5$$

$$z^2 - 2z + 5 = 0$$

$$z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4.1.5}}{2.1} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$\therefore$ *The invariant points are* $1 + 2i$ *and* $1 - 2i$.

5. Find the bilinear transformation which maps the points z = 1, i, –1 into the points w = 0, 1, ∞. Show that this transformation maps the interior of the unit circle of the z-plane onto the upper half of the w-plane.

Sol. *Given* $z_1 = 1,\ z_2 = i,\ z_3 = -1\ ;\ w_1 = 0,\ w_2 = 1,\ w_3 = \infty$

The bilinear (or) Mobius transformation is

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

$$(i.e.) \quad \frac{(w - w_1)\left(\dfrac{w_2}{w_3} - 1\right)}{(w_2 - w_1)\left(\dfrac{w}{w_3} - 1\right)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

$$\frac{(w - 0)\left(\dfrac{1}{\infty} - 1\right)}{(1 - 0)\left(\dfrac{w}{\infty} - 1\right)} = \frac{(z - 1)(i + 1)}{(z + 1)(i - 1)}$$

$$\frac{i+1}{i-1} = \frac{(i+1)(i+1)}{(i-1)(i+1)} = \frac{i^2 + 2i + 1}{i^2 - 1} = \frac{-1 + 2i + 1}{-1 - 1} = \frac{2i}{-2} = -i$$

$$\frac{(w)(-1)}{(-1)} = \frac{(z - 1)}{(z + 1)}(-i)$$

$$(i.e.) \quad w = i\left(\frac{1 - z}{1 + z}\right)$$

$$w = i\left(\frac{1 - z}{1 + z}\right)$$
$$w + wz = i - zi$$
$$wz + zi = i - w$$
$$z(w + i) = i - w$$
$$z = \frac{i - w}{i + w}$$

When $|z| < 1$ (*interior of the unit circle*), *we have*

$$\left|\frac{i - w}{i + w}\right| < 1$$

$$|i - w| < |i + w|$$

$$|i-(u+iv)| < |i+(u+iv)|$$
$$|-u+i(1-v)| < |u+i(1+v)|$$
$$\sqrt{(-u)^2+(1-v)^2} < \sqrt{(u)^2+(1+v)^2}$$
$$u^2+(1-v)^2 < u^2+(1+v)^2$$
$$u^2+1-2v+v^2 < u^2+1+2v+v^2$$
$$-4v<0$$
$$-v<0 \ (or) \ v>0$$
which is the upper half of the w− plane.

6. Find the bilinear transformation that maps the points 1 + *i*, –*i*, 2 – *i* of the z-plane into the points 0, 1, *i* of the *w*-plane.

Sol. *Given* $z_1=1+i,\ z_2=-i,\ z_3=2-i\ ;\ w_1=0,\ w_2=1,\ w_3=i$

The bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$
$$(i.e.)\quad \frac{(w-0)(1-i)}{(w-i)(1-0)}=\frac{[z-(1+i)][-i-(2-i)]}{[z-(2-i)][-i-(1+i)]}$$
$$\left(\frac{w}{w-i}\right)(1-i)=\frac{(z-1-i)(-2)}{(z-2+i)(-1-2i)}$$
$$\frac{w}{w-i}=\frac{(z-1-i)(-2)}{(z-2+i)(-1-2i)(1-i)}$$
$$\frac{w}{w-i}=\frac{(z-1-i)(-2)}{(z-2+i)(-3-i)}$$
$$\frac{w}{w-i}=\frac{(z-1-i)}{(z-2+i)}\left(\frac{3-i}{5}\right)$$
$$\frac{w+(w-i)}{w-(w-i)}=\frac{(z-1-i)(3-i)+5(z-2+i)}{(z-1-i)(3-i)-5(z-2+i)}$$
$$\frac{2w-i}{i}=\frac{3z-zi-3+i-3i-1+5z-10+5i}{3z-zi-3+i-3i-1-5z+10-5i}$$
$$\frac{2w-i}{i}=\frac{8z-zi+3i-14}{-2z-zi-7i+6}$$

$$\boxed{\begin{aligned}&(-1-2i)(1-i)\\&=-1+i-2i+2i^2\\&=-1-i-2\\&=-3-i\end{aligned}}$$

$$\boxed{\begin{aligned}\frac{-2}{-3-i}&=\frac{-2(-3+i)}{(-3-i)(-3+i)}\\&=\frac{6-2i}{9-i^2}\\&=\frac{6-2i}{10}=\frac{3-i}{5}\end{aligned}}$$

$$\frac{2w-i}{i}=\frac{8z-zi+3i-14}{-2z-zi-7i+6}$$

$$2w-i=\frac{8zi+z-3-14i}{-2z-zi-7i+6}$$

$$2w=\frac{8zi+z-3-14i}{-2z-zi-7i+6}+i$$

$$=\frac{8zi+z-3-14i-2zi+z+7+6i}{-2z-zi-7i+6}$$

$$=\frac{6zi+2z+4-8i}{-2z-zi-7i+6}$$

$$2w=\frac{2z(3i+1)+(4-8i)}{-z(2+i)+(6-7i)}$$

$$(i.e.)\ w=\frac{z(3i+1)+(2-4i)}{-z(2+i)+(6-7i)}$$

7. *Show that under the mapping* $w=\dfrac{i-z}{i+z}$, *the image of the circle* $x^2+y^2<1$ *is the entire half of the* $w-$ *plane to the right of the imaginary axis.*

Sol. $w=\dfrac{i-z}{i+z} \Rightarrow z=\dfrac{i(1-w)}{1+w}$

$$w=\frac{i-z}{i+z}$$
$$wi+wz=i-z$$
$$wz+z=i-wi$$
$$z(w+1)=i(1-w)$$
$$z=\frac{i\,(1-w)}{1+w}$$

Given $x^2+y^2<1$

$(i.e.)\ |z|<1$

$$\Rightarrow \left|\frac{i(1-w)}{1+w}\right|<1$$

$$|1-w|<|1+w| \quad [\text{since } |i|=1]$$

$$|1-(u+iv)|<|1+(u+iv)|$$

$$|(1-u)-iv|<|(1+u)+iv|$$

$$\sqrt{(1-u)^2+(-v)^2}<\sqrt{(1+u)^2+(v)^2}$$

$$(1-u)^2+(-v)^2<(1+u)^2+(v)^2$$

$$1-2u+u^2+v^2<1+2u+u^2+v^2$$

$-4u<0 \Rightarrow u>0$, *which is entire half of* $w-$ *plane to the right of the imaginary axis*.

8. *Show that in a transformation* $w = \dfrac{z-i}{z+i}$, *the axis of reals in the* $z-$ *plane transforms into the circle* $|w|=1$. *Find the portion of the* $z-$ *plane corresponding to the interior of the circle* $|w|=1$.

Sol. We have $z = x + iy$

When $y = 0$ (axis of reals in $z-$ plane), we have $z = x$

$$w = \frac{z-i}{z+i} = \frac{x-i}{x+i}$$

$$|w| = \left|\frac{x-i}{x+i}\right| = \frac{\sqrt{x^2+(-1)^2}}{\sqrt{x^2+(1)^2}} = \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} = 1$$

Hence the axis of reals in $z-$ plane is mapped into the circle $|w|=1$.

Now, $|w| < 1$ (interior of the circle $|w|=1$)

$$\left|\frac{z-i}{z+i}\right| < 1$$

$$|z-i| < |z+i|$$

$$|(x+iy)-i| < |(x+iy)+i|$$

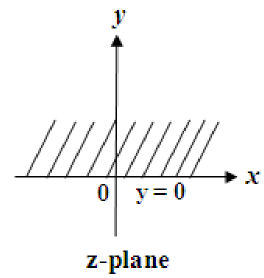
$$|x+i(y-1)| < |x+i(y+1)|$$

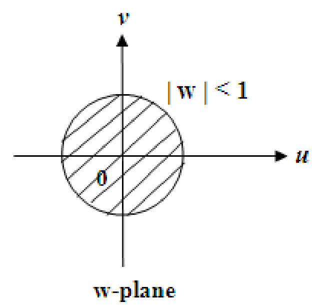
$$\sqrt{x^2+(y-1)^2} < \sqrt{x^2+(y+1)^2}$$

$$x^2+(y-1)^2 < x^2+(y+1)^2$$

$$x^2+y^2-2y+1 < x^2+y^2+2y+1$$

$$-4y < 0 \Rightarrow y > 0$$





Hence the upper half of $z-$ plane is mapped into $|w| < 1$.

9. *Show that* $w = \dfrac{z-1}{z+1}$, *maps the imaginary axis in the* $z-$ *plane onto the circle* $|w|=1$. *What portion of the* $z-$ *plane corresponds to the interior of the circle* $|w|=1$.

Sol. We have $z = x + iy$

When $x = 0$ (imaginary axis in $z-$ plane), we have $z = iy$

$$w = \frac{z-1}{z+1} = \frac{iy-1}{iy+1}$$

$$|w| = \left|\frac{iy-1}{iy+1}\right| = \frac{\sqrt{(-1)^2+(y)^2}}{\sqrt{1^2+(y)^2}} = \frac{\sqrt{1+y^2}}{\sqrt{1+y^2}} = 1$$

Hence the imaginary axis in $z-$plane is mapped into the circle $|w|=1$.

Now, $|w| < 1$ (interior of the circle $|w|=1$)

$$\left|\frac{z-1}{z+1}\right| < 1$$

$$|z-1| < |z+1|$$

$$|(x+iy)-1| < |(x+iy)+1|$$

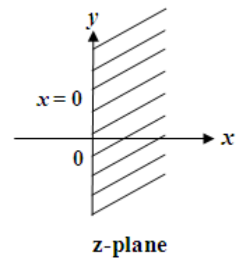
$$|(x-1)+i\,y| < |(x+1)+i\,y|$$

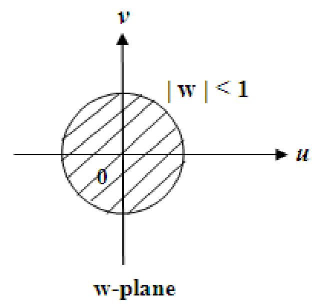
$$\sqrt{(x-1)^2+y^2} < \sqrt{(x+1)^2+y^2}$$

$$(x-1)^2+y^2 < (x+1)^2+y^2$$

$$x^2-2x+1+y^2 < x^2+2x+1+y^2$$

$$-4x < 0 \Rightarrow x > 0$$





Hence the right half of the $z-$plane is mapped into $|w| < 1$.

10. *Show that the transformation* $w = \dfrac{z-i}{1-i\,z}$ *maps*

i) the interior of the circle $|z|=1$ onto the lower half of the $w-$plane.

ii) the upper half of the $z-$plane onto the interior of the circle $|w|=1$.

Sol. $w = \dfrac{z-i}{1-i\,z} \Rightarrow z = \dfrac{w+i}{1+wi}$

$$w = \frac{z-i}{1-i\,z}$$
$$w - wi\,z = z-i$$
$$wi\,z + z = w+i$$
$$z(wi+1) = w+i$$
$$z = \frac{w+i}{1+wi}$$

(*i*) *When* $|z| < 1$ (interior of the circle $|z|=1$), *we have*

$$\left|\frac{w+i}{1+wi}\right| < 1$$

$$|w+i| < |1+wi|$$

$$|(u+iv)+i| < |1+i(u+iv)|$$

$$|u+i(v+1)| < |(1-v)+iu|$$

$$\sqrt{u^2+(v+1)^2} < \sqrt{(1-v)^2+u^2}$$

$$u^2+(v+1)^2 < (1-v)^2+u^2$$

$$u^2+v^2+2v+1 < 1-2v+v^2+u^2$$

$4v < 0 \Rightarrow v < 0$, *which is lower half of the $w-$plane.*

$(ii)\quad We \;\; have \;\; z = \dfrac{w+i}{1+wi}$

$$x + i\,y = \frac{(u+i\,v)+i}{1+i\,(u+i\,v)}$$

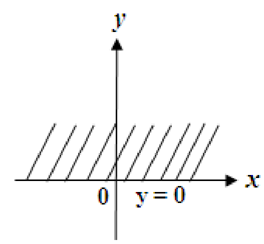
$$= \frac{u+i\,(v+1)}{(1-v)+i\,u}$$

$$= \frac{[u+i\,(v+1)]}{[(1-v)+i\,u]}\cdot\frac{[(1-v)-i\,u]}{[(1-v)-i\,u]}$$

$$= \frac{[u(1-v)+u(v+1)]+i\,[(v+1)(1-v)-u^2]}{(1-v)^2-i^2u^2}$$

$$= \frac{2u+i\,(1-v^2-u^2)}{(1-v)^2+u^2}$$

$$\Rightarrow \;\; x = \frac{2u}{(1-v)^2+u^2}, \qquad y = \frac{1-v^2-u^2}{(1-v)^2+u^2}$$



z-plane

$Now, \;\; given \;\; y > 0 \; (upper \;\; half \;\; of \;\; the \;\; z-plane)$

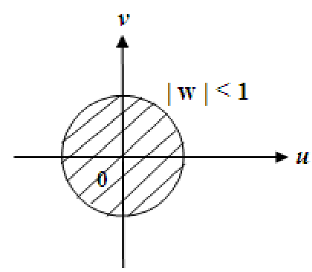
$$(i.e.)\;\; \frac{1-v^2-u^2}{(1-v)^2+u^2} > 0$$

$$1-v^2-u^2 > 0$$

$$1 > u^2+v^2$$

$$u^2+v^2 < 1$$

$$(i.e.)\;\; |w| < 1$$



w-plane

$which \;\; is \;\; the \;\; interior \;\; of \;\; the \;\; circle \; |w| = 1.$

Home Work

Find the bilinear transformation that maps the points

1. $z = 0, -i, -1$ into $w = i, 1, 0$
2. $z = -2, 0, 2$ into $w = 0, i, -i$
3. $z = -i, 0, i$ into $w = -1, i, 1$
4. $z = 1, i, -1$ into $w = 2, i, -2$
5. $z = 0, 1, i$ into $w = 1+i, -i, 2-i$
6. $z = 1, i, -1$ into $w = i, 0, -i$. *Hence find the image of* $|z| < 1$.

Answers

Page No. 26

1. $f(z) = z^2 e^{-z} + c$ 2. $f(z) = i z e^{-z} + c$ 3. $f(z) = z^4 + c$

4. $f(z) = i z e^{-z} + c$ 5. $f(z) = e^z + c$ 6. $f(z) = \left(\frac{5i-1}{13}\right) e^z + c$

Page No. 36

1. $u^2 + v^2 = 36$ 2. $\left(u - \frac{1}{2k}\right)^2 + v^2 = \frac{1}{4k^2}$

3. $x = 1 \Rightarrow \left(u - \frac{1}{2}\right)^2 + v^2 = \frac{1}{4}$; $y = 1 \Rightarrow u^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{4}$

$x = 3 \Rightarrow \left(u - \frac{1}{6}\right)^2 + v^2 = \frac{1}{36}$; $y = 3 \Rightarrow u^2 + \left(v + \frac{1}{6}\right)^2 = \frac{1}{36}$

4. $|w - i| > \frac{1}{2}$

5. $(i)\ u = -\frac{1}{2}$ $(ii)\ u = \frac{1}{2}$ $(iii)\ v = \frac{1}{6}$ $(iv)\ v < -\frac{1}{4}$

Page No. 45

1. $w = \frac{i(1+z)}{1-z}$ 2. $w = \frac{-i(z+2)}{3z-2}$ 3. $w = \frac{i(1-z)}{1+z}$ 4. $w = \frac{-6z+2i}{iz-3}$

5. $w = \frac{z(5-3i)+(2i-2)}{z(1+i)+2i}$

6. $w = \frac{-iz-1}{iz-1}$, $|z| < 1$ *maps* into *the half plane* $u > 0$ *in the* $w-$ *plane.*

UNIT – V Complex Integration

Cauchy's Integral Theorem [Cauchy's fundamental theorem (or) Cauchy's theorem]

If $f(z)$ *is analytic within and on a simple closed curve* C, *then* $\int_C f(z)\,dz = 0$.

Cauchy's Integral Formula

If $f(z)$ *is analytic within and on a simple closed curve* C *and if* z_0 *is any point inside* C, *then* $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$

the integration round C *being taken in the positive sense.*

Cauchy's Integral Formula for the derivatives of an analytic function

If $f(z)$ *is analytic within and on a simple closed curve* C *and if* z_0 *is any point inside* C, *then* $f'(z_0) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz$

Similarly, $f''(z_0) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^3}\,dz$

$$f'''(z_0) = \frac{3!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^4}\,dz$$

In general, $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$

Problem

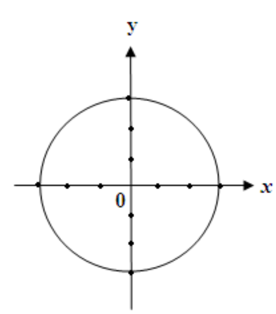
1. *Evaluate* $\int_C (z^2-2z-3)dz$ *where* C *is the circle* $|z|=3$.

Sol. Since the function $f(z) = z^2-2z-3$ being a polynomial, is analytic inside and on the circle C : $|z|=3$

Hence by Cauchy's theorem,

$$\int_C f(z)dz = 0$$

(*i.e.*) $\int_C (z^2-2z-3)dz = 0$



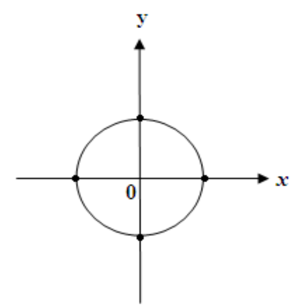
2. *From the integral* $\int_C \frac{dz}{z+2}$ *where C is the circle* $|z|=1$, *show that*

$$\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = 0$$

Sol. The function $f(z) = \frac{1}{z+2}$ is not analytic at $z=-2$

But $z=-2$ lies outside the circle C: | z | = 1

$\therefore f(z) = \frac{1}{z+2}$ is analytic inside and on the circle C.



Hence by Cauchy's theorem,

$$\int_C f(z)dz = 0 \quad (i.e.) \int_C \frac{dz}{z+2} dz = 0 \quad ----(1)$$

Any point on C: | z | = 1 is $z = e^{i\theta}$

$$dz = ie^{i\theta} d\theta, \quad (0 < \theta < 2\pi)$$

Equation (1) becomes,

$$\int_0^{2\pi} \frac{ie^{i\theta} d\theta}{e^{i\theta}+2} = 0$$

$$\int_0^{2\pi} \frac{\cos\theta + i\sin\theta}{\cos\theta + i\sin\theta + 2} d\theta = \frac{0}{i}$$

$$\int_0^{2\pi} \frac{\cos\theta + i\sin\theta}{(\cos\theta+2) + i\sin\theta} d\theta = 0$$

$$\int_0^{2\pi} \frac{[\cos\theta + i\sin\theta][(\cos\theta+2) - i\sin\theta]}{[(\cos\theta+2) + i\sin\theta][(\cos\theta+2) - i\sin\theta]} d\theta = 0$$

$$\int_0^{2\pi} \frac{[\cos\theta(\cos\theta+2) + \sin^2\theta] + i[\sin\theta(\cos\theta+2) - \sin\theta\cos\theta]}{[(\cos\theta+2)^2 + \sin^2\theta]} d\theta = 0$$

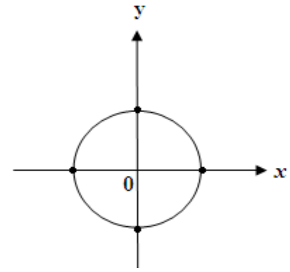
$$\int_0^{2\pi} \frac{(1+2\cos\theta) + i2\sin\theta}{4+4\cos\theta+\cos^2\theta+\sin^2\theta} d\theta = 0$$

Equating R.P we get

$$\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = 0$$

$$2\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = 0 \quad (i.e.) \int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = \frac{0}{2} = 0$$

3. $Evaluate \int\limits_{|z|=1} \frac{e^z}{z} dz$



$$Sol. \int\limits_{|z|=1} \frac{e^z}{z} dz = \int\limits_{|z|=1} \frac{e^z}{z-0} dz$$

$$= \int\limits_C \frac{f(z)}{z-z_0} dz \qquad (where\ f(z) = e^z\ is\ analytic\ inside\ and\ on\ C\ and\ z_0 = 0)$$

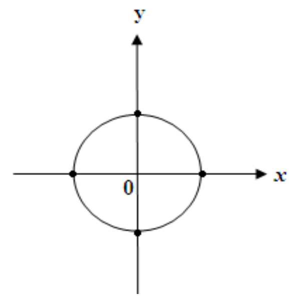
$$= 2\pi i\, f(z_0)$$

$$= 2\pi i\, f(0)$$

$$= 2\pi i\, (e^0)$$

$$= 2\pi i$$

4. $Evaluate \int\limits_{|z|=1} \frac{e^z}{z-a} dz$



Sol. $f(z) = e^z, \quad z_0 = a\ , \quad C: |z|=1$

If a < 1, z_0 lies inside C

$$\int\limits_{|z|=1} \frac{e^z}{z-a} dz = \int\limits_C \frac{f(z)}{z-z_0} dz \qquad (where\ f(z) = e^z\ is\ analytic\ inside\ and\ on\ C\ and\ z_0 = a)$$

$$= 2\pi i\, f(z_0)$$

$$= 2\pi i\, f(a)$$

$$= 2\pi i\ e^a \quad (a < 1)$$

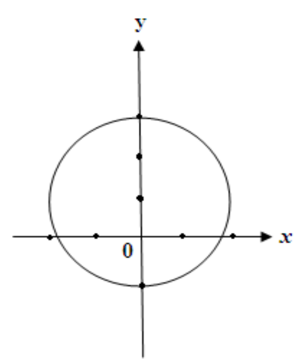
If a > 1, z_0 lies outside C and $f(z) = \frac{e^z}{z-a}$ is analytic inside and on C

Hence by Cauchy's theorem,

$$\int\limits_C f(z) dz = 0$$

$$(i.e.) \int\limits_C \frac{e^z}{z-a} dz = 0 \quad (a > 1)$$

5. $Evaluate \int\limits_C \frac{dz}{z^2+4}\ where\ C\ is\ the\ circle\ |z-i| = 2.$



$$Sol. \int\limits_C \frac{dz}{z^2+4} = \int\limits_C \frac{dz}{(z+2i)(z-2i)}$$

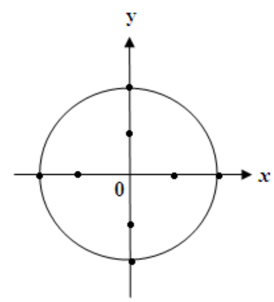
$$= \int_C \frac{\cancel{1}\!/\!\cancel{z+2i}}{z-2i}\, dz$$

$$= \int_C \frac{f(z)}{z-z_0}\, dz \quad \text{where } f(z) = \frac{1}{z+2i} \text{ and } z_0 = 2i$$

$$= 2\pi i\, f(z_0)$$

$$= 2\pi i\, f(2i) = 2\pi i\left(\frac{1}{2i+2i}\right) = 2\pi i\left(\frac{1}{4i}\right) = \frac{\pi}{2}$$

6. *Evaluate* $\int_C \frac{e^{iz}}{z^3} dz$, *where* C *is the circle* $|z| = 2$.



Sol. $\int_C \frac{e^{iz}}{z^3} dz = \int_C \frac{e^{iz}}{(z-0)^3} dz$

$z_0 = 0$ *lies inside the circle* $|z| = 2$

$$f(z) = e^{iz}, \quad z_0 = 0$$

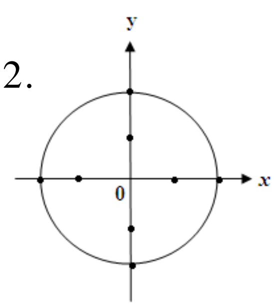
By Cauchy's integral formula, we have

$$\int_C \frac{e^{iz}}{z^3} dz = \frac{2\pi i}{2!} f''(0)$$

$$= \pi i(-e^0) = \pi i(-1) = -\pi i$$

$$\boxed{\begin{aligned} f(z) &= e^{iz} \\ f'(z) &= i e^{iz} \\ f''(z) &= i^2 e^{iz} = -e^{iz} \end{aligned}}$$

7. *Show that* $\int_C \frac{z\, dz}{(9-z^2)(z+i)} = \frac{\pi}{5}$ *where* C *is the circle* $|z| = 2$.



Sol. $\int_C \frac{z\, dz}{(9-z^2)(z+i)} = \int_C \frac{\cancel{z}\!/\!\cancel{9-z^2}}{z+i}\, dz$

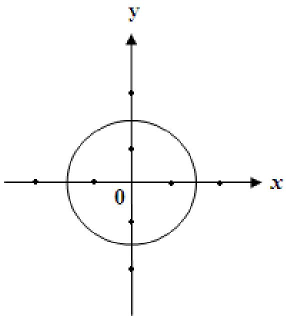
$$= \int_C \frac{f(z)}{z-z_0}\, dz \quad \text{where } f(z) = \frac{z}{9-z^2} \text{ and } z_0 = -i$$

$$= 2\pi i\, f(z_0)$$

$$= 2\pi i\, f(-i) = 2\pi i\left(\frac{-i}{9-(-i)^2}\right) = 2\pi i\left(\frac{-i}{10}\right) = \frac{\pi}{5}$$

8. *Evaluate* $\int_C \frac{z e^z}{(z-1)^2(z-2)}\, dz$ *where* C *is the circle* $|z| = 1.5$

$$Sol.\quad \int_C \frac{z\,e^z}{(z-1)^2(z-2)}\,dz = \int_C \frac{\frac{z\,e^z}{z-2}}{(z-1)^2}\,dz$$



$$= \int_C \frac{f(z)}{(z-z_0)^2}\,dz \quad where\ f(z) = \frac{z\,e^z}{z-2}\ \ and\ \ z_0 = 1$$

$$= \frac{2\pi i}{1!} f'(z_0)$$

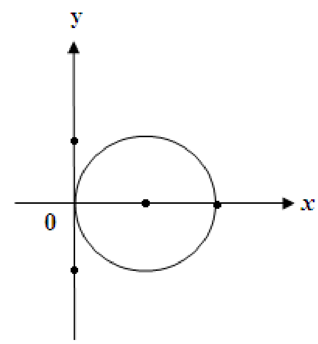
$$= 2\pi i\, f'(1)$$

$$= 2\pi i\,(-3e)$$

$$= -6\pi i\, e$$

$$\boxed{\begin{aligned} f'(z) &= \frac{(z-2)[\,z\,e^z + e^z.1] - z\,e^z\,(1)}{(z-2)^2} \\ f'(1) &= \frac{(-1)[\,e+e] - e}{(-1)^2} \\ &= -3e \end{aligned}}$$

9. *Evaluate* $\int \frac{2z^2+z}{z^2-1}\,dz$ *where C is a circle* $|z-1| = 1$.



$$Sol.\quad \int \frac{2z^2+z}{z^2-1}\,dz = \int \frac{2z^2+z}{(z-1)(z+1)}\,dz$$

$$= \int \frac{\frac{2z^2+z}{z+1}}{z-1}\,dz$$

$$= \int_C \frac{f(z)}{z-z_0}\,dz \quad where\ f(z) = \frac{2z^2+z}{z+1}\ \ and\ \ z_0 = 1$$

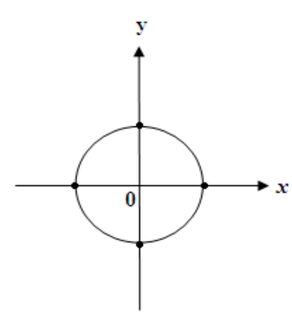
$$= 2\pi i\, f(z_0)$$

$$= 2\pi i\, f(1) = 2\pi i\left(\frac{2+1}{1+1}\right) = 2\pi i\left(\frac{3}{2}\right) = 3\pi i$$

10. *Evaluate* $\int_C \frac{3z-1}{z^3-z}\,dz$ *where C is* $(i)\ |z| = \frac{1}{2}\quad (ii)\ |z| = 2.$

$$Sol.\quad \int_C \frac{3z-1}{z^3-z}\,dz = \int_C \frac{3z-1}{z(z^2-1)}\,dz = \int_C \frac{3z-1}{(z-0)(z-1)(z+1)}\,dz$$

i) On C: $|\,z\,| = ½$, $z_0 = 0$ lies inside C.



$$\int_C \frac{3z-1}{z^3-z}\,dz = \int_C \frac{\frac{3z-1}{z^2-1}}{z-0}\,dz$$

$$= \int_C \frac{f(z)}{z-z_0}\,dz \quad where\ f(z) = \frac{3z-1}{z^2-1}\ \ and\ \ z_0 = 0$$

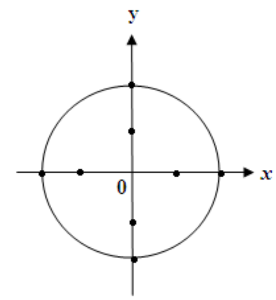
$$= 2\pi i\, f(z_0)$$

$$= 2\pi i\, f(0) = 2\pi i \left(\frac{0-1}{0-1}\right) = 2\pi i\,(1) = 2\pi i$$

ii) On C: | z | = 2, $z_0 = 0, 1, -1$ lies inside C.

$$\frac{3z-1}{z(z-1)(z+1)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z+1}$$

$$3z-1 = A(z-1)(z+1) + B\,z(z+1) + C\,z(z-1)$$



$$put \quad z=0, \quad -1 = A(-1)(1)+0+0$$

$$A = 1$$

$$put \quad z=1, \quad 2 = 0 + B(1)(2) + 0$$

$$B = 1$$

$$put \quad z=-1, \quad -4 = 0+0+C(-1)(-2)$$

$$C = -2$$

$$\frac{3z-1}{z(z-1)(z+1)} = \frac{1}{z} + \frac{1}{z-1} - \frac{2}{z+1}$$

$$\int_C \frac{3z-1}{z^3-z}dz = \int_C \frac{dz}{z-0} + \int_C \frac{dz}{z-1} - 2\int_C \frac{dz}{z+1}$$

where $f(z)=1$ *in all* 3 *integrands*
and $z_0 = 0,\ z_0 = 1,\ z_0 = -1$

$$= 2\pi i\, f(0) + 2\pi i\ f(1) - 2[2\pi i\, f(-1)]$$

$$= 2\pi i\ (1) + 2\pi i\,(1) - 4\pi i\,(1)$$

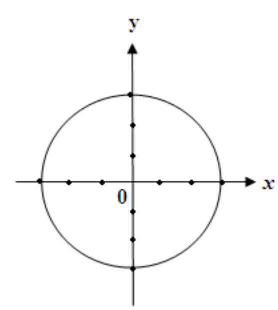
$$= 4\pi i - 4\pi i$$

$$= 0$$

$$\boxed{\begin{array}{l} f(z)=1 \\ f(0)=1 \\ f(1)=1 \\ f(-1)=1 \end{array}}$$

11. *Evaluate* $\displaystyle\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)}\,dz$ *around* $|z| = 3$.

Sol. On C: | z | = 3, $z_0 = 1, 2$ lies inside C.



$$\frac{1}{(z-1)^2(z-2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z-2}$$

$$1 = A(z-1)(z-2) + B(z-2) + C(z-1)^2$$

$$put \quad z=1, \quad 1 = 0 + B(-1) + 0$$

$$B = -1$$

$$put \quad z=2, \quad 1 = 0+0+C(1)^2$$

$$C = 1$$

$$Coeff.\ of\ \ z^2, \quad 0 = A + C$$

$$0 = A + 1 \ \Rightarrow\ A = -1$$

$$\frac{1}{(z-1)^2(z-2)} = \frac{-1}{z-1} - \frac{1}{(z-1)^2} + \frac{1}{z-2}$$

$$\frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)} = -\frac{\sin\pi z^2 + \cos\pi z^2}{z-1} - \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2} + \frac{\sin\pi z^2 + \cos\pi z^2}{z-2}$$

$$\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2(z-2)}\,dz = -\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-1}\,dz - \int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)^2}\,dz + \int_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-2}\,dz$$

$$= -2\pi i\, f(1) - \frac{2\pi i}{1!} f'(1) + 2\pi i\, f(2) \quad \textit{where } f(z) = \sin\pi z^2 + \cos\pi z^2 \textit{ and } z_0 = 1,\ z_0 = 1,\ z_0 = 2$$

$$= -2\pi i(-1) - 2\pi i(-2\pi) + 2\pi i(1)$$

$$= 2\pi i + 4\pi^2 i + 2\pi i$$

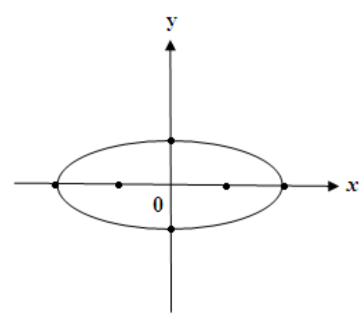
$$= 4\pi^2 i + 4\pi i$$

$$= 4\pi(\pi+1)i$$

$$\boxed{\begin{aligned} &f(z) = \sin\pi z^2 + \cos\pi z^2 \\ &f'(z) = 2\pi z\cos\pi z^2 - 2\pi z\sin\pi z^2 \\ &f(1) = \sin\pi + \cos\pi = 0 - 1 = -1 \\ &f'(1) = 2\pi\cos\pi - 2\pi\sin\pi = -2\pi - 0 = -2\pi \\ &f(2) = \sin 4\pi + \cos 4\pi = 0 + 1 = 1 \end{aligned}}$$

12. *Evaluate* $\int_C \frac{7z-1}{z^2-3z-4}\,dz$ *where C is the ellipse* $x^2 + 4y^2 = 4$.

Sol. *Given* $C:\ x^2 + 4y^2 = 4 \Rightarrow \frac{x^2}{4} + \frac{y^2}{1} = 1$



$$\int_C \frac{7z-1}{z^2-3z-4}\,dz = \int_C \frac{7z-1}{(z-4)(z+1)}\,dz$$

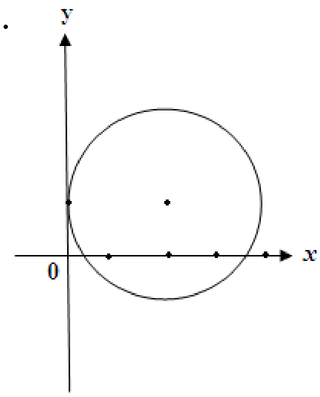
$$= \int_C \frac{\frac{7z-1}{z-4}}{z+1}\,dz$$

$$= \int_C \frac{f(z)}{z-z_0}\,dz \quad \textit{where } f(z) = \frac{7z-1}{z-4} \textit{ and } z_0 = -1$$

$$= 2\pi i\, f(z_0) = 2\pi i\, f(-1)$$

$$= 2\pi i\left[\frac{-7-1}{-1-4}\right] = 2\pi i\left(\frac{-8}{-5}\right) = \frac{16\pi i}{5}$$

13. *Evaluate* $\int_C \frac{z+1}{z^3-2z^2}\,dz$ *where C is the circle* $|z-2-i|=2$.



Sol. *Given* $C: |z-2-i| = 2 \Rightarrow |z-(2+i)| = 2$

$$\int_C \frac{z+1}{z^3-2z^2}\,dz = \int_C \frac{z+1}{z^2(z-2)}\,dz$$

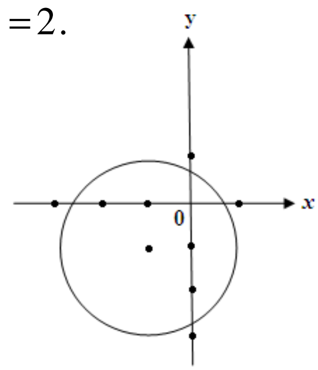
$$= \int_C \frac{{}^{z+1}/_{z^2}}{z-2}\,dz$$

$$= \int_C \frac{f(z)}{z-z_0}\,dz \quad where\ f(z)=\frac{z+1}{z^2}\ and\ z_0 = 2$$

$$= 2\pi i\, f(z_0) = 2\pi i\, f(2)$$

$$= 2\pi i\left[\frac{2+1}{4}\right] = 2\pi i\left(\frac{3}{4}\right) = \frac{3\pi i}{2}$$

14. *Evaluate* $\int_C \frac{z+4}{z^2+2z+5}\,dz$ *where C is the circle* $|z+1+i|=2$.



Sol. *Given* $C: |z+1+i| = 2 \Rightarrow |z-(-1-i)| = 2$

$$\int_C \frac{z+4}{z^2+2z+5}\,dz = \int_C \frac{z+4}{[z-(-1-2i)][z-(-1+2i)]}\,dz$$

$$= \int_C \frac{{}^{z+4}/_{z+1-2i}}{z-(-1-2i)}\,dz$$

$$= \int_C \frac{f(z)}{z-z_0}\,dz \quad where\ f(z)=\frac{z+4}{z+1-2i}\ and\ z_0 = -1-2i$$

$$= 2\pi i\, f(z_0)$$

$$= 2\pi i\, f(-1-2i)$$

$$= 2\pi i\left[\frac{-1-2i+4}{-1-2i+1-2i}\right]$$

$$= 2\pi i\left(\frac{3-2i}{-4i}\right)$$

$$= \frac{\pi}{2}(2i-3)$$

$$z^2+2z+5=0$$

$$z = \frac{-2\pm\sqrt{4-4.1.5}}{2.1}$$

$$= \frac{-2\pm\sqrt{-16}}{2}$$

$$= \frac{-2\pm 4i}{2} = -1\pm 2i$$

The factors are

$[z-(-1+2i)][z-(-1-2i)]$

Home Work

1. *Evaluate* $\int_C \frac{\cos \pi z}{z-1}\, dz$ *where C is the circle* $|z| = 2$.
2. *Evaluate* $\int_C \frac{z}{(z-1)(z-2)}\, dz$ *where C is the circle* $|z-2| = \frac{1}{2}$.
3. *Evaluate* $\int_C \frac{1}{(z^2+4)}\, dz$ *where C is the circle* $|z-i| = 2$.
4. *Evaluate* $\int_C \frac{e^{-z}}{(z-1)^2}\, dz$ *where C is the circle* $|z| = 2$.
5. *Evaluate* $\int_C \frac{z^2+2z}{(z+3)(z-1)}\, dz$ *where C is the circle* $|z| = 2$.
6. *Evaluate* $\int_C \frac{z^2+1}{z^2-1}\, dz$ *if C is a circle of unit radius with centre at* $z = -1$.
7. *Evaluate* (*i*) $\int_C \frac{z}{2z+1}\, dz$ (*ii*) $\int_C \frac{e^{-z}}{z - \frac{\pi i}{2}}\, dz$ *where C is the boundary of the square whose side lie along the line* $x = \pm 2,\ y = \pm 2$.
8. *Evaluate* $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}\, dz$ *where C is the circle* $|z| = 3$.
9. *Evaluate* $\int_C \frac{4z^2+4z+5}{z-3}\, dz$ *where C is the ellipse* $9x^2 + 4y^2 = 36$.
10. *Evaluate* $\int_C \frac{z+1}{z^2+2z+4}\, dz$ *where C is the circle* $|z+1+i| = 2$.

Taylor's Theorem

Let $f(z)$ be analytic everywhere inside a circle C with centre at z_0 and radius R. Then at each point z inside C, we have

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \dots\dots\dots + \frac{f^n(z_0)}{n!}(z-z_0)^n + \dots\dots\dots$$

$$(i.e.)\quad f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!}(z-z_0)^n$$

<u>Note</u>: When $z_0 = 0$, Taylor's series becomes Maclaurin's series which is

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \dots\dots\dots + \frac{f^n(z_0)}{n!}z^n + \dots\dots\dots$$

$$(i.e.)\quad f(z) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}z^n$$

Some Important Result:

1. $\sin z = z - \frac{z}{3!} + \frac{z^5}{5!} - \dots\dots\dots\dots\dots\infty$

2. $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots\dots\dots\dots\infty$

3. $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\dots\dots\dots\infty$

4. $e^{-z} = 1 - \frac{z}{1!} + \frac{z^2}{2!} - \frac{z^3}{3!} + \dots\dots\dots\dots\infty$

5. $\sinh z = \frac{e^z - e^{-z}}{2} = z + \frac{z}{3!} + \frac{z^5}{5!} + \dots\dots\dots\dots\infty$

6. $\cosh z = \frac{e^z + e^{-z}}{2} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots\dots\dots\dots\infty$

<u>n^th derivative</u>

1. $If \quad y = e^{ax} \quad then \quad y_n = a^n e^{ax}$

2. $If \quad y = \frac{1}{ax+b} \quad then \quad y_n = \frac{(-1)^n\, n!\, a^n}{(ax+b)^{n+1}}$

3. $If \quad y = \frac{1}{(ax+b)^2} \quad then \quad y_n = \frac{(-1)^n\,(n+1)\,!\; a^n}{(ax+b)^{n+2}}$

4. $If \;\; y = \log(a\,x + b) \;\; then \;\; y_n = \dfrac{(-1)^n \,(n-1)!\; a^n}{(a\,x+b)^n}$

5. $If \;\; y = \sin(a\,x+b) \;\; then \;\; y_n = a^n \sin\left(\dfrac{n\pi}{2} + a\,x + b\right)$

6. $If \;\; y = \cos(a\,x+b) \;\; then \;\; y_n = a^n \cos\left(\dfrac{n\pi}{2} + a\,x + b\right)$

Binomial Series

$If \;|x| < 1 \;\; (i.e.) \;\; -1 < x < 1 \;\;\; we \;\; have$

1. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots + x^n + \ldots\ldots\infty = \sum_{n=0}^{\infty} x^n$

2. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots + (-1)^n x^n + \ldots\ldots\infty = \sum_{n=0}^{\infty} (-1)^n x^n$

3. $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots + (n+1)\,x^n + \ldots\ldots\infty = \sum_{n=0}^{\infty} (n+1)\,x^n$

4. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots + (-1)^n (n+1)\,x^n + \ldots\ldots\infty = \sum_{n=0}^{\infty} (-1)^n (n+1)\,x^n$

Problems

1. Find the Taylor's expansion of $\dfrac{1}{z}$ about the point $z = 1$

Sol. Taylor's series of $f(z)$ about $z = z_0$ is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!}(z - z_0)^n \quad ------ \; (1)$$

$$f(z) = \frac{1}{z}, \quad z_0 = 1$$

$$f^n(z) = \frac{(-1)^n \, n!\,(1)^n}{z^{n+1}}$$

$$\therefore \; f^n(z_0) = f^n(1) = \frac{(-1)^n \, n!}{(1)^{n+1}} = (-1)^n \, n!$$

Hence equation (1) *becomes*

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \, n!}{n!}\,(z-1)^n$$

$$(i.e.) \;\; \frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n \;(z-1)^n \qquad (|z-1| < 1)$$

2. *Show that* $e^z = e \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$

Sol. Taylor's series of $f(z)$ about $z = z_0$ is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!}(z-z_0)^n \quad ------- (1)$$

$$f(z) = e^z, \quad z_0 = 1$$

$$f^n(z) = 1^n e^z = e^z$$

$$\therefore f^n(z_0) = f^n(1) = e^1 = e$$

Hence equation (1) *becomes*

$$f(z) = \sum_{n=0}^{\infty} \frac{e}{n!}(z-1)^n$$

$$(i.e.)\ e^z = e \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \quad (|z-1| < 1)$$

3. Find the Taylor's series for the function log(1 – z) at z = 0. State the region in which the expansion valid.

Sol. Taylor's series of $f(z)$ about $z = z_0$ is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!}(z-z_0)^n \quad ------- (1)$$

$$f(z) = \log(1-z), \quad z_0 = 0$$

$$f^n(z) = \frac{(-1)^{n-1}(n-1)!\,(-1)^n}{(1-z)^n} = \frac{(-1)^{2n-1}(n-1)!}{(1-z)^n}$$

$$\therefore f^n(z_0) = f^n(0) = \frac{(-1)^{2n-1}(n-1)!}{(1)^n} = (-1)^{2n-1}(n-1)! = -(n-1)!$$

Hence equation (1) *becomes*

$$f(z) = \sum_{n=0}^{\infty} \frac{-(n-1)!}{n!}(z-0)^n$$

$$(i.e.)\ \log(1-z) = -\sum_{n=0}^{\infty} \frac{z^n}{n} \quad (|z| < 1)$$

4. *Obtain the Taylor's series expansion of* $f(z)=\dfrac{z^2-1}{(z+2)(z+3)}$ *in* $|z|<2$.

Sol. $f(z)=\dfrac{z^2-1}{(z+2)(z+3)}=A+\dfrac{B}{z+2}+\dfrac{C}{z+3}$

$$z^2-1=A(z+2)(z+3)+B(z+3)+C(z+2)$$

put $z=-2,\quad 4-1=0+B(1)+0$

$$B=3$$

put $z=-3,\quad 9-1=0+0+C(-1)$

$$C=-8$$

coeff. of $z^2,\qquad 1=A$

$$f(z)=\frac{z^2-1}{(z+2)(z+3)}=1+\frac{3}{z+2}-\frac{8}{z+3}$$

In $|z|<2$, *we have*

$$f(z)=1+\frac{3}{2\left(1+\frac{z}{2}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

If $|z|<k$, then take k outside.
If $|z|>k$, then take z outside.
(where k is constant)

$$=1+\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=1+\frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n-\frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

$$=1+3\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^{n+1}}-8\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{3^{n+1}}$$

5. *Show that when* $0<|z-1|<2$, $\dfrac{z}{(z-1)(z-3)}=\dfrac{-1}{2(z-1)}-3\sum_{n=0}^{\infty}\dfrac{(z-1)^n}{2^{n+2}}$

Sol. $\dfrac{z}{(z-1)(z-3)}=\dfrac{A}{z-1}+\dfrac{B}{z-3}$

$$z=A(z-3)+B(z-1)$$

put $z=1,\quad 1=A(-2)+0$

$$A=-1/2$$

put $z=3,\quad 3=0+B(2)$

$$B=3/2$$

$$\frac{z}{(z-1)(z-3)} = \frac{-1/2}{z-1} + \frac{3/2}{z-3}$$

In $0 < |z-1| < 2$, *we have*

$$\frac{z}{(z-1)(z-3)} = -\frac{1}{2}\frac{1}{z-1} + \frac{3}{2}\frac{1}{[(z-1)-2]}$$

$$= -\frac{1}{2(z-1)} + \frac{3}{-4}\frac{1}{\left[1-\frac{z-1}{2}\right]}$$

$$= -\frac{1}{2(z-1)} - \frac{3}{4}\left[1-\frac{z-1}{2}\right]^{-1}$$

$$= -\frac{1}{2(z-1)} - \frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$$

6. Find the Maclaurin's series expansion of sinz.

Sol. Maclaurin's series is

$$f(z) = \sum_{n=0}^{\infty}\frac{f^n(0)}{n!} z^n \quad ------ (1)$$

$$f(z) = \sin z$$

$$f^n(z) = 1^n \sin\left(\frac{n\pi}{2}+z\right) = \sin\left(\frac{n\pi}{2}+z\right)$$

$$\therefore f^n(0) = \sin\left(\frac{n\pi}{2}+0\right) = \sin\left(\frac{n\pi}{2}\right)$$

Hence equation (1) *becomes*

$$f(z) = \sum_{n=0}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{n!} z^n$$

$$= 0 + \frac{z}{1!} + 0 - \frac{z^3}{3!} + 0 + \frac{z^5}{5!} + 0 - ....\infty$$

$$(i.e.)\ \sin z = \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - ....\infty$$

Laurent's Series

If $f(z)$ is analytic on two concentric circles C_1 and C_2 with centre at z_0 and radii R_1 and R_2 ($R_1 > R_2$) and also in the annular region R bounded by C_1 and C_2, then at any point z in R, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

$$where \quad a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{(w-z_0)^{n+1}}\, dw, \quad n = 0, 1, 2...$$

$$b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{(w-z_0)^{-n+1}}\, dw, \quad n = 1, 2, 3....$$

Note: The series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is called the regular part and $\sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$ is called the principal part of $f(z)$.

Problems

1. *Find the Laurent's expansion of* $\frac{1}{(z-1)(z-2)}$ *in*

$i)\ 1 < |z| < 2 \quad ii)\ 0 < |z-1| < 1$

Sol.

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z-1)$$

$$put\ z = 1, \quad 1 = A(-1) + 0$$

$$A = -1$$

$$put\ z = 2, \quad 1 = 0 + B(1)$$

$$B = 1$$

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

(i) *In* $1 < |z| < 2$, *we have*

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{-2\left(1-\frac{z}{2}\right)}$$

$$= -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$\frac{1}{(z-1)(z-2)} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$= -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

(ii) We have $\dfrac{1}{(z-1)(z-2)} = \dfrac{-1}{z-1} + \dfrac{1}{z-2}$

In $0 < |z-1| < 1$, we have

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{(z-1)-1}$$

$$= \frac{-1}{z-1} + \frac{1}{-1[1-(z-1)]}$$

$$= \frac{-1}{z-1} - [1-(z-1)]^{-1}$$

$$= -\frac{1}{z-1} - \sum_{n=0}^{\infty}(z-1)^n$$

2. *Expand* $\dfrac{1}{(z+1)(z+3)}$ *in the regions* $i)\ |z| < 1 \quad ii)\ 1 < |z| < 3 \quad iii)\ |z| > 3.$

Sol. $\dfrac{1}{(z+1)(z+3)} = \dfrac{A}{z+1} + \dfrac{B}{z+3}$

$$1 = A(z+3) + B(z+1)$$

put $z = -1$, $1 = A(2) + 0$

$A = 1/2$

put $z = -3$, $1 = 0 + B(-2)$

$B = -1/2$

$$\frac{1}{(z+1)(z+3)} = \frac{1/2}{z+1} - \frac{1/2}{z+3}$$

(i) In $|z| < 1$, we have

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\frac{1}{1+z} - \frac{1}{2}\frac{1}{3\left(1+\frac{z}{3}\right)}$$

$$= \frac{1}{2}(1+z)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n z^n - \frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

$$(ii)\ \ We\ \ have\ \ \frac{1}{(z+1)(z+3)} = \frac{1/2}{z+1} - \frac{1/2}{z+3}$$

$$In\ \ 1 < |z| < 3,\ \ we\ \ have$$

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{2}\frac{1}{3\left(1+\frac{z}{3}\right)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n - \frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

$$(iii)\ \ We\ \ have\ \ \frac{1}{(z+1)(z+3)} = \frac{1/2}{z+1} - \frac{1/2}{z+3}$$

$$In\ \ |z| > 3,\ \ we\ \ have$$

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{2}\frac{1}{z\left(1+\frac{3}{z}\right)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{2z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n - \frac{1}{2z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{n+1}} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{z^{n+1}}$$

3. *Expand in Laurent's series* $\frac{1}{z(z-1)^2}$ *at the point* $i)\ z=0\ \ ii)\ z=1.$

Sol. $$\frac{1}{z(z-1)^2} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$1 = A(z-1)^2 + Bz(z-1) + Cz$$

$$put\ \ z=0,\ \ 1 = A(-1)^2 + 0 + 0$$

$$A = 1$$

$$put\ \ z=1,\ \ 1 = 0 + 0 + C(1)$$

$$C = 1$$

$$Coeff.\ of\ \ z^2,\ \ 0 = A + B$$

$$0 = 1 + B \Rightarrow B = -1$$

$$\frac{1}{z(z-1)^2} = \frac{1}{z} - \frac{1}{z-1} + \frac{1}{(z-1)^2}$$

(i) $At\ \ z = 0\ \ (i.e.)\ \ when\ \ |z| < 1,\ \ we\ \ have$

$$\frac{1}{z(z-1)^2} = \frac{1}{z} - \frac{1}{-(1-z)} + \frac{1}{(1-z)^2}$$

$$= \frac{1}{z} + (1-z)^{-1} + (1-z)^{-2}$$

$$= \frac{1}{z} + \sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} (n+1)\, z^n$$

(ii) $We\ have\ \dfrac{1}{z(z-1)^2} = \dfrac{1}{z} - \dfrac{1}{z-1} + \dfrac{1}{(z-1)^2}$

$At\ \ z = 1\ \ (i.e.)\ \ when\ \ |z-1| < 1,\ \ we\ \ have$

$$\frac{1}{z(z-1)^2} = \frac{1}{(z-1)+1} - \frac{1}{z-1} + \frac{1}{(z-1)^2}$$

$$= [1 + (z-1)]^{-1} - \frac{1}{z-1} + \frac{1}{(z-1)^2}$$

$$= \sum_{n=0}^{\infty} (-1)^n (z-1)^n - \frac{1}{z-1} + \frac{1}{(z-1)^2}$$

4. $Expand\ \ \dfrac{z+2}{z(z-2)}\ \ as\ a\ Laurent's\ \ series\ about\ \ z = 2.$

$Sol.$ $$\frac{z+2}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2}$$

$$z + 2 = A(z-2) + B(z)$$

$put\ \ z = 0,\ \ 2 = A(-2) + 0$

$$A = -1$$

$put\ \ z = 2,\ \ 4 = 0 + B(2)$

$$B = 2$$

$$\frac{z+2}{z(z-2)} = \frac{-1}{z} + \frac{2}{z-2}$$

$$\frac{z+2}{z(z-2)} = \frac{-1}{z} + \frac{2}{z-2}$$

$At \;\; z = 2 \;\; (i.e.) \;\; when \;\; |z-2| < 1, \;\; we \;\; have$

$$\frac{z+2}{z(z-2)} = \frac{-1}{(z-2)+2} + \frac{2}{z-2}$$

$$= \frac{-1}{2\left[1+\frac{z-2}{2}\right]} + \frac{2}{z-2}$$

$$= -\frac{1}{2}\left[1+\frac{z-2}{2}\right]^{-1} + \frac{2}{z-2}$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2}{2}\right)^n + \frac{2}{z-2}$$

5. *Find the Laurent's series for* $\frac{z}{(z+1)(z+2)}$ *about* $z = -2$

Sol. $$\frac{z}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2}$$

$$z = A(z+2) + B(z+1)$$

$put \;\; z = -1, \;\; -1 = A(1) + 0$

$$A = -1$$

$put \;\; z = -2, \;\; -2 = 0 + B(-1)$

$$B = 2$$

$$\frac{z}{(z+1)(z+2)} = \frac{-1}{z+1} + \frac{2}{z+2}$$

$At \;\; z = -2 \;\; (i.e.) \;\; when \;\; |z+2| < 1, \;\; we \;\; have$

$$\frac{z}{(z+1)(z-2)} = \frac{-1}{(z+2)-1} + \frac{2}{z+2}$$

$$= \frac{-1}{-[1-(z+2)]} + \frac{2}{z+2}$$

$$= [1-(z+2)]^{-1} + \frac{2}{z+2}$$

$$= \sum_{n=0}^{\infty}(z+2)^n + \frac{2}{z+2}$$

6. *Find the Laurent's series of* $f(z) = \dfrac{1}{z(1-z)}$ *valid in the region*

$i)\ |z+1| < 1 \quad ii)\ 1 < |z+1| < 2 \quad iii)\ |z+1| > 2$

Sol. $$\frac{1}{z(1-z)} = \frac{A}{z} + \frac{B}{1-z}$$

$$1 = A(1-z) + B(z)$$

$$put\ z = 0,\quad 1 = A(1) + 0$$

$$A = 1$$

$$put\ z = 1,\quad 1 = 0 + B(1)$$

$$B = 1$$

$$\frac{1}{z(1-z)} = \frac{1}{z} + \frac{1}{1-z}$$

(i) $$\frac{1}{z(1-z)} = \frac{1}{(z+1)-1} + \frac{1}{2-(z+1)}$$ [Since Laurent's series in powers of (z + 1) are required]

In $|z+1| < 1$, *we have*

$$\frac{1}{z(1-z)} = \frac{1}{-[1-(z+1)]} + \frac{1}{2\left[1-\frac{z+1}{2}\right]}$$

$$= -\left[1-(z+1)\right]^{-1} + \frac{1}{2}\left[1-\frac{z+1}{2}\right]^{-1}$$

$$= -\sum_{n=0}^{\infty}(z+1)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z+1}{2}\right)^n$$

(ii) $$\frac{1}{z(1-z)} = \frac{1}{(z+1)-1} + \frac{1}{2-(z+1)}$$

In $1 < |z+1| < 2$, *we have*

$$\frac{1}{z(1-z)} = \frac{1}{(z+1)\left[1-\frac{1}{z+1}\right]} + \frac{1}{2\left[1-\frac{z+1}{2}\right]}$$

$$= \frac{1}{z+1}\left[1-\frac{1}{z+1}\right]^{-1} + \frac{1}{2}\left[1-\frac{z+1}{2}\right]^{-1}$$

$$= \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z+1}{2}\right)^n = \sum_{n=0}^{\infty}\frac{1}{(z+1)^{n+1}} + \sum_{n=0}^{\infty}\frac{(z+1)^n}{2^{n+1}}$$

$$(iii) \quad \frac{1}{z(1-z)} = \frac{1}{(z+1)-1} + \frac{1}{2-(z+1)}$$

$In \ |z+1| > 2, \ we \ have$

$$\frac{1}{z(1-z)} = \frac{1}{(z+1)\left[1-\frac{1}{z+1}\right]} + \frac{1}{-(z+1)\left[1-\frac{2}{z+1}\right]}$$

$$= \frac{1}{z+1}\left[1-\frac{1}{z+1}\right]^{-1} - \frac{1}{z+1}\left[1-\frac{2}{z+1}\right]^{-1}$$

$$= \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n - \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{2}{z+1}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{1}{(z+1)^{n+1}} - \sum_{n=0}^{\infty}\frac{2^n}{(z+1)^{n+1}}$$

Home Work

1. *Find the Taylor's series expansion of* $f(z) = \dfrac{1}{(z+1)(z+3)}$ *in* $|z| < 1$.

2. *Find the Taylor's series expansion of* $\dfrac{(z-2)(z+2)}{(z+1)(z+4)}$ *in* $|z| < 1$.

3. *Find the Laurent's series expansion of* $\dfrac{1}{z(z^2-3z+2)}$ *in the region*

   i) $1 < |z| < 2$ *ii)* $0 < |z| < 1$ *iii)* $|z| > 2$

4. *Find the Laurent's series expansion of* $\dfrac{1}{(z-4)(z-5)}$ *in the region*

   i) $0 < |z| < 4$ *ii)* $4 < |z| < 5$ *iii)* $|z| > 5$

5. *Find the Laurent's series expansion of* $\dfrac{z^2-1}{(z+2)(z-3)}$ *in the region*

   i) $|z| < 2$ *ii)* $2 < |z| < 3$ *iii)* $|z| > 3$.

6. *Find the Laurent's series expansion of* $\dfrac{7z-2}{(z+1)z(z-2)}$ *valid in the region*

   $1 < |z+1| < 3$.

Singularities

Zeros of an analytic function

1) A zero of an analytic function $f(z)$ is a value of z such that $f(z) = 0$.
2) An analytic function $f(z)$ is said to have a zero of order 'm' if $f(z)$ is expressible as $f(z) = (z - z_0)^m \phi(z)$ where $\phi(z)$ is analytic and $\phi(z_0) \neq 0$.

$f(z)$ is said to have a simple zero at $z = z_0$ if $z = z_0$ is a zero of order one.

Poles of an analytic function

If $f(z) = \dfrac{\phi(z)}{\psi(z)}$ then the poles of $f(z)$ are $\psi(z) = 0$.

An analytic function $f(z)$ is said to have a pole of order 'm' if $f(z)$ is expressible as $f(z) = (z - z_0)^{-m} \phi(z)$ where $\phi(z)$ is analytic and $\phi(z_0) \neq 0$.

$f(z)$ is said to have a simple pole at $z = z_0$ if $z = z_0$ is a pole of order one.

Singular Point

A point $z = z_0$ is called a singular point (or a singularity) of $f(z)$, if $f(z)$ is not analytic at z_0.

For example, if $f(z) = \dfrac{1}{z-2}$ then $z = 2$ is a singularity of $f(z)$.

Isolated Singular point

A singular point z_0 is said to be an isolated singular point if there is some neighbourhood of z_0 throughout which 'f' is analytic except at the point itself.

Example:

1. $f(z) = \dfrac{1}{z}$ is analytic everywhere except at $z = 0$.

   $\therefore$ $z = 0$ is an isolated singularity of $f(z)$.

2. $f(z) = \dfrac{z+1}{z^3(z^2+1)}$ has three isolated singular points $z = 0$, $z = \pm i$.

3. $f(z) = \dfrac{1}{\sin\left(\dfrac{\pi}{z}\right)}$

   The singularity of $f(z)$ are given by

$$\sin\left(\frac{\pi}{z}\right) = 0 = \sin n\pi$$

$$\Rightarrow \frac{\pi}{z} = n\pi$$

$$\Rightarrow z = \frac{1}{n} \quad (n = \pm 1, \pm 2, \ldots\ldots\ldots)$$

Limit Point
Limit of the sequence is called the limit point.

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

$z = 0$ *is the singular point*

The singular point z = 0 is not an isolated singular point because every neighbourhood of the origin contains other singular points.

Classification of the Singular point

If $z = z_0$ is an isolated singularity of $f(z)$ then we have by Laurent's series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n} \quad ------ (1)$$

The second portion of the series, namely the series involving negative powers of $z = z_0$ is known as the principal part of 'f' at $z = z_0$.

This principal part can be used to distinguish between three types of isolated singular points.

<u>Case (i)</u> : If the principal part contains only a finite number of terms (i.e.) there exists a positive integer m such that $b_m \neq 0,\ but\ b_{m+1} = 0, b_{m+2} = 0, ...$

Then the expansion (1) becomes

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + ... + \frac{b_m}{(z - z_0)^m}$$

where $b_m \neq 0$. In this case the isolated singularity $z = z_0$ is called a pole of order m. A pole of order m = 1 is called a single pole (simple pole). A pole of order m = 2 is called a double pole.

<u>Case (ii)</u> : When the principal part of 'f' at z_0 has infinite number of non-zero terms, the point $z = z_0$ is called an essential singular point. In this case the Laurent's series (1) contains infinite terms in the negative powers of $z - z_0$.

<u>Case (iii)</u> : When all the coefficients $b_n = 0$ (i.e.) if the principal part contains no terms, then the point $z = z_0$ is called removable singularity of 'f'. In this case the Laurent's series (1) contains only non-negative powers of $z - z_0$.

$$(i.e.)\ f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

<u>Example</u>

1. If $f(z) = \frac{1}{z}$, z = 0 is a simple pole
2. If $f(z) = e^{1/z}$, z = 0 is an essential singularity
3. If $f(z) = \frac{\sin z}{z}$, z = 0 is an removable singularity.

Problems

Find the singular point of the following function and specify their nature.

1. $f(z) = \frac{e^z}{z}$

Sol. *Given* $f(z) = \frac{e^z}{z}$

The singular point is z = 0

$$f(z) = \frac{e^z}{z}$$

$$= \frac{1}{z}\left[1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\ldots\ldots\right] = \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \ldots\ldots\ldots\ldots$$

Its principal part is $\frac{1}{z}$

$\therefore$ z = 0 is a simple pole.

2. $f(z) = \frac{e^z}{z^2}$

Sol. *Given* $f(z) = \frac{e^z}{z^2}$

The singular point is z = 0

$$f(z) = \frac{e^z}{z^2}$$

$$= \frac{1}{z^2}\left[1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\ldots\ldots\right] = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \ldots\ldots\ldots\ldots$$

Its principal part is $\frac{1}{z^2} + \frac{1}{z}$. $\therefore$ z = 0 is a pole of order 2.

3. *Find the singularity of* $\frac{\sin z}{z}$ *and specify their nature.*

Sol. *Let* $f(z) = \frac{\sin z}{z}$

Singularity of $f(z)$ *is* $z = 0$

$$f(z) = \frac{\sin z}{z} = \frac{\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right]}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots$$

Since $f(z)$ *contains only non-negative powers of* z, *the point* $z = 0$ *is a removable singularity.*

4. *Find the singularity of* $e^{1/z+1}$ *and specify their nature.*

Sol. *Let* $f(z) = e^{1/z+1}$

Singularity of $f(z)$ *is* $z+1=0 \Rightarrow z=-1$

$$f(z) = e^{1/z+1} = 1 + \frac{1}{(z+1)1!} + \frac{1}{(z+1)^2\,2!} + \frac{1}{(z+1)^3\,3!} + \ldots\ldots\ldots\ldots$$

Since $f(z)$ *contains an infinite series of negative powers of* $z+1$, *the point* $z=-1$ *is an essential singularity.*

5. *Find the singularity of* $\frac{z-\sin z}{z^3}$ *and specify their nature.*

Sol. *Let* $f(z) = \frac{z-\sin z}{z^3}$

Singularity of $f(z)$ *is* $z=0$

$$f(z) = \frac{z-\sin z}{z^3} = \frac{z-\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ldots\ldots\right]}{z^3} = \frac{1}{3!} - \frac{z^2}{5!} - \ldots\ldots$$

Since $f(z)$ *contains only non-negative powers of* z, *the point* $z=0$ *is a removable singularity.*

Definition

1. Limit point of zeros is an isolated essential singularity.
2. Limit point of poles is an non-isolated essential singularity.

Problems

Identify the type of the singularity of the following function.

1. $f(z) = \sin\left(\frac{1}{1-z}\right)$

Zeros of $f(z)$ *are* $\sin\left(\frac{1}{1-z}\right) = 0$

$$\sin\left(\frac{1}{1-z}\right) = \sin n\pi$$

$$\Rightarrow \frac{1}{1-z} = n\pi$$

$$\Rightarrow 1 - z = \frac{1}{n\pi} \Rightarrow z = 1 - \frac{1}{n\pi} \ (n = 0, \pm 1, \pm 2, ...)$$

$z = 1$ *is the limit point of zeros.*

$\therefore$ $z = 1$ *is an isolated essential singularity.*

2. $f(z) = \tan\left(\frac{1}{z}\right) = \frac{\sin(1/z)}{\cos(1/z)}$

Poles of $f(z)$ *are* $\cos\left(\frac{1}{z}\right) = 0$

$$\cos\left(\frac{1}{z}\right) = \cos(2n+1)\frac{\pi}{2}$$

$$\Rightarrow \frac{1}{z} = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow z = \frac{2}{(2n+1)\pi} \quad (n = 0, \pm 1, \pm 2, ...)$$

$z = 0$ *is the limit point of poles.*

$\therefore$ $z = 0$ *is an non-isolated essential singularity.*

3. $f(z) = \tan z = \frac{\sin z}{\cos z}$

Poles of $f(z)$ *are* $\cos z = 0$

$$\cos z = \cos(2n+1)\frac{\pi}{2}$$

$$\Rightarrow z = (2n+1)\frac{\pi}{2} \ (n = 0, \pm 1, \pm 2, ...)$$

$z = \infty$ *is the limit point of poles.*

$\therefore$ $z = \infty$ *is an non-isolated essential singularity.*

4. $f(z) = z \operatorname{cosec} z = \frac{z}{\sin z}$

Poles of $f(z)$ *are* $\sin z = 0$

$$\sin z = \sin n\pi$$

$$\Rightarrow z = n\pi$$

$z = \infty$ *is the limit point of poles.*

$\therefore$ $z = \infty$ *is an non-isolated essential singularity.*

5. $f(z) = \dfrac{1}{\cos z - \sin z}$

$$Poles\ of\ f(z)\ are\ \cos z - \sin z = 0$$
$$\sin z = \cos z$$
$$\Rightarrow \tan z = 1$$
$$\Rightarrow \tan z = \tan\left(n\pi + \frac{\pi}{4}\right)$$
$$\Rightarrow z = n\pi + \frac{\pi}{4} \quad (n = 0, \pm 1, \pm 2, \ldots..)$$
$$z = \infty\ is\ the\ \lim it\ po\text{int}\ of\ poles.$$
$$\therefore z = \infty\ is\ an\ non-isolated\ essential\ \sin gularity.$$

6. $f(z) = \sin z - \cos z$

$$Zeros\ of\ f(z)\ are\ \sin z - \cos z = 0$$
$$\sin z = \cos z$$
$$\Rightarrow \tan z = 1$$
$$\Rightarrow \tan z = \tan\left(n\pi + \frac{\pi}{4}\right)$$
$$\Rightarrow z = n\pi + \frac{\pi}{4} \quad (n = 0, \pm 1, \pm 2, \ldots..)$$
$$z = \infty\ is\ the\ \lim it\ po\text{int}\ of\ zeros.$$
$$\therefore z = \infty\ is\ an\ isolated\ essential\ \sin gularity.$$

7. $f(z) = \cot z = \dfrac{\cos z}{\sin z}$

$$Poles\ of\ f(z)\ are\ \sin z = 0$$
$$\sin z = \sin n\pi$$
$$\Rightarrow z = n\pi \quad (n = 0, \pm 1, \pm 2, \ldots\ldots.)$$
$$z = \infty\ is\ the\ \lim it\ po\text{int}\ of\ poles.$$
$$\therefore z = \infty\ is\ an\ non-isolated\ essential\ \sin gularity.$$

8. $f(z) = \operatorname{cosec}\left(\frac{1}{z}\right) = \frac{1}{\sin(1/z)}$

$$\textit{Poles of } f(z) \textit{ are } \sin\left(\frac{1}{z}\right) = 0$$

$$\sin\left(\frac{1}{z}\right) = \sin n\pi$$

$$\Rightarrow \frac{1}{z} = n\pi$$

$$\Rightarrow z = \frac{1}{n\pi} \quad (n = 0, \pm 1, \pm 2, ...)$$

$z = 0$ *is the limit point of poles.*

$\therefore$ $z = 0$ *is an non-isolated essential singularity.*

Residue at a pole

The residue of f(z) at z = z_0 is $\frac{1}{2\pi i}\int_C f(z)\,dz$

(i.e.) Res (z = z_0) = $\frac{1}{2\pi i}\int_C f(z)\,dz$

where C is a closed contour containing the only singularity at z = z_0 and the integration along C being taken anti-clockwise direction.

Evaluation of Residues

1. If z = z_0 is a simple pole of f(z) then

$$\operatorname{Res}(z = z_0) = \lim_{z \to z_0} (z - z_0)\, f(z)$$

2. If z = z_0 is a pole of order 2, then

$$\operatorname{Res}(z = z_0) = \frac{1}{1!} \lim_{z \to z_0} \left[\frac{d}{dz} \{(z - z_0)^2\, f(z)\} \right]$$

3. If z = z_0 is a pole of order 3, then

$$\operatorname{Res}(z = z_0) = \frac{1}{2!} \lim_{z \to z_0} \left[\frac{d^2}{dz^2} \{(z - z_0)^3\, f(z)\} \right]$$

4. In general, if z = z_0 is a pole of order m, then

$$\operatorname{Res}(z = z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \{(z - z_0)^m\, f(z)\} \right]$$

5. *If* $f(z) = \dfrac{\phi(z)}{\psi(z)}$ *has a simple pole at* $z = z_0$ [*where* $\psi(z_0) = 0$ *but* $\phi(z_0) \neq 0$]

then $\operatorname{Re}s\,(z = z_0) = \dfrac{\phi(z_0)}{\psi'(z_0)}$

6. $\operatorname{Re}s\,(z = \infty) = -\left[\text{Coefficient of } \dfrac{1}{z} \text{ in the expansion of } f(z)\right]$

7. $\operatorname{Re}s\,(z = \infty) = \lim_{z \to \infty} [-z\, f(z)]$ *provided* $f(z)$ *is analytic at* $z = \infty$.

Problems

1. *Find the residue of* $\dfrac{z^2}{z^2 + a^2}$ *at* $z = i\,a$.

Sol. *Let* $f(z) = \dfrac{z^2}{z^2 + a^2} = \dfrac{z^2}{(z + a\,i)(z - a\,i)}$

Poles of $f(z)$ *are* $z^2 + a^2 = 0$

$$z^2 = -a^2$$

$$z = \pm\, a\, i$$

$z = a\,i$ *is a simple pole.*

$$\operatorname{Re}s\,(z = a\,i) = \lim_{z \to ai} (z - a\,i)\, f(z)$$

$$= \lim_{z \to ai} (z - a\,i) \frac{z^2}{(z + a\,i)(z - a\,i)}$$

$$= \frac{(a\,i)^2}{2\,a\,i}$$

$$= \frac{-a^2}{2\,a\,i} = \frac{-a}{2\,i} = \frac{a\,i}{2} \qquad \left(\text{since } \frac{1}{i} = -i\right)$$

2. *Find the residue of* $\dfrac{1}{(z^2 + 1)^3}$ *at* $z = i$.

Sol. *Let* $f(z) = \dfrac{1}{(z^2 + 1)^3} = \dfrac{1}{(z + i)^3\,(z - i)^3}$

Poles of $f(z)$ *are* $z^2 + 1 = 0$

$$z = \pm\, i$$

$z = i$ *is a pole of order* 3.

$$\operatorname{Re}s\,(z=i) = \frac{1}{2!}\lim_{z\to i}\left[\frac{d^2}{dz^2}\{(z-i)^3 f(z)\}\right]$$

$$= \frac{1}{2}\lim_{z\to i}\left[\frac{d^2}{dz^2}\{(z-i)^3 \frac{1}{(z+i)^3 (z-i)^3}\}\right]$$

$$= \frac{1}{2}\lim_{z\to i}\left[\frac{d}{dz}\left\{\frac{-3}{(z+i)^4}\right\}\right]$$

$$= \frac{1}{2}\lim_{z\to i}\left[\frac{12}{(z+i)^5}\right]$$

$$= \frac{1}{2}\left[\frac{12}{(2i)^5}\right] = \frac{6}{32i} = \frac{3}{16i}$$

3. *Find the residue of* $f(z) = \tan z$ *at* $z = \frac{\pi}{2}$.

Sol. *Let* $f(z) = \tan z = \frac{\sin z}{\cos z} = \frac{\phi(z)}{\psi(z)}$

$$\phi(z_0) = \phi(\pi/2) = \sin\left(\frac{\pi}{2}\right) = 1 \neq 0$$

$$\psi(z_0) = \psi(\pi/2) = \cos\left(\frac{\pi}{2}\right) = 0$$

$\therefore\ z = \frac{\pi}{2}$ *is a simple pole of* $f(z)$.

$$\operatorname{Re}s\,(z = z_0) = \frac{\phi(z_0)}{\psi'(z_0)}$$

$$\psi(z) = \cos z$$
$$\psi'(z) = -\sin z$$

$$\operatorname{Re}s\,(z = \pi/2) = \frac{\phi(\pi/2)}{\psi'(\pi/2)}$$

$$= \frac{\sin(\pi/2)}{-\sin(\pi/2)} = \frac{1}{-1} = -1$$

4. *Find the residue of* $f(z) = \cot z$ *at* $z = 0$.

Sol. *Let* $f(z) = \cot z = \frac{\cos z}{\sin z} = \frac{\phi(z)}{\psi(z)}$

$$\phi(z_0) = \phi(0) = \cos 0 = 1 \neq 0$$
$$\psi(z_0) = \psi(0) = \sin 0 = 0$$

$\therefore\ z = 0$ *is a simple pole of* $f(z)$.

$$\operatorname{Re}s\,(z = z_0) = \frac{\phi(z_0)}{\psi'(z_0)}$$

$$\psi(z) = \sin z$$
$$\psi'(z) = \cos z$$

$$\operatorname{Re}s\,(z = 0) = \frac{\phi(0)}{\psi'(0)} = \frac{\cos 0}{\cos 0} = \frac{1}{1} = 1$$

5. *Find the residue of* $\dfrac{z e^{iz}}{z^2+a^2}$ *at each of its poles.*

Sol. *Let* $f(z)=\dfrac{z e^{iz}}{z^2+a^2}=\dfrac{z e^{iz}}{(z+ai)(z-ai)}$

Poles of $f(z)$ *are* $z^2+a^2=0$

$$z^2=-a^2$$

$$z=\pm\, ai$$

$z=ai,\,-ai$ *are simple poles.*

$$\operatorname{Res}(z=ai)=\lim_{z\to ai}(z-ai)\,f(z)$$

$$=\lim_{z\to ai}(z-ai)\frac{z e^{iz}}{(z+ai)(z-ai)}$$

$$=\frac{(ai)e^{i(ai)}}{2ai}$$

$$=\frac{ai\,e^{-a}}{2ai}=\frac{e^{-a}}{2}$$

$$\operatorname{Res}(z=-ai)=\lim_{z\to -ai}(z+ai)\,f(z)$$

$$=\lim_{z\to -ai}(z+ai)\frac{z e^{iz}}{(z+ai)(z-ai)}$$

$$=\frac{(-ai)e^{i(-ai)}}{-2ai}$$

$$=\frac{e^{a}}{2}$$

6. *Find the residue of* $\dfrac{z^3}{(z-1)^4(z-2)(z-3)}$ *at each of its poles.*

Sol. *Let* $f(z)=\dfrac{z^3}{(z-1)^4(z-2)(z-3)}$

Poles of $f(z)$ *are* $z=1,\ 2,\ 3.$

$z=1$ *is a pole of order* 4 *and* $z=2,3$ *are simple poles.*

$$\operatorname{Re}s\,(z=2) = \lim_{z\to 2}\ (z-2)\,f(z)$$

$$= \lim_{z\to 2}\ (z-2)\frac{z^3}{(z-1)^4\,(z-2)(z-3)} \ = \frac{8}{(1)(-1)} = -8$$

$$\operatorname{Re}s\,(z=3) = \lim_{z\to 3}\ (z-3)\,f(z)$$

$$= \lim_{z\to 3}\ (z-3)\frac{z^3}{(z-1)^4\,(z-2)(z-3)} \ = \frac{27}{(2)^4(1)} = \frac{27}{16}$$

$$\operatorname{Re}s\,(z=1) = \frac{1}{3!}\ \lim_{z\to 1}\left[\frac{d^3}{dz^3}\ \{(z-1)^4\ \ f(z)\}\right]$$

$$= \frac{1}{6}\ \lim_{z\to 1}\left[\frac{d^3}{dz^3}\left\{(z-1)^4\ \frac{z^3}{(z-1)^4(z-2)(z-3)}\right\}\right]$$

$$\operatorname{Re}s\,(z=1) = \frac{1}{6}\ \lim_{z\to 1}\left[\frac{d^3}{dz^3}\left\{\frac{z^3}{(z-2)(z-3)}\right\}\right] \text{------- (1)}$$

$$\frac{z^3}{(z-2)(z-3)} = A\,z + B + \frac{C}{z-2} + \frac{D}{z-3}$$

$$z^3 = (A\,z+B)(z-2)(z-3) + C(z-3) + D(z-2)$$

$$put\ \ z=2, \qquad 8 = 0 + C(-1) + 0$$

$$C = -8$$

$$put\ \ z=3, \qquad 27 = 0 + 0 + D(1)$$

$$D = 27$$

$$Coeff.\ of\ \ z^3, \quad 1 = A$$

$$Coeff.\ of\ \ z^2, \quad 0 = -5A + B$$

$$0 = -5(1) + B \ \ \Rightarrow\ \ B = 5$$

$$\frac{z^3}{(z-2)(z-3)} = z + 5 + \frac{-8}{z-2} + \frac{27}{z-3}$$

$$\frac{d^3}{dz^3}\left[\frac{z^3}{(z-2)(z-3)}\right] = \frac{d^3}{dz^3}\left[z + 5 - \frac{8}{z-2} + \frac{27}{z-3}\right]$$

$$= \frac{d^2}{dz^2}\left[1 + 0 + \frac{8}{(z-2)^2} - \frac{27}{(z-3)^2}\right]$$

$$= \frac{d}{dz}\left[0 - \frac{16}{(z-2)^3} + \frac{54}{(z-3)^3}\right]$$

$$= \frac{48}{(z-2)^4} - \frac{162}{(z-3)^4}$$

Hence equation (1) *becomes*

$$\operatorname{Re}s\,(z=1) = \frac{1}{6}\lim_{z\to 1}\left[\frac{48}{(z-2)^4} - \frac{162}{(z-3)^4}\right]$$

$$= \frac{1}{6}\left[\frac{48}{(-1)^4} - \frac{162}{(-2)^4}\right]$$

$$= \frac{1}{6}\left[\frac{768-162}{16}\right]$$

$$= \frac{1}{6}\left[\frac{606}{16}\right] = \frac{101}{16}$$

7. *Find the residue of* $f(z) = \dfrac{\sin z}{z\cos z}$ (or) $\dfrac{\tan z}{z}$ *at each of its poles inside the circle* $|z|=2$.

Sol. *Given* $f(z) = \dfrac{\sin z}{z\cos z} = \dfrac{\phi(z)}{\psi(z)}$

Poles of $f(z)$ *are* $z\cos z = 0$

$$z = 0 \ (or)\ \cos z = 0$$

$$\cos z = \cos(2n+1)\frac{\pi}{2}$$

$$z = (2n+1)\frac{\pi}{2} \quad \left(n = 0, \pm\frac{\pi}{2}, ...\right)$$

$$\therefore\ z = 0, \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, ...$$

$z = 0, \pm\dfrac{\pi}{2}$ *lies within the circle* $|z|=2$, *which are simple poles.*

$$\operatorname{Re}s\,(z=0) = \lim_{z\to 0}\,(z-0)\,f(z)$$

$$= \lim_{z\to 0}\,(z-0)\frac{\sin z}{z\cos z} = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$$

Now, $z = \frac{\pi}{2}, -\frac{\pi}{2}$ *are simple poles of* $f(z)$.

$$\phi(z_0) = \phi(\pi/2) = \sin\left(\frac{\pi}{2}\right) = 1 \neq 0$$

$$\psi(z_0) = \psi(\pi/2) = \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) = 0$$

$$\text{Re}\, s\,(z = z_0) = \frac{\phi(z_0)}{\psi'(z_0)}$$

$$\text{Re}\, s\,(z = \pi/2) = \frac{\phi(\pi/2)}{\psi'(\pi/2)}$$

$$= \frac{\sin(\pi/2)}{-\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right)}$$

$$= \frac{1}{-\frac{\pi}{2}(1) + 0}$$

$$= -\frac{2}{\pi}$$

$$\phi(z_0) = \phi(-\pi/2) = \sin\left(-\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1 \neq 0$$

$$\psi(z_0) = \psi(-\pi/2) = -\frac{\pi}{2}\cos\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) = 0$$

$$\psi(z) = z\cos z$$

$$\psi'(z) = -z\sin z + \cos z.1$$

$$\text{Re}\, s\,(z = -\pi/2) = \frac{\phi(-\pi/2)}{\psi'(-\pi/2)}$$

$$= \frac{\sin(-\pi/2)}{-\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) + \cos\left(-\frac{\pi}{2}\right)}$$

$$= \frac{-1}{\frac{\pi}{2}(-1) + 0}$$

$$= \frac{2}{\pi}$$

8. *Find the residue of* $\frac{z^3}{z^2 - 1}$ *at* $z = \infty$.

Sol. *Let* $f(z) = \frac{z^3}{z^2 - 1} = \frac{z^3}{z^2\left(1 - \frac{1}{z^2}\right)} = z\left(1 - \frac{1}{z^2}\right)^{-1}$

$$= z\left(1 + \frac{1}{z^2} + \frac{1}{z^4} + \frac{1}{z^6} + \ldots\ldots\ldots\right)$$

$$= z + \frac{1}{z} + \frac{1}{z^3} + \frac{1}{z^5} + \ldots\ldots\ldots$$

$$\text{Re}\, s\,(z = \infty) = -\mathit{coeff.\ of}\ \frac{1}{z} = -(1) = -1$$

9. *Find the residue of* $\dfrac{z^2}{(z-1)(z-2)(z-3)}$ *at* $z=\infty$.

Sol. $\operatorname{Res}(z=\infty) = \lim\limits_{z\to\infty}[-z\,f(z)]$

$$= \lim_{z\to\infty}\left[-z.\frac{z^2}{(z-1)(z-2)(z-3)}\right]$$

$$= \lim_{z\to\infty}\left[\frac{-z^3}{z^3\left(1-\frac{1}{z}\right)\left(1-\frac{2}{z}\right)\left(1-\frac{3}{z}\right)}\right]$$

$$= \frac{-1}{(1-0)(1-0)(1-0)}$$

$$= -1$$

Home Work

1. *Find the residue of* $\dfrac{z\,e^z}{(z-1)^2}$ *at* $z=1$.

2. *Find the residue of* $\dfrac{\cos z}{\left(z-\frac{\pi}{2}\right)^2}$ *at* $z=\dfrac{\pi}{2}$.

3. *Find the residue of* $\dfrac{1}{(z+1)^3}$ *at its poles*.

4. *Find the residue of the following function at its poles.*

(i) $\dfrac{1-e^{2z}}{z^4}$ (ii) $\dfrac{e^z}{z^2+\pi^2}$ (iii) $\dfrac{z^2-1}{(z^2+1)^2}$ (iv) $\dfrac{z}{(z^2+1)(z-2)}$

(v) $\dfrac{e^{2z}}{(z-1)^2}$ (vi) $\dfrac{z^2-2z}{(z+1)^2(z^2+4)}$ (vii) $\dfrac{z}{z^2+1}$ $(viii)$ $\dfrac{z+1}{z^2(z-2)}$

Cauchy's Residue Theorem

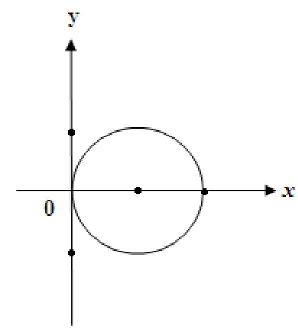
If $f(z)$ is analytic at all points inside and on a simple closed curve C, except at a finite number of poles $z_1, z_2, z_3, \ldots\ldots, z_n$ within C, then

$$\oint_C f(z)\,dz = 2\pi i\,[Sum\ of\ the\ residues\ of\ f(z)\ at\ its\ poles]$$

$$= 2\pi i\,[\operatorname{Res}(z=z_1) + \operatorname{Res}(z=z_2) + \ldots\ldots\ldots + \operatorname{Res}(z=z_n)]$$

Problems

1. *Using Residue theorem evaluate* $\int_C \frac{2z^2+z}{z^2-1}\,dz$, *where C is* $|z-1|=1$.



Sol. $Let\ f(z) = \frac{2z^2+z}{z^2-1}$

Poles of $f(z)$ *are* $z^2-1=0$

$$z^2 = 1$$

$$z = \pm 1$$

$z=1$ *is a simple pole, which lies inside the circle* $|z-1|=1$

$z=-1$ *is a simple pole, which lies outside the circle* $|z-1|=1$

$$\operatorname{Res}(z=1) = \lim_{z\to 1}(z-1)\,f(z)$$

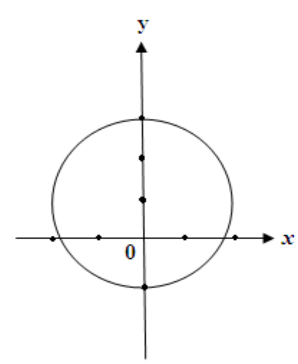
$$= \lim_{z\to 1}\cancel{(z-1)}\frac{2z^2+z}{\cancel{(z-1)}(z+1)}$$

$$= \frac{2+1}{2} = \frac{3}{2}$$

$\therefore$ *By Cauchy Residue theorem,*

$$\int_C f(z)dz = 2\pi i\,\operatorname{Res}(z=1) = 2\pi i\left(\frac{3}{2}\right) = 3\pi i.$$

2. *Evaluate* $\int_C \frac{dz}{(z^2+4)^2}$, *where C is the circle* $|z-i|=2$.



Sol. $Let\ f(z) = \frac{1}{(z^2+4)^2} = \frac{1}{(z+2i)^2(z-2i)^2}$

Poles of $f(z)$ *are* $z^2+4=0$

$$z = \pm\,2i$$

$z=2i$ *is a pole of order* 2, *which lies inside the circle* $|z-i|=2$

$$\operatorname{Res}(z=2i) = \frac{1}{1!} \lim_{z \to 2i} \frac{d}{dz}[(z-2i)^2 \; f(z)]$$

$$= \lim_{z \to 2i} \frac{d}{dz}\left[\cancel{(z-2i)^2} \; \frac{1}{\cancel{(z-2i)^2}\,(z+2i)^2}\right]$$

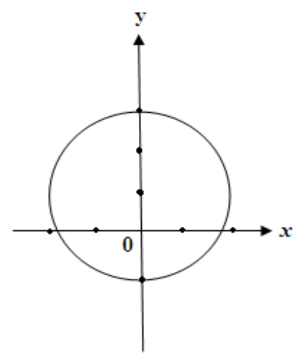
$$= \lim_{z \to 2i} \left[\frac{-2}{(z+2i)^3}\right]$$

$$= \frac{-2}{(4i)^3} = \frac{-2}{-64i} = \frac{1}{32i}$$

$\therefore$ *By Cauchy Residue theorem,*

$$\int_C f(z)dz = 2\pi i \operatorname{Res}(z=2i) = 2\pi i \left(\frac{1}{32i}\right) = \frac{\pi}{16}$$

3. *Evaluate* $\displaystyle\int_C \frac{z-1}{(z+1)^2(z-2)}\, dz$, *where* C *is* $|z-i|=2$.



Sol. *Let* $f(z) = \dfrac{z-1}{(z+1)^2(z-2)}$

Poles of $f(z)$ *are* $(z+1)^2(z-2)=0$

$$z=-1,\ 2$$

$z=-1$ *is a pole of order* 2, *which lies inside the circle* $|z-i|=2$

$$\operatorname{Res}(z=-1) = \frac{1}{1!} \lim_{z \to -1} \frac{d}{dz}[(z+1)^2 \; f(z)]$$

$$= \lim_{z \to -1} \frac{d}{dz}\left[\cancel{(z+1)^2} \; \frac{z-1}{\cancel{(z+1)^2}\,(z-2)}\right]$$

$$= \lim_{z \to -1} \left[\frac{(z-2).(1)-(z-1).(1)}{(z-2)^2}\right]$$

$$= \frac{(-3)-(-2)}{(-3)^2} = \frac{-3+2}{9} = -\frac{1}{9}$$

$\therefore$ *By Cauchy Residue theorem,*

$$\int_C f(z)dz = 2\pi i \operatorname{Res}(z=-1) = 2\pi i \left(\frac{-1}{9}\right) = -\frac{2\pi i}{9}$$

4. *Evaluate* $\int_C \frac{\sin z}{(z-1)^2(z^2+9)}\, dz$, *where C is the circle* $|z-3i|=1$.

Sol. *Let* $f(z)=\frac{\sin z}{(z-1)^2(z^2+9)}=\frac{\sin z}{(z-1)^2(z+3i)(z-3i)}$

Poles of $f(z)$ *are* $z=1,\ 3i,\ -3i$

$z=3i$ *is a simple pole, which lies inside the circle* $|z-3i|=1$.

$$\operatorname{Res}(z=3i)=\lim_{z\to 3i}(z-3i)\,f(z)$$

$$=\lim_{z\to 3i}(z-3i)\frac{\sin z}{(z-1)^2\,(z+3i)(z-3i)}$$

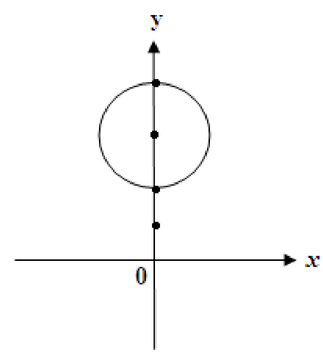
$$=\frac{\sin(3i)}{(3i-1)^2\,(6i)}$$

$$\boxed{\sin(ix)=i\sinh x}$$

$$=\frac{i\sinh 3\ (3i+1)^2}{(6i)(3i-1)^2\,(3i+1)^2}$$

$$=\frac{\sinh 3\ (-9+1+6i)}{(6)(-9-1)^2}$$

$$=\frac{(6i-8)\sinh 3}{600}$$



$\therefore$ *By Cauchy Residue theorem,*

$$\int_C f(z)dz = 2\pi i\ \operatorname{Res}(z=3i)$$

$$=2\pi i\left[\frac{(6i-8)\sinh 3}{600}\right]$$

$$=\frac{\pi i(6i-8)\sinh 3}{300}=\frac{\pi(-6-8i)\sinh 3}{300}$$

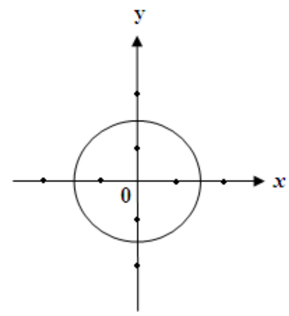
5. *Evaluate* $\int_C \frac{4-3z}{z(z-1)(z-2)}\, dz$, *where C is the circle* $|z|=\frac{3}{2}$.

Sol. *Let* $f(z)=\frac{4-3z}{z(z-1)(z-2)}$

Poles of $f(z)$ *are* $z=0,\ 1,\ 2$

$z=0,\ 1$ *are simple poles, which lies inside the circle* $|z|=\frac{3}{2}$.

$$\operatorname{Res}(z=0) = \lim_{z \to 0} (z-0) f(z)$$



$$= \lim_{z \to 0} (z-0) \frac{4-3z}{z(z-1)(z-2)}$$

$$= \frac{4-0}{(-1)(-2)}$$

$$= 2$$

$$\operatorname{Res}(z=1) = \lim_{z \to 1} (z-1) f(z)$$

$$= \lim_{z \to 1} (z-1) \frac{4-3z}{z(z-1)(z-2)}$$

$$= \frac{4-3}{1(-1)}$$

$$= -1$$

$\therefore$ *By Cauchy* Re*sidue theorem,*

$$\int_C f(z)dz = 2\pi i \, [\operatorname{Res}(z=0) + \operatorname{Res}(z=1)]$$

$$= 2\pi i \, [2-1]$$

$$= 2\pi i$$

Home Work

1. *Evaluate* $\int_C \frac{e^z}{(z+1)^2} \, dz$, *around the circle* $|z-1|=3$.

2. *Evaluate* $\int_C \frac{3z^2+2}{(z-1)(z^2+9)} \, dz$, *where C is the circle* $|z|=4$.

CONTOUR INTEGRATION

Type 1 ***Integration round the unit circle***

We proceed to evaluate the integrals of the form $\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$

$Using\ the\ formula\ \cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}\ and\ \sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}\ in\ the\ denominator$

$and\ if\ we\ take\ z = e^{i\theta}\ then\ dz = ie^{i\theta}\, d\theta\ \Rightarrow d\theta = \dfrac{dz}{iz}$

then the above takes the form

$$\int_C \phi(z)\, dz = 2\pi i\, [Sum\ of\ residues\ of\ \phi(z)\ at\ its\ poles\ inside\ C]$$

where C is the unit circle $|z| = 1.$

Problems

1. $Evaluate\ \displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$

$$z = e^{i\theta}$$
$$dz = ie^{i\theta} d\theta$$
$$\Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$$

$$Sol.\ Let\ I = \int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$$

$$= \int_0^{2\pi} \frac{d\theta}{5+4\left(\dfrac{e^{i\theta}-e^{-i\theta}}{2i}\right)}$$

$$= \int_C \frac{dz/iz}{5+2\left(\dfrac{z-z^{-1}}{i}\right)}\ ,\quad where\ C\ is\ the\ unit\ circle\ |z| = 1$$

$$= \int_C \frac{dz/iz}{5i+2\left(z-\dfrac{1}{z}\right)}$$

$$= \int_C \frac{dz}{5zi+2z^2-2}$$

$$= \frac{1}{2}\int_C \frac{dz}{z^2+\dfrac{5}{2}zi-1}$$

$$= \frac{1}{2}\int_C f(z)\, dz \quad ------ (1)$$

$$where\ \ f(z)=\frac{1}{z^2+\frac{5}{2}zi-1}$$

Poles of $f(z)$ are $z^2+\frac{5i}{2}z-1=0$

$$z=\frac{-\frac{5i}{2}\pm\sqrt{-\frac{25}{4}-4.1.(-1)}}{2.1}$$

$$=\frac{-\frac{5i}{2}\pm\sqrt{\frac{-25+16}{4}}}{2}=\frac{-\frac{5i}{2}\pm\sqrt{\frac{-9}{4}}}{2}$$

$$=\frac{-\frac{5i}{2}\pm\frac{3i}{2}}{2}$$

$$=\frac{-\frac{5i}{2}+\frac{3i}{2}}{2},\ \frac{-\frac{5i}{2}-\frac{3i}{2}}{2}$$

$$=\frac{-2i}{4},\frac{-8i}{4}$$

$$=\frac{-i}{2},-2i$$

$z=-2i$ lies outside the circle $|z|=1$

$z=-\frac{i}{2}$ lies inside the circle $|z|=1$, which is the simple pole .

$$\operatorname{Res}\left(z=-\frac{i}{2}\right)=\lim_{z\to-\frac{i}{2}}\left(z+\frac{i}{2}\right)f(z)$$

$$=\lim_{z\to-\frac{i}{2}}\left(z+\frac{i}{2}\right)\frac{1}{\left(z+\frac{i}{2}\right)(z+2i)}=\frac{1}{\left(-\frac{i}{2}+2i\right)}=\frac{2}{3i}$$

$\therefore$ *By Cauchy* Re*sidue theorem*,

$$\int_C f(z)dz=2\pi i\operatorname{Res}\left(z=-\frac{i}{2}\right)=2\pi i\left(\frac{2}{3i}\right)=\frac{4\pi}{3}$$

$$(1)\Rightarrow\quad I=\frac{1}{2}\int_C f(z)\,dz$$

$$=\frac{1}{2}\left(\frac{4\pi}{3}\right)=\frac{2\pi}{3}$$

2. *Evaluate* $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta$

$$z = e^{i\theta}$$
$$dz = i e^{i\theta} d\theta$$
$$\Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$$

Sol. Let $I = \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta$

$$= R.P. \int_0^{2\pi} \frac{e^{i2\theta}}{5+4\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)} d\theta$$

$$= R.P. \int_C \frac{z^2}{5+2(z+z^{-1})} \frac{dz}{iz}, \quad \textit{where C is the unit circle } |z| = 1$$

$$= R.P. \frac{1}{i} \int_C \frac{z}{5+2\left(z+\frac{1}{z}\right)} dz$$

$$= R.P. \frac{1}{i} \int_C \frac{z^2}{5z+2z^2+2} dz$$

$$= R.P \frac{1}{2i} \int_C \frac{z^2}{z^2+\frac{5}{2}z+1} dz$$

$$= R.P. \frac{1}{2i} \int_C f(z)\, dz \quad ----- (1)$$

$$\textit{where } f(z) = \frac{z^2}{z^2+\frac{5}{2}z+1}$$

Poles of $f(z)$ are $z^2 + \frac{5}{2}z + 1 = 0$

$$z = \frac{-\frac{5}{2} \pm \sqrt{\frac{25}{4} - 4.1.1}}{2.1}$$

$$= \frac{-\frac{5}{2} \pm \sqrt{\frac{25-16}{4}}}{2} = \frac{-\frac{5}{2} \pm \sqrt{\frac{9}{4}}}{2}$$

$$= \frac{-\frac{5}{2} \pm \frac{3}{2}}{2}$$

$$= \frac{-\frac{5}{2}+\frac{3}{2}}{2}, \ \frac{-\frac{5}{2}-\frac{3}{2}}{2}$$

$$= \frac{-2}{4}, \frac{-8}{4}$$

$$= \frac{-1}{2}, -2$$

$z = -2$ lies outside the circle $|z| = 1$

$z = -\frac{1}{2}$ lies inside the circle $|z| = 1$, which is the simple pole .

$$\operatorname{Res}\left(z = -\frac{1}{2}\right) = \lim_{z \to -\frac{1}{2}} \left(z + \frac{1}{2}\right) f(z)$$

$$= \lim_{z \to -\frac{1}{2}} \left(z + \frac{1}{2}\right) \frac{z^2}{\left(z+\frac{1}{2}\right)(z+2)} = \frac{1/4}{\left(-\frac{1}{2}+2\right)} = \frac{1/4}{3/2} = \frac{1}{6}$$

$\therefore$ *By Cauchy Residue theorem,*

$$\int_C f(z)dz = 2\pi i \operatorname{Res}\left(z = -\frac{1}{2}\right) = 2\pi i \left(\frac{1}{6}\right) = \frac{\pi i}{3}$$

$$(1) \Rightarrow \quad I = R.P. \frac{1}{2i} \int_C f(z)\, dz$$

$$= R.P. \frac{1}{2i}\left(\frac{\pi i}{3}\right) = \frac{\pi}{6}$$

3. *Evaluate* $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta}$, $|a| < 1$ (or) $-1 < a < 1$

$$z = e^{i\theta}$$
$$dz = i e^{i\theta} d\theta$$
$$\Rightarrow d\theta = \frac{dz}{i e^{i\theta}} = \frac{dz}{iz}$$

Sol. *Let* $I = \displaystyle\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta}$

$$= \int_0^{2\pi} \frac{d\theta}{1 + a\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)}$$

$$= \int_C \frac{dz/iz}{1 + a\left(\frac{z - z^{-1}}{2i}\right)}, \quad \textit{where } C \textit{ is the unit circle } |z| = 1$$

$$= \int_C \frac{dz/iz}{1+\frac{a}{2i}\left(z-\frac{1}{z}\right)}$$

$$= \int_C \frac{2\,dz}{2zi + az^2 - a}$$

$$= \frac{2}{a}\int_C \frac{dz}{z^2 + \frac{2}{a}zi - 1}$$

$$= \frac{2}{a}\int_C f(z)\,dz \quad ------ (1)$$

$$where \quad f(z) = \frac{1}{z^2 + \frac{2i}{a}z - 1}$$

Poles of $f(z)$ are $z^2 + \frac{2i}{a}z - 1 = 0$

$$z = \frac{-\frac{2i}{a} \pm \sqrt{-\frac{4}{a^2} - 4.1.(-1)}}{2.1}$$

$$= \frac{-\frac{2i}{a} \pm \sqrt{\frac{-4+4a^2}{a^2}}}{2} = \frac{-\frac{2i}{a} \pm 2\,\frac{\sqrt{-(1-a^2)}}{a}}{2}$$

$$= \frac{-i \pm i\sqrt{1-a^2}}{a}$$

$$Let \quad z_1 = \frac{-i + i\sqrt{1-a^2}}{a}, \quad z_2 = \frac{-i - i\sqrt{1-a^2}}{a}$$

z_2 lies outside the circle $|z| = 1$

z_1 lies inside the circle $|z| = 1$, which is the simple pole .

$$\operatorname{Res}(z=z_1) = \lim_{z \to z_1} (z-z_1)\, f(z)$$

$$= \lim_{z \to z_1} (z-z_1)\, \frac{1}{(z-z_1)(z-z_2)}$$

$$= \frac{1}{z_1 - z_2}$$

$$= \frac{1}{\left(\frac{-i+i\sqrt{1-a^2}}{a}\right) - \left(\frac{-i-i\sqrt{1-a^2}}{a}\right)}$$

$$= \frac{a}{2i\sqrt{1-a^2}}$$

$\therefore$ *By Cauchy Residue theorem,*

$$\int_C f(z)dz = 2\pi i \operatorname{Res}(z = z_1)$$

$$= 2\pi i \left(\frac{a}{2i\sqrt{1-a^2}}\right) = \frac{\pi a}{\sqrt{1-a^2}}$$

$$(1) \Rightarrow \quad I = \frac{2}{a}\int_C f(z)\,dz$$

$$= \frac{2}{a}\left(\frac{\pi a}{\sqrt{1-a^2}}\right) = \frac{2\pi}{\sqrt{1-a^2}}$$

4. *Evaluate* $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta$

Sol. Let $\displaystyle I = \int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta$

$$z = e^{i\theta}$$
$$dz = ie^{i\theta}d\theta$$
$$\Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$$

$$= \int_0^{2\pi} \frac{\left(\frac{1-\cos 2\theta}{2}\right)}{a+b\cos\theta}\,d\theta$$

$$= \frac{1}{2} R.P. \int_0^{2\pi} \frac{1-e^{i2\theta}}{a+b\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)}\,d\theta$$

$$= R.P. \int_C \frac{1-z^2}{2a+b(z+z^{-1})}\frac{dz}{iz}, \quad \textit{where C is the unit circle } |z| = 1$$

$$= R.P. \int_C \frac{1-z^2}{2a+b\left(z+\frac{1}{z}\right)}\frac{dz}{iz}$$

$$= R.P.\ \frac{1}{i} \int_C \frac{1-z^2}{2az+bz^2+b}\, dz$$

$$= R.P\ \frac{1}{bi} \int_C \frac{1-z^2}{z^2+\frac{2a}{b}z+1}\, dz$$

$$= R.P.\ \frac{1}{bi} \int_C f(z)\, dz \quad ------ (1) \quad where\ \ f(z) = \frac{1-z^2}{z^2+\frac{2a}{b}z+1}$$

Poles of $f(z)$ are $z^2+\frac{2a}{b}z+1=0$

$$z = \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2}{b^2}-4.1.1}}{2.1}$$

$$= \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2-4b^2}{b^2}}}{2} = \frac{-\frac{2a}{b} \pm 2\,\frac{\sqrt{a^2-b^2}}{b}}{2}$$

$$= \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

$$Let\ \ z_1 = \frac{-a+\sqrt{a^2-b^2}}{b}, \quad z_2 = \frac{-a-\sqrt{a^2-b^2}}{b}$$

z_2 lies outside the circle $|z|=1$

z_1 lies inside the circle $|z|=1$, which is the simple pole.

$$\operatorname{Re}s\,(z=z_1) = \lim_{z\to z_1} (z-z_1)\, f(z)$$

$$= \lim_{z\to z_1} (z-z_1)\, \frac{1-z^2}{(z-z_1)(z-z_2)}$$

$$= \frac{1-z_1^{\ 2}}{z_1-z_2}$$

$$= \frac{1-\left(\frac{-a+\sqrt{a^2-b^2}}{b}\right)^2}{\left(\frac{-a+\sqrt{a^2-b^2}}{b}\right)-\left(\frac{-a-\sqrt{a^2-b^2}}{b}\right)}$$

$$= \frac{b^2 - (a^2 + a^2 - b^2 - 2a\sqrt{a^2 - b^2})}{b^2} \times \frac{b}{2\sqrt{a^2 - b^2}}$$

$$= \frac{-2a^2 + 2b^2 + 2a\sqrt{a^2 - b^2}}{2b\sqrt{a^2 - b^2}}$$

$$= \frac{-(a^2 - b^2) + a\sqrt{a^2 - b^2}}{b\sqrt{a^2 - b^2}}$$

$$= \frac{-\sqrt{a^2 - b^2}}{b} + \frac{a}{b}$$

$$= \frac{a - \sqrt{a^2 - b^2}}{b}$$

$\therefore$ *By Cauchy Residue theorem,*

$$\int_C f(z)dz = 2\pi i \operatorname{Res}(z = z_1)$$

$$= 2\pi i \left(\frac{a - \sqrt{a^2 - b^2}}{b} \right)$$

$$(1) \Rightarrow \quad I = R.P. \frac{1}{bi} \int_C f(z)\, dz$$

$$= R.P. \frac{1}{bi} \left[\frac{2\pi i\, (a - \sqrt{a^2 - b^2})}{b} \right]$$

$$= \frac{2\pi}{b^2} [a - \sqrt{a^2 - b^2}]$$

5. *Evaluate* $\displaystyle\int_0^{\pi} \frac{a\, d\theta}{a^2 + \sin^2\theta}$ *where* $a > 0$.

Sol. *Let* $\displaystyle I = \int_0^{\pi} \frac{a\, d\theta}{a^2 + \sin^2\theta} = \int_0^{\pi} \frac{a\, d\theta}{a^2 + \left(\frac{1 - \cos 2\theta}{2}\right)}$

$$= \int_0^{\pi} \frac{2a\, d\theta}{2a^2 + 1 - \cos 2\theta}$$

$$= \int_0^{2\pi} \frac{a\, dt}{2a^2 + 1 - \cos t}$$

put $2\theta = t$

$2d\theta = dt$

When $\theta = 0 \Rightarrow t = 0$

When $\theta = \pi \Rightarrow t = 2\pi$

$$= \int_0^{2\pi} \frac{a\,dt}{2a^2+1-\left(\dfrac{e^{it}+e^{-it}}{2}\right)}$$

$$z = e^{it}$$

$$dz = i e^{it}\, dt$$

$$\Rightarrow dt = \frac{dz}{ie^{it}} = \frac{dz}{iz}$$

$$= \int_C \frac{a}{2a^2+1-\left(\dfrac{z+z^{-1}}{2}\right)} \frac{dz}{iz} \quad , \quad \textit{where C is the unit circle } |z| = 1$$

$$= \frac{1}{i} \int_C \frac{2a}{4a^2+2-\left(z+\dfrac{1}{z}\right)} \frac{dz}{z}$$

$$= \frac{1}{i} \int_C \frac{2a}{(4a^2+2)\,z - z^2 - 1}\, dz$$

$$= -\frac{2a}{i} \int_C \frac{1}{z^2-(4a^2+2)\,z+1}\, dz$$

$$= -\frac{2a}{i} \int_C f(z)\; dz \quad ------ (1)$$

$$where \quad f(z) = \frac{1}{z^2-(4a^2+2)z+1}$$

Poles of $f(z)$ are $z^2-(4a^2+2)z+1=0$

$$z = \frac{(4a^2+2) \pm \sqrt{(4a^2+2)^2-4.1.1}}{2.1}$$

$$= \frac{(4a^2+2) \pm \sqrt{16a^4+16a^2+4-4}}{2}$$

$$= \frac{(4a^2+2) \pm \sqrt{16a^2(a^2+1)}}{2}$$

$$= \frac{(4a^2+2) \pm 4a\ \sqrt{a^2+1}}{2}$$

$$= (2a^2+1) \pm 2a\ \sqrt{a^2+1}$$

$$Let \quad z_1 = (2a^2+1) + 2a\sqrt{a^2+1}\ , \quad z_2 = (2a^2+1) - 2a\sqrt{a^2+1}$$

z_1 lies outside the circle $|z| = 1$

z_2 lies inside the circle $|z| = 1$, which is the simple pole .

$$\operatorname{Re}s\,(z=z_2) = \lim_{z \to z_2} (z-z_2)\, f(z)$$

$$= \lim_{z \to z_2} (z-z_2) \frac{1}{(z-z_1)(z-z_2)}$$

$$= \frac{1}{z_2 - z_1}$$

$$= \frac{1}{[(2a^2+1) - 2a\sqrt{a^2+1}] - [(2a^2+1) + 2a\sqrt{a^2+1}]}$$

$$= \frac{1}{-4a\sqrt{a^2+1}}$$

$\therefore$ *By Cauchy* Re*sidue theorem,*

$$\int_C f(z)dz = 2\pi i \operatorname{Re}s\,(z=z_2)$$

$$= 2\pi i \left(\frac{1}{-4a\sqrt{a^2+1}}\right) = \frac{-\pi i}{2a\sqrt{a^2+1}}$$

$$(1) \Rightarrow \quad I = -\frac{2a}{i}\int_C f(z)\,dz$$

$$= -\frac{2a}{i}\left[\frac{-\pi i}{2a\sqrt{a^2+1}}\right]$$

$$= \frac{\pi}{\sqrt{a^2+1}}$$

6. *Evaluate* $\displaystyle\int_{-\pi}^{\pi} \frac{a\cos\theta}{a+\cos\theta}\, d\theta$ *where* $a > 1$.

Sol. Let $\displaystyle I = \int_{-\pi}^{\pi} \frac{a\cos\theta}{a+\cos\theta}\, d\theta$

$$= \int_0^{2\pi} \frac{a\cos\theta}{a+\cos\theta}\, d\theta$$

$$= R.P. \int_0^{2\pi} \frac{a\,e^{i\theta}}{a + \left(\dfrac{e^{i\theta}+e^{-i\theta}}{2}\right)}\, d\theta$$

Since $\displaystyle\int_{-\pi}^{\pi} f(\cos\theta)\, d\theta = 2\int_0^{\pi} f(\cos\theta)\, d\theta$

$$= 2 \times \frac{1}{2}\int_0^{2\pi} f(\cos\theta)\, d\theta$$

$$= \int_0^{2\pi} f(\cos\theta)\, d\theta$$

$$= R.P. \int_C \frac{a\,z}{a + \left(\frac{z + z^{-1}}{2}\right)} \frac{d\,z}{i\,z} \quad , \ where\ C\ is\ the\ unit\ circle\ |z| = 1$$

$$z = e^{i\theta}$$
$$d\,z = i\,e^{i\theta} d\theta$$
$$\Rightarrow d\theta = \frac{d\,z}{i\,e^{i\theta}} = \frac{d\,z}{i\,z}$$

$$= R.P. \frac{1}{i} \int_C \frac{2\,a}{2\,a + \left(z + \frac{1}{z}\right)} d\,z$$

$$= R.P. \frac{1}{i} \int_C \frac{2a\,z}{2\,a\,z + z^2 + 1} d\,z$$

$$= R.P. \frac{2\,a}{i} \int_C \frac{z}{z^2 + 2\,a\,z + 1} d\,z$$

$$= R.P. \frac{2\,a}{i} \int_C f(z)\ d\,z \ ------ (1)$$

$$where \ \ f(z) = \frac{z}{z^2 + 2\,a\,z + 1}$$

Poles of $f(z)$ are $z^2 + 2\,a\,z + 1 = 0$

$$z = \frac{-2\,a \pm \sqrt{4a^2 - 4.1.1}}{2.1}$$

$$= \frac{-2\,a \pm 2\sqrt{a^2 - 1}}{2}$$

$$= -a \pm \sqrt{a^2 - 1}$$

$$Let \ \ z_1 = -a + \sqrt{a^2 - 1} \ , \quad z_2 = -a - \sqrt{a^2 - 1}$$

z_2 lies outside the circle $|z| = 1$

z_1 lies inside the circle $|z| = 1$, which is the simple pole .

$$\operatorname{Re}s\,(z = z_1) = \lim_{z \to z_1} (z - z_1)\ f(z)$$

$$= \lim_{z \to z_1} (z - z_1) \frac{z}{(z - z_1)(z - z_2)}$$

$$= \frac{z_1}{z_1 - z_2}$$

$$= \frac{-a + \sqrt{a^2 - 1}}{[-a + \sqrt{a^2 - 1}\ ] - [-a - \sqrt{a^2 - 1}\ ]} = \frac{-a + \sqrt{a^2 - 1}}{2\sqrt{a^2 - 1}}$$

$\therefore$ *By Cauchy Residue theorem,*

$$\int_C f(z)dz = 2\pi i \operatorname{Res}(z = z_1)$$

$$= 2\pi i \left(\frac{-a+\sqrt{a^2-1}}{2\sqrt{a^2-1}} \right) = \frac{\pi i \, (-a+\sqrt{a^2-1}\,)}{\sqrt{a^2-1}}$$

$$(1) \Rightarrow \quad I = R.P. \; \frac{2a}{i} \int_C f(z)\, dz$$

$$= R.P. \; \frac{2a}{i} \left[\frac{\pi i \, (-a+\sqrt{a^2-1}\,)}{\sqrt{a^2-1}} \right]$$

$$= 2\pi a \left(1 - \frac{a}{\sqrt{a^2-1}} \right)$$

7. *Evaluate* $\displaystyle\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$, $|a| < 1$

$$z = e^{i\theta}$$
$$dz = ie^{i\theta} d\theta$$
$$\Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$$

Sol. Let $\displaystyle I = \int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$

$$= \int_0^{2\pi} \frac{d\theta}{1-2a\left(\dfrac{e^{i\theta}+e^{-i\theta}}{2}\right)+a^2}$$

$$= \int_C \frac{dz/iz}{1-a(z+z^{-1})+a^2}, \quad \text{where } C \text{ is the unit circle } |z| = 1$$

$$= \int_C \frac{dz/iz}{1-a\left(z+\dfrac{1}{z}\right)+a^2}$$

$$= \frac{1}{i} \int_C \frac{dz}{z-a(z^2+1)+a^2 z}$$

$$= \frac{1}{i} \int_C \frac{dz}{-az^2+(a^2+1)z-a}$$

$$= \frac{1}{-ai} \int_C \frac{dz}{z^2-\left(\dfrac{a^2+1}{a}\right)z+1} = \frac{1}{-ai} \int_C f(z)\, dz \; ------ (1)$$

$$where \quad f(z) = \frac{1}{z^2 - \left(\frac{a^2+1}{a}\right) z + 1}$$

Poles of $f(z)$ are $z^2 - \left(\frac{a^2+1}{a}\right) z + 1 = 0$

$$z = \frac{\frac{a^2+1}{a} \pm \sqrt{\left(\frac{a^2+1}{a}\right)^2 - 4.1.1}}{2.1}$$

$$= \frac{\frac{a^2+1}{a} \pm \sqrt{\frac{(a^2+1)^2 - 4a^2}{a^2}}}{2}$$

$$= \frac{\frac{a^2+1}{a} \pm \sqrt{\frac{a^4+2a^2+1-4a^2}{a^2}}}{2}$$

$$= \frac{\frac{a^2+1}{a} \pm \sqrt{\frac{a^4-2a^2+1}{a^2}}}{2}$$

$$= \frac{\frac{a^2+1}{a} \pm \sqrt{\frac{(a^2-1)^2}{a^2}}}{2} \quad = \frac{\frac{a^2+1}{a} \pm \frac{a^2-1}{a}}{2}$$

$$= \frac{\frac{a^2+1}{a} + \frac{a^2-1}{a}}{2}, \quad \frac{\frac{a^2+1}{a} - \frac{a^2-1}{a}}{2}$$

$$= \frac{2a^2}{2a}, \quad \frac{2}{2a}$$

$$= a, \ \frac{1}{a}$$

$z = \frac{1}{a}$ lies outside the circle $|z| = 1$

z = a lies inside the circle $|z| = 1$, which is the simple pole .

$$\operatorname{Res}(z=a) = \lim_{z \to a} (z-a) f(z)$$

$$= \lim_{z \to a} (z-a) \frac{1}{(z-a)\left(z-\frac{1}{a}\right)} = \frac{1}{\left(a-\frac{1}{a}\right)} = \frac{a}{a^2-1}$$

$\therefore$ *By Cauchy Residue theorem,*

$$\int_C f(z)dz = 2\pi i \operatorname{Res}(z=a) = 2\pi i \left(\frac{a}{a^2-1}\right) = \frac{2\pi i a}{a^2-1}$$

$$(1) \Rightarrow \quad I = \frac{1}{-ai} \int_C f(z)\, dz$$

$$= \frac{1}{-ai}\left(\frac{2\pi i a}{a^2-1}\right) = \frac{2\pi}{1-a^2}$$

Home Work

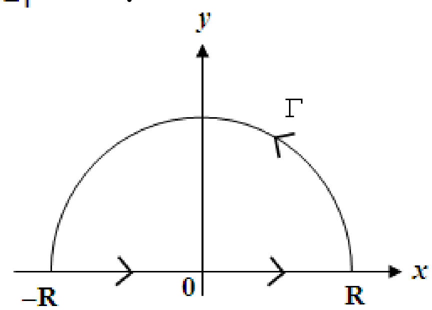
1. *Show that* $\displaystyle\int_0^{2\pi} \frac{4\, d\theta}{4+\sin\theta} = \frac{8\pi}{\sqrt{15}}$
2. *Show that* $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta = \frac{\pi}{12}$
3. *Show that* $\displaystyle\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta} = \frac{\pi}{6}$
4. *Show that* $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5-4\cos\theta}\, d\theta = \frac{\pi}{4}$
5. *Show that* $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+k\cos\theta} = \frac{2\pi}{\sqrt{1-k^2}} \quad (k^2 < 1)$

Type 2

Evaluation of integrals of the type $\int_{-\infty}^{\infty} f(z)\,dz$ where the function f(z) is such that no pole of f(z) lies on the real line, but poles lie in the upper half of z-plane. We evaluate the above integrals by considering them along a closed contour C consisting of

i) Semi circle | z | = R in the upper half plane.
ii) Real axis from –R to R.

Then we try to show that integral along Γ vanishes as $|z| \to \infty$.



Thus $\int_c f(z)dz = \int_\Gamma f(z)dz + \int_{-R}^{R} f(x)dx$

Taking limit as $R \to \infty$, we have

$$\int_c f(z)dz = \int_{-\infty}^{\infty} f(x)dx$$

By Cauchy's Residue theorem, this becomes

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\,[Sum\ \ of\ \ the\ \ residues\ \ within\ \ C]$$

Cauchy's Lemma:

If f(z) is a continuous function such that $|zf(z)| \to 0$ uniformly as $|z| \to \infty$ on Γ, then $\lim_{R\to\infty}\int_\Gamma f(z)dz = 0$ where Γ is the semi circle $|z| = R$ above the real axis.

Jordan's lemma:

If f(z) is analytic except at finite number of singularities and if $f(z) \to 0$ uniformly as $z \to \infty$ then $\lim_{R\to\infty}\int_\Gamma e^{imz} f(z)\,dz = 0\ (m > 0)$ where Γ is the semi circle $|z| = R$ above the imaginary axis.

Problems

1. *Evaluate* $\int_0^{\infty} \frac{dx}{1+x^2}$

Sol. *Let* $f(z) = \frac{1}{1+z^2}$

Consider the integral $\int_C f(z)dz$, *where C is the closed contour consisting of* Γ, *the upper half of the large circle* $|z| = R$ *and the real axis from* $-R$ *to* R

$$Poles\ of\ f(z)\ are\ z^2+1=0$$

$$z=\pm i$$

$z=i$ *lies within* C, *which is the simple pole.*

$$\operatorname{Res}(z=i)=\lim_{z\to i}[(z-i)\,f(z)]$$

$$=\lim_{z\to i}\left[\cancel{(z-i)}\,\frac{1}{\cancel{(z-i)}(z+i)}\right]$$

$$=\frac{1}{2i}$$

$\therefore$ *By Cauchy Residue theorem,*

$$\int_C f(z)\,dz=2\pi i\operatorname{Res}(z=i)$$

$$=2\pi i\left(\frac{1}{2i}\right)$$

$$\int_C f(z)\,dz=\pi$$

$$(i.e.)\int_{-\infty}^{\infty} f(x)\,dx=\pi\quad(by\ cauchy's\ lemma)$$

$$(i.e.)\int_{-\infty}^{\infty}\frac{dx}{1+x^2}=\pi$$

$$2\int_0^{\infty}\frac{dx}{1+x^2}=\pi$$

$$(i.e.)\int_0^{\infty}\frac{dx}{1+x^2}=\frac{\pi}{2}$$

2. *Evaluate* $\displaystyle\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}\,dx$

Sol. *Let* $\displaystyle f(z)=\frac{z^2-z+2}{z^4+10z^2+9}$

Consider the integral $\int_C f(z)dz$, *where* C *is the closed contour consisting of* Γ, *the upper half of the large circle* $|z|=R$ *and the real axis from* $-R$ *to* R

$$Poles\ of\ f(z)\ are\ z^4+10z^2+9=0$$

$$(z^2+1)(z^2+9)=0$$

$$z=\pm i,\ \pm 3i$$

$$z=i,\ 3i\ \ lies\ within\ C,\ which\ are\ the\ simple\ poles.$$

$$\operatorname{Res}(z=i)=\lim_{z\to i}\ [(z-i)\ f(z)]$$

$$=\lim_{z\to i}\left[\cancel{(z-i)}\ \frac{z^2-z+2}{\cancel{(z-i)}(z+i)(z^2+9)}\right]$$

$$=\frac{i^2-i+2}{(2i)(i^2+9)}=\frac{-1-i+2}{(2i)(-1+9)}=\frac{1-i}{(2i)(8)}=\frac{1-i}{16i}$$

$$\operatorname{Res}(z=3i)=\lim_{z\to 3i}\ [(z-3i)\ f(z)]$$

$$=\lim_{z\to 3i}\left[\cancel{(z-3i)}\ \frac{z^2-z+2}{\cancel{(z-3i)}(z+3i)(z^2+1)}\right]$$

$$=\frac{(3i)^2-3i+2}{(6i)[(3i)^2+1]}=\frac{-9-3i+2}{(6i)(-9+1)}=\frac{-7-3i}{(6i)(-8)}=\frac{7+3i}{48i}$$

$$\therefore\ By\ Cauchy\ \operatorname{Re}sidue\ theorem,$$

$$\int_C f(z)\,dz=2\pi i\,[\operatorname{Res}(z=i)+\operatorname{Res}(z=3i)]$$

$$=2\pi i\left(\frac{1-i}{16i}+\frac{7+3i}{48i}\right)$$

$$=2\pi i\left(\frac{3-3i+7+3i}{48i}\right)$$

$$=2\pi i\left(\frac{10}{48i}\right)$$

$$\int_C f(z)\,dz=\frac{5\pi}{12}$$

$$(i.e.)\int_{-\infty}^{\infty} f(x)\,dx=\frac{5\pi}{12}\quad (by\ cauchy's\ lemma)$$

$$(i.e.)\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}\,dx=\frac{5\pi}{12}$$

3. *Evaluate* $\int_0^{\infty} \frac{dx}{(1+x^2)^2}$

Sol. *Let* $f(z) = \frac{1}{(1+z^2)^2}$

Consider the integral $\int_C f(z)dz$, *where* C *is the closed contour consisting of* Γ, *the upper half of the large circle* $|z| = R$ *and the real axis from* $-R$ *to* R

Poles of $f(z)$ *are* $z^2 + 1 = 0$

$$z = \pm i$$

$z = i$ *lies within* C, *which is the pole of order* 2.

$$\operatorname{Res}(z=i) = \frac{1}{1!} \lim_{z \to i} \frac{d}{dz}[(z-i)^2 \ f(z)]$$

$$= \lim_{z \to i} \frac{d}{dz}\left[\cancel{(z-i)^2} \frac{1}{\cancel{(z-i)^2} (z+i)^2} \right]$$

$$= \lim_{z \to i} \left[\frac{-2}{(z+i)^3} \right]$$

$$= \frac{-2}{(2i)^3} = \frac{-2}{-8i} = \frac{1}{4i}$$

$\therefore$ *By Cauchy Residue theorem,*

$$\int_C f(z)\,dz = 2\pi i \operatorname{Res}(z=i)$$

$$= 2\pi i \left(\frac{1}{4i} \right)$$

$$\int_C f(z)\,dz = \frac{\pi}{2}$$

$$(i.e.) \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{2} \quad (by\ cauchy's\ lemma)$$

$$(i.e.) \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}$$

$$2\int_0^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}$$

$$(i.e.) \int_0^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}$$

4. *Evaluate* $\displaystyle\int_0^\infty \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx$

Sol. Let $f(z) = \dfrac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$

Consider the integral $\int_C f(z)dz$, where C is the closed contour consisting of Γ, the upper half of the large circle $|z| = R$ and the real axis from $-R$ to R

Poles of $f(z)$ are $(z^2+a^2)(z^2+b^2)=0$

$$z = \pm ai,\ \pm bi$$

$z = ai, bi$ lies within C, which are the simple poles.

$$\operatorname{Res}(z=ai) = \lim_{z\to ai}\,[(z-ai)\,f(z)]$$

$$= \lim_{z\to ai}\left[\cancel{(z-ai)}\,\frac{e^{iz}}{\cancel{(z-ai)}(z+ai)(z^2+b^2)}\right]$$

$$= \frac{e^{-a}}{(2ai)(-a^2+b^2)} = -\frac{e^{-a}}{(2ai)(a^2-b^2)}$$

$$\operatorname{Res}(z=bi) = \lim_{z\to bi}\,[(z-bi)\,f(z)]$$

$$= \lim_{z\to bi}\left[\cancel{(z-bi)}\,\frac{e^{iz}}{\cancel{(z-bi)}(z+bi)(z^2+a^2)}\right]$$

$$= \frac{e^{-b}}{(2bi)(-b^2+a^2)} = -\frac{e^{-b}}{(2bi)(a^2-b^2)}$$

$\therefore$ By Cauchy Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,[\operatorname{Res}(z=ai)+\operatorname{Res}(z=bi)]$$

$$= 2\pi i\left[\frac{-e^{-a}}{(2ai)(a^2-b^2)} + \frac{e^{-b}}{(2bi)(a^2-b^2)}\right]$$

$$= \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

$$\int_C f(z)\,dz = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

$$(i.e.) \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right) \quad (by\ Jordan's\ lemma)$$

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right)$$

Equating Real part, we have

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right)$$

$$2\int_{0}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right)$$

$$\int_{0}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{2(a^2-b^2)}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right)$$

5. *Evaluate* $\displaystyle\int_0^{\infty} \frac{x \sin x}{x^2+a^2}\,dx$

Sol. *Let* $f(z) = \dfrac{z e^{iz}}{z^2+a^2}$

Consider the integral $\int_C f(z)dz$, *where C is the closed contour consisting of* Γ, *the upper half of the large circle* $|z| = R$ *and the real axis from* $-R$ *to* R

Poles of $f(z)$ *are* $z^2+a^2=0$

$$z = \pm a i$$

$z = ai$ *lies within C, which is a simple pole.*

$$\operatorname{Res}(z=ai) = \lim_{z\to ai} [(z-ai)\, f(z)]$$

$$= \lim_{z\to ai}\left[\cancel{(z-ai)}\,\frac{z e^{iz}}{\cancel{(z-ai)}(z+ai)}\right]$$

$$= \frac{a i e^{-a}}{(2ai)}$$

$$= \frac{e^{-a}}{2}$$

$\therefore$ *By Cauchy Residue theorem,*

$$\int_C f(z)\,dz = 2\pi i \operatorname{Res}(z = ai)$$

$$= 2\pi i\left[\frac{e^{-a}}{2}\right]$$

$$= i\pi e^{-a}$$

$$\int_C f(z)\,dz = i\pi e^{-a}$$

$$(i.e.)\quad \int_{-\infty}^{\infty} f(x)\,dx = i\pi e^{-a} \quad (by\ Jordan's\ lemma)$$

$$\int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2 + a^2}\,dx = i\pi e^{-a}$$

Equating Imaginary part, we have

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2}\,dx = \pi e^{-a}$$

$$2\int_{0}^{\infty} \frac{x \sin x}{x^2 + a^2}\,dx = \pi e^{-a}$$

$$\int_{0}^{\infty} \frac{x \sin x}{x^2 + a^2}\,dx = \frac{\pi}{2} e^{-a}$$

6. *Show that* $\displaystyle\int_0^{\infty} \frac{dx}{x^4 + a^4} = \frac{\pi\sqrt{2}}{4a^3}$ *where* $a > 0.$

Sol. *Let* $f(z) = \dfrac{1}{z^4 + a^4}$

Consider the integral $\int_C f(z)dz$, *where* C *is the closed contour consisting of* Γ, *the upper half of the large circle* $|z| = R$ *and the real axis from* $-R$ *to* R

Poles of $f(z)$ *are* $z^4 + a^4 = 0$

$$z^4 = -a^4$$

$$= (\cos\pi + i\sin\pi)\,a^4$$

$$= [\cos(2n\pi + \pi) + i\sin(2n\pi + \pi)]\,a^4$$

$$= e^{i(2n+1)\pi}.a^4$$

$$z = a\, e^{i\,(2n+1)\frac{\pi}{4}}\,,\ n = 0, 1, 2, 3.$$

Poles of $f(z)$ *are* $a\,e^{i\frac{\pi}{4}},\ a\,e^{i\frac{3\pi}{4}},\ a\,e^{i\frac{5\pi}{4}},\ a\,e^{i\frac{7\pi}{4}}$.

But only two simple poles $z = a\,e^{i\frac{\pi}{4}},\ a\,e^{i\frac{3\pi}{4}}$ *lies within C*

We know that if $f(z) = \dfrac{\phi(z)}{\psi(z)}$ *then* $\operatorname{Re}s\ (z = z_0) = \dfrac{\phi(z_0)}{\psi'(z_0)}$

$$\boxed{\begin{aligned} f(z) &= \frac{1}{z^4 + a^4} = \frac{\phi(z)}{\psi(z)} \\ \phi(z) &= 1 \\ \psi(z) &= z^4 + a^4 \\ \psi'(z) &= 4z^3 \end{aligned}}$$

$$\begin{aligned} \operatorname{Re}s\ (z = \alpha) &= \lim_{z \to \alpha} \frac{\phi(z)}{\psi'(z)} \\ &= \lim_{z \to \alpha} \frac{1}{4z^3} \\ &= \lim_{z \to \alpha} \frac{z}{4z^4} \\ &= \lim_{z \to \alpha} \frac{z}{-4a^4} \quad [\text{since } z^4 + a^4 = 0] \\ &= -\frac{\alpha}{4a^4} \end{aligned}$$

$$\begin{aligned} \operatorname{Re}s\left(z = a\,e^{i\frac{\pi}{4}}\right) + \operatorname{Re}s\left(z = a\,e^{i\frac{3\pi}{4}}\right) &= -\frac{1}{4a^4}\left[a\,e^{i\frac{\pi}{4}} + a\,e^{i\frac{3\pi}{4}}\right] \\ &= -\frac{1}{4a^3}\left[e^{i\frac{\pi}{4}} + e^{i\pi}\,e^{-i\frac{\pi}{4}}\right] \\ &= -\frac{1}{4a^3}\left[e^{i\frac{\pi}{4}} - e^{-i\frac{\pi}{4}}\right] \\ &= -\frac{1}{4a^3}\left[2i\sin\frac{\pi}{4}\right] \\ &= \frac{-i}{2a^3}\,\frac{1}{\sqrt{2}} = \frac{-i}{2\sqrt{2}\ a^3} \end{aligned}$$

$\therefore$ *By Cauchy* Re*sidue theorem,*

$$\int_C f(z)\,dz = 2\pi i\ (Sum\ of\ residues\ within\ C)$$

$$= 2\pi i\left[\frac{-i}{2\sqrt{2}\ a^3}\right] = \frac{\pi}{\sqrt{2}\ a^3}$$

$$\int_C f(z)\,dz = \frac{\pi\sqrt{2}}{2a^3}$$

$$(i.e.)\quad \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi\sqrt{2}}{2a^3}$$

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+a^4} = \frac{\pi\sqrt{2}}{2a^3}$$

$$2\int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{\pi\sqrt{2}}{2a^3}$$

$$(i.e.)\quad \int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{\pi\sqrt{2}}{4a^3}$$

Home Work

1. *Show that* $\displaystyle\int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{6}$

2. *Show that* $\displaystyle\int_0^{\infty} \frac{x^2+2}{x^4+24x^2+144} = \frac{7\pi}{48\sqrt{3}}$

3. *Show that* $\displaystyle\int_0^{\infty} \frac{dx}{(x^2+1)^3}\,dx = \frac{3\pi}{16}$

UNIT – IV Ordinary Differential Equations

Higher order Linear Differential Equations with constant coefficients

To solve the equation $(aD^2 + bD + c)y = 0$ where $D = \frac{d}{dx}$

1) Write the Auxiliary Equation (A.E)
2) Find the roots of the A.E say m_1 and m_2.
3) The solution is
   (i) $y = Ae^{m_1 x} + Be^{m_2 x}, \ \ if \ \ m_1 \neq m_2.$
   (ii) $y = (Ax + B)e^{m_1 x}, \ \ if \ \ m_1 = m_2.$
   (iii) $y = e^{\alpha x}[A\cos\beta x + B\sin\beta x],$ if the roots are imaginary.

In general to solve n^{th} order linear differential equation

$D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \ldots\ldots + a_n y = 0$ where $D = \frac{d}{dx}$

1) Write the Auxiliary Equation (A.E)
2) Find the roots of the A.E say $m_1, m_2, m_3, \ldots\ldots m_n$
3) The solution is

<u>Case (i)</u> If all the roots are real and different, then the solution is

$$y = Ae^{m_1 x} + Be^{m_2 x} + Ce^{m_3 x} + De^{m_4 x} + \ldots\ldots\ldots\ldots$$

<u>Case(ii)</u> If two roots are equal and other roots are real and different, then $y = (Ax + B)e^{m_1 x} + Ce^{m_3 x} + De^{m_4 x} + \ldots\ldots\ldots\ldots$

<u>Case(iii)</u> If $m_1 = m_2$ and $m_3 = m_4$ and other roots are real and different, then

$$y = (Ax + B)e^{m_1 x} + (Cx + D)e^{m_3 x} + E\,e^{m_5 x} + \ldots\ldots\ldots\ldots$$

<u>Case(iv)</u> If three roots are equal & real and other roots are real and different, then $y = (Ax^2 + Bx + C)e^{m_1 x} + De^{m_4 x} + \ldots\ldots\ldots\ldots$

<u>Case(v)</u> If two roots are imaginary but other roots are real and different, then

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x) + Ce^{m_3 x} + \ldots\ldots\ldots\ldots$$

<u>Case(vi)</u> If $m_1 = \alpha + i\beta = m_2$ and $m_3 = \alpha - i\beta = m_4$ but other roots are real and different, then

$$y = e^{\alpha x}[(Ax + B)\cos\beta x + (Cx + D)\sin\beta x] + Ee^{m_5 x} + \ldots\ldots\ldots\ldots$$

Problems

1. Solve : $\frac{d^2 y}{dx^2} - 5\frac{dy}{dx} + 4y = 0 \quad (or) \quad y'' - 5y' + 4y = 0$

Sol. The given equation can be written as $(D^2 - 5D + 4)\,y = 0$

A.E is $m^2 - 5m + 4 = 0$

$$(m-1)(m-4) = 0$$

$$m = 1,\ 4$$

The solution is $y = A\,e^{x} + B\,e^{4x}$

2. Solve: $(D^2 + 2D + 1)\,y = 0$

Sol. A.E is $m^2 + 2m + 1 = 0$

$$(m+1)(m+1) = 0$$

$$m = -1,\ -1$$

The solution is $y = (Ax + B)\,e^{-x}$

3. Solve: $(D^2 - 3D + 5)\,y = 0$

Sol. A.E is $m^2 - 3m + 5 = 0$

$$m = \frac{-(-3) \pm \sqrt{(-3)^2 - 4.1.5}}{2.1}$$

$$= \frac{3 \pm \sqrt{-11}}{2}$$

$$= \frac{3 \pm i\sqrt{11}}{2} = \frac{3}{2} \pm i\,\frac{\sqrt{11}}{2} = \alpha \pm i\,\beta$$

The solution is $y = e^{\frac{3}{2}x}\left[A\cos\frac{\sqrt{11}}{2}x + B\sin\frac{\sqrt{11}}{2}x\right]$

4. Solve: $(D^3 + 2D^2 - D - 2)\,y = 0$

Sol. A.E is $m^3 + 2m^2 - m - 2 = 0$

1	1	2	–1	–2
	0	1	3	2
	1	3	2	0

m = 1 is a root.

The other roots are $m^2 + 3m + 2 = 0$

$$(m+1)(m+2) = 0$$

$$m = -1,\ -2$$

$$\therefore\ m = 1,\ -1,\ -2$$

The solution is $y = A\,e^{x} + B\,e^{-x} + C\,e^{-2x}$

5. Solve: $(D^3 - 3D + 2)\,y = 0$

Sol. A.E is $m^3 - 3m + 2 = 0$

1	1	0	–3	2
	0	1	1	–2
	1	1	–2	0

m = 1 is a root.

The other roots are $m^2+m-2=0$

$$(m-1)(m+2)=0$$

$$m=1,\,-2$$

$$\therefore\ m=1,1,-2$$

The solution is $y=(Ax+B)e^{x}+Ce^{-2x}$

6. Solve: $(D^4-2D^3+D^2)y=0$

Sol. A.E is $m^4-2m^3+m^2=0$

$$m^2(m^2-2m+1)=0$$

$$m^2=0 \quad or \quad m^2-2m+1=0$$

$$m=0,0 \qquad (m-1)(m-1)=0$$

$$m=1,\,1$$

The solution is $y=(Ax+B)e^{0x}+(Cx+D)e^{x}$

$$y=(Ax+B)+(Cx+D)e^{x}$$

7. Solve: $(D^3-3D^2+3D-1)y=0$

Sol. A.E is $m^3-3m^2+3m-1=0$

$$\begin{array}{r|rrrr} 1 & 1 & -3 & 3 & -1 \\ & 0 & 1 & -2 & 1 \\ \hline & 1 & -2 & 1 & \boxed{0} \end{array}$$

m = 1 is a root.

The other roots are $m^2-2m+1=0$

$$(m-1)(m-1)=0$$

$$m=1,\,1$$

$$\therefore\ m=1,1,1$$

The solution is $y=(Ax^2+Bx+C)e^{x}$

8. Solve: $(D^3+2D^2+4D+8)y=0$

Sol. A.E is $m^3+2m^2+4m+8=0$

$$\begin{array}{r|rrrr} -2 & 1 & 2 & 4 & 8 \\ & 0 & -2 & 0 & -8 \\ \hline & 1 & 0 & 4 & \boxed{0} \end{array}$$

m = –2 is a root.

The other roots are $m^2+0m+4=0$

$$m^2+4=0$$

$$m^2=-4$$

$$m=\pm 2i$$

$$\therefore\ m=-2,\ \pm 2i$$

The solution is $y=e^{0x}(A\cos 2x+B\sin 2x)+Ce^{-2x}$

$$y=A\cos 2x+B\sin 2x+Ce^{-2x}$$

$$(or) \qquad y=Ae^{-2x}+e^{0x}(B\cos 2x+C\sin 2x)$$

$$y=Ae^{-2x}+B\cos 2x+C\sin 2x$$

9. Solve: $(D^4-1)y=0$

Sol. A.E is $m^4-1=0$

$$(m^2-1)(m^2+1)=0$$
$$m^2=1 \quad or \quad m^2=-1$$
$$m=\pm 1 \qquad m=\pm i$$
$$m=1,-1,\ \pm i$$

The solution is $y=A e^{x}+B e^{-x}+C\cos x+D\sin x$

10. Solve: $(D^4+8D^2+16)y=0$

Sol. A.E is $m^4+8m^2+16=0$

$$(m^2+4)(m^2+4)=0$$
$$m^2=-4 \quad or \quad m^2=-4$$
$$m=\pm 2i \qquad m=\pm 2i$$

The solution is $y=e^{0x}\left[(Ax+B)\cos 2x+(Cx+D)\sin 2x\right]$

$$y=(Ax+B)\cos 2x+(Cx+D)\sin 2x$$

11. Solve: $(6D^2-5D-6)y=0$

Sol. A.E is $6m^2-5m-6=0$

$$6m^2-9m+4m-6=0$$
$$3m(2m-3)+2(2m-3)=0$$
$$(2m-3)(3m+2)=0$$
$$m=\frac{3}{2},\ -\frac{2}{3}$$

The solution is $y=A e^{3/2\,x}+B e^{-2/3\,x}$

Home Work

1. $Solve:\ (D^3+6D^2+11D+6)y=0$
2. $Solve:\ (D^3-4D^2+5D-2)y=0$
3. $Solve:\ (D^4+2D^3+D^2)y=0$
4. $Solve:\ (D^3-3D^2+4)y=0$
5. $Solve:\ (D^3-1)y=0$
6. $Solve:\ (D^3-D)y=0$
7. $Solve:\ (D^4+4D^3+8D^2+8D+4)y=0$

Particular Integral (P.I)

Suppose the differential equation is $(aD^2 + bD + c)y = X$

(or) $f(D)y = X$ where $f(D) = aD^2 + bD + c$

Then the complete solution is $y = C.F + P.I$

C.F (Complementary Function) :-

This is the solution of the given equation assuming R.H.S to be zero.

P.I (Particular Integral)

$$P.I. = \frac{1}{f(D)} X$$

$$= \frac{1}{aD^2 + bD + c} X$$

Type – 1

Let X be of the form e^{ax}

$$P.I. = \frac{1}{f(D)} e^{ax}$$

$$= \frac{1}{f(a)} e^{ax}, \quad if \quad f(a) \neq 0 \ [put \ \ D = a]$$

$$If \ f(a) = 0, \ then \ \ P.I. = \frac{1}{f(D)} e^{ax}$$

$$= \frac{x}{Diff.\ Coeff.\ of \ \ Dr.} e^{ax}$$

Problems

1. Solve: $(D^2 - 3D + 2)y = e^{5x}$

Sol. A.E is $m^2 - 3m + 2 = 0$

$$(m-1)(m-2) = 0$$

$$m = 1, 2$$

$$C.F = A e^x + B e^{2x}$$

$$P.I. = \frac{1}{D^2 - 3D + 2} e^{5x}$$

$$= \frac{1}{25 - 15 + 2} e^{5x}$$

$$= \frac{e^{5x}}{12}$$

$$y = C.F + P.I$$

$$(i.e.) \ \ y = A e^x + B e^{2x} + \frac{e^{5x}}{12}$$

2. Solve: $(D^2-4D-5)y=e^{5x}$

Sol. A.E is $m^2-4m-5=0$

$$(m-5)(m+1)=0$$

$$m=5,\ -1$$

$$C.F = A\,e^{5x}+B\,e^{-x}$$

$$P.I.=\frac{1}{D^2-4D-5}\,e^{5x}$$

$$=\cancel{\frac{1}{25-20-5}}\,e^{5x}$$ [Since f(5) = 0, multiply x on Nr. and Diff. Dr. w.r.to 'D']

$$=\frac{x}{2D-4}e^{5x}$$

$$=\frac{x}{10-4}e^{5x}\ =\frac{x}{6}e^{5x}$$

$$y=C.F+P.I$$

$$(i.e.)\ \ y=A\,e^{5x}+B\,e^{-x}+\frac{x}{6}e^{5x}$$

3. Solve: $(D^3-D^2-D+1)y=2e^{x}$

Sol. A.E is $m^3-m^2-m+1=0$

$$m^2(m-1)-1(m-1)=0$$

$$(m-1)(m^2-1)=0$$

$$m=1,\ m=\pm 1$$

$$\therefore\ m=1,1,-1$$

$$C.F=(Ax+B)e^{x}+C\,e^{-x}$$

$$P.I.=2\ \frac{1}{D^3-D^2-D+1}\,e^{x}$$

$$=2\ \cancel{\frac{1}{1-1-1+1}}\,e^{x}$$ [Since f(1) = 0, multiply x on Nr. and Diff. Dr. w.r.to 'D']

$$=2\ \frac{x}{3D^2-2D-1}e^{x}$$

$$=2\ \cancel{\frac{x}{3-2-1}}e^{x}$$ [Again, since f(1) = 0, multiply x on Nr. and Diff. Dr. w.r.to 'D']

$$=2\ \frac{x^2}{6D-2}\,e^{x}$$

$$=2\ \frac{x^2}{4}e^{x}\ =\frac{x^2}{2}e^{x}$$

$$y=C.F+P.I$$

$$(i.e.)\ \ y=(Ax+B)\,e^{x}+C\,e^{-x}+\frac{x^2}{2}\,e^{x}$$

4. Solve: $(6D^2-5D-6)y = 2e^{\frac{3}{2}x} - 5$

Sol. A.E is $6m^2-5m-6=0$

$$6m^2-9m+4m-6=0$$

$$3m(2m-3)+2(2m-3)=0$$

$$(2m-3)(3m+2)=0$$

$$m=\frac{3}{2},\ -\frac{2}{3}$$

$$C.F = A\,e^{3/2\,x} + B\,e^{-2/3\,x}$$

$$P.I. = \frac{1}{6D^2-5D-6}(2e^{3/2x}-5)$$

$$= 2\frac{1}{6D^2-5D-6}\,e^{3/2x} - 5\,\frac{1}{6D^2-5D-6}\,e^{0x}$$

$$= 2\frac{x}{12D-5}\,e^{3/2x} - 5\,\frac{1}{-6}$$

[Since f(3/2) = 0, multiply x on Nr. and Diff. Dr. w.r.to 'D']

$$= 2\frac{x}{18-5}e^{3/2x} + \frac{5}{6}$$

$$= \frac{2}{13}\,xe^{3/2x} + \frac{5}{6}$$

$$(i.e.)\ y = A\,e^{3/2x} + B\,e^{-2/3x} + \frac{2}{13}\,x\,e^{3/2x} + \frac{5}{6}$$

5. Solve: $(D^2-5D+6)y = \sinh x$

Sol. A.E is $m^2-5m+6=0$

$$(m-2)(m-3)=0$$

$$m=2,\ 3$$

$$C.F = A\,e^{2x} + B\,e^{3x}$$

$$P.I. = \frac{1}{D^2-5D+6}\sinh x$$

$$= \frac{1}{D^2-5D+6}\left(\frac{e^x-e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2-5D+6}\,e^x - \frac{1}{D^2-5D+6}\,e^{-x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-5+6}\,e^x - \frac{1}{1+5+6}\,e^{-x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}e^x - \frac{1}{12}e^{-x}\right] = \frac{e^x}{4} - \frac{e^{-x}}{24}$$

$$(i.e.)\ y = A\,e^{2x} + B\,e^{3x} + \frac{e^x}{4} - \frac{e^{-x}}{24}$$

6. Solve: $(D^3 - 12D + 16)y = (e^x + e^{-2x})^2$

Sol. A.E is $m^3 - 12m + 16 = 0$

2	1	0	–12	16
	0	2	4	–16
	1	2	–8	0

m = 2 is a root.

The other roots are $m^2 + 2m - 8 = 0$

$$(m+4)(m-2) = 0$$

$$m = -4,\ 2$$

$$\therefore\ m = 2,\ 2,\ -4$$

$$C.F = (Ax + B)e^{2x} + Ce^{-4x}$$

$$P.I. = \frac{1}{D^3 - 12D + 16}(e^x + e^{-2x})^2$$

$$= \frac{1}{D^3 - 12D + 16}(e^{2x} + e^{-4x} + 2e^x e^{-2x})$$

$$= \frac{1}{D^3 - 12D + 16}\ e^{2x} + \frac{1}{D^3 - 12D + 16}\ e^{-4x} + \frac{1}{D^3 - 12D + 16}\ 2e^{-x}$$

$$= \frac{x}{3D^2 - 12}\ e^{2x} + \frac{x}{3D^2 - 12}\ e^{-4x} + 2\frac{1}{-1 + 12 + 16}\ e^{-x}$$

[Since f(2) = 0 & f(– 4) = 0, multiply x on Nr. and Diff. Dr. w.r.to 'D']

$$= \frac{x^2}{6D}\ e^{2x} + \frac{x}{48 - 12}\ e^{-4x} + 2\frac{1}{27}\ e^{-x}$$

[Again, since f(2) = 0, multiply x on Nr. and Diff. Dr. w.r.to 'D']

$$= \frac{x^2}{12}\ e^{2x} + \frac{x}{36}\ e^{-4x} + \frac{2}{27}\ e^{-x}$$

$$(i.e.)\quad y = (Ax + B)e^{2x} + Ce^{-4x} + \frac{x^2}{12}e^{2x} + \frac{x}{36}e^{-4x} + \frac{2}{27}e^{-x}$$

7. Solve: $(D^3 - 1)y = (e^x + 1)^2$

Sol. A.E is $m^3 - 1 = 0$

1	1	0	0	–1
	0	1	1	1
	1	1	1	0

m = 1 is a root.

The other roots are $m^2 + m + 1 = 0$

$$m = \frac{-1 \pm \sqrt{1^2 - 4.1.1}}{2.1}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm i\sqrt{3}}{2} = \frac{-1}{2} \pm i\,\frac{\sqrt{3}}{2}$$

$$C.F = e^{-1/2x}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right] + Ce^x$$

$$P.I. = \frac{1}{D^3-1}(e^x+1)^2$$

$$= \frac{1}{D^3-1}(e^{2x}+1+2e^x)$$

$$= \frac{1}{D^3-1}\ e^{2x} + \frac{1}{D^3-1}\ e^{0x} + 2\frac{1}{D^3-1}\ 2e^x$$

$$= \frac{1}{8-1}e^{2x} + \frac{1}{0-1} + 2\ \frac{x}{3D^2}\ e^x$$ [Since f(1) = 0, multiply x on Nr. and Diff. Dr. w.r.to 'D']

$$= \frac{1}{7}e^{2x} - 1 + 2\frac{x}{3}e^x$$

$$= \frac{e^{2x}}{7} + \frac{2}{3}xe^x - 1$$

$$(i.e.)\quad y = e^{-1/2x}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right] + Ce^x + \frac{e^{2x}}{7} + \frac{2}{3}xe^x - 1$$

8. Solve: $(D^2-3D+2)y = e^{3x}$ **given that y = 3 and Dy = 3 when x = 0.**

Sol. A.E is $m^2-3m+2=0$

$$(m-1)(m-2)=0$$

$$m=1,\ 2$$

$$C.F = Ae^x + Be^{2x}$$

$$P.I. = \frac{1}{D^2-3D+2}\ e^{3x}$$

$$= \frac{1}{9-9+2}\ e^{3x}$$

$$= \frac{e^{3x}}{2}$$

$$(i.e.)\ y = Ae^x + Be^{2x} + \frac{e^{3x}}{2} \quad ------ (1)$$

$$Dy = \frac{dy}{dx} = Ae^x + 2Be^{2x} + \frac{3e^{3x}}{2} \quad ------ (2)$$

When y = 3 and x = 0, equation (1) becomes

$$3 = A + B + \frac{1}{2}$$

$$A + B = 3 - \frac{1}{2}$$

$$A + B = \frac{5}{2} \quad ------ (3)$$

When Dy = 3 and x = 0, equation (2) becomes

$$3 = A + 2B + \frac{3}{2}$$

$$A + 2B = 3 - \frac{3}{2}$$

$$A + 2B = \frac{3}{2} \text{ ------ } (4)$$

(4) – (3) implies

$$2B - B = \frac{3}{2} - \frac{5}{2}$$

$$B = -\frac{2}{2} = -1$$

Sub. B = –1 in equation (3) we have

$$A - 1 = \frac{5}{2}$$

$$A = \frac{5}{2} + 1 = \frac{7}{2}$$

Sub. A = 7/2, B = –1 in equation (1) we get

$$y = \frac{7}{2}e^{x} - e^{2x} + \frac{e^{3x}}{2}$$

<u>Home Work</u>

1. $Solve: \; (D^2 - 5D + 6)y = e^{3x}$
2. $Solve: \; (3D^2 + D - 4)y = 13e^{2x}$
3. $Solve: \; (3D^2 - 4D + 2)y = e^{x}$
4. $Solve: \; (4D^2 - 12D + 9)y = 5e^{3/2x} - 1$
5. $Solve: \; (D^2 - 2D + 4)y = e^{2x} + 3$
6. $Solve: \; (D^2 - 2D + 1)y = e^{2x} + e^{x}$
7. $Solve: \; (D^2 - 5D + 6)y = \cosh^2 x$
8. $Solve: \; (D^2 + 4D + 8)y = (1 + e^{x})^2$
9. $Solve: \; (D^2 - 3D + 2)y = e^{3x} \;\; given\ that\ \ y = 0\ when\ \ x = 0\ and\ also$ $when\ \ x = \log_e 2.$

Type – 2

Let X is of the form sin*ax* (or) cos*ax*

$$P.I. = \frac{1}{f(D^2)} \sin ax \ (or) \ \cos ax$$

$$= \frac{1}{f(-a^2)} \sin ax \ (or) \ \cos ax, \quad if \quad f(-a^2) \neq 0 \ [put \ \ D^2 = -a^2]$$

$$If \ f(-a^2) = 0, \ then \ \ P.I. = \frac{x}{Diff.\, Coeff.\, of \ \ Dr.} \sin ax \ (or) \ \cos ax$$

Note: $D(x^2) = 2x, \quad D(\sin ax) = a \cos ax$

$$\frac{1}{D}(x) = \frac{x^2}{2}, \quad \frac{1}{D}(\sin ax) = \frac{-\cos ax}{a}$$

Problems

1. Solve: $(D^2 - 3D + 2)y = \sin 3x$

Sol. A.E is $m^2 - 3m + 2 = 0$

$$(m-1)(m-2) = 0$$

$$m = 1,\ 2$$

$$C.F = A e^{x} + B e^{2x}$$

$$P.I. = \frac{1}{D^2 - 3D + 2} \sin 3x$$

$$= \frac{1}{-9 - 3D + 2} \sin 3x$$

$$= \frac{1}{-3D - 7} \sin 3x$$

$$= \frac{(-3D + 7)}{(-3D - 7)(-3D + 7)} \sin 3x$$

$$= \frac{(-3D + 7)}{9D^2 - 49} \sin 3x$$

$$= \frac{(-3D + 7)}{9(-9) - 49} \sin 3x$$

$$= \frac{-3D(\sin 3x) + 7\sin 3x}{-81 - 49} = \frac{-3(3\cos 3x) + 7\sin 3x}{-130}$$

$$= \frac{-9\cos 3x + 7\sin 3x}{-130}$$

$$= \frac{9\cos 3x - 7\sin 3x}{130}$$

$$(i.e.) \ \ y = A e^{x} + B e^{2x} + \frac{9\cos 3x - 7\sin 3x}{130}$$

2. Solve: $(D^2+16)y = e^{-3x} + \cos 4x$

Sol. A.E is $m^2+16=0$

$$m^2=-16$$
$$m=\pm 4i$$
$$C.F = e^{0x}(A\cos 4x + B\sin 4x) = A\cos 4x + B\sin 4x$$
$$P.I_1 = \frac{1}{D^2+16}\,e^{-3x}$$
$$=\frac{1}{9+16}\,e^{-3x}$$
$$=\frac{e^{-3x}}{25}$$
$$P.I_2 = \frac{1}{D^2+16}\cos 4x$$
$$=\cancel{\frac{1}{-16+16}\cos 4x}$$

[Since f(–16) = 0, multiply x on Nr. and Diff. Dr. w.r.to 'D']

$$=\frac{x}{2D}\cos 4x$$
$$=\frac{x}{2}\left(\frac{\sin 4x}{4}\right)$$
$$=\frac{x\sin 4x}{8}$$
$$(i.e.)\quad y = A\cos 4x + B\sin 4x + \frac{e^{-3x}}{25} + \frac{x\sin 4x}{8}$$

3. Solve: $(D^2-D+3)y = e^{2x} + \cos^2 x$

Sol. A.E is $m^2-m+3=0$

$$m=\frac{-(-1)\pm\sqrt{(-1)^2-4.1.3}}{2.1}$$
$$=\frac{1\pm\sqrt{-11}}{2}$$
$$=\frac{1\pm i\sqrt{11}}{2}=\frac{1}{2}\pm i\,\frac{\sqrt{11}}{2}$$
$$C.F = e^{\frac{1}{2}x}\left[A\cos\frac{\sqrt{11}}{2}x + B\sin\frac{\sqrt{11}}{2}x\right]$$
$$P.I_1 = \frac{1}{D^2-D+3}\,e^{2x}$$
$$=\frac{1}{4-2+3}\,e^{2x} \quad =\frac{e^{2x}}{5}$$

$$P.I_2 = \frac{1}{D^2 - D + 3}\cos^2 x = \frac{1}{D^2 - D + 3}\left(\frac{1+\cos 2x}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{D^2 - D + 3}e^{0x} + \frac{1}{D^2 - D + 3}\cos 2x\right)$$

$$= \frac{1}{2}\left(\frac{1}{3} + \frac{1}{-4 - D + 3}\cos 2x\right)$$

$$= \frac{1}{2}\left(\frac{1}{3} + \frac{1}{-D - 1}\cos 2x\right)$$

$$= \frac{1}{2}\left(\frac{1}{3} + \frac{-D + 1}{(-D-1)(-D+1)}\cos 2x\right)$$

$$= \frac{1}{2}\left(\frac{1}{3} + \frac{-D + 1}{D^2 - 1}\cos 2x\right)$$

$$= \frac{1}{2}\left(\frac{1}{3} + \frac{-D + 1}{(-4) - 1}\cos 2x\right)$$

$$= \frac{1}{2}\left(\frac{1}{3} + \frac{-D(\cos 2x) + \cos 2x}{-5}\right)$$

$$= \frac{1}{2}\left(\frac{1}{3} + \frac{2\sin 2x + \cos 2x}{-5}\right)$$

$$= \frac{1}{6} - \frac{2\sin 2x + \cos 2x}{10}$$

$$(i.e.)\ y = e^{\frac{1}{2}x}\left[A\cos\frac{\sqrt{11}}{2}x + B\sin\frac{\sqrt{11}}{2}x\right] + \frac{e^{2x}}{5} + \frac{1}{6} - \frac{2\sin 2x + \cos 2x}{10}$$

4. Solve $(D^2 - 4D + 3)y = \sin 3x \cos 2x$

Sol. A.E is $m^2 - 4m + 3 = 0$

$$(m-1)(m-3) = 0$$

$$m = 1, 3$$

$$C.F = A e^x + B e^{3x}$$

2sinAcosB = sin(A + B) + sin(A – B)

sinAcosB = $\frac{1}{2}$ [sin(A + B) + sin(A – B)]

$$P.I. = \frac{1}{D^2 - 4D + 3}\sin 3x \cos 2x$$

$$= \frac{1}{D^2 - 4D + 3}\,\frac{1}{2}[\sin(3x + 2x) + \sin(3x - 2x)]$$

$$= \frac{1}{2}\frac{1}{D^2 - 4D + 3}(\sin 5x + \sin x)$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - 4D + 3}\sin 5x + \frac{1}{D^2 - 4D + 3}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-25 - 4D + 3}\sin 5x + \frac{1}{-1 - 4D + 3}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-4D-22}\sin 5x + \frac{1}{-4D+2}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{-4D+22}{(-4D-22)(-4D+22)}\sin 5x + \frac{-4D-2}{(-4D+2)(-4D-2)}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{-4D+22}{16D^2-484}\sin 5x + \frac{-4D-2}{16D^2-4}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{-4D+22}{16(-25)-484}\sin 5x + \frac{-4D-2}{16(-1)-4}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{-4D(\sin 5x)+22\sin 5x}{-884} + \frac{-4D(\sin x)-2\sin x}{-20}\right]$$

$$= \frac{1}{2}\left[\frac{-20\cos 5x+22\sin 5x}{-884} + \frac{-4\cos x-2\sin x}{-20}\right]$$

$$= \frac{10\cos 5x - 11\sin 5x}{884} + \frac{2\cos x+\sin x}{20}$$

$$\therefore\ y = A e^{x} + B e^{3x} + \frac{10\cos 5x - 11\sin 5x}{884} + \frac{2\cos x + \sin x}{20}$$

5. Solve $(D^3-3D^2+4D-2)y = e^x + \cos x$

Sol. A.E is $m^3-3m^2+4m-2=0$

$$\begin{array}{c|cccc} 1 & 1 & -3 & 4 & -2 \\ & 0 & 1 & -2 & 2 \\ \hline & 1 & -2 & 2 & \boxed{0} \end{array}$$

m = 1 is a root.

The other roots are $m^2-2m+2=0$

$$m = \frac{-(-2)\pm\sqrt{(-2)^2-4.1.2}}{2.1}$$

$$= \frac{2\pm\sqrt{-4}}{2}$$

$$= \frac{2\pm 2i}{2}$$

$$= 1 \pm i$$

$$C.F = e^{x}\left[A\cos x + B\sin x\right] + C e^{x}$$

$$P.I_1 = \frac{1}{D^3-3D^2+4D-2}\, e^{x} = \cancel{\frac{1}{1-3+4-2}}\, e^{x}$$

$$= \frac{x}{3D^2-6D+4}\, e^{x}$$

$$= \frac{x}{3-6+4} e^{x}$$

$$= x e^{x}$$

$$P.I_2 = \frac{1}{D^3-3D^2+4D-2}\cos x$$

$D^3 = D.D^2$
$=D(-1)$

$$= \frac{1}{D(-1)-3(-1)+4D-2}\cos x$$

$$= \frac{1}{3D+1}\cos x$$

$$= \frac{(3D-1)}{(3D+1)(3D-1)}\cos x$$

$$= \frac{(3D-1)}{9D^2-1}\cos x$$

$$= \frac{(3D-1)}{9(-1)-1}\cos x$$

$$= \frac{3D(\cos x)-\cos x}{-10}$$

$$= \frac{3(-\sin x)-\cos x}{-10}$$

$$= \frac{3\sin x+\cos x}{10}$$

$$(i.e.)\ \ y = e^x\left[A\cos x + B\sin x\right] + C\,e^x + x\,e^x + \frac{3\sin x+\cos x}{10}$$

Home Work

1. $Solve:\ \ (D^2+4D+4)y = e^{-2x}+\cos x$
2. $Solve:\ \ (D^2-10D+25)y = 7\,e^{5x}+\cos 5x$
3. $Solve:\ \ (D^2+16)y = \sin 4x + e^{-4x}+3$
4. $Solve:\ \ (D^2-6D+9)y = \dfrac{1}{e^{3x}}+\sin x$
5. $Solve:\ \ (D^2-4D+1)y = 2\cos 4x\cos 2x$
6. $Solve:\ \ (D^3+1)y = \sin 3x - \cos^2\left(\dfrac{x}{2}\right)$
7. $Solve:\ \ (D^2+5D-6)y = \sin 4x\sin x$
8. $Solve:\ \ \dfrac{d^2x}{dt^2}+2\dfrac{dx}{dt}+4x = 2\cos 2t\ \ given\ that\ \ t=0,\ x=0=\dfrac{dx}{dt}.$

<u>Type – 3</u>

Let X is of the form x^m

$$P.I. = \frac{1}{f(D)}\ x^m$$

To evaluate this, exp*and* $\frac{1}{f(D)}$ (*i.e.*) $[f(D)]^{-1}$ *in powers of D by Binomial series upto* D^m.

Then operate each term on x^m.

<u>Note : 1</u> Before applying the Binomial series, first make the constant term unity.

<u>Note : 2</u> Some standard Binomial series are

$$(1-D)^{-1} = 1 + D + D^2 + D^3 + \ldots\ldots\ldots$$
$$(1+D)^{-1} = 1 - D + D^2 - D^3 + \ldots\ldots\ldots$$
$$(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \ldots\ldots\ldots$$
$$(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \ldots\ldots\ldots$$

<u>Problems</u>

1. Solve $(D^2 + 3D + 2)y = x^2$

Sol. A.E is $m^2 + 3m + 2 = 0$

$$(m+1)(m+2) = 0$$
$$m = -1, -2$$
$$C.F = A\,e^{-x} + B\,e^{-2x}$$

$$\text{P.I} = \frac{1}{D^2+3D+2}\ x^2 = \frac{1}{2\left(1+\frac{D^2+3D}{2}\right)}\ x^2 = \frac{1}{2}\left(1+\frac{D^2+3D}{2}\right)^{-1} x^2$$

[Since R.H.S function is x^2, we can neglect D^3 and higher powers of D]

$$= \frac{1}{2}\left[1-\left(\frac{D^2+3D}{2}\right)+\left(\frac{D^2+3D}{2}\right)^2\right] x^2$$
$$= \frac{1}{2}\left[1-\frac{D^2}{2}-\frac{3D}{2}+\frac{9D^2}{4}\right] x^2$$
$$= \frac{1}{2}\left[1-\frac{3D}{2}+\frac{7D^2}{4}\right] x^2$$
$$= \frac{1}{2}\left[x^2-\frac{3D}{2}(x^2)+\frac{7D^2}{4}(x^2)\right]$$
$$= \frac{1}{2}\left[x^2-\frac{3}{2}(2x)+\frac{7}{4}(2)\right]$$
$$= \frac{1}{2}\left[x^2-3x+\frac{7}{2}\right]$$

$$(i.e.)\ \ y = A\,e^{-x} + B\,e^{-2x} + \frac{1}{2}\left[x^2-3x+\frac{7}{2}\right]$$

2. Solve $(D^3 - D^2 - 6D)y = x^2 + 1$

Sol. A.E is $m^3 - m^2 - 6m = 0$

$$m(m^2 - m - 6) = 0$$

$$m = 0 \;\; or \;\; m^2 - m - 6 = 0$$

$$(m-3)(m+2) = 0$$

$$m = 3, -2$$

$$m = 0, 3, -2$$

$$C.F = A + B e^{3x} + C e^{-2x}$$

$$P.I. = \frac{1}{D^3 - D^2 - 6D}(x^2+1) = \frac{1}{-6D\left(1 + \frac{D^3 - D^2}{-6D}\right)}(x^2+1)$$

$$= \frac{1}{-6D}\left(1 - \frac{D^2 - D}{6}\right)^{-1}(x^2+1)$$

[Since R.H.S function is x^2, we can neglect D^3 and higher powers of D]

$$= \frac{1}{-6D}\left[1 + \left(\frac{D^2 - D}{6}\right) + \left(\frac{D^2 - D}{6}\right)^2\right](x^2+1)$$

$$= \frac{1}{-6D}\left[1 + \frac{D^2}{6} - \frac{D}{6} + \frac{D^2}{36}\right](x^2+1)$$

$$= \frac{1}{-6D}\left[1 - \frac{D}{6} + \frac{7D^2}{36}\right](x^2+1)$$

$$= \frac{1}{-6D}\left[(x^2+1) - \frac{D}{6}(x^2+1) + \frac{7D^2}{36}(x^2+1)\right]$$

$$= \frac{1}{-6D}\left[x^2 + 1 - \frac{1}{6}(2x) + \frac{7}{36}(2)\right]$$

$$= \frac{1}{-6D}\left[x^2 - \frac{x}{3} + \frac{25}{18}\right]$$

$$= -\frac{1}{6}\left[\frac{x^3}{3} - \frac{x^2}{6} + \frac{25}{18}x\right]$$

$$(i.e.)\;\; y = A + B e^{3x} + C e^{-2x} - \frac{1}{6}\left[\frac{x^3}{3} - \frac{x^2}{6} + \frac{25}{18}x\right]$$

3. Solve $(D^2 + 4D + 5)y = e^x + x^3 + \cos 2x$

Sol. A.E is $m^2 + 4m + 5 = 0$

$$m = \frac{-4 \pm \sqrt{(4)^2 - 4.1.5}}{2.1}$$

$$= \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

$$C.F = e^{-2x}[A\cos x + B\sin x]$$

$$P.I_1 = \frac{1}{D^2+4D+5}\, e^x = \frac{1}{1+4+5}\, e^x = \frac{e^x}{10}$$

$$P.I_2 = \frac{1}{D^2+4D+5}\, x^3 = \frac{1}{5\left(1+\dfrac{D^2+4D}{5}\right)}\, x^3$$

$$= \frac{1}{5}\left(1+\frac{D^2+4D}{5}\right)^{-1} x^3$$

$$= \frac{1}{5}\left[1-\left(\frac{D^2+4D}{5}\right)+\left(\frac{D^2+4D}{5}\right)^2-\left(\frac{D^2+4D}{5}\right)^3\right] x^3$$

[Since R.H.S function is x^3, we can neglect D^4 and higher powers of D]

$$= \frac{1}{5}\left[1-\frac{D^2}{5}-\frac{4D}{5}+\frac{16D^2}{25}-\frac{64D^3}{125}\right] x^3$$

$$= \frac{1}{5}\left[1-\frac{4D}{5}+\frac{11D^2}{25}-\frac{64D^3}{125}\right] x^3$$

$$= \frac{1}{5}\left[x^3-\frac{4D}{5}(x^3)+\frac{11D^2}{25}(x^3)-\frac{64D^3}{125}(x^3)\right]$$

$$= \frac{1}{5}\left[x^3-\frac{4}{5}(3x^2)+\frac{11}{25}(6x)-\frac{64}{125}(6)\right]$$

$$= \frac{1}{5}\left[x^3-\frac{12}{5}x^2+\frac{66}{25}x-\frac{144}{125}\right]$$

$$P.I_3 = \frac{1}{D^2+4D+5}\cos 2x = \frac{1}{-4+4D+5}\cos 2x = \frac{1}{4D+1}\cos 2x$$

$$= \frac{(4D-1)}{(4D+1)(4D-1)}\cos 2x$$

$$= \frac{(4D-1)}{16D^2-1}\cos 2x$$

$$= \frac{4D(\cos 2x)-\cos 2x}{16(-4)-1}$$

$$= \frac{4(-2\sin 2x)-\cos 2x}{-65}$$

$$= \frac{8\sin 2x+\cos 2x}{65}$$

$$(i.e.)\ y = e^{-2x}[A\cos x + B\sin x]+\frac{e^x}{10}+\frac{1}{5}\left[x^3-\frac{12}{5}x^2+\frac{66}{25}x-\frac{144}{125}\right]+\frac{8\sin 2x+\cos 2x}{65}$$

4. Solve $(D^4 + D^3 + D^2)y = 5x^2 + \cos x$

Sol. A.E is $m^4 + m^3 + m^2 = 0$

$$m^2(m^2 + m + 1) = 0$$

$$m^2 = 0 \quad or \quad m^2 + m + 1 = 0$$

$$m = 0, 0 \qquad m = \frac{-1 \pm \sqrt{1^2 - 4.1.1}}{2.1}$$

$$= \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$$C.F = (Ax + B) + e^{-x/2}\left[C\cos\frac{\sqrt{3}}{2}x + D\sin\frac{\sqrt{3}}{2}x\right]$$

$$P.I_1 = 5\,\frac{1}{D^4 + D^3 + D^2}\,x^2 = 5\,\frac{1}{D^2[1 + (D^2 + D)]}\,x^2$$

$$= \frac{5}{D^2}[1 + (D^2 + D)]^{-1}\,x^2$$

$$= \frac{5}{D^2}[1 - (D^2 + D) + (D^2 + D)^2]\,x^2$$

$$= \frac{5}{D^2}[1 - D^2 - D + D^2]\,x^2$$

$$= \frac{5}{D^2}[1 - D]\,x^2$$

$$= \frac{5}{D^2}[x^2 - D(x^2)]$$

$$= \frac{5}{D^2}[x^2 - 2x]$$

$$= \frac{5}{D}\left[\frac{x^3}{3} - 2\frac{x^2}{2}\right]$$

$$= 5\left[\frac{x^4}{12} - \frac{x^3}{3}\right]$$

$$P.I_2 = \frac{1}{D^4 + D^3 + D^2}\cos x = \frac{1}{(-1)^2 + D(-1) - 1}\cos x = \frac{1}{1 - D + 1}\cos x$$

$$= \frac{1}{-D}\cos x$$

$$= -\sin x$$

$$(i.e.)\ y = (Ax + B) + e^{-x/2}\left[C\cos\frac{\sqrt{3}}{2}x + D\sin\frac{\sqrt{3}}{2}x\right] + 5\left[\frac{x^4}{12} - \frac{x^3}{3}\right] - \sin x$$

5. Solve: $(D^3 + D^2 + D + 1)y = 2x^3 + 3x^2 - 4x + 5$

Sol. A.E is $m^3 + m^2 + m + 1 = 0$

$$m^2(m+1) + 1(m+1) = 0$$

$$(m+1)(m^2+1) = 0$$

$$m = -1,\ m^2 = -1$$

$$m = \pm i$$

$$C.F = A\cos x + B\sin x + C\,e^{-x}$$

$$P.I = \frac{1}{D^3 + D^2 + D + 1}\ (2x^3 + 3x^2 - 4x + 5)$$

$$= \frac{1}{[1 + (D^3 + D^2 + D)]}\ (2x^3 + 3x^2 - 4x + 5)$$

$$= [1 + (D^3 + D^2 + D)]^{-1}\ (2x^3 + 3x^2 - 4x + 5)$$

$$= [1 - (D^3 + D^2 + D) + (D^3 + D^2 + D)^2 - (D^3 + D^2 + D)^3]\ (2x^3 + 3x^2 - 4x + 5)$$

$$= [1 - D^3 - D^2 - D + D^2 + 2D^3 - D^3]\ (2x^3 + 3x^2 - 4x + 5)$$

$$= [1 - D]\ (2x^3 + 3x^2 - 4x + 5)$$

$$= (2x^3 + 3x^2 - 4x + 5) - D(2x^3 + 3x^2 - 4x + 5)$$

$$= (2x^3 + 3x^2 - 4x + 5) - (6x^2 + 6x - 4)$$

$$= 2x^3 - 3x^2 - 10x + 9$$

$$(i.e.)\ y = A\cos x + B\sin x + C\,e^{-x} + 2x^3 - 3x^2 - 10x + 9$$

<u>Home Work</u>

1. *Solve*: $(D^2 + D + 1)y = x^2$
2. *Solve*: $(D^3 + 3D^2 + 2D)y = x^2$
3. *Solve*: $(D^2 - 1)y = e^{x} + x^3 + \cos 2x$
4. *Solve*: $(D^3 - D)y = e^x + \sin x + x$
5. *Solve*: $(6D^2 - D - 2)y = e^{4x} + x^2$
6. *Solve*: $(D^2 - 4D + 1)y = x^2 + e^{2x}$
7. *Solve*: $(D^2 + 1)y = x^4$
8. *Solve*: $\dfrac{d^2y}{dx^2} = a + bx + cx^2$ *given that* $\dfrac{dy}{dx} = 0$ *when* $x = 0$ *and* $y = d$ *when* $x = 0.$

Type – 4

Let X is of the form $e^{ax}V$, where V is any function of x.

$$P.I. = \frac{1}{f(D)}\, e^{ax}V$$

$$= e^{ax}\, \frac{1}{f(D+a)}\, V$$

Note: When X = xV(x) where V(x) is of the form sinax (or) cosax

$$P.I. = \frac{1}{f(D)}\, xV$$

$$= x\, \frac{1}{f(D)}\, V - \frac{f'(D)}{[f(D)]^2}\, V$$

Problems

1. Solve: $(D^2 - 4D + 3)y = e^{-x}\sin x$

Sol. A.E is $m^2 - 4m + 3 = 0$

$$(m-1)(m-3) = 0$$

$$m = 1,\ 3$$

$$C.F = Ae^{x} + Be^{3x}$$

$$P.I. = \frac{1}{D^2 - 4D + 3}\, e^{-x}\sin x$$

$$= e^{-x}\, \frac{1}{(D-1)^2 - 4(D-1) + 3}\, \sin x$$

$$= e^{-x}\, \frac{1}{D^2 - 2D + 1 - 4D + 4 + 3}\, \sin x$$

$$= e^{-x}\, \frac{1}{D^2 - 6D + 8}\, \sin x$$

$$= e^{-x}\, \frac{1}{-1 - 6D + 8}\, \sin x$$

$$= e^{-x}\, \frac{(-6D - 7)}{(-6D + 7)(-6D - 7)}\, \sin x$$

$$= e^{-x}\, \frac{-6D - 7}{36D^2 - 49}\, \sin x$$

$$= e^{-x}\, \frac{-6D - 7}{36(-1) - 49}\, \sin x$$

$$= e^{-x}\left[\frac{-6D(\sin x) - 7\sin x}{-85}\right] = e^{-x}\left[\frac{-6\cos x - 7\sin x}{-85}\right] = e^{-x}\left[\frac{6\cos x + 7\sin x}{85}\right]$$

$$(i.e.)\ \ y = Ae^{x} + Be^{3x} + e^{-x}\left[\frac{6\cos x + 7\sin x}{85}\right]$$

2. Solve: $(D^2+2D+5)y = xe^x$

Sol. A.E is $m^2+2m+5=0$

$$m = \frac{-2\pm\sqrt{(2)^2-4.1.5}}{2.1}$$

$$= \frac{-2\pm\sqrt{-16}}{2} = \frac{-2\pm 4i}{2} = -1\pm 2i$$

$$C.F = e^{-x}[A\cos 2x + B\sin 2x]$$

$$P.I = \frac{1}{D^2+2D+5}\, x\, e^x = e^x \frac{1}{(D+1)^2+2(D+1)+5}\, x$$

$$= e^x \frac{1}{D^2+2D+1+2D+2+5}\, x$$

$$= e^x \frac{1}{D^2+4D+8}\, x$$

$$= e^x \frac{1}{8\left(1+\dfrac{D^2+4D}{8}\right)}\, x$$

$$= \frac{e^x}{8}\left(1+\frac{D^2+4D}{8}\right)^{-1} x$$

$$= \frac{e^x}{8}\left[1-\left(\frac{D^2+4D}{8}\right)\right] x$$

$$= \frac{e^x}{8}\left[1-\frac{4D}{8}\right] x$$

$$= \frac{e^x}{8}\left[x-\frac{1}{2}D(x)\right] = \frac{e^x}{8}\left[x-\frac{1}{2}\right]$$

$$(i.e.)\quad y = e^{-x}[A\cos 2x + B\sin 2x] + \frac{e^x}{8}\left(x-\frac{1}{2}\right)$$

3. Solve: $(D^2+4)y = x\sin x$

Sol. A.E is $m^2+4=0$

$$m^2 = -4$$

$$m = \pm 2i$$

$$C.F = e^{0x}(A\cos 2x + B\sin 2x) = A\cos 2x + B\sin 2x$$

$$P.I. = \frac{1}{D^2+4}\, x\sin x$$

$$= x\,\frac{1}{D^2+4}\,\sin x - \frac{2D}{[D^2+4]^2}\,\sin x$$

$$P.I. = x\,\frac{1}{-1+4}\sin x - \frac{2D}{[-1+4]^2}\sin x$$

$$= x\,\frac{1}{3}\sin x - \frac{2D}{9}\sin x$$

$$= \frac{x}{3}\sin x - \frac{2}{9}\cos x$$

$$(i.e.)\;\; y = A\cos 2x + B\sin 2x + \frac{x}{3}\sin x - \frac{2}{9}\cos x$$

4. Solve: $(D^2 - 2D + 1)y = xe^x \sin x$

Sol. A.E is $m^2 - 2m + 1 = 0$

$$(m-1)(m-1) = 0$$

$$m = 1, 1$$

$$C.F = (Ax + B)e^x$$

$$P.I. = \frac{1}{D^2 - 2D + 1}\; xe^x \sin x$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 1}\; x\sin x$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 2D - 2 + 1}\; x\sin x$$

$$= e^x \frac{1}{D^2}\; x\sin x$$

$$= e^x \left[x\,\frac{1}{D^2}\sin x - \frac{2D}{(D^2)^2}\sin x \right]$$

$$= e^x \left[x\,\frac{1}{-1}\sin x - \frac{2D}{(-1)^2}\sin x \right]$$

$$= e^x [-x\sin x - 2\cos x]$$

$$= -e^x [x\sin x + 2\cos x]$$

$$(i.e.)\;\; y = (Ax + B)e^x - e^x(x\sin x + 2\cos x)$$

5. Solve: $(D^3 - 2D + 4)y = e^x \cos x$

Sol. A.E is $m^3 - 2m + 4 = 0$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -2 & 4 \\ & 0 & -2 & 4 & -4 \\ \hline & 1 & -2 & 2 & \boxed{0} \end{array}$$

m = –2 is a root.

The other roots are $m^2 - 2m + 2 = 0$

$$m = \frac{-(-2) \pm \sqrt{(-2)^2 - 4.1.2}}{2.1}$$

$$= \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm 2i}{2}$$

$$= 1 \pm i$$

$$C.F = e^{x}\left[A\cos x + B\sin x\right] + C\,e^{-2x}$$

$$P.I. = \frac{1}{D^3 - 2D + 4}\ e^{x}\cos x$$

$$= e^{x}\ \frac{1}{(D+1)^3 - 2(D+1) + 4}\ \cos x$$

$$= e^{x}\ \frac{1}{D^3 + 3D^2 + 3D + 1 - 2D - 2 + 4}\ \cos x$$

$$= e^{x}\ \frac{1}{D^3 + 3D^2 + D + 3}\ \cos x$$

$$= e^{x}\ \frac{1}{\cancel{D(-1)} + \cancel{3(-1)} + D + 3}\ \cos x$$

$$= e^{x}\ \frac{x}{3D^2 + 6D + 1}\ \cos x$$

$$= e^{x}\ \frac{x}{3(-1) + 6D + 1}\ \cos x$$

$$= e^{x}\ \frac{x}{6D - 2}\ \cos x$$

$$= x\,e^{x}\left[\frac{6D+2}{(6D-2)(6D+2)}\ \cos x\right]$$

$$= x\,e^{x}\left[\frac{6D+2}{36D^2 - 4}\ \cos x\right]$$

$$= x\,e^{x}\left[\frac{6D+2}{36(-1) - 4}\ \cos x\right]$$

$$= x\,e^{x}\left[\frac{6D(\cos x) + 2\cos x}{-40}\right]$$

$$= x\,e^{x}\left[\frac{-6\sin x + 2\cos x}{-40}\right] = x\,e^{x}\left[\frac{3\sin x - 2\cos x}{20}\right]$$

$$\therefore\ \ y = e^{x}\left[A\cos x + B\sin x\right] + C\,e^{-2x} + x\,e^{x}\left[\frac{3\sin x - 2\cos x}{20}\right]$$

6. Solve: $(D^2+4)y = x^2 \sin x$

Sol. A.E is $m^2+4=0$

$$m^2=-4$$

$$m=\pm 2i$$

$$C.F = e^{0x}(A\cos 2x + B\sin 2x) = A\cos 2x + B\sin 2x$$

$$P.I. = \frac{1}{D^2+4}\, x^2 \sin x$$

$$= I.P. \frac{1}{D^2+4}\, x^2 e^{ix}$$

$$= I.P. e^{ix} \frac{1}{(D+i)^2+4}\, x^2$$

$$= I.P. e^{ix} \frac{1}{D^2+2Di+i^2+4}\, x^2$$

$$= I.P. e^{ix} \frac{1}{D^2+2Di-1+4}\, x^2$$

$$= I.P. e^{ix} \frac{1}{D^2+2Di+3}\, x^2$$

$$= I.P. e^{ix} \frac{1}{3\left[1+\frac{D^2+2Di}{3}\right]}\, x^2$$

$$= \frac{1}{3}\ I.P. e^{ix} \left[1+\frac{D^2+2Di}{3}\right]^{-1} x^2$$

$$= \frac{1}{3}\ I.P. e^{ix} \left[1-\left(\frac{D^2+2Di}{3}\right)+\left(\frac{D^2+2Di}{3}\right)^2\right] x^2$$

$$= \frac{1}{3}\ I.P. e^{ix} \left[1-\frac{D^2}{3}-\frac{2Di}{3}-\frac{4D^2}{9}\right] x^2$$

$$= \frac{1}{3}\ I.P. e^{ix} \left[1-\frac{2Di}{3}-\frac{7D^2}{9}\right] x^2$$

$$= \frac{1}{3}\ I.P. e^{ix} \left[x^2-\frac{2i}{3}D(x^2)-\frac{7}{9}D^2(x^2)\right]$$

$$= \frac{1}{3}\ I.P. e^{ix} \left[x^2-\frac{2i}{3}(2x)-\frac{7}{9}(2)\right]$$

$$= \frac{1}{3}\ I.P.\ (\cos x + i\sin x)\left[\left(x^2-\frac{14}{9}\right)-i\frac{4x}{3}\right]$$

$$=\frac{1}{3}\ I.P.\left[\left\{\left(x^2-\frac{14}{9}\right)\cos x+\frac{4x\sin x}{3}\right\}+i\left\{\left(x^2-\frac{14}{9}\right)\sin x-\frac{4x\cos x}{3}\right\}\right]$$

$$=\frac{1}{3}\left[\left(x^2-\frac{14}{9}\right)\sin x-\frac{4x\cos x}{3}\right]$$

$$\therefore\ \ y=A\cos 2x+B\sin 2x+\frac{1}{3}\left[\left(x^2-\frac{14}{9}\right)\sin x-\frac{4x\cos x}{3}\right]$$

<u>Home Work</u>

1. $Solve:\ \ (D^2+2D-3)y=e^x\cos x+e^{-2x}$
2. $Solve:\ \ (D^3-3D^2+3D-1)y=x^2\,e^x$
3. $Solve:\ \ (D-1)^2\,y=e^{\,x}-x\,e^{2x}-2\sin x$
4. $Solve:\ \ (D^3-1)\,y=x\,e^x+\cos^2 x$
5. $Solve:\ \ D^2\,y=x^2\sin x$
6. $Solve:\ \ (D^2-4D+4)\,y=(x+1)\,e^{\,x}$
7. $Solve:\ \ (D^2+4)\,y=x\,e^{2x}$
8. $Solve:\ \ y''+2y'+y=e^{-x}\log x+e^x$

Homogeneous Linear Equations with Variable coefficients

A homogeneous linear equation is of the form

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + P_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + ... + P_n y = X \quad ------(1)$$

where $P_1, P_2, \ldots P_n$ are constants and X is a function of *x*.

To solve this equation put $z = \log x$

$$\Rightarrow e^z = x$$

$$We \;\; have \quad \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{dy}{dz} \cdot \frac{1}{x} \quad -----(2)$$

$$x \frac{dy}{dx} = Dy \quad where \;\; D = \frac{d}{dz}$$

$$Diff.\ (2)\ w.r.t.\ x$$

$$\frac{d^2 y}{dx^2} = \frac{1}{x} \frac{d^2 y}{dz^2} \cdot \frac{dz}{dx} + \frac{dy}{dz}\left(\frac{-1}{x^2}\right)$$

$$= \frac{1}{x} \frac{d^2 y}{dz^2} \cdot \frac{1}{x} - \frac{dy}{dz}\left(\frac{1}{x^2}\right)$$

$$x^2 \frac{d^2 y}{dx^2} = D^2 y - Dy \quad where \;\; D = \frac{d}{dz}$$

$$x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

$$Similarly, \quad x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y \;\; and \;\; so \;\; on.$$

Substitute these values of $x\frac{dy}{dx}, x^2 \frac{d^2 y}{dx^2}, etc$ in equation (1) we will get a linear equation with constant coefficients which can be solved.

<u>Note:</u>

$$\frac{1}{\theta - \alpha} X = x^{\alpha} \int x^{-\alpha - 1} X \, dx$$

If R.H.S. is unknown function (other than the previous four types), we have to use the above formula to find P.I. in variable coefficients.
[when we apply this formula change the operator D to θ].

Problems

1. Solve: $x^3\dfrac{d^3y}{dx^3}+3x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=x+\log x$

(or) $x^3y'''+3x^2y''+xy'+y=x+\log x$ **(or)** $(x^3D^3+3x^2D^2+xD+1)y=x+\log x$

Sol. put $z=\log x$

$\Rightarrow e^z=x$

Then the equation becomes

$$D(D-1)(D-2)y+3D(D-1)y+Dy+y=e^z+z$$

$$[D(D^2-3D+2)+3D^2-3D+D+1]y=e^z+z$$

$$[D^3-3D^2+2D+3D^2-3D+D+1]y=e^z+z$$

$$(D^3+1)=e^z+z$$

A.E is $m^3+1=0$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & 0 & 1 \\ & 0 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & \boxed{0} \end{array}$$

m = –1 is a root.

The other roots are $m^2-m+1=0$

$$m=\frac{-(-1)\pm\sqrt{(-1)^2-4.1.1}}{2.1}$$

$$=\frac{1\pm\sqrt{-3}}{2}$$

$$=\frac{1\pm i\sqrt{3}}{2}=\frac{1}{2}\pm i\,\frac{\sqrt{3}}{2}$$

$$C.F=e^{\frac{1}{2}z}\left[A\cos\frac{\sqrt{3}}{2}z+B\sin\frac{\sqrt{3}}{2}z\right]+Ce^{-z}$$

$$=\sqrt{x}\left[A\cos\left(\frac{\sqrt{3}}{2}\log x\right)+B\sin\left(\frac{\sqrt{3}}{2}\log x\right)\right]+\frac{C}{x}$$

$$e^{\frac{1}{2}z}=(e^z)^{1/2}=x^{1/2}=\sqrt{x}$$

$$e^{-z}=(e^z)^{-1}=x^{-1}=\frac{1}{x}$$

$$P.I_1=\frac{1}{D^3+1}e^z=\frac{1}{1+1}e^z=\frac{e^z}{2}=\frac{x}{2}$$

$$P.I_2=\frac{1}{D^3+1}z=(1+D^3)^{-1}z$$

$$=(1-D^3)z$$

$D^3(z) = 0$

$$=z-0$$

$$=\log x$$

$$\therefore\ y=\sqrt{x}\left[A\cos\left(\frac{\sqrt{3}}{2}\log x\right)+B\sin\left(\frac{\sqrt{3}}{2}\log x\right)\right]+\frac{C}{x}+\frac{x}{2}+\log x$$

2. Solve: $(x^2D^2 + 4xD + 2)y = x\log x$

Sol. put $z = \log x$

$\Rightarrow e^z = x$

Then the equation becomes

$$D(D-1)y + 4Dy + 2y = ze^z$$

$$(D^2 - D + 4D + 2)y = ze^z$$

$$(D^2 + 3D + 2)y = ze^z$$

A.E. is $m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0$

$m = -1, -2$

$$C.F. = Ae^{-z} + Be^{-2z}$$

$$= \frac{A}{x} + \frac{B}{x^2}$$

$$\boxed{e^{-2z} = (e^z)^{-2} = x^{-2} = \frac{1}{x^2}}$$

$$P.I. = \frac{1}{D^2 + 3D + 2} ze^z = e^z \frac{1}{(D+1)^2 + 3(D+1) + 2} z$$

$$= e^z \frac{1}{D^2 + 2D + 1 + 3D + 3 + 2} z$$

$$= e^z \frac{1}{D^2 + 5D + 6} z$$

$$= e^z \frac{1}{6\left[1 + \frac{D^2 + 5D}{6}\right]} z$$

$$= \frac{e^z}{6}\left[1 + \left(\frac{D^2 + 5D}{6}\right)\right]^{-1} z$$

$$= \frac{e^z}{6}\left[1 - \left(\frac{D^2 + 5D}{6}\right)\right] z$$

$$= \frac{e^z}{6}\left[1 - \frac{5D}{6}\right] z$$

$$= \frac{e^z}{6}\left[z - \frac{5}{6}\right] \qquad D(z) = 1$$

$$= \frac{x}{6}\left(\log x - \frac{5}{6}\right)$$

$$\therefore\ y = \frac{A}{x} + \frac{B}{x^2} + \frac{x}{6}\left(\log x - \frac{5}{6}\right)$$

3. Solve: $(x^2D^2 - 4xD + 6)y = x(1+x)$

Sol. put $z = \log x \Rightarrow x = e^z$

Then the equation becomes

D(D – 1)y – 4Dy + 6 y = $e^z(1+e^z)$

$$(D^2 - D - 4D + 6)y = e^z(1+e^z)$$

$$(D^2 - 5D + 6)y = e^z + e^{2z}$$

A.E. is $m^2 - 5m + 6 = 0$

$(m-2)(m-3) = 0$

$m = 2, 3$

$$C.F. = Ae^{2z} + Be^{3z} = Ax^2 + Bx^3$$

$$\text{P.I.} = \frac{1}{D^2 - 5D + 6}(e^z + e^{2z})$$

$$= \frac{1}{D^2 - 5D + 6}e^z + \frac{1}{D^2 - 5D + 6}e^{2z}$$

$$= \frac{1}{1 - 5 + 6}e^z + \frac{z}{2D - 5}e^{2z}$$ [Since f(2) = 0, multiply z on Nr. and Diff. Dr. w.r.to 'D']

$$= \frac{e^z}{2} + \frac{z}{-1}e^{2z}$$

$$= \frac{x}{2} - x^2 \log x$$

$$\therefore \; y = Ax^2 + Bx^3 + \frac{x}{2} - x^2 \log x$$

4. Solve: $x^2 y'' + xy' + y = 4\sin(\log x)$

Sol. put $z = \log x \Rightarrow x = e^z$

Then the equation becomes

D(D – 1)y + Dy + y = 4sinz

$$(D^2 - D + D + 1)y = 4\sin z$$

$$(D^2 + 1)y = 4\sin z$$

A.E. is $m^2 + 1 = 0$

$m = \pm i$

$$C.F. = A\cos z + B\sin z = A\cos(\log x) + B\sin(\log x)$$

$$\text{P.I.} = 4\frac{1}{D^2 + 1}\sin z = 4\frac{z}{2D}\sin z$$ [Since f(–1) = 0, multiply z on Nr. and Diff. Dr. w.r.to 'D']

$$= 2z(-\cos z) = -2\log x\,[\cos(\log x)]$$

$$\therefore \; y = A\cos(\log x) + B\sin(\log x) - 2\log x[\cos(\log x)]$$

5. Solve: $(x^2D^2 - 2xD - 4)y = 32(\log x)^2$

Sol. put $z = \log x \Rightarrow x = e^z$

Then the equation becomes

$$D(D-1)y - 2Dy - 4y = 32z^2$$

$$(D^2 - 3D - 4)y = 32z^2$$

A.E. is $m^2 - 3m - 4 = 0$

$(m-4)(m+1) = 0$

$m = 4, -1$

$$C.F. = Ae^{4z} + Be^{-z} = Ax^4 + \frac{B}{x}$$

$$\text{P.I.} = \frac{1}{D^2 - 3D - 4} 32z^2 = 32\frac{1}{-4\left[1 + \frac{D^2 - 3D}{-4}\right]} z^2$$

$$= -8\left[1 - \left(\frac{D^2 - 3D}{4}\right)\right]^{-1} z^2$$

$$= -8\left[1 + \left(\frac{D^2 - 3D}{4}\right) + \left(\frac{D^2 - 3D}{4}\right)^2\right] z^2$$

$$= -8\left[1 + \frac{D^2}{4} - \frac{3D}{4} + \frac{9D^2}{16}\right] z^2$$

$$= -8\left[1 - \frac{3D}{4} + \frac{13D^2}{16}\right] z^2$$

$$= -8\left[z^2 - \frac{3D}{4}(z^2) + \frac{13D^2}{16}(z^2)\right]$$

$$= -8\left[z^2 - \frac{3}{4}(2z) + \frac{13}{16}(2)\right]$$

$$= -8\left[z^2 - \frac{3z}{2} + \frac{13}{8}\right]$$

$$= -8\left[(\log x)^2 - \frac{3(\log x)}{2} + \frac{13}{8}\right]$$

$$\therefore\ y = Ax^4 + \frac{B}{x} - 8\left[(\log x)^2 - \frac{3(\log x)}{2} + \frac{13}{8}\right]$$

6. Solve: $(x^2D^2 + 4xD + 2)y = e^x$

Sol. put $z = \log x \Rightarrow x = e^z$

Then the equation becomes

$$D(D-1)y + 4Dy + 2y = e^{e^z}$$

$$(D^2 - D + 4D + 2)y = e^{e^z}$$

[This function is different from the previous four types]

$$(D^2 + 3D + 2)y = e^x$$

A.E. is $m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0$

$m = -1, -2$

$$C.F. = Ae^{-z} + Be^{-2z}$$

$$= \frac{A}{x} + \frac{B}{x^2}$$

$$\boxed{e^{-2z} = (e^z)^{-2} = x^{-2} = \frac{1}{x^2}}$$

[R.H.S function is e^{e^z} To find P.I. we have to use the formula

$$\frac{1}{\theta - \alpha} X = x^{\alpha} \int x^{-\alpha-1} X \, dx \Big]$$

$$P.I. = \frac{1}{\theta^2 + 3\theta + 2} e^x$$

[When we apply this formula, change the operator D to θ]

$$= \frac{1}{(\theta+2)(\theta+1)} e^x$$

$$= \frac{1}{(\theta+2)} \left[x^{-1} \int x^{1-1} e^x \, dx\right]$$

$$= \frac{1}{(\theta+2)} \left[\frac{1}{x} \int e^x \, dx\right]$$

$$= \frac{1}{(\theta+2)} \left[\frac{e^x}{x}\right]$$

$$= x^{-2} \int x^{2-1} \frac{e^x}{x} \, dx$$

$$= \frac{1}{x^2} \int x \frac{e^x}{x} \, dx = \frac{1}{x^2} \int e^x \, dx = \frac{e^x}{x^2}$$

$$\therefore \; y = \frac{A}{x} + \frac{B}{x^2} + \frac{e^x}{x^2}$$

7. Solve: $(x^2D^2+3xD+1)y = \dfrac{1}{(1-x)^2}$

Sol. put $z=\log x \Rightarrow x=e^z$

Then the equation becomes

$$D(D-1)y + 3Dy + y = \frac{1}{(1-e^z)^2}$$

$$(D^2 - D + 3D + 1)y = (1-e^z)^{-2}$$

[On expansion using Binomial series, we have infinite number of terms]

$$(D^2+2D+1)y = \frac{1}{(1-x)^2}$$

A.E. is $m^2 + 2m + 1 = 0$

$(m+1)(m+1) = 0$

$m = -1, -1$

$$C.F. = (Az+B)e^{-z}$$

$$= \frac{A\log x + B}{x}$$

$$P.I. = \frac{1}{\theta^2+2\theta+1}\cdot\frac{1}{(1-x)^2}$$

$$= \frac{1}{(\theta+1)(\theta+1)}\cdot\frac{1}{(1-x)^2}$$

$$= \frac{1}{(\theta+1)}\left[x^{-1}\int x^{1-1}\frac{1}{(1-x)^2}\,dx\right]$$

$$= \frac{1}{(\theta+1)}\left[\frac{1}{x}\int\frac{1}{(1-x)^2}\,dx\right]$$

$$\int\frac{dx}{(1-x)^2} = \int(1-x)^{-2}dx = \frac{(1-x)^{-1}}{-(-1)} = \frac{1}{1-x}$$

$$= \frac{1}{(\theta+1)}\left[\frac{1}{x(1-x)}\right]$$

$$= x^{-1}\int x^{1-1}\frac{1}{x(1-x)}\,dx$$

$$= \frac{1}{x}\int\frac{1}{x(1-x)}\,dx$$

$$\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x}$$

$$1 = A(1-x) + B.x$$

$put\ x=0,\quad 1 = A(1)+0$

$A = 1$

$put\ x=1,\quad 1 = 0 + B(1)$

$B = 1$

$$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$$

$$= \frac{1}{x}\int\left[\frac{1}{x}+\frac{1}{1-x}\right]dx = \frac{1}{x}\left[\log x + \frac{\log(1-x)}{-1}\right]$$

$$= \frac{1}{x}\log\left(\frac{x}{1-x}\right)$$

$$\therefore\ y = \frac{A\log x + B}{x} + \frac{1}{x}\log\left(\frac{x}{1-x}\right)$$

8. Solve: $(x^2D^2 - 3xD + 4)y = x^2$ *given that* $y(1) = 1,\ y'(1) = 0$

Sol. put $z = \log x \Rightarrow x = e^z$

Then the equation becomes

$$D(D-1)y - 3Dy + 4y = e^{2z}$$

$$(D^2 - 4D + 4)y = e^{2z}$$

A.E. is $m^2 - 4m + 4 = 0$

$(m-2)(m-2) = 0$

$m = 2, 2$

$$C.F. = (Az + B)e^{2z} = [A\log x + B]x^2$$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 4}e^{2z} = \frac{z}{2D-4}e^{2z} = \frac{z^2}{2}e^{2z} = \frac{x^2(\log x)^2}{2}$$

$$\therefore\ y = (A\log x + B)x^2 + \frac{x^2}{2}(\log x)^2 \quad \text{----------} \ (1)$$

$$y' = (A\log x + B)(2x) + x^2\left(\frac{A}{x}\right) + \frac{x^2}{2}2(\log x).\frac{1}{x} + (\log x)^2.\frac{2x}{2}$$

$$= (A\log x + B)(2x) + Ax + x(\log x) + (\log x)^2.x \quad \text{--------} \ (2)$$

When $y(1) = 1$, Equation (1) becomes

$$1 = (A\log 1 + B)(1) + \frac{1}{2}(\log 1)^2$$

$$1 = 0 + B + 0$$

$$B = 1$$

When $y'(1) = 0$, Equation (2) becomes

$$0 = (A\log 1 + B)(2) + A + \log 1 + (\log 1)^2$$

$$0 = 0 + 2B + A + 0 + 0$$

$$0 = 2(1) + A$$

$$A = -2$$

$$\therefore\ y = (1 - 2\log x)x^2 + \frac{x^2}{2}(\log x)^2$$

<u>Home Work</u>

1. *Solve*: $(x^2D^2 + xD + 1)y = \log x$
2. *Solve*: $(x^3D^3 + 3x^2D^2 + xD + 1)y = x^2 + \log x$
3. *Solve*: $x^3y''' + 3x^2y'' + xy' + y = \sin(\log x)$
4. *Solve*: $(3x^2D^2 + xD + 1)y = x$
5. *Solve*: $x^2y'' - xy' - 3y = x^2\log x$

6. $Solve: \ x^4 y''' + 2x^3 y'' - x^2 y' + xy = 1 \qquad (H\text{int}: Divide \ \ by \ \ x)$

7. $Solve: \ x^2 \dfrac{d^5 y}{dx^5} - 4x \dfrac{d^4 y}{dx^4} + 6 \dfrac{d^3 y}{dx^3} = 4 \qquad (H\text{int}: put \ \ \dfrac{d^3 y}{dx^3} = u)$

8. $Solve: \ x^2 y'' + xy' + y = \log x . \sin(\log x)$

9. $Solve: \ (x^2 D^2 + 4xD + 2) y = x^2 + \dfrac{1}{x^2}$

10. $Solve: \ (x^2 D^2 + 8xD + 13) y = x^3$

Equations reducible to homogeneous linear differential equation

The equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + P_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ... + P_n y = X \ \ ------(1)$$

can be reduced to homogeneous linear differential equation by using the substitution

$$ax + b = e^z$$

$$\Rightarrow \log(ax+b) = z$$

$$We \ \ have \ \ \frac{dy}{dx} = \frac{dy}{dz} . \frac{dz}{dx}$$

$$= \frac{dy}{dz} . \frac{a}{ax+b}$$

$$(ax+b) \frac{dy}{dx} = a\, Dy \qquad where \ \ D = \frac{d}{dz}$$

Similarly, $(ax+b)^2 \dfrac{d^2 y}{dx^2} = a^2 \, D(D-1) y$

$$(ax+b)^3 \frac{d^3 y}{dx^3} = a^3 \, D(D-1)(D-2) y \ \ and \ \ so \ \ on.$$

Substitute these values in equation (1) we will get a linear equation with constant coefficients which can be solved.

Problems

1. Solve: $(5+2x)^2 y'' - 6(5+2x)y' + 8y = 6x$

Sol. put $5+2x = e^z$

$\Rightarrow z = \log(5+2x)$

Then the equation becomes

$$2^2.D(D-1)y - 6(2Dy) + 8y = 6\left(\frac{e^z-5}{2}\right)$$

$\div$ by 4, we get $(D^2 - D - 3D + 2)y = \frac{3}{4}(e^z - 5)$

$$(D^2 - 4D + 2)y = \frac{3}{4}(e^z - 5)$$

A.E. is $m^2 - 4m + 2 = 0$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4.1.2}}{2.1}$$

$$= \frac{4 \pm \sqrt{8}}{2}$$

$$= \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$$

$$C.F = Ae^{(2+\sqrt{2})z} + Be^{(2-\sqrt{2})z}$$

$$= A(5+2x)^{(2+\sqrt{2})} + B(5+2x)^{(2-\sqrt{2})}$$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 2}\frac{3}{4}(e^z - 5)$$

$$= \frac{3}{4}\left[\frac{1}{D^2 - 4D + 2}e^z - 5\frac{1}{D^2 - 4D + 2}e^{0z}\right]$$

$$= \frac{3}{4}\left[\frac{1}{1-4+2}e^z - 5\frac{1}{2}\right]$$

$$= \frac{3}{4}\left[\frac{e^z}{-1} - \frac{5}{2}\right] = \frac{3}{4}\left[\frac{5+2x}{-1} - \frac{5}{2}\right]$$

$$= \frac{3}{4}\left[-5 - 2x - \frac{5}{2}\right]$$

$$= \frac{3}{4}\left[-2x - \frac{15}{2}\right] = -\frac{3x}{2} - \frac{45}{8}$$

$$\therefore\ y = A(5+2x)^{(2+\sqrt{2})} + B(5+2x)^{(2-\sqrt{2})} - \frac{3x}{2} - \frac{45}{8}$$

2. Solve: $(1+x)^2 y'' + (1+x)y' + y = 2\sin[\log(1+x)]$

Sol. put $1+x = e^z$

$\Rightarrow z = \log(1+x)$

Then the equation becomes

$$D(D-1)y + Dy + y = 2\sin z$$

$$(D^2 - D + D + 1)y = 2\sin z$$

$$(D^2+1)\,y = 2\sin z$$

A.E. is $m^2 + 1 = 0$

$m = \pm i$

$$C.F. = A\cos z + B\sin z$$

$$= A\cos\{\log(1+x)\} + B\sin\{\log(1+x)\}$$

$$\text{P.I.} = 2\frac{1}{D^2+1}\sin z = 2\frac{z}{2D}\sin z$$

$$= z(-\cos z)$$

$$= -\log(1+x).\cos\{\log(1+x)\}$$

$$\therefore\ y = A\cos\{\log(1+x)\} + B\sin\{\log(1+x)\} - \log(1+x).\cos\{\log(1+x)\}$$

Home Work

1. *Solve*: $(2x-1)^2 y'' - 4(2x-1)y' + 8y = 8x$
2. *Solve*: $(2x+3)^2 y'' - 2(2x+3)y' - 12y = 6x$
3. *Solve*: $(x+a)^2 y'' - 4(x+a)y' + 6y = x$
4. *Solve*: $(x+1)^2 y'' + (x+1)y' + y = \sin[2\log(x+1)]$
5. *Solve*: $(x+1)^2 y'' + (x+1)y' + y = 4\cos\log(1+x)$

Simultaneous Linear Differential Equations

Problems

1. Solve $2\frac{dx}{dt}+\frac{dy}{dt}-3x=e^t, \quad \frac{dx}{dt}+\frac{dy}{dt}+2y=\cos 2t$

Sol. The given equation can be written as

$$(2D-3)x+D\,y=e^t \quad -----(1)$$

$$and \quad D\,x+(D+2)y=\cos 2t \quad -----(2)$$

(1)×(D + 2) – (2) ×D, we get

$$(2D-3)(D+2)\,x-D^2\,x=(D+2)\,e^t-D(\cos 2t)$$

$$(2D^2+4D-3D-6-D^2)\,x=e^t+2\,e^t+2\sin 2t$$

$$(D^2+D-6)x=3e^t+2\sin 2t$$

A.E. is $m^2 + m - 6 = 0$

$$(m-2)(m+3)=0$$

$$m = 2, -3$$

$$C.F.=Ae^{2t}+Be^{-3t}$$

$$P.I_1 = 3\frac{1}{D^2+D-6}\,e^t = 3\frac{1}{1+1-6}\,e^t = -\frac{3e^t}{4}$$

$$P.I_2 = 2\frac{1}{D^2+D-6}\sin 2t = 2\frac{1}{-4+D-6}\sin 2t$$

$$= 2\frac{1}{D-10}\sin 2t$$

$$= 2\frac{D+10}{(D-10)(D+10)}\sin 2t$$

$$= 2\frac{D+10}{D^2-100}\sin 2t$$

$$= 2\frac{D+10}{-4-100}\sin 2t$$

$$= 2\frac{D(\sin 2t)+10\sin 2t}{-104}$$

$$= \frac{2\cos 2t+10\sin 2t}{-52}$$

$$= \frac{\cos 2t+5\sin 2t}{-26}$$

$$\therefore\ x = Ae^{2t}+Be^{-3t}-\frac{3e^t}{4}-\frac{\cos 2t+5\sin 2t}{26}$$

Now, $(2D-3)\,x + Dy = e^t$

$$\Rightarrow\; Dy = e^t - 2D\,x + 3\,x$$

$$= e^t - 2D\left[Ae^{2t} + Be^{-3t} - \frac{3e^t}{4} - \frac{\cos 2t + 5\sin 2t}{26}\right]$$

$$+3\left[Ae^{2t} + Be^{-3t} - \frac{3e^t}{4} - \frac{\cos 2t + 5\sin 2t}{26}\right]$$

$$= e^t - 2\left[2Ae^{2t} - 3Be^{-3t} - \frac{3e^t}{4} - \frac{-2\sin 2t + 10\cos 2t}{26}\right]$$

$$+3\left[Ae^{2t} + Be^{-3t} - \frac{3e^t}{4} - \frac{\cos 2t + 5\sin 2t}{26}\right]$$

$$D\,y = -Ae^{2t} + 9Be^{-3t} + \frac{e^t}{4} + \frac{17\cos 2t - 19\sin 2t}{26}$$

$$y = -A\frac{e^{2t}}{2} + 9B\frac{e^{-3t}}{-3} + \frac{e^t}{4} + \frac{17\sin 2t + 19\cos 2t}{52}$$

$$\therefore\; y = Ae^{2t} + Be^{-3t} + \frac{e^t}{4} + \frac{19\cos 2t + 17\sin 2t}{52}$$

$$where\; A = -\frac{A}{2},\quad B = -3B$$

2. Solve $\dfrac{dx}{dt} + y = e^t, \quad x - \dfrac{dy}{dt} = t$

Sol. The given equation can be written as

$$D\,x + y = e^t \;-----(1)$$

$$and\quad x - D\,y = t \;-----(2)$$

(1)×D + (2), we get

$$D^2\,x + x = D(e^t) + t$$

$$(D^2+1)\,x = e^t + t$$

A.E. is $m^2 + 1 = 0$

$$m^2 = -1$$

$$m = \pm\, i$$

$$C.F. = A\cos t + B\sin t$$

$$\text{P.I}_1 = \frac{1}{D^2+1}e^t = \frac{1}{1+1}e^t = \frac{e^t}{2}$$

$$\text{P.I}_2 = \frac{1}{D^2+1}\,t = (1+D^2)^{-1}\,t$$
$$= (1-D^2)\,t$$
$$= t-0$$
$$= t$$
$$\therefore\ x = A\cos t + B\sin t + \frac{e^t}{2} + t$$

Now, $\dfrac{dx}{dt} + y = e^t \Rightarrow y = e^t - \dfrac{dx}{dt}$

$$\Rightarrow y = e^t - \left[-A\sin t + B\cos t + \frac{e^t}{2} + 1\right]$$
$$\therefore\ y = A\sin t - B\cos t + \frac{e^t}{2} - 1$$

3. Solve $\dfrac{dx}{dt} = 3x+8y,\quad \dfrac{dy}{dt} = -x-3y,\quad x(0)=6,\quad y(0)=-2$

Sol. The given equation can be written as

$$Dx - 3x - 8y = 0$$
$$\textit{and}\quad x + Dy + 3y = 0$$
$$\textit{i.e.}\ (D-3)x - 8y = 0 \ ------ (1)$$
$$\textit{and}\quad x + (D+3)y = 0 \ ------ (2)$$

(1)×(D + 3) + (2) ×8, we get

$$(D-3)(D+3)\,x + 8\,x = 0$$
$$(D^2-9+8)\,x = 0$$
$$(D^2-1)\,x = 0$$

A.E. is $m^2 - 1 = 0$

$$(m+1)(m-1) = 0$$
$$m = 1, -1$$
$$x = A e^t + B e^{-t}$$

Now, $\dfrac{dx}{dt} = 3x + 8y$

$$\Rightarrow 8y = \frac{dx}{dt} - 3x$$
$$= (A e^t - B e^{-t}) - 3(A e^t + B e^{-t})$$
$$8y = -2A e^t - 4B e^{-t}$$
$$y = -\frac{1}{4} A e^t - \frac{1}{2} B e^{-t}$$

Hence $x(t) = A e^{t} + B e^{-t}$ --------- (3)

$y(t) = -\frac{1}{4} A e^{t} - \frac{1}{2} B e^{-t}$ --------- (4)

Given $x(0) = 6, \quad y(0) = -2$

When $x(0) = 6$, equation (3) becomes

$6 = A + B$ -------- (5)

When y(0) = –2, equation (4) becomes

$-2 = -\frac{A}{4} - \frac{B}{2} \Rightarrow -8 = -A - 2B$ --------- (6)

Adding (5) & (6), we get

6 – 8 = B – 2B

B = 2

$\therefore$ A = 6 – B

= 6 – 2 = 4

$\therefore \quad x = 4e^{t} + 2e^{-t}, \quad y = -e^{t} - e^{-t}$

Home Work

1. $Solve: \frac{dx}{dt} + 2x - 3y = t, \quad \frac{dy}{dt} - 3x + 2y = e^{2t}$
2. $Solve: \frac{dx}{dt} + y = \sin t, \quad \frac{dy}{dt} + x = \cos t \quad given \;\; that \quad x = 2 \;\; and \quad y = 0 \;\; at \;\; t = 0.$
3. $Solve: \frac{dx}{dt} + 2x + 3y = 2e^{2t}, \quad \frac{dy}{dt} + 3x + 2y = 0$
4. $Solve: \frac{dx}{dt} - y = t, \quad \frac{dy}{dt} + x = t^{2}$
5. $Solve: \frac{dx}{dt} + 2y = -\sin t, \quad \frac{dy}{dt} - 2x = \cos t$

Variation of Parameters

Let $a\dfrac{d^2y}{dx^2}+b\dfrac{dy}{dx}+cy=X$ where X is a function of x be the given differential equations.

Let the C.F. be $c_1f_1 + c_2f_2$ where c_1 and c_2 are constants and f_1 and f_2 are functions of x.

$$P.I = Pf_1 + Qf_2$$

$$where \quad P=-\int \frac{f_2X}{f_1 f_2' - f_1' f_2}dx, \qquad Q=\int \frac{f_1X}{f_1 f_2' - f_1' f_2}dx$$

Then the complete solution is y = C.F + P.I

Problems

1. Solve $\dfrac{d^2y}{dx^2}+y=\sec x$ using variation of parameters.

Sol. A.E. is $m^2 + 1 = 0$

$$m^2 = -1$$

$$m = \pm i$$

$$C.F. = A\cos x + B\sin x$$

P.I. = $Pf_1 + Qf_2$ where $f_1 = \cos x$, $f_2 = \sin x$

$f_1' = -\sin x$, $f_2' = \cos x$

$$f_1 f_2' - f_1' f_2 = \cos x(\cos x) - (-\sin x)(\sin x)$$

$$= \cos^2 x + \sin^2 x$$

$$= 1$$

$$P = -\int \frac{f_2X}{f_1 f_2' - f_1' f_2}dx = -\int \frac{\sin x \sec x}{1}\,dx$$

$$= -\int \frac{\sin x}{\cos x}\,dx \quad = -\int \tan x\,dx$$

$$= -\log(\sec x)$$

$$Q = \int \frac{f_1X}{f_1 f_2' - f_1' f_2}dx = \int \frac{\cos x \sec x}{1}\,dx$$

$$= \int \frac{\cos x}{\cos x}\,dx \quad = \int dx \quad = x$$

$$\therefore\ P.I. = P\,f_1 + Q\,f_2$$

$$= -\cos x \log(\sec x) + x\sin x$$

$$\therefore\ y = A\cos x + B\sin x - \cos x\log(\sec x) + x\sin x$$

2. Solve: $(D^2 - 3D + 2)y = e^{4x}$ **using variation of parameters.**

Sol. A.E. is $m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0$

$m = 1, 2$

$C.F. = Ae^{x} + Be^{2x}$

P.I. = $Pf_1 + Qf_2$ where $f_1 = e^{x}$, $f_2 = e^{2x}$

$f_1' = e^{x}$, $f_2' = 2e^{2x}$

$$f_1 f_2' - f_1' f_2 = e^{x} 2e^{2x} - e^{x}e^{2x} = 2e^{3x} - e^{3x} = e^{3x}$$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx = -\int \frac{e^{2x}e^{4x}}{e^{3x}} dx$$

$$= -\int \frac{e^{6x}}{e^{3x}} dx = -\int e^{3x} dx = -\frac{e^{3x}}{3}$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx = \int \frac{e^{x}e^{4x}}{e^{3x}} dx$$

$$= \int \frac{e^{5x}}{e^{3x}} dx = \int e^{2x} dx = \frac{e^{2x}}{2}$$

$$\therefore\ P.I. = P f_1 + Q f_2$$

$$= -\frac{e^{3x}}{3} e^{x} + \frac{e^{2x}}{2} e^{2x}$$

$$= -\frac{e^{4x}}{3} + \frac{e^{4x}}{2}$$

$$= \frac{e^{4x}}{6}$$

$$\therefore\ y = Ae^{z} + Be^{2z} + \frac{e^{4x}}{6}$$

3. Solve $\frac{d^2y}{dx^2} + 4y = \tan^2 2x$ **using variation of parameters.**

Sol. A.E. is $m^2 + 4 = 0$

$m^2 = -4$

$m = \pm 2i$

$C.F. = A\cos 2x + B\sin 2x$

P.I. = $Pf_1 + Qf_2$ where $f_1 = \cos 2x$, $f_2 = \sin 2x$

$f_1' = -2\sin 2x$, $f_2' = 2\cos 2x$

$$f_1 f_2' - f_1' f_2 = \cos 2x(2\cos 2x) - (-2\sin 2x)(\sin 2x)$$
$$= 2\cos^2 2x + 2\sin^2 2x$$
$$= 2(\cos^2 2x + \sin^2 2x) = 2(1) = 2$$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx = -\int \frac{\sin 2x \tan^2 2x}{2} dx$$
$$= -\frac{1}{2}\int \frac{\sin 2x \ \sin^2 2x}{\cos^2 2x} dx$$
$$= -\frac{1}{2}\int \frac{\sin 2x(1-\cos^2 2x)}{\cos^2 2x} dx$$
$$= -\frac{1}{2}\int \left(\frac{\sin 2x}{\cos^2 2x} - \sin 2x\right) dx$$
$$= -\frac{1}{2}\int \left(\frac{1}{\cos 2x} . \frac{\sin 2x}{\cos 2x} - \sin 2x\right) dx$$
$$= -\frac{1}{2}\int (\sec 2x \tan 2x - \sin 2x) dx$$
$$= -\frac{1}{2}\left[\frac{\sec 2x}{2} + \frac{\cos 2x}{2}\right]$$
$$= -\frac{\sec 2x}{4} - \frac{\cos 2x}{4}$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx = \int \frac{\cos 2x \tan^2 2x}{2} dx$$
$$= \frac{1}{2}\int \frac{\cos 2x \ \sin^2 2x}{\cos^2 2x} dx$$
$$= \frac{1}{2}\int \frac{(1-\cos^2 2x)}{\cos 2x} dx$$
$$= \frac{1}{2}\int \left(\frac{1}{\cos 2x} - \frac{\cos^2 2x}{\cos 2x}\right) dx$$
$$= \frac{1}{2}\int (\sec 2x - \cos 2x) dx$$
$$= \frac{1}{2}\left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right]$$
$$= \frac{\log(\sec 2x + \tan 2x)}{4} - \frac{\sin 2x}{4}$$

$$\therefore\ P.I. = P f_1 + Q f_2$$

$$= \left(-\frac{1}{4\cos 2x} - \frac{\cos 2x}{4}\right)\cos 2x + \left(\frac{\log(\sec 2x + \tan 2x)}{4} - \frac{\sin 2x}{4}\right)\sin 2x$$

$$= -\frac{1}{4} - \frac{1}{4}\cos^2 2x - \frac{1}{4}\sin^2 2x + \frac{1}{4}\sin 2x\ \log(\sec 2x + \tan 2x)$$

$$= -\frac{1}{4} - \frac{1}{4}(1) + \frac{1}{4}\sin 2x\ \log(\sec 2x + \tan 2x)$$

$$= -\frac{1}{2} + \frac{1}{4}\sin 2x\ \log(\sec 2x + \tan 2x)$$

$$\therefore\ y = A\cos 2x + B\sin 2x + \frac{1}{4}\sin 2x \log(\sec 2x + \tan 2x) - \frac{1}{2}$$

4. Solve $(D^2 + 2D + 5)y = e^{-x}\tan x$ using variation of parameters.

Sol. A.E. is $m^2 + 2m + 5 = 0$

$$m = \frac{-2 \pm \sqrt{4 - 4.1.(5)}}{2.1}$$

$$= \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$C.F. = e^{-x}(A\cos 2x + B\sin 2x)$$

P.I. = $Pf_1 + Qf_2$ where $f_1 = e^{-x}\cos 2x$, $f_2 = e^{-x}\sin 2x$

$$f_1' = -2e^{-x}\sin 2x - e^{-x}\cos 2x, \quad f_2' = 2e^{-x}\cos 2x - e^{-x}\sin 2x$$

$$f_1 f_2' - f_1' f_2 = e^{-x}\cos 2x(2e^{-x}\cos 2x - e^{-x}\sin 2x) - (e^{-x}\sin 2x)(-2e^{-x}\sin 2x - e^{-x}\cos 2x)$$

$$= 2e^{-2x}\cos^2 2x - e^{-2x}\sin 2x\cos 2x + 2e^{-2x}\sin^2 2x + e^{-2x}\sin 2x\cos 2x$$

$$= 2e^{-2x}(\cos^2 2x + \sin^2 2x) = 2e^{-2x}$$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2}dx = -\int \frac{e^{-x}\sin 2x\ e^{-x}\tan x}{2e^{-2x}}dx$$

$$= -\frac{1}{2}\int \sin 2x \tan x\ dx$$

$$= -\frac{1}{2}\int 2\sin x\cos x.\frac{\sin x}{\cos x}\ dx$$

$$= -\int \sin^2 x\ dx$$

$$= -\int \frac{1-\cos 2x}{2}\ dx$$

$$= -\frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] = -\frac{x}{2} + \frac{\sin 2x}{4}$$

$$Q=\int \frac{f_1 X}{f_1 f_2' - f_1' f_2}\,dx = \int \frac{e^{-x}\cos 2x\; e^{-x}\tan x}{2e^{-2x}}dx$$

$$=\frac{1}{2}\int (2\cos^2 x - 1).\frac{\sin x}{\cos x}\,dx$$

$$=\frac{1}{2}\int (2\sin x\cos x - \tan x)\,dx$$

$$=\frac{1}{2}\int (\sin 2x - \tan x)\,dx$$

$$=\frac{1}{2}\left[\frac{-\cos 2x}{2} - \log(\sec x)\right]$$

$$=-\frac{\cos 2x}{4} - \frac{1}{2}\log(\sec x)$$

$$\therefore\; P.I. = P f_1 + Q f_2$$

$$=\left(-\frac{x}{2}+\frac{\sin 2x}{4}\right)e^{-x}\cos 2x + \left(-\frac{\cos 2x}{4} - \frac{1}{2}\log(\sec x)\right)e^{-x}\sin 2x$$

$$=-\frac{xe^{-x}\cos 2x}{2} + \frac{e^{-x}\sin 2x\cos 2x}{4} - \frac{e^{-x}\sin 2x\cos 2x}{4} - \frac{e^{-x}\sin 2x\log(\sec x)}{2}$$

$$=-\frac{xe^{-x}\cos 2x}{2} - \frac{e^{-x}\sin 2x\log(\sec x)}{2}$$

$$\therefore\; y = e^{-x}(A\cos 2x + B\sin 2x) - \frac{xe^{-x}\cos 2x}{2} - \frac{e^{-x}\sin 2x\log(\sec x)}{2}$$

Home Work

Using variation of parameters solve

1. $(D^2+4)y = 4\tan 2x$
2. $(D^2+4)y = \sec 2x$
3. $(D^2+1)y = \tan x$
4. $(D^2+4)y = \cos ec\, 2x$
5. $(D^2+1)y = \sec x \cos ec\, x$
6. $\dfrac{d^2 y}{d x^2} + 121 y = \tan 11x$
7. $(D^2+4)y = \cot 2x$

Answers

Page No. 4

1. $y = A e^{-x} + B e^{-2x} + C e^{-3x}$
2. $y = (Ax + B) e^{x} + C e^{2x}$
3. $y = A x + B + (Cx + D) e^{-x}$
4. $y = (Ax + B) e^{2x} + C e^{-x}$
5. $y = e^{-x/2}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x \right] + C e^{x}$
6. $y = A + B e^{x} + C e^{-x}$
7. $y = e^{-x}\left[(Ax + B)\cos x + (Cx + D)\sin x \right]$

Page No. 10

1. $y = A e^{2x} + B e^{3x} + x e^{3x}$
2. $y = A e^{x} + B e^{-4/3\,x} + \frac{13}{10} e^{2x}$
3. $y = e^{2/3\,x}\left[A\cos\frac{\sqrt{2}}{3}x + B\sin\frac{\sqrt{2}}{3}x \right] + e^{x}$
4. $y = (Ax + B) e^{3/2\,x} + \frac{5}{8}x^2 e^{3/2\,x} - \frac{1}{9}$
5. $y = e^{x}\left[A\cos\sqrt{3}\,x + B\sin\sqrt{3}\,x \right] + \frac{1}{4}(e^{2x} + 3)$
6. $y = (Ax + B) e^{x} + e^{2x} + \frac{x^2}{2} e^{x}$
7. $y = A e^{2x} + B e^{3x} + \frac{1}{4}\left(-x e^{2x} + \frac{e^{-2x}}{20} + \frac{1}{3} \right)$
8. $y = e^{-2x}\left[A\cos 2x + B\sin 2x \right] + \frac{1}{8} + \frac{e^{2x}}{20} + \frac{2}{13} e^{x}$
9. $y = e^{x} - \frac{3}{2} e^{2x} + \frac{e^{3x}}{2}$

Page No. 15

1. $y = (Ax + B) e^{-2x} + \frac{x^2}{2} e^{-2x} + \frac{4\sin x + 3\cos x}{25}$
2. $y = (Ax + B) e^{5x} + \frac{7x^2}{2} e^{5x} - \frac{\sin 5x}{50}$
3. $y = A\cos 4x + B\sin 4x - \frac{x}{8}\cos 4x + \frac{e^{-4x}}{32} + \frac{3}{16}$
4. $y = (Ax + B) e^{3x} + \frac{e^{-3x}}{36} + \frac{4\sin x + 3\cos x}{50}$
5. $y = A e^{(2+\sqrt{3})x} + B e^{(2-\sqrt{3})x} - \frac{(24\sin 6x + 35\cos 6x)}{1801} - \frac{(8\sin 2x + 3\cos 2x)}{73}$
6. $y = e^{x/2}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x \right] + C e^{-x} + \frac{\sin 3x + 27\cos 3x}{730} - \frac{1}{2} - \frac{(\cos x - \sin x)}{4}$
7. $y = A e^{x} + B e^{-6x} + \frac{\sin 3x - \cos 3x}{60} + \frac{31\cos 5x - 25\sin 5x}{3172}$
8. $x = e^{-t}\left[-\frac{1}{\sqrt{3}}\sin\sqrt{3}\,t \right] + \frac{\sin 2t}{2}$

Page No. 20

1. $y = e^{-x/2}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right] + x^2 - 2x$
2. $y = A + B\,e^{-x} + C\,e^{-2x} + \frac{x^3}{6} - \frac{3x^2}{4} + \frac{7x}{4}$
3. $y = A e^{x} + B e^{-x} + \frac{x e^{x}}{2} - \frac{\cos 2x}{5} - x^3 - 6x$
4. $y = A + B\,e^{x} + C\,e^{-x} + \frac{1}{2}(x e^{x} + \cos x - x^2)$
5. $y = A\,e^{-x/2} + B\,e^{2/3\,x} + \frac{e^{4x}}{90} - \frac{1}{2}\left(x^2 - x + \frac{13}{2}\right)$
6. $y = A\,e^{(2+\sqrt{3})x} + B\,e^{(2-\sqrt{3})x} - \frac{e^{2x}}{3} + x^2 + 8x + 30$
7. $y = A\cos x + B\sin x + x^4 - 12x^2 + 24$
8. $y = d + \frac{a\,x^2}{2} + \frac{b\,x^3}{6} + \frac{c\,x^4}{12}$

Page No. 26

1. $y = A\,e^{x} + B\,e^{-3x} + e^{x}\left(\frac{4\sin x - \cos x}{17}\right) - \frac{e^{-2x}}{3}$
2. $y = (A x^2 + B x + C)\,e^{x} + \frac{e^{x}\,x^5}{60}$
3. $y = (Ax + B)\,e^{x} + \frac{x^2}{2}e^{x} - e^{2x}(x-2) - \cos x$
4. $y = e^{-x/2}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right] + C\,e^{x} + \frac{e^{x}}{3}\left(\frac{x^2}{2} - x\right) - \frac{(8\sin 2x + \cos 2x)}{130} - \frac{1}{2}$
5. $y = (Ax + B) + (6 - x^2)\sin x - 4x\cos x$
6. $y = (Ax + B)\,e^{2x} + e^{x}(x+3)$
7. $y = A\cos 2x + B\sin 2x + \frac{e^{2x}}{8}\left(x - \frac{1}{2}\right)$
8. $y = (Ax + B)\,e^{-x} + e^{-x}\left(\frac{x^2}{2}\log x - \frac{3x^2}{4}\right) + \frac{e^{x}}{4}$

Page No. 34 & 35

1. $y = A\cos(\log x) + B\sin(\log x) + \log x$
2. $y = \sqrt{x}\left[A\cos\left(\frac{\sqrt{3}}{2}\log x\right) + B\sin\left(\frac{\sqrt{3}}{2}\log x\right)\right] + \frac{C}{x} + \frac{x^2}{9} + \log x$

3. $y=\sqrt{x}\left[A\cos\left(\frac{\sqrt{3}}{2}\log x\right)+B\sin\left(\frac{\sqrt{3}}{2}\log x\right)\right]+\frac{C}{x}+\frac{1}{2}[\cos(\log x)+\sin(\log x)]$

4. $y=x^{1/3}\left[A\cos\left(\frac{\sqrt{2}}{3}\log x\right)+B\sin\left(\frac{\sqrt{2}}{3}\log x\right)\right]+\frac{x}{2}$

5. $y=\frac{A}{x}+B\,x^3-\frac{x^2}{3}\left(\log x+\frac{2}{3}\right)$

6. $y=(A\log x+B)+\frac{C}{x}+\frac{\log x}{4x}$

7. $y=\frac{A\,x^5}{60}+\frac{B\,x^6}{120}+\frac{x^3}{9}+\frac{C\,x^2}{2}+D\,x+E$

8. $y=A\cos(\log x)+B\sin(\log x)+\frac{1}{4}\left[\log x.\sin(\log x)-(\log x)^2\cos(\log x)\right]$

9. $y=\frac{A}{x}+\frac{B}{x^2}+\frac{x^2}{12}-\frac{\log x}{x^2}$

10. $y=x^{-7/2}\left(A\cos\frac{\sqrt{3}}{2}\log x+B\sin\frac{\sqrt{3}}{2}\log x\right)+\frac{x^3}{43}$

Page No. 37

1. $y=A(2x-1)+B(2x-1)^2+\frac{1}{2}-(2x-1)\log(2x-1)$

2. $y=A(2x+3)^3+\frac{B}{2x+3}-\frac{3x}{8}+\frac{3}{16}$

3. $y=A(x+a)^2+B(x+a)^3+\frac{3x+2a}{6}$

4. $y=A\cos\log(1+x)+B\sin\log(1+x)-\frac{1}{3}\sin[2\log(1+x)]$

5. $y=A\cos\log(1+x)+B\sin\log(1+x)+2\log(1+x).\sin\log(1+x)$

Page No. 41

1. $x=Ae^{-5t}+Be^{t}-\frac{2t}{5}+\frac{3}{7}e^{2t}-\frac{13}{25}$, $\quad y=-Ae^{-5t}+Be^{t}-\frac{3t}{5}+\frac{4}{7}e^{2t}-\frac{12}{25}$

2. $x=e^{t}+e^{-t}$, $\quad y=-e^{t}+e^{-t}+\sin t$

3. $x=Ae^{t}+Be^{-5t}+\frac{8}{7}e^{2t}$, $\quad y=-Ae^{t}+Be^{-5t}-\frac{6}{7}e^{2t}$

4. $x=A\cos t+B\sin t+t^2-1$, $\quad y=-A\sin t+B\cos t+t$

5. $x=A\cos 2t+B\sin 2t-\cos t$, $\quad y=A\sin 2t-B\cos 2t-\sin t$

Page No. 46

1. $y = A\cos 2x + B\sin 2x - \cos 2x \log(\sec 2x + \tan 2x)$
2. $y = A\cos 2x + B\sin 2x - \dfrac{\cos 2x}{4}\log(\sec 2x) + \dfrac{x}{2}\sin 2x$
3. $y = A\cos x + B\sin x - \cos x \log(\sec x + \tan x)$
4. $y = A\cos 2x + B\sin 2x + \dfrac{\sin 2x}{4}\log(\sin 2x) - \dfrac{x}{2}\cos 2x$
5. $y = A\cos x + B\sin x - \cos x \log(\sec x + \tan x) - \sin x \log(\cos ecx + \cot x)$
6. $y = A\cos 11x + B\sin 11x - \dfrac{\cos 11x}{121}\log(\sec 11x + \tan 11x)$
7. $y = A\cos 2x + B\sin 2x - \dfrac{\sin 2x}{4}\log(\cos ec 2x + \cot 2x)$