DEPARTMENT OF HUMANITIES AND SCIENCE-MATHEMATICS

(ACADEMIC YEAR: 2022-2023)

MA3251 –STATISTICS & NUMERICAL METHODS

(Regulation 2021)

COMMON TO ALL BRANCHES OF B.E/B.TECH

I-YEAR-Semester-2

NAME-

REG NO-

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Golden Jubilee Year
1973 - 2022
ACCREDITED WITH GRADE A NAAC

MA3251 **STATISTICS AND NUMERICAL METHODS** **L T P C**

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OBJECTIVES:

- This course aims at providing the necessary basic concepts of a few statistical and numerical methods and give procedures for solving numerically different kinds of problems occurring in engineering and technology.
- To acquaint the knowledge of testing of hypothesis for small and large samples which plays an important role in real life problems.
- To introduce the basic concepts of solving algebraic and transcendental equations.
- To introduce the numerical techniques of interpolation in various intervals and numerical techniques of differentiation and integration which plays an important role in engineering and technology disciplines.
- To acquaint the knowledge of various techniques and methods of solving ordinary differential equations.

UNIT I TESTING OF HYPOTHESIS 12

Sampling distributions - Estimation of parameters - Statistical hypothesis - Large sample tests based on Normal distribution for single mean and difference of means -Tests based on t, Chi-square and F distributions for mean, variance and proportion - Contingency table (test for independent) - Goodness of fit.

UNIT II DESIGN OF EXPERIMENTS 12

One way and two way classifications - Completely randomized design - Randomized block design - Latin square design - 2^2 factorial design.

UNIT III SOLUTION OF EQUATIONS AND EIGENVALUE PROBLEMS 12

Solution of algebraic and transcendental equations - Fixed point iteration method - Newton Raphson method - Solution of linear system of equations - Gauss elimination method - Pivoting - Gauss Jordan method - Iterative methods of Gauss Jacobi and Gauss Seidel - Eigenvalues of a matrix by Power method and Jacobi's method for symmetric matrices.

UNIT IV INTERPOLATION, NUMERICAL DIFFERENTIATION AND NUMERICAL INTEGRATION 12

Lagrange's and Newton's divided difference interpolations - Newton's forward and backward difference interpolation - Approximation of derivates using interpolation polynomials - Numerical single and double integrations using Trapezoidal and Simpson's 1/3 rules.

UNIT V NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS 12

Single step methods : Taylor's series method - Euler's method - Modified Euler's method - Fourth order Runge-Kutta method for solving first order equations - Multi step methods : Milne's and Adams - Bash forth predictor corrector methods for solving first order equations.

TOTAL : 60 PERIODS

OUTCOMES:

Upon successful completion of the course, students will be able to:

- Apply the concept of testing of hypothesis for small and large samples in real life problems.
- Apply the basic concepts of classifications of design of experiments in the field of agriculture.
- Appreciate the numerical techniques of interpolation in various intervals and apply the numerical techniques of differentiation and integration for engineering problems.
- Understand the knowledge of various techniques and methods for solving first and second order ordinary differential equations.
- Solve the partial and ordinary differential equations with initial and boundary conditions by using certain techniques with engineering applications

TEXT BOOKS:

1. Grewal. B.S. and Grewal. J.S., "Numerical Methods in Engineering and Science", 10 th Edition, Khanna Publishers, New Delhi, 2015.
2. Johnson, R.A., Miller, I and Freund J., "Miller and Freund's Probability and Statistics for Engineers", Pearson Education, Asia, 8 th Edition, 2015.

REFERENCES:

1. Burden, R.L and Faires, J.D, "Numerical Analysis", 9 th Edition, Cengage Learning, 2016.
2. Devore. J.L., "Probability and Statistics for Engineering and the Sciences", Cengage Learning, New Delhi, 8 th Edition, 2014.
3. Gerald. C.F. and Wheatley. P.O. "Applied Numerical Analysis" Pearson Education, Asia, New Delhi, 2006.
4. Spiegel. M.R., Schiller. J. and Srinivasan. R.A., "Schaum's Outlines on Probability and Statistics ", Tata McGraw Hill Edition, 2004.
5. Walpole. R.E., Myers. R.H., Myers. S.L. and Ye. K., "Probability and Statistics for Engineers and Scientists", 8 th Edition, Pearson Education, Asia, 2007.

STATISTICS

UNIT- I TESTING OF HYPOTHESIS

Population:

A population consists of collection of individual units, which may be person's or experimental outcomes, whose characteristics are to be studied.

Sample:

A sample is proportion of the population that is studied to learn about the characteristics of the population.

Random sample:

A random sample is one in which each item of a population has an equal chance of being selected.

Sampling:

The process of drawing a sample from a population is called sampling.

Sample size:

The number of items selected in a sample is called the sample size and it is denoted by 'n'. If $n \geq 30,$ the sample is called large sample and if $n \leq 30,$ it is called small sample

Sampling distribution:

Consider all possible samples of size' n' drawn from a given population at random. We calculate mean values of these samples.

If we group these different means according to their frequencies, the frequency distribution so formed is called sampling distribution.

The statistic is itself a random variate. Its probability distribution is often called sampling distribution.

All possible samples of given size are taken from the population and for each sample, the statistic is calculated. The values of the statistic form its sampling distribution.

Standard error:

The standard deviation of the sampling distribution is called the standard error.

Notation:

Pop. mean = μ; Pop. S.D = σ; P - Pop. proportion sample mean = $\bar{x}$; sample S.D = s; P' = sample Proportion

Note

Statistic	S.E (Standard Error)
$\bar{x}$	$\dfrac{\sigma}{\sqrt{n}}$
$p_1' - p_2'$ (Difference of sample proportions)	$\sqrt{pq\left(\dfrac{1}{n_1}+\dfrac{1}{n_2}\right)}$
$\bar{x}_1 - \bar{x}_2$ (Difference of sample means)	$\sqrt{\dfrac{\sigma_1^2}{n_1}+\dfrac{\sigma_2^2}{n_2}}$
p' (Sample proportion)	$\sqrt{\dfrac{pq}{n}}$

Null Hypothesis (H_0)

The hypothesis tested for possible rejection under the assumption that it is true is usually called null hypothesis. The null hypothesis is a hypothesis which reflects no change or no difference. It is usually denoted by H_0

Alternative Hypothesis (H_1)

The Alternative hypothesis is the statement which reflects the situation anticipated to be correct if the null hypothesis is wrong. It is usually denoted by H_1.

For example:

If $H_0 : \mu_1 = \mu_2$ (There is no diff' bet' the means) then the formulated alternative hypothesis is

$H_1 : \mu_1 \neq \mu_2$

ie., either $H_1 : \mu_1 < \mu_2$ (or) $\mu_1 > \mu_2$

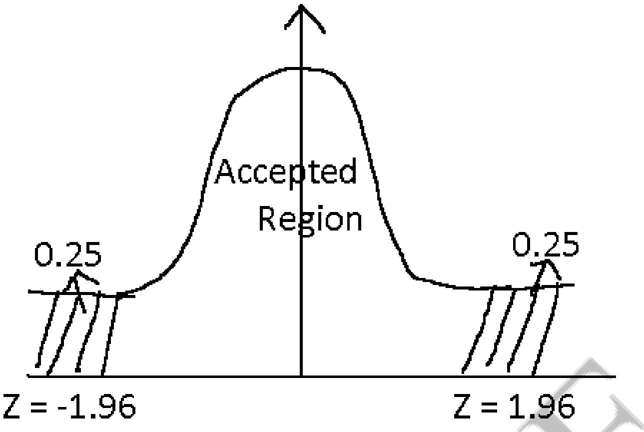
Level of significance

It is the probability level below which the null hypothesis is rejected. Generally, 5% and 1% level of significance are used.

Critical Region (or) Region of Rejection

The critical region of a test of statistical hypothesis is that region of the normal curve which corresponds to the rejection of null hypothesis.

The shaded portion in the following figure is the critical region which corresponds to 5% LOS



Critical values (or) significant values

The sample values of the statistic beyond which the null hypothesis will be rejected are called critical values or significant values

	Level of significance		
Types of test	1%	5%	10%
Two tailed test	2.58	1.96	1.645
One tailed test	2.33	1.645	1.28

Two tailed test and one-tailed tests:

When two tails of the sampling distribution of the normal curve are used, the relevant test is called two tailed test.

The alternative hypothesis $H_1 : \mu_1 \neq \mu_2$ is taken in two tailed test for $H_0 : \mu_1 = \mu_2$

When only one tail of the sampling distribution of the normal curve is used, the test is described as one tail test $H_1 : \mu_1 < \mu_2$ (or) $\mu_1 > \mu_2$

$$\left.\begin{aligned} H_0 = \mu_1 = \mu_2 \\ H_1 = \mu_1 \neq \mu_2 \end{aligned}\right\} \text{two tailed test}$$

Type I and type II Error

Type I Error : Rejection of null hypothesis when it is correct

Type II Error : Acceptance of null hypothesis when it is wrong

Procedure for testing Hypothesis:

1. Formulate H_0 and H_1
2. Choose the level of significance α
3. Compute the test statistic Z, using the data available in the problem
4. Pick out the critical value at α % level say Z_α
5. Draw conclusion: If $|Z| < Z_\alpha$, accept H_0 at $\alpha\%$ level. Otherwise reject H_0 at $\alpha\%$ level

Test of Hypothesis (Large Sample Tests)

Large sample tests (Test based in Normal distribution.)

Type - I: (Test of significance of single mean)

Let $\{x_1, x_2, \ldots x_n\}$ be a sample of size $(n \geq 30)$ taken from a population with mean μ and S.D σ. Let $\bar{x}$ be the sample mean. Assume that the population is Normal.

To test whether the difference between Population mean μ and sample mean $\bar{x}$ is significant or not and this sample comes from the normal population whose mean is μ or not.

$H_0 : \mu =$ a specified value

$H_1 : \mu \neq$ a specified value

we choose $\alpha = 0.05(5\%)$ (or) $0.01(1\%)$ as the Level of significance

the test statistic is

$$Z = \frac{\bar{x} - \mu}{S.E(\bar{x})} = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0.1) \text{ for large n.}$$

Note:

1. If σ is not known, for large n, S.E $\bar{x} = \frac{s}{\sqrt{n}}$ where 's' is the sample S.D

Problems:

1. A sample of 900 members is found to have a mean 3.5cm. Can it reasonably regarded as a simple sample from a large population whose mean is 3.38 and a standard deviation 2.4cm?

Solution:

We formulate the null hypothesis that the sample is drawn from population whose mean is 3.38cm.

$i.e., H_0 : \mu = 3.38$

$H_1 : \mu \neq 3.38$

Hence it is a two-tailed test

Level of significance $\alpha = 0.05$

Test statistic $Z = \dfrac{\bar{x} - \mu}{\dfrac{\sigma}{\sqrt{n}}}$

Given $\bar{x} = 3.5$, μ=3.38, n = 900, σ=2.4

$$\therefore \ Z = \frac{3.5 - 3.32}{\dfrac{2.4}{\sqrt{900}}} = 1.5$$

Critical value:

At 5% level, the tabulated value of Z is 1.96

Conclusion:

Since $|Z| = 1.5 < 1.96$, H_0 is accepted at 5% level of significance

i.e., the sample comes from a population with mean 3.38cm

2. A manufacturer claims that his synthetic fishing line has a mean breaking strength of 8kg and S.D 0.5kg. Can we accept his claim if a random sample of 50 lines yield a mean breaking of 7.8kg. Use 1% level of significance.

Solution:

We formulate $H_0 : \mu = 8$

$H_1 : \mu \neq 8$

L.O.S $\alpha = 0.01$

$$\text{Test statistic } Z = \frac{\overline{x}-\mu}{\frac{\sigma}{\sqrt{n}}}$$

Given $\overline{x} = 7.8,\ \mu = 8,\ n = 50,\ \sigma = 0.5$

$$\therefore\ Z = \frac{7.8-8}{\frac{0.5}{\sqrt{50}}} = -2.828$$

$\therefore\ |Z| = 2.828$

Critical value:

At 1% level of significance the table of Z = 2.58

Conclusion:

Since |Z| > 2.58, H_0 is rejected at 1% level

i.e., the manufacturer's claim is not accepted

3. A random sample of 200 Employee's at a large corporation showed their average age to be 42.8 years, with a S.D of 6.8 years. Test the hypothesis $H_0 : \mu = 40$ versus $H_1 : \mu > 40$ at $\alpha = 0.01$ level of significance.

Solution:

We set up $H_0 : \mu = 40$

$H_1 : \mu \neq 40$

It is one tailed test.

L.O.S $\alpha = 0.01$

$$\text{Test statistic } Z = \frac{\overline{x}-\mu}{\frac{\sigma}{\sqrt{n}}}$$

Given $\overline{x} = 42.8,\ \mu = 40,\ n = 200,\ \sigma = 6.89$

$$\therefore\ Z = \frac{42.8 - 40}{\frac{6.89}{\sqrt{200}}} = 5.747$$

Critical value:

For one-tail test, the table value of Z at 1% level = 2.33

Conclusion:

Since |Z| =5.747> 2.33, H_0 is rejected at 1% level.

i.e., The hypothesis $\mu = 40$ is accepted at this level.

Type - II:

Test of significance of difference of two means

Consider two samples of sizes n_1 and n_2 taken from two different populations with population means μ_1 and μ_1 and S.D's σ_1 and σ_2

Let $\overline{x_1}$ and $\overline{x_2}$ be the sample means and S_1 and S_2 be the S.D's of the samples

The formulated null and alternative hypothesis is,

$H_0 : \mu_1 = \mu_2$

$H_1 : \mu_1 \neq \mu_2$

The test statistic 'Z' is defined by

$$Z = \frac{\overline{x_1} - \overline{x_2}}{S.E(\overline{x_1} - \overline{x_2})}$$

$$\text{ie., } Z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

We use the los is $\alpha = 0.05\ (or)\ 0.01$

If $|Z| < Z_\alpha$, H_0 is accepted at $\alpha\ \%Los$

otherwise, H_0 is rejected at $\alpha\ \%Los$

Note:

In many situations, we do not know the S.D's of the populations (or) population from which the samples are drawn.

In such cases, we can subs the S.D's are of samples S_1 *and* S_2 in place of σ_1 *and* σ_2

$$\therefore \text{ The test statistic Z } = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

Problems

The mean of two sample large samples of 1000 and 200 members are 67.5 inches and 68 inches respectively. Can the samples be regard as drawn from the population of standard deviation of 2.5 inches? Test at 5% Los

Solution

we set up $H_0 : \mu_1 = \mu_2$

ie., the samples are drawn from the sample population

$H_1: \mu_1 \neq \mu_2$

$$\text{The test statistic Z } = \frac{\overline{x_1} - \overline{x_2}}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

Given $\overline{x_1} = 67.5; \quad n_1 = 1000$

$\overline{x_2} = 68; \quad n_2 = 2000; \quad \sigma=2.5$

$$\therefore \quad Z = \frac{67.5 - 68}{2.5\sqrt{\frac{1}{1000} + \frac{1}{2000}}} = -5.164$$

$\therefore$ |Z| = 5.164

We choose the Los α=0.05

Critical value:

The table values of Z at 5% Los is Z = 1.96

Conclusion:

Since |Z| > 1.96, H_0 is rejected at 5% Los.

∴ The sample cannot be regards as drawn from the same population.

2. Samples of students were drawn from two universities and from the weights is kilogram. The means and S.D's are calculated. Test the significance of the difference between the means of two samples

	Mean	S.D	Sample Size
University A	55	10	400
University B	57	15	100

Solution:

we set up $H_0 : \mu_1 = \mu_2$

ie., there is no significant difference between the sample means

$H_1: \mu_1 \neq \mu_2; \quad \alpha = 0.05$

The test statistic $Z = \dfrac{\overline{x_1} - \overline{x_2}}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$ `

Given $\overline{x_1} = 55; \quad s_1 = 10; \quad n_1 = 400$

$\overline{x_2} = 57; \quad s_2 = 15; \quad n_2 = 100$

$$\therefore \quad Z = \frac{55 - 57}{\sqrt{\dfrac{10^2}{400} + \dfrac{15^2}{100}}} = -1.265$$

∴ |Z| = 1.265

Critical value:

The table values of Z at 5% Los is Z = 1.96

Conclusion:

Since |Z| < 1.96, H_0 is accepted at 5% Los. We conclude that the difference between the means is not significant.

3. The average hourly wage of a sample of 150 workers is plant A was Rs. 2.56 with a S.D of Rs.1.08. The average wage of a sample of 200 workers in plant B was Rs. 2.87 with a S.D of Rs. 1.28. Can an applicant safety assume that the hourly wages paid by plant B are greater than those paid by plant A?

Solution:

Let x_1 and x_2 denote the hourly wages paid to workers in plant A and plant B respectively.

We set up $H_0: \mu_1 < \mu_2$ (Plant Bnot greater than Plant A)

$H_1: \mu_1 < \mu_2$ (one tailed test)

$\alpha = 0.05$

$$Z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

Given $\overline{x_1} = 2.56$; $s_1 = 1.08$; $n_1 = 150$

$\overline{x_2} = 2.87$; $s_2 = 1.28$; $n_2 = 200$

$$\therefore \quad Z = \frac{2.56 - 2.87}{\sqrt{\frac{(1.08)^2}{150} + \frac{(1.28)^2}{200}}} = -2.453$$

$\therefore$ |Z| = 2.453

Critical value:

The table values of Z at 5% in case of one-tailed test is Z = 1.645

Conclusion:

Since |Z| > 1.643, H_0 is rejected at 5% Los.

$\therefore$ The hourly wage paid by Plant B are greater than those paid by Plant A

4. A sample of size 30 from a normal population yielded 80 and variance 150. A sample of size 40 from a second normal population yielded the sample mean 71 and variance 200. Test $H_0 : \mu_1 - \mu_2 = 2.$ Versus $H_1 : \mu_1 > \mu_2 = 2$

Solution:

$H_0 : \mu_1 - \mu_2 = 2.$

ie., the diff 'bet the means of two population is 2

Versus $H_1 : \mu_1 > \mu_2 = 2$ (one tailed)

$$\text{Test Statistic Z } = \frac{\overline{x_1} - \overline{x_2} \quad \mu_1 - \mu_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

$$\therefore \ Z = \frac{(80-71)-2}{\sqrt{\frac{150}{30} + \frac{200}{40}}} = 2.215$$

Critical value:

For one tail test, at 5% Los the table value of z = 1.645

Conclusion:

Since |Z| > 1.645, H_0 is rejected.

$\therefore$ The formulated null hypothesis $H_0 : \mu_1 - \mu_2 = 2$ is wrong

5. A buyer of electric bulbs purchases 400 bulbs; 200 bulbs of each brand. Upon testing these bulbs be found that brand A has an average of 1225 hrs with a S.D of 42 hrs. where as brand B had a mean life of 1265 hrs with a S.D of 60 hrs. Can the buyer be certain that brand B is Superior than brand A in quality?

Solution:

$H_0 : \mu_1 = \mu_2;$

ie., the two brands of bulbs do not differ in quality

ie., they have the same mean life

$H_1: \mu_1 < \mu_2$ (one tailed)

L.o.s : $\alpha = 0.05$

Test Statistic $Z = \dfrac{\overline{x_1} - \overline{x_2}}{S.E\ \overline{x_1} - \overline{x_2}} = \dfrac{\overline{x_1} - \overline{x_2}}{\sqrt{\dfrac{s_1{}^2}{n_1} + \dfrac{s_2{}^2}{n_2}}}$

Here, $\overline{x_1} = 1225;$ $s_1 = 42;$ $n_1 = 200$

$\overline{x_2} = 1265;$ $s_2 = 60;$ $n_2 = 200$

$\therefore\ Z = \dfrac{1225 - 1265}{\sqrt{\dfrac{(42)^2}{200} + \dfrac{(60)^2}{200}}} = \dfrac{-40}{5.18} = -7.72$

$\Rightarrow\ |Z| = 7.72$

Critical value:

The critical value of Z at 5% Los Z = 1.645.

Conclusion:

Since $|z| < 1.645$ H_0 is rejected.

$\therefore$ The brand B is superior to brand A in equality.

Type - III:

Test of significance of single proportion:

If ‘x’ is the number of times possessing a certain attribute in a sample of n items, The sample proportion $p' = \frac{x}{n}$

p' : sample porportion;

p: population proportion.

The hypothesis $\mathrm{H}_0 : p = \mathrm{p}'$

ie., p has a specified value

Alternative hyp: $\mathrm{H}_1 : p \neq \mathrm{p}'$

$$\text{Test statistic } \mathrm{Z} = \frac{p' - p}{\sqrt{\frac{\mathrm{pq}}{n}}}$$

Since the sample is large Z $\sim$ N(0,1)

Problems

1. A coin is tossed 400 times and it turns up head 216 times. Discuss whether the coin may be regarded as unbiased one.

Solution

we set up H_0 : coin is unbiased

ie., $\mathrm{p} = \frac{1}{2} \Rightarrow \mathrm{q} = 1\text{-}\ \mathrm{p} = \frac{1}{2}$

H_1 : coin is biased

$\alpha = 0.05$

$$\text{Test statistic } \mathrm{Z} = \frac{p' - p}{\sqrt{\frac{\mathrm{pq}}{n}}}$$

Here $\mathrm{p}' = \frac{216}{400}; n = 400$

$$\therefore \quad Z = \frac{0.54 - 0.5}{\sqrt{\frac{1}{600}}} = 1.6$$

Table value of Z $= 1.96$

Conclusion:

Since $|z| < 1.96$, H_0 is accepted at 5% Los

Hence the coin may be regarded as unbiased

2. In a city of sample of 500 people, 280 are tea drinkers and the rest are coffee drinkers. Can we assume that both coffee and tea are equally popular in this city at 5% Los.

Solution:

we set up $H_0 : p = \frac{1}{2}$

ie., the coffee and tea are equally popular

$H_1 : p \neq \frac{1}{2}$

Test statistic $Z = \frac{p' - p}{\sqrt{\frac{pq}{n}}}$

Here $p' = \frac{280}{500} = 0.56; n = 500;$ p=0.5

$\Rightarrow$ q = 1-p $= 0.5$

$$\therefore \quad Z = \frac{0.56 - 0.5}{\sqrt{\frac{0.5 \times 0.5}{500}}} = 2.68$$

Conclusion:

Since $|z| > 1.96$, H_0 is rejected at 5% level

Both type of drinkers are not popular at 5% Los.

3. A manfacturing company claims that atleast 95% of its products supplied confirm to the specifications out of a sample of 200 prodcuts, 18 are defective. Test the claim at 5% Los.

Solution

we set up H_0 : The proportion of the products confirming to specification is 95%

ie., p = 0.95

H_1 : p < 0.95 (one tailed test)

$$Z = \frac{p^{'} - p}{\sqrt{\frac{pq}{n}}}$$

$$\text{Here } p^{'} = \frac{200-18}{200} = 0.91; n = 200$$

$p = 0.95 \Rightarrow q = 1 - p = 0.05$

$$\therefore \quad Z = \frac{0.91-0.95}{\sqrt{\frac{0.95\times0.05}{200}}} = -2.595 \Rightarrow |Z| = 2.595$$

Critical value : at 5%Los $Z_\alpha = 1.645$

Conclusion:

$|z|$=2.595 > 1.645, H_0 is rejected at 5% Los(Level of significance)

4. A manfacturer claims that only 4% of his products supplied by him are defective. Sample of 600 products contained 36 defectives. Test the claim of the manufactrer.

Solution:

we set up H_0 : p = 0.04

H_1 : p > 0.04 (one tailed test)

$$\text{Test Statistic } Z = \frac{p^{'} - p}{\sqrt{\frac{pq}{n}}}$$

Here p =0.04 $\Rightarrow$ q = 1-p = 0.96

$$p^{'} = \frac{36}{600} = 0.06; \quad n = 500$$

$$\therefore \quad Z = \frac{0.06-0.04}{\sqrt{\frac{0.04\times0.96}{600}}} = 2.5$$

Critical value :

The table value of Z =1.645 at 5%L.o.s

Conclusion:

$|Z|$ = 2.5 > 1.645, H_0 is rejected

$\therefore$ Manufacturer's claim is not acceptable

Type - IV: Test of significance for Difference of proportion of success in two samples:

To test the significance of the difference between the sample proportions $p_1^{'}$ and $p_2^{'}$.

We formulate the null hypothesis H_0: $p_1 = p_2$

ie., the population proportions are equal

The alternative hypothesis is H_1: $p_1 \neq p_2$

The standard error of $p_1' - p_2' = \sqrt{pq\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}$

Where p= $\frac{x_1+x_2}{n_1+n_2} = \frac{n_1 p_1' + n_2 p_2'}{n_1+n_2}$

The test statistic is $Z = \frac{p_1' - p_2'}{\sqrt{pq\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} \sim N(0,1)$

Problems:

1. If a sample of 300 units of a manufactured product 65 units were found to be defective and in another sample of 200 units, there were 35 defectives. Is there significant difference in the proportion of defectives in the samples at 5% Los.

Solution:

H_0: $p_1 = p_2$ (ie., There is no significant difference in the proportion defectives in the samples)

The alternative hypothesis H_1: $p_1 \neq p_2$

Los: $\alpha = 0.05$

The test statistic is $Z = \frac{p_1' - p_2'}{\sqrt{pq\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}}$

$p_1' = \frac{65}{300} = 0.22$; p_2'=0.175

p= $\frac{100}{500} = \frac{1}{5} \Rightarrow q = \frac{4}{5}$

$$\sqrt{pq\left(\frac{1}{n_1}+\frac{1}{n_2}\right)} = \sqrt{\frac{4}{25}\left(\frac{1}{300}+\frac{1}{200}\right)} = 0.0365$$

$$\therefore\ Z = \frac{0.22-0.175}{0.0365} = 1.233$$

Critical value :

The table value of Z at 5% Level =1.96

Conclusion:

$|Z| < 1.96$, H_0 is accepted at 5% Los.

$\therefore$ The difference in the porportion of defectives in the samples is not significant

2. A machine puts out 16 imperfect articles in a sample of 500. After the machine is over-hauled in puts out 3 imperfect articles in a batch of 100. Has the machine improved?

Solution:

H_0: Machine has not been improved

ie., $H_0: p_1 = p_2$

The alternative hypothesis H_1: $p_1 > p_2$ (one-tailed)

Los: $\alpha = 0.05$

The test statistic is $Z = \dfrac{p_1^{'} - p_2^{'}}{\sqrt{pq\left(\dfrac{1}{n_1}+\dfrac{1}{n_2}\right)}}$

Here $p_1^{'} = \dfrac{16}{500} = 0.032;\ p_2^{'}=0.03$

$n_1 = 500;\ n_2 = 100$

$p = \dfrac{19}{600}$ and $q = \dfrac{581}{600}$

$$\therefore\ Z = \frac{0.032-0.03}{\sqrt{\frac{19}{600}\times\frac{581}{600}\left(\frac{1}{500}+\frac{1}{100}\right)}} = 0.104$$

$| Z | = 0.104$

Critical value :

The table value of Z for one tailed test Z = 1.645 at 5 Los

Conclusion:

$|Z| < 1.645$, H_0 is accepted at 5% Los.

The Machine has not improved due to overhaulding.

3. Before an increase in excise duty on tea, 800 perons out of a sample of 1000 persons were found to be tea drinkers. After an increse is excise duty. 800 people were tea drinkers in a sample of 1200 people. Test whether there is a significant decrease in the consumption of tea after the increase in excise duty at 5% Los

Solution:

H_0: the proportion of tea drinkers before and after the increase in excise duty are equal

ie., $p_1 = p_2$

H_1: $p_1 > p_2$

Los: $\alpha = 0.05$

The test statistic is $Z = \dfrac{p_1' - p_2'}{\sqrt{pq\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$

Here $x_1=800$; $x_2=800$; $n_1= 1000$;

$n_2=1200$; $p_1' = \dfrac{800}{1000} = 0.8$; $p_2' = \dfrac{800}{1200} = 0.67$

$$p = \frac{x_1 + x_2}{n_1 + n_2} = \frac{1600}{2200} = \frac{8}{11} \Rightarrow q = \frac{3}{11}$$

$$\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = \sqrt{\frac{24}{121}\ 0.001 + 0.0008} = 0.0189$$

$$\therefore\ Z = \frac{0.13}{0.0189} = 6.88 \Rightarrow\ |Z|\ =\ 6.88$$

Critical value: At 5% Los 1.645

Conclusion:

$|Z| > 1.645$, H_0 is rejected.

$\therefore$ There is a significance decrease in the consumption of tea due to increase in excise duty.

Type - V: (Test of significance for the difference of S.D's of two large samples)

Let S_1 and S_2 be the S.D's of two indepedent samples of sizes n_1 and n_2 respectively

The null hypothesis H_0: $\mu_1 = \mu_2$;

ie., the sample S.D's do not differ significantly.

The Alternative Hypothesis H_1: $\mu_1 \neq \mu_2$

the test statistic is $Z = \dfrac{S_1 - S_2}{S.E(S_1 - S_2)} \sim N(0,1)$ for large 'n'

ie., If σ_1 and σ_2 are known,

$$Z = \frac{S_1 - S_2}{\sqrt{\dfrac{\sigma_1^2}{2n_1} + \dfrac{\sigma_2^2}{2n_2}}} \sim N(0,1)$$

(or)If σ_1 and σ_2 are not known,

$$Z = \frac{S_1 - S_2}{\sqrt{\dfrac{S_1^2}{2n_1} + \dfrac{S_2^2}{2n_2}}}$$

If $|Z| > Z_\alpha$, H_0 is rejected at $\alpha\%$ level, otherwise H_0 is accepted

Problems:

1. The sample of sizes 1000 and 800 gave the following results

	Mean	S.D
Sample I	17.5	2.5
Sample II	18	2.7

Assuming that the samples are indepedent, test whether the two samples may be regarded as drawn from the universe with same S.D's at 1% Level.

Solution:

We set up $H_0 : \sigma_1 = \sigma_2;$

ie., two samples maybe regarded as drawn from the universe with same S.D’s

$H_1 : \sigma_1 \neq \sigma_2$

Test statistic $Z = \dfrac{S_1 - S_2}{\sqrt{\dfrac{S_1{}^2}{2n_1} + \dfrac{S_2{}^2}{2n_2}}}$

Here $n_1 = 1000$; $n_2 = 800$; $S_1 = 2.5$; $S_2 = 2.7$

$$\therefore\ Z = \frac{2.5 - 2.7}{\sqrt{\dfrac{(2.5)^2}{2000} + \dfrac{(2.7)^2}{1600}}} = \frac{-0.2}{\sqrt{0.3125 + 0.455625}}$$

$$\Rightarrow\ |Z| = 2.282$$

Critical value :

At 1% Los, the tabulated value is 2.58

Conclusion:

Since $|Z| < 2.58$, H_0 is accepted at 1% Los.

$\therefore$ The two samples may be regarded as drawn from the universe with the same S.D's

2. In a survey of incomes of two classes of workers, two random samples gave the following results. Examine whether the differences between (i) the means and (ii) the S.D's are significant.

Sample	Size	Mean annualincome (Rs)	S.D in Rs
I	100	582	24

II	100	546	28

Examine also whether the samples have been drawn from a population with same S.D

Solution:

(i) We set up $H_0 : \mu_1 = \mu_2$;

ie., the difference is not significant

$H_1 : \mu_1 \neq \mu_2$

Here it is two tailed test

$$\text{Test statistic } Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} = \frac{582 - 546}{\sqrt{\frac{(24)^2}{100} + \frac{(28)^2}{100}}}$$

$$\therefore \; Z = \frac{360}{\sqrt{(24)^2 + (28)^2}} = 9.76$$

$$\Rightarrow \; |Z| = 9.76$$

Critical value :

At 5% Los, the table value of Z is 1.96

Conclusion:

Since $|Z| > 1.96$, H_0 is rejected at 5% Los.

$\therefore$ There is a significant difference in the means in the two samples.

(ii) $H_0 : \sigma_1 = \sigma_2$

$H_1 : \sigma_1 \neq \sigma_2$

Here it is two tailed test

Los: $\alpha = 0.05$

$$\text{Test statistic } Z = \frac{S_1 - S_2}{\sqrt{\frac{S_1^2}{2n_1} + \frac{S_2^2}{2n_2}}} = \frac{24 - 28}{\sqrt{\frac{(24)^2}{200} + \frac{(28)^2}{200}}}$$

$$\therefore \quad Z = \frac{-40}{\sqrt{288+392}} = -1.53$$

$\Rightarrow \quad |Z| = 1.53$

Critical value :

At 5% Los, the table value of Z is 1.96

Conclusion:

Since $|Z| < 1.96$, H_0 is accepted at 5% Los.

$\therefore$ The difference between the sample S.D's is not significant.

Hence we conclude that the two samples have been drawn from population with the same S.D's

3. Two machines A and B produced 200 and 250 items on the average per day with a S.D of 20 and 25 items reply on the basis of records of 50 day's production. Can you regard both machine's equally efficient at 1% Los.

Solution:

(i) $H_0 : \sigma_1 = \sigma_2$; ie., the two machines aer equally efficient

$H_1 : \sigma_1 \neq \sigma_2$

Los: $\alpha = 0.05$

$$\text{Test statistic } Z = \frac{S_1 - S_2}{\sqrt{\frac{S_1^{\,2}}{n_1} + \frac{S_2^{\,2}}{n_2}}}$$

n_1=200 $\times$50; S_2= 25

n_2=250 $\times$50; $S_1 = 20$

$$\therefore \quad Z = \frac{(20-25)\times\sqrt{50}}{\sqrt{\frac{400}{400}+\frac{625}{500}}} = \frac{-5\sqrt{50}}{\sqrt{1+1.25}} = -23.57$$

$\Rightarrow \quad |Z| = 23.57$

Critical value :

At 1% Los, the table value of Z is 2.58

Conclusion:

Since $|Z| > 2.58$, H_0 is rejected at 1% Los.

We conclude that the both machines are not equally efficient at 1% Los

Small sample Tests (t - Test):

Definition:

Consider a random sample $\{x_1, x_2, ...x_n\}$ of size 'n' drawn from a Normal population with mean μ and variance σ^2.

$$\text{Sample mean } \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

The unbiased estimate of the pop.variance σ^2 is denoted as s^2.

$$s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}$$

The student's t-statistic is defined as $t = \frac{|\bar{x} - \mu|}{s}\sqrt{n}$, Where n = sample size

The degree of freedon of this statistic

$V = n - 1$

Type I:

To test the significance of a single mean (For small samples)

$$\text{Test Statistic } t = \frac{\bar{x} - \mu}{\frac{S.D}{\sqrt{n-1}}} = \frac{\bar{x} - \mu}{\frac{S}{\sqrt{n}}}$$

s = sample S.D and

$$S^2 = \frac{1}{n-1}\sum (x - \bar{x})^2 \text{ (or) } S = \sqrt{\frac{ns^2}{n-1}}$$

If the computed value of t is greater than the critical value t_α, H_o is rejected

(or) if $|t| < t_\alpha$, the null hypothesis H_o is accepted at α level.

1. A machinist is making engine parts with axle diameter of 0.700 inch. A random sample of 10 parts shows a mean diamter of 0.742 inch with a S.D of 0.40. Test whether the work is meeting the specification at 5% Los

Solution:

Given that n =10; $\overline{x} = 0.742$ inches

$\mu = 0.700$ inches $\quad$ S=$\sqrt{\frac{ns^2}{n-1}} = \sqrt{\frac{10\times(0.40)^2}{9}} = 0.4216$

s = 0.40 inches $\quad$ S = 0.42

Null hypothesis H_0 : the product is confirming to specification ie., there is no significant difference between $\overline{x}$ and μ

H_0 : $\mu = 0.700$ inches

H_1 : $\mu \neq 0.700$ inches

Test Statistic t = $\frac{|\overline{x}-\mu|}{s}\sqrt{n}$ =0.316

degrees of freedom = n-1 = 9

Table value of t at 5% level = 2.26

$\therefore$ the product is meeting the specification.

2. Ten individuals are chosen at random from a population and their heights are found to be in inches 63, 63, 66, 67, 68, 69, 70, 70, 71, 71. In the light of this data, discuss the suggestion that the mean height in the universe is 66 inches.

Solution:

x :	63	63	66	67	68	69	70	70	71	71
$(x-\overline{x})^2$:	23.04	23.04	3.24	0.64	0.04	1.44	4.84	4.84	10.24	10.24

$\therefore \sum x = 678$ and $\sum (x - \overline{x})^2 = 81.6$

$$\bar{x} = \frac{\sum x}{n} = \frac{678}{10} = 67.8$$

$$S = \sqrt{\frac{\sum (x - \bar{x})^2}{9}} = \sqrt{\frac{81.6}{9}} = 3.011$$

Let $H_0 : \mu = 66$ the mean and height if the universe is 66 inches

and $H_1 : \mu \neq 66$

Los $\alpha = 0.05$

Test Statistic $t = \frac{|\bar{x} - \mu|}{s}\sqrt{n} = \frac{67.8 - 66}{3.011}\sqrt{10} = 1.89$

Table value of t for 9 d.f at 5% Los is $t_0 = 2.2$

Since $|t| < t_0, H_0$ is accepted at 5% level.

$\therefore$ The mean height of universe of 66 is accepted.

Type II: (Test of significance of difference of mean)

$$\text{Test Statistic } t = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

Where $S^2 = \frac{n_1 s_1{}^2 + n_2 s_2{}^2}{n_1 + n_2 - 2}$ (or)

$$S^2 = \frac{\sum (x_1 - \bar{x}_1)^2 + \sum (x_2 - \bar{x}_2)^2}{n_1 + n_2 - 2}$$

The number of degrees of freedom = $V = n_1 + n_2 - 2$

The calculated value of t is less than the table value of t for d.f $= n_1 + n_2 - 2$, H_0 is accepted

Otherwise H_0 is rejected at the selected Los

1. Two independent samples from normal pop's with equal variances gave the following results

Sample	Size	Mean	S.D

1	16	23.4	2.5
2	12	24.9	2.8

Test for the equations of means.

Solution:

(i) We set up $H_0 : \mu_1 = \mu_2$; ie., there is no significant difference between their means

$H_1 : \mu_1 \neq \mu_2$

Los: $\alpha = 0.05$

Test Statistic $t = \dfrac{\overline{x_1} - \overline{x_2}}{S\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$ Where $S^2 = \dfrac{n_1 s_1^{\ 2} + n_2 s_2^{\ 2}}{n_1 + n_2 - 2}$

Given $\overline{x_1} = 23.4$; $n_1 = 16$; $s_1 = 2.5$

$\overline{x_2} = 24.9$; $n_2 = 12$; $s_2 = 2.8$

$$S^2 = \frac{n_1 s_1^{\ 2} + n_2 s_2^{\ 2}}{n_1 + n_2 - 2} = \frac{16(2.5)^2 + 12(2.8)^2}{16 + 12 - 2}$$

$$= \frac{100 + 94.08}{26} = 7.465$$

$S = 2.732$

$$\therefore\ t = \frac{23.4 - 24.9}{2.732\sqrt{\frac{1}{16} + \frac{1}{12}}} = -1.438$$

$\therefore\ |t| = 1.438$

Number of degrees of freedom $= n_1 + n_2 - 2 = 26$

Critical value :

The table value of t for 26 d.f at 5% Los is

$t_{0.05} = 2.056$

Conclusion:

Since the calculated value of t is less than table value of t,

H_0 is accepted at 5% Los.

$\therefore$ There is no significant difference between their means

2. Two independent samples of 8 and 7 items respectively had the following values

Sample I :	9	13	11	11	15	9	12	14
Sample II :	10	12	10	14	9	8	10	

Is the difference between the means of the samples significant?

Solution:

We set up $H_0 : \mu_1 = \mu_2$

$H_1 : \mu_1 \neq \mu_2$

Hence it is a two tailed test

Los: $\alpha = 0.05$

Test Statistic $t = \dfrac{\overline{x_1} - \overline{x_2}}{S\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$

Where $S^2 = \dfrac{\sum (x_1 - \overline{x_1})^2 + \sum (x_2 - \overline{x}_2)^2}{n_1 + n_2 - 2}$

Computation of t:

x_1	$d_1=(x_1-\overline{x_1})$ $d_1=x_1-11.75$	$d_1{}^2=(x_1-\overline{x_1})^2$	x_2	$d_2=(x_2-\overline{x_2})$ $d_2=x_2-10.43$	$d_2{}^2=(x_2-\overline{x_2})^2$
9	-2.75	7.5625	10	-0.43	0.1849
13	1.25	1.5625	12	1.57	2.4649
11	-0.75	0.5625	10	-0.43	0.1849
11	-0.75	0.5625	14	3.57	12.7449
15	3.25	10.5625	9	-1.43	2.0449
9	-2.75	7.5625	8	-2.43	5.9049
12	0.25	0.0625	10	-0.43	0.1849
14	2.25	5.0625			
	$\sum d_1=3.5$	$\sum d_1{}^2=33.5$		$\sum d_2=-0.01$	$\sum d_2{}^2=23.7143$

$$\overline{x_1}=11+\frac{6}{8}=11.75$$

$$\overline{x_2}=10+\frac{3}{7}=10.43$$

$$\sum(x_1-\overline{x_1})^2=\sum d_1{}^2-\frac{\sum d_1{}^2}{n_1}=38-\frac{36}{8}=33.5$$

$$\sum(x_2-\overline{x_2})^2=\sum d_2{}^2-\frac{\sum d_2{}^2}{n_2}=25-\frac{9}{7}=33.5$$

$$\therefore S^2=\frac{33.5+23.71}{8+7-2} \quad \Rightarrow \quad S=2.097$$

$$\therefore t=\frac{\overline{x_1}-\overline{x_2}}{S\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$$

$$= \frac{11.75 - 10.43}{2.097\sqrt{\frac{1}{8} + \frac{1}{7}}}$$

$t = 1.218$

d.f = 8 + 7 - 2 = 13

Critical value:

The table value of t for 13 d.f at 5% level is 2.16

Conclusion:

Since $| t | < 2.16$, H_0 is accepted

$\therefore$ There is no significant difference between the means of the two samples.

Type III:

Testing of significance of the difference in means paired data.

When the two samples are of the same sizes and the data are paired

$$\text{the test statistic is } t = \frac{\overline{d}}{\frac{S}{\sqrt{n}}}$$

Where $\overline{d} =$ mean of differences

$$\text{and } S = \sqrt{\frac{\sum (d - \overline{d})^2}{n-1}}$$

Degrees of freedom = n-1

1. Elevan school boys were given a test in painting. They were given a month's further tution and a second test of equal difficulty was held at the end of the month. Do the marks give evidance that the students have beneifit by extra coaching?

Boys:	1	2	3	4	5	6	7	8	9	10	11
First Test (marks)	25	23	19	22	21	19	22	21	25	18	20
Second test (marks)	26	22	22	19	23	21	24	24	25	22	18

Solution:

$H_0 : \mu =$ the student have not been benefited by extra coaching.

ie., The mean of the difference between the marks of the two tests is zero

ie., $H_0 : \bar{d} = 0$

$H_1 : \bar{d} > 0$

Los: $\alpha = 0.05$ (or) 5%

the test statistic is t = $\dfrac{\bar{d}}{\frac{S}{\sqrt{n}}}$

S. No	1	2	3	4	5	6	7	8	9	10	11
d = x - y	-1	1	-3	3	-2	-2	-2	-3	0	-4	2
d - $\bar{d}$	0	2	-2	4	-1	-1	-1	-2	1	-3	3
d - $\bar{d}\ ^2$	0	4	4	16	1	1	1	4	1	9	9

$$\sum d = -11; \quad \bar{d} = \frac{\sum d}{n} = \frac{-11}{11} = -1$$

$$\sum \ d - \bar{d}\ ^2 = 50$$

$$S = \sqrt{\frac{\sum (d - \bar{d})^2}{n-1}} = \sqrt{\frac{50}{10}} = \sqrt{5} = 2.236$$

$$\therefore \text{t} = \frac{\bar{d}}{\frac{S}{\sqrt{n}}} = \frac{-1}{\frac{2.236}{\sqrt{11}}}$$

$$\therefore \quad |\text{t}| = \frac{1}{0.625} = 1.48$$

No. of d.f = 11-1 = 10

Critical value:

At 5% Los, the table value of t at 10 degree freedom is 1.812

Conclusion:

$|t| < 1.812$, H_0 is accepted at 5% Los.

$\therefore$ The students have not been benefited by extra-coaching.

2. The scores of 10 candidates prior and after training are given below,

Prior	:	84	48	36	37	54	69	83	96	90	65
After	:	90	58	56	49	62	81	84	86	84	75

Is the training effective?

Solution:

We set up H_0 : the training is not effective

ie., $H_0: \ \bar{d} = 0$

$H_1: \ \bar{d} > 0$

the test statistic is $\text{t} = \dfrac{\bar{d}}{\frac{S}{\sqrt{n}}}$

S. No	1	2	3	4	5	6	7	8	9	10
d = x - y	-6	-10	-20	-12	-8	-12	-1	10	6	-10
$d - \bar{d}$	0.3	-3.7	-13.7	-5.7	-1.7	-5.7	5.3	16.3	12.3	-3.7
$d - \bar{d}\ ^{2}$	0.09	13.69	187.69	32.49	2.89	32.49	28.09	265.69	151.29	13.69

$$\sum d = -63; \quad \bar{d} = \frac{\sum d}{n} = \frac{-63}{10} = -6.3$$

$$\sum \; d - \bar{d} \;^{2} = 728.1$$

$$S = \sqrt{\frac{\sum (d-\bar{d})^2}{n-1}} = \sqrt{\frac{728.1}{9}} = \sqrt{80.9} = 8.994$$

$$S = 8.994$$

$$\therefore t = \frac{-6.3}{\frac{8.994}{\sqrt{10}}} = \frac{-6.3}{2.844} = -2.21$$

$$| t | = 2.21$$

Degrees of freedom V= n-1 = 10-1 = 9

Critical value:

At 5% Los, the table value of t at 9 degree freedom is 2.262

Conclusion:

$| t | < 2.262$, H_0 is accepted at 5% Los.

$\therefore$ There is no effective in the training.

Variance Ratio Test (or) F-test for equality of variances

This test is used to test the significance of two or more sample estimates of population variance

The F-statistic is defined as a ratio of unbiased estimate of population variance

$$F = \frac{S_1^2}{S_2^2}; \quad \text{Where } S_1^2 = \frac{\sum \; x_1 - \bar{x}_1 \;^{2}}{n_1 - 1}$$

$$S_2^2 = \frac{\sum \; x_2 - \bar{x}_2 \;^{2}}{n_2 - 1}$$

$\therefore$ The distribution of $F = \frac{S_1^{\,2}}{S_2^{\,2}}$ $\quad S_1^{\,2} > S_2^{\,2}$ is given by the following p.d.f

If $S_1^{\,2}$ and $S_2^{\,2}$ are the variances of two sample of sizes n_1 and n_2 respectively, the estimate of the population variances based on these samples are respectively

$$S_1^{\,2} = \frac{n_1 s_1^{\,2}}{n_1 - 1}; \qquad S_2^{\,2} = \frac{n_2 s_2^{\,2}}{n_2 - 1}$$

$$\text{d.f } V_1 = n_1 - 1 \ \& V_2 = n_2 - 1$$

While defining the statistic F, the large oftwo variances is always placed in the numerator and smaller in the denominator

Test of significance for equality of population variances

Consider two independent R, samples $x_1, x_2, ...x_{n_1}$ & $y_1, y_2, ...y_{n_2}$ from normal populations

The hypothesis to be tested is

"The population variances are same".

we set up: $H_0 : \sigma_1^{\,2} = \sigma_2^{\,2}$

& $H_1 : \sigma_1^{\,2} \neq \sigma_2^{\,2}$

The test statistic $\quad F = \frac{S_1^{\,2}}{S_2^{\,2}} \quad S_1^{\,2} > S_2^{\,2}$

$$S_1^{\,2} = \frac{1}{n_1 - 1}\sum_{i=1}^{n} x_i - \bar{x}^{\,2} \text{ and } S_1^{\,2} = \frac{1}{n_1 - 1}\sum_{j=1}^{n} y_j - \bar{y}^{\,2}$$

F distribution with d.f $V_1 = n_1 - 1 \ \& V_2 = n_2 - 1$

Problems:

1. It is known that the mean diameters o rivets produced by two firms A and B are practically the same but the standard deviations may differ.

For 22 rivets produced by A, the S.D is 2.9 m, while for 16 rivets manufactured by B, the S.D is 3.8 m. Test whether the products of A have the same variability as those of B

Solution:

$H_0 : \sigma_1^{\,2} = \sigma_2^{\,2}$

ie., variability for the two types of products are same.

Los: $\alpha = 0.05$ (or) 5%

The test statistic $F = \frac{S_1^2}{S_2^2} \quad S_1^2 > S_2^2$

Given, $n_1 = 22; \quad n_2 = 16$

$S_1 = 2.9; \quad S_2 = 3.8$

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{22(2.9)^2}{22-1} = 8.81$$

$$S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{16(3.8)^2}{16-1} = 15.40$$

$$F = \frac{S_2^2}{S_1^2} \quad S_2^2 > S_1^2$$

$$= \frac{15.40}{8.81}$$

$F = 1.748$

Number of degrees of freedom are $V_1 = 16 - 1 = 15$

$V_2 = 22 - 1 = 21$

Critical value:

At 5% Los, the table value of F at d.f (15,21) is F = 2.18

Conclusion:

F< 2.18, H_0 is accepted at 5% Los.

$\therefore$ Variability for two types of products may be same.

2. Two random samples of sizes 8 and 11, drawn from two normal populations are characterized as follows

	Size	Sum of observations	Sum of square of observations
Sample I	8	9.6	61.52
Sample II	11	16.5	73.26

You are to decide if the two populations can be taken to have the same variance.

Solution:

Let x and y be the observations of two samples

we set up: $H_0 : \sigma_1^2 = \sigma_2^2$

& $H_1 : \sigma_1^2 \neq \sigma_2^2$

For sample I

$$s_1^2 = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2$$

$$= \frac{61.52}{8} - \left(\frac{9.6}{8}\right)^2$$

$$= 7.69 - (1.2)^2 = 7.69 - 1.44$$

$$s_1^2 = 6.25$$

For sample II

$$s_2^2 = \frac{\sum y^2}{n} - \left(\frac{\sum y}{n}\right)^2$$

$$= \frac{73.26}{11} - \left(\frac{16.5}{11}\right)^2$$

$$= 6.66 - (1.5)^2 = 6.66 - 2.25$$

$$s_2^2 = 4.41$$

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{8(6.25)}{7} = 7.143$$

$$S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{11(4.41)}{10} = 4.851$$

$$F = \frac{S_2^2}{S_1^2} \quad S_2^2 > S_1^2$$

$$= \frac{7.143}{4.851} = 1.472$$

$$F = 1.472$$

Number of degrees of freedom are $V_1 = n_1 - 1 = 8 - 1 = 7$

$$V_2 = n_2 - 1 = 11 - 1 = 10$$

Critical value:

The table value of F for (7,10) d.f at 5% Los is 3.14

Conclusion:

Since $|F| < 3.14$, H_0 is accepted at 5% level

$\therefore$ Variances of two populations may be same.

Variability for two types of products may be same.

Chi-Square Test

Definition

If O_i (i = 1, 2, ... n) are set of observed (experimental) frequencies and $E_i (i = 1, 2, ...n)$

are the corresponding set of expected frequncies, then the statistic

χ^2 is defined as

$$\chi^2 = \sum_{i=1}^{n} \frac{O_i - E_i^{\ 2}}{E_i}$$

The degree of freedom is $v = n - 1$

For fitting Binomial distribution $v = n - 1$

For fitting Poisson distribution $v = n - 2$

For fitting Normal distribution $v = n - 3$

Chi-square Test of Goodness of fit

If the calculated value of χ^2 is less than the table value at a specified Los.
The fit is considered to be good
otherwise the fit is considered to be poor.

Conditions for applying χ^2 Test

For the validity of chi-square test of "goodness of fit" between theory and experiment following Conditions must be satisfied.

(i) The sample of observations should be independent
(ii) Constraints on the cell frequncies. If any, should be linear.
(iii) N, the total frequency should be reasonably large, say greater than 50.
(iv) N_0 theoretical cell frequency should be less than 5, If any theoretical cell frequency less than 5, the for application χ^2 test
It is pooled with the preceeding or succeeding frequency so that the pooled frequency is greater than 5 and finally adjust for the d.f lost in pooling.

Problems

1. The following table gives the number of aircraft accident that occured during the various days of the week. Test whether the acidents are uniformly distributed over the week.

Days :	Mon	Tue	Wed	Thu	Fri	Sat	Total
No.of accidents :	14	18	12	11	15	14	84

Solution:

We set up H_0 : The accidents are uniformly distributed over the week

Los $\alpha = 0.05$

Test Statistic $\chi^2 = \sum_{i=1}^{n} \frac{\mathrm{O_i} - \mathrm{E_i}^{\;2}}{\mathrm{E_i}}$

Under the nul hypothesis,

The expected frequency of the on each day = $\frac{84}{6} = 14$

$\mathrm{O_i}$: 14 18 12 11 15 14

$\mathrm{E_i}$: 14 14 14 14 14 14

$$\chi^2 = \frac{14-14^{\;2}}{14} + \frac{18-14^{\;2}}{14} + \frac{12-14^{\;2}}{14} + \frac{11-14^{\;2}}{14} + \frac{15-14^{\;2}}{14} + \frac{14-14^{\;2}}{14}$$

= $1.143 + 0.286 + 0.643 + 0.071$

= 2.143

Number of degrees of freedom V = n – 1=7 – 1=6

Critical value:

The tablulated value of χ^2 at 5% for 6 d.f is 12.59

Conclusion:

Since $\chi^2 < 12.59$, we accept the null hypothesis

$\therefore$ We conclude that the accidents are uniformly distributed over the week.

2. The theory predicts the population of beans in the four groups A, B , C and D should be

9:3:3:1. In an experiment among 1600 beans, the number in the four groups were 882, 313, 287 and 118. Does the experimental result support the theory?

Solution:

We set up the null hypothesis

H_0 : The theory fits well into the experiment

ie., the experimental results supports the theory

Total Number of beans = 1600

Divide these beans in the ratio 9:3:3:1

To calculate the expected frequencies

$$E(882) = \frac{9}{16}\times 1600 = 900$$

$$E(313) = \frac{3}{16}\times 1600 = 300$$

$$E(287) = \frac{3}{16}\times 1600 = 300$$

$$E(118) = \frac{1}{16}\times 1600 = 100$$

O_i : 882 313 287 118

E_i : 900 300 300 100

Test Statistic $\chi^2 = \sum_{i=1}^{n} \frac{{O_i - E_i}^{2}}{E_i}$

$$\chi^2 = \frac{{882-900}^{2}}{900} + \frac{{313-300}^{2}}{300} + \frac{{287-300}^{2}}{300} + \frac{{118-100}^{2}}{100}$$

$$= 0.36 + 0.563 + 0.563 + 3.24$$

$\chi^2 = 4.726$

Critical Value:

The table value of χ^2 at 5% for 3 d.f is 7.815

Conclusion:

Since $\chi^2 < 7.815$, H_0 is accepted at 5% Los.

$\therefore$ We conclude that there is a very good correspondent between theory and experiment

3. 4 coins were tossed 160 times and the following results were obtained.

No. of heads :	0	1	2	3	4	
Frequency :	19	50	52	30	9	160
	0	50	104	90	36	280

Test the goodness of fit with the help of χ^2 on the assumption that the coins are unbiased

Solution:

We set up, the null hypothesis, the coins are unbiased:

The probability if getting the success of heads is p = $\frac{1}{2}$

q = 1- p = $\frac{1}{2}$

When 4 coins are tossed, the probability of getting 'r' heads is given by,

$$P(x = r) = n_{C_r}\ p^r q^{n-r}\ ;\quad r = 0, 1, 2, 3, 4$$

$$= 4_{C_r}\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{4-r}$$

$$= 4_{C_r}\left(\frac{1}{2}\right)^4$$

$$\therefore\ P(x = r) = 4_{C_r}\ \frac{1}{16}\quad r = 0, 1, 2, 3, 4$$

The expected frequencies of getting 0, 1, 2, 3, 4 heads are given by $1604_{C_r}\ \frac{1}{16}$

$= 104_{C_r}$, r = 0, 1, 2, 3, 4

= 10, 40, 60, 40, 10

O_i : 19 50 52 30 9

E_i : 10 40 60 40 10

26 48 43 26 12

Test Statistic $\chi^2 = \sum_{i=1}^{n} \frac{O_i - E_i\ ^2}{E_i}$

$$\chi^2 = \frac{19-10\ ^2}{10} + \frac{50-40\ ^2}{40} + \frac{52-60\ ^2}{60} + \frac{30-40\ ^2}{40} + \frac{9-10\ ^2}{40}$$

$= 8.1 + 2.5 + 1.067 + 2.5 + 0.1$

$\chi^2 = 14.267$

D.f V= n-1=5-1=4

Critical value:

The table value of χ^2 for 4 d.f at 5% Los is 9.488

Conclusion:

Since $\chi^2 > 9.488$, H_0 is rejected at 5% Los

$\therefore$ The coins are biased

4. The follwoing table shows the distribution of goals in a football match

No. of goals :	0	1	2	3	4	5	6	7
No. of mistakes :	95	158	108	63	40	9	5	2

Fit a poisson distribution and test the goodness of fit.

Solution:

Fitting of poisson distribution

x :	0	1	2	3	4	5	6	7
f :	95	158	108	63	40	9	5	2

$\sum fx = 812$ and $\sum f = 480$

$$\therefore \quad \bar{x} = \lambda = \frac{\sum fx}{\sum f} = \frac{812}{480} = 1.7$$

$\therefore$ The expected frequencies are computed by

$$= 480 \times \frac{e^{-1.7}(1.7)^r}{r!} \qquad r = 0, 1, 2, 3, 4, 5, 6, 7$$

= 88, 150, 126, 72, 30, 10, 3, 1

We set up H_0 : The fit is good

$$\text{Test Statistic } \chi^2 = \sum_{i=1}^{n} \frac{{O_i - E_i}^{2}}{E_i}$$

$$\begin{array}{lllllll} O_i & : 95 & 158 & 108 & 63 & 40 & \underbrace{9 \quad 5 \quad 2}_{16} \\ E_i & : 88 & 150 & 126 & 72 & 30 & \underbrace{10 \quad 3 \quad 1}_{14} \end{array}$$

$$\chi^2 = \frac{{O - E}^{2}}{E} = \frac{{95-88}^{2}}{88} + \frac{{158-150}^{2}}{150} + \frac{{108-126}^{2}}{126} + \frac{{40-30}^{2}}{30} + \frac{{16-14}^{2}}{14} + \frac{{63-72}^{2}}{72}$$

= 0.56 + 0.43 + 2.57 + 3.33 + 1.12 +0.29

$\chi^2 \ = 8.30$

Number of degrees of freedom V= n-2 = 6 - 2 =4

Critical value:

The table value of χ^2 at 5% Los for 4 d.f is 9.483

Conclusion:

Since $\chi^2 < 9.483$, H_0 is accepted at 5% Los.

$\therefore$ The fit is good

5. Apply the χ^2 test of goodness of fit to the follwoing data

$O_i \quad : \quad 1 \quad 5 \quad 20 \quad 28 \quad 42 \quad 22 \quad 15 \quad 5 \quad 2$

$E_i \quad : \quad 1 \quad 6 \quad 18 \quad 25 \quad 40 \quad 25 \quad 18 \quad 6 \quad 1$

Solution:

H_0 : The fit is good

$\alpha = 0.05$ (or) 5%

$$O_i \quad : \quad \underbrace{1 \qquad 5}_{6} \quad 20 \quad 28 \quad 42 \quad 22 \quad 15 \quad \underbrace{5 \qquad 2}_{7}$$

$$E_i \quad : \quad \underbrace{1 \qquad 6}_{7} \quad 18 \quad 25 \quad 40 \quad 25 \quad 18 \quad \underbrace{6 \qquad 1}_{7}$$

n = 7

Test Statistic $\chi^2 = \sum_{i=1}^{n} \frac{O_i - E_i^{\quad 2}}{E_i}$

$$\chi^2 = \frac{6-7^{\quad 2}}{7} + \frac{20-18^{\quad 2}}{18} + \frac{28-25^{\quad 2}}{25} + \frac{42-40^{\quad 2}}{40} + \frac{22-25^{\quad 2}}{25} + \frac{15-18^{\quad 2}}{18} + \frac{7-7^{\quad 2}}{7}$$

$= 0.143 + 0.222 + 0.36 + 0.1 + 0.36 + 0.5 + 0$

$\chi^2 \ = 1.685$

d.f V= n-1 = 7 - 1 = 6

Critical value:

At 5% Los, the table value of χ^2 for 6 d.f is 12.592

Conclusion:

Since $\chi^2 < 12.592$, H_0 is accepted at 5% Los.

$\therefore$ The fit is good

6. The follwoing table shows the number of electricity failures in a town for a period of 180 days

Failures :	0	1	2	3	4	5	6	7
No. of days :	12	39	47	40	20	17	3	2

Use χ^2, examine whether the data are poisson distributed.

Solution:

Fitting of poisson distribution

x :	0	1	2	3	4	5	6	7
f :	12	39	47	40	20	17	3	2
fx :	0	39	94	120	80	85	18	14

$\sum fx = 450$ and $\sum f = 180$

$$\therefore \quad \bar{x} = \lambda = \frac{\sum fx}{\sum f} = \frac{450}{180} = 2.5$$

$\therefore$ The expected frequencies are computed by

$$= 180 \times \frac{e^{-2.5}(2.5)^r}{r!} \qquad r = 0, 1, 2, 3, 4, 5, 6, 7$$

E_i = 15, 37, 46, 38, 24, 12, 5, 2

r = 0 1 2 3 4 5 6 7

We set up H_0 : The fit is good

O_i : 12 39 47 40 20 17 $\underbrace{3 \qquad 2}_{5}$

$E_i \quad : \quad 15 \quad 37 \quad 46 \quad 38 \quad 24 \quad 12 \quad \underbrace{5 \qquad 2}_{7}$

$\therefore$ Test Statistic χ^2

$$\chi^2 = \frac{12-15\;^{2}}{15}+\frac{39-37\;^{2}}{37}+\frac{47-46\;^{2}}{46}+\frac{40-38\;^{2}}{38}+\frac{20-24\;^{2}}{24}+\frac{17-12\;^{2}}{12}+\frac{5-7\;^{2}}{7}$$

$= 0.6 + 0.108 + 0.022 + 0.105 + 0.667 + 2.083 + 0.5 + 1$

$\chi^2 = 4.156$

d.f V = n-1 = 7 - 1 = 6

Critical value:

At 5% Los, the table value of χ^2 for 6 d.f is 12.592

Conclusion:

Since $\chi^2 < 12.592$, H_0 is accepted at 5% Los.

$\therefore$ The fit is good

Test for Independence of Attributes

		Attribute A		
		A_1	A_2................A_j.................. A_t	Total
Attribute	B_1	O_{11}	O_{12}................O_{1j}.................. O_{1t}	(B_1)
B	B_2			
	.			
	.			
	B_i	O_{i1}	O_{i2}................O_{ij}.................. O_{it}	(B_i)
	.	.		
	.	.		

B_s	O_{s1}	O_{s2}................O_{sj}.................. O_{st}	(B_s)
Total	(A_1)	(A_2)................(A_i)................(A_t)	N

			Attribute A	
		A_1	A_2................A_j.................. A_t	Total
Attribute	B_1	O_{11}	O_{12}................O_{1j}.................. O_{1t}	(B_1)
B	B_2			
	.			
	.			
	.			
	B_i	O_{i1}	O_{i2}................O_{ij}.................. O_{it}	(B_i)
	.	.		
	.	.		
	.	.		
	B_s	O_{s1}	O_{s2}................O_{sj}.................. O_{st}	(B_s)
Total		(A_1)	(A_2)................(A_i)................(A_t)	N

Such a table is called (s × t) consistency table

Here, N → Total Frequency

O_{ij} → Observed frequency of $(i, j)^{th}$ cell

The expected frequency e_{ij} obtained by the rule

$$e_{ij} = \frac{\text{row total } B_i \quad \text{Column total } A_j}{N}$$ Where i = 1, 2, 3 .... s

j = 1, 2 .... t

Degrees of freedom associated with s × t consistency table = (s - 1) × (t - 1)

Chi-square table for 2×2 consistency table

In a 2×2 consistency table where in the frequencies are $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the value of χ^2 is

$$\chi^2 = \frac{(a+b+c+d)(ad-bc)^2}{(a+b)(a+c)(c+d)(b+d)}$$

Problems :

1. An opinion poll was conducted to find the reaction to a proposed civic reform in 100 members of each of the two political parties theinformation is tabulated below

	Favorable	**Unfavorable**	**Indifferent**
Party A	**40**	**30**	**30**
Party B	**42**	**28**	**30**

Test for Indepedence of reduction with the party affiliations.

Solution:

We set up H_0 : Reactions and party affiliations are independent

The expected frequencies are calculated by

				Total
	40	30	30	100
	42	28	30	100
Total	82	58	60	200

	Favorable	Unfavorable	Indifferent
Party A	$\frac{82\times100}{200}=41$	$\frac{58\times100}{200}=29$	$\frac{60\times100}{200}=30$
Party B	$\frac{82\times100}{200}=41$	$\frac{58\times100}{200}=29$	$\frac{60\times100}{200}=30$

$\therefore$ Test Statistic χ^2

$$\chi^2=\sum_{i=1}^{n}\frac{\mathrm{O_i}-\mathrm{E_i}\ ^{2}}{\mathrm{E_i}}$$

O_i	:	40	30	30	42	28	30
E_i	:	41	29	30	41	29	30

$$\chi^2 = \frac{40-41\;^{2}}{41}+\frac{30-29\;^{2}}{29}+\frac{30-30\;^{2}}{30}+\frac{42-41\;^{2}}{41}+\frac{28-29\;^{2}}{29}+\frac{30-30\;^{2}}{30}$$

$= 0.024 + 0.024 + 0.034 + 0.034$

$\chi^2 = 0.116$

Number of degrees of freedom =(2-1)(3-1) = 2

Critical value:

At 5% Los, the table value of χ^2 for 2 d.f is 5.99

Conclusion:

Since $\chi^2 < 5.99$, H_0 is accepted at 5% Los.

$\therefore$ The independence of reactions with the party affiliations may be correct.

2. In a locality 100 persons were randomly selected and asked about their educational achievements. The results are given below.

	Education		
	Middle	**High School**	**College**
Male	**10**	**15**	**25**
Female	**25**	**10**	**15**

Can you say that education depends on sex?

3. The following table gives the classification of 100 workers according to sex and the nature of work. Test whether nature of work is independent of the sec of the worker.

		Skilled	Unskilled	Total
	Male	40	20	60
Sex	Female	10	30	40
	Total	50	50	

Solution:

H_0 : Nature of work is independent of the sex of the worker

Under H_0, the expected frequencies are

$$E(40) = \frac{60\times 50}{100} = 30; \qquad E(20) = \frac{60\times 50}{100} = 30$$

$$E(10) = \frac{40\times 50}{100} = 20; \qquad E(30) = \frac{40\times 50}{100} = 20$$

$\therefore$ Test Statistic χ^2

$$\chi^2 = \sum_{i=1}^{n} \frac{O_i - E_i \ ^2}{E_i}$$

O_i : 40 20 10 30

E_i : 30 30 20 20

$$\chi^2 = \frac{40-30 \ ^2}{30} + \frac{20-30 \ ^2}{30} + \frac{10-20 \ ^2}{20} + \frac{30-20 \ ^2}{20}$$

$= 3.333 + 3.333 + 5 + 5$

$\chi^2 = 16.67$

Number of degrees of freedom =(2-1)(2-1) = 1

Critical value:

The table value of χ^2 at 5% Los, for 1 d.f is 3.841

Conclusion:

Since $\chi^2 > 3.841$, H_0 is rejected at 5% Los.

$\therefore$ We conclude that the nature of work is dependent on sex of the worker.

4. From the following data, test whether there is any association between intelligency and economics conditions

Intelligences						
	Excellent		**Good**	**Medium**	**Dull**	**Total**
Economic	**Good**	**48**	**200**	**150**	**80**	**478**
Conditions	**Not Good**	**52**	**180**	**190**	**100**	**522**
	Total	**100**	**380**	**340**	**180**	**1000**

Solution:

H_0 : There is no association between intelligency and economic conditions.

Los : $\alpha = 0.05$ (or) 5%

Under H_0, the expected frequencies are obtained as follows

$E(48) = \frac{100 \times 478}{1000} = 47.8;$ $\quad E(52) = \frac{100 \times 522}{1000} = 52.2$

$E(200) = \frac{380 \times 478}{1000} = 181.64;$ $\quad E(180) = \frac{380 \times 522}{1000} = 198.36$

$E(150) = \frac{478 \times 340}{1000} = 162.52;$ $\quad E(190) = \frac{340 \times 522}{1000} = 177.48$

$E(80) = \frac{180 \times 478}{1000} = 86.04;$ $\quad E(100) = \frac{180 \times 522}{1000} = 93.96$

O_i : 48 200 150 80 52 180 190 100

E_i : 47.8 181.64 162.52 86.04 52.2 198.36 177.48 93.96

$\therefore$ Test Statistic χ^2

$$\chi^2 = \sum_{i=1}^{n} \frac{O_i - E_i{}^{\;2}}{E_i}$$

$$= \frac{48-47.8^{\;2}}{47.8} + \frac{150-162.52^{\;2}}{162.52} + \frac{52-52.2^{\;2}}{52.2} + \frac{190-177.48^{\;2}}{177.48} + \frac{200-181.64^{\;2}}{181.64}$$

$$+ \frac{80-86.04^{\;2}}{86.04} + \frac{180-198.36^{\;2}}{198.36} + \frac{100-93.96^{\;2}}{93.96}$$

$= 0.0008 + 0.9645 + 0.0008 + 0.8832 + 1.8558 + 0.4240 + 1.6994 + 0.3883$

$\chi^2 = 6.2168$

Number of degrees of freedom = (s-1)(t-1) = (2-1)(4-1) = 3

Critical value:

The table value of χ^2 at 5% Los for 3 d.f is 7.815

Conclusion:

Since $\chi^2 < 7.815$, H_0 is accepted at 5% Los.

$\therefore$ We conclude that there is no association between intelligency and economic conditions

5. From the following data, test the hypothesis that the flower color is independent of flatness of leaf

	Flat leaves	Curved leaves	Total
White Flowers	**99**	**36**	**135**
Red Flowers	**20**	**5**	**25**
Total	**119**	**41**	**160**

Solution:

We set up: H_0 : flower color is independent of flatness of leaf. Los α=0.05 (or) 5%

The given probelm is a 2 x 2 consistency table

$\therefore$ we use the formula to find χ^2 is

$$\chi^2 = \frac{(a+b+c+d)(ad-bc)^2}{(a+b)(a+c)(c+d)(b+d)}$$

Here, a = 99; b = 36; e = 20; d = 5

$$\chi^2 = \frac{160(495-720)^2}{(135)(119)(25)(41)} = \frac{160(50625)}{16,466,625}$$

$$\chi^2 = 0.4919$$

Number of degrees of freedom = (s-1)(t-1) = (2-1)(2-1) = 1

Critical value:

The table value of χ^2 at 5% Los for 1 d.f is 3.841

Conclusion:

Since $\chi^2 < 3.841$, H_0 is accepted at 5% Los.

$\therefore$ Flower colour is independent of flatness of leaf.

Test for single variance

Chi-square test for population variance

In this method, we set up the null hypothesis $H_0 : \sigma^2 = \sigma_0{}^2$ (with a specified variance)

The test statistic $\chi^2 = \dfrac{ns^2}{\sigma^2}$

Where n = sample size

s = sample variance

σ = population variance

Note:

$*$ If the sample size n is large (>30)

The test statistic $z = \sqrt{2\chi^2} - \sqrt{2n-1} \sim N(0,1)$

We use the usual normal test.

1. A random sample of size 9 from a normal population have the following values 72, 68, 74, 77, 61, 63, 63, 73, 71. Test the hypothesis that the population variance is 36.

Solution:

Null hypothesis $H_0 : \sigma^2 = 36$

Alternative hypothesis $H_1 : \sigma^2 \neq 36$

Los α : 0.05 (or) 5%

$\therefore$ The test statistic $\chi^2 = \dfrac{ns^2}{\sigma^2}$

x : 72 68 74 77 61 63 63 73 71

$\sum x = 622$; $\bar{x} = \frac{\sum x}{n} = \frac{622}{9} = 69.11$

$x-\bar{x}$: 2.9 -1.1 4.9 7.9 -8.1 -6.1 -6.1 3.9 1.9

$x-\bar{x}^{2}$: 8.41 1.21 24.01 62.41 65.61 37.21 37.21 15.21 3.61

$\sum x-\bar{x}^{2} = 254.89$

$\chi^2 = \frac{ns^2}{\sigma^2} = \frac{254.89}{36} = 7.08$

d. f n-1 = 9-1 = 8

Critical value:

The table value of χ^2 for 8 d.f at 5% Los is 15.51

Conclusion:

Since $\chi^2 < 15.51$, H_0 is accepted at 5% Los.

$\therefore$ We conclude that the hypothesis of population variance is 36 is accepted

2. Test the hypothesis that $\sigma = 10$, given that s = 15 for a random sample of size 50 from a normal population

Solution:

Null hypothesis $H_0: \sigma = 10$

Alternative hypothesis $H_1: \sigma \neq 36$

We are given n = 50; s = 15

$\chi^2 = \frac{ns^2}{\sigma^2} = \frac{50 \times 225}{100} = 112.5$

Since 'n' is large (n > 30, the test statistic z = $\sqrt{2\chi^2} - \sqrt{2n-1}$

$= \sqrt{225} - \sqrt{99} = 15 - 9.95$

$$z = 5.05$$

This statistic z follows N(0,1)

Critical value:

At 5% Los, the table value of z is 3

Conclusion:

Since $|z| > 3$, H_0 is rejected.

$\therefore$ We conclude that $\sigma \neq 10$

3. The standard deviation of the distribution of times taken by 12 workers for performing a Job is 11 sec. Can it be taken 1 as a sample from a popualation whose S.D is 10 sec.

Solution:

Let $H_0 : \sigma = 10$

ie., the population standard deviation $\sigma = 10$

$H_1 : \sigma \neq 10$

Los α : 0.05 (or) 5% Los

Given n = 12;   s = 11

$\therefore$ The test statistic is

$$\chi^2 = \frac{ns^2}{\sigma^2}$$

$$= \frac{12 \times 121}{100} = 14.52$$

$$\chi^2 = 14.52$$

Degrees of freedom = n - 1 = 12 - 1 = 11

Critical value:

The table value of χ^2 for 11 d.f at 5%Los is 19.675.

Conclusion:

Since $\chi^2 < 19.675$, H_0 is accepted at 5% level

$\therefore$ The S.D of the time element is 10 sec is supported.

ie., the population standard deviation $\sigma = 10$

UNIT -2 DESIGNS OF EXPERIMENT

Define Anova?

Anova is separation of variance ascribable to one group of causes from the variance ascribable to other group

Some Important Abbreviations:

- SSC- Between sum of squares (Column)
- TSS- Total sum of squares
- SST- Sum of squares due to Treatments
- MSS- Mean Sum of squares
- SSE- Error Sum of squares (or) Within Sum of squares
- RSS- Row Sum of squares
- CF- Correction Factor
- CD- Critical Difference
- SSR- Sum of squares between Rows
- MSC- Mean Sum of squares (Between Columns)
- MSE- Mean Sum of squares (within Columns)
- MSR-Mean Sum of squares (Between Rows)
- N1- Number of Elements in each Column
- N2- Number of Elements in each Row

Anova Table For One way Classification (C.R.D):

Source of Variance	Sum of Squares	Degree of Freedom	Mean sum of Squares	Variance Ratio
Between Columns	SSC	C-1	$MSC = \frac{SSC}{C-1}$	$F = \frac{MSC}{MSE}$ (OR) $F = \frac{MSE}{MSC}$
Within Columns	SSE	N-C	$MSE = \frac{SSE}{N-C}$	
Total	TSS	N-1	Condition Always F >1	

Anova Table For Two way Classification :

Randomized Block Design (R.B.D)

Source of Variance	Sum of Squares	Degree of Freedom	Mean sum of Squares	Variance Ratio
Column Treatment	SSC	C-1	$MSC = \frac{SSC}{C-1}$	$F_c = \frac{MSC}{MSE}$ $F_R = \frac{MSR}{MSE}$
Row Treatment	SSR	R-1	$MSR = \frac{SSR}{R-1}$	
Error	SSE	N-C-R+1	$MSE = \frac{SSE}{(R-1)(C-1)}$	
Total	TSS		Condition Always F >1	

Anova Table For Three way Classification :

Latin Square Design (L.S.D)

Source of Variance	Sum of Squares	Degree of Freedom	Mean sum of Squares	Variance Ratio
Between Columns	SSC	K-1	$\text{MSC} = \frac{\text{SSC}}{\text{K}-1}$	$\text{F}_c = \frac{\text{MSC}}{\text{MSE}}$
Between Rows	SSR	K-1	$\text{MSR} = \frac{\text{SSR}}{\text{K}-1}$	$\text{F}_R = \frac{\text{MSR}}{\text{MSE}}$
Between Treatments	SST	K-1	$\text{MST} = \frac{\text{SST}}{\text{K}-1}$	
Error	SSE	(K-1)(K-2)	$\text{MSE} = \frac{\text{SSE}}{(\text{K}-1)(\text{K}-2)}$	$\text{F}_T = \frac{\text{MST}}{\text{MSE}}$
Total	TSS	$\text{K}^2 - 1$	Condition Always F >1	

Merits & Demerits of C.R.D:

- Merits
    - It has a simple layout
    - The analysis the design as it results in a one way classification analysis of variance
    - There is complete flexibility as the number of replication is not fixed
    - Analysis can be performed if some observations are missing
- Demerits:
    - The Experimental error is large as compare to other designs because homogeneity of the units is not taken into consideration

Merits & Demerits of R.B.D:

- Merits
    - It has a simple layout but it is more efficient than CRD because of reduction of experimental error
    - Analysis is possible even if some observations are missing
    - It is flexible and so any number of treatments and any number of replication may be used
    - The analysis of design is simple as its results in a two way classification of analysis of variance
    - This is more popular design with experiments because of its simplicity, flexibility and validity
- Demerits:
    - If the number of treatments is large, then the size of the block will increase this may causes heterogeneity within the blocks
    - The shape of experimental material should be Rectangle
    - If the interation are large, the experiment may yields misreading results

Merits & Demerits of L.S.D:

- Merits
    - The analysis of design is simple as its results in a three way classification of analysis of variance
    - LSD controls variations in two directions of the experimental materials as row and column resulting in the reduction of experimental error

- ➢ The analysis of remains relatively simple even with missing data

❖ Demerits:

- ➢ The number of treatments should be equal to the number of rows and columns as the area should be in square form
- ➢ It is suitable only for smaller number of treatments say between 5 to 12
- ➢ 2x2 Latin square is not possible
- ➢ The process of randomization is not as simple as RBD

Some Important Formulas:

- Correction Factor (CF) = $\frac{T^2}{N}$, Where N=Number of Data's given in the problem, T = Total
- TSS = $[(\sum_{i=1,2,3\ldots} X_i) - \frac{T^2}{N}]$, i ranges from number of colums given in the Pb
- SSC = $[(\frac{(\sum_{i=1,2,3\ldots} X_i)^2}{N_1}) - \frac{T^2}{N}]$, i ranges from number of colums given in the Pb
- SSR = $[(\frac{(\sum_{i=1,2,3\ldots} Y_i)^2}{N_2}) - \frac{T^2}{N}]$, i ranges from number of rows given in the Pb
- SSE = TSS – SSC (OR) SSE = TSS – SSR, Based on the Problem (For CRD)
- SSE = TSS – SSR- SSC (For RBD)
- SSE = TSS – SSR-SSC-SST (For LSD)

Note:

- ✓ N_1 = number of elements in each Column
- ✓ N_2 = number of elements in each Row

- **Compare RBD and LSD:**

S.No.	LSD	RBD
1	The number of replication of each treatment is equal to the number of treatments in LSD	There are no such restrictions on treatments and replication in RBD.
2	LSD can be performed on a square field.	While RBD can be performed either on a square field or a rectangle field.
3	LSD is known to be suitable for the case when the number of treatments is between 5 and 12	RBD can be used for any number of treatments.
4	The main advantage of LSD is that it controls the effect of two extraneous variables.	RBD controls the effect of only one extraneous variable. Hence the experimental error is reduced to a larger extent in LSD than in RBD.

- **Name the basic principles of experimental design.**

There are three basic principles of experimental design. They are:

(i) Randomization
(ii) Replication
(iii) Local Control

- **Define Randomization:**

It ensures that each treatment gets an equal chance of being allocated. Consequently randomization eliminates the bias of any form.

- **Define Replication:**
  By replication we mean, the repetition of the treatments under investigation. Due to replication more reliable estimates can be made available. To be more precise as the replication increases the experimental error decreases.
- **Define Local Control:**
  It is a process of reducing the experimental error by dividing the heterogeneous experimental area into homogeneous blocks.
- **Define "Analysis of Variance" (or) ANOVA.**
  According to R.A. Fisher, Analysis Of Variance (ANOVA) is the separation of variance ascribable to one group of causes from the variance ascribable to other groups.
- **Define "experimental error".**
  The estimation of the amount of variation due to each of the independent factors separately and then comparing these estimates due to assignable factors with the estimate due to the chance factor is known as experimental error or simple error.
- **What do you mean by one-way classification in analysis of variance?**
  In one-way classification the data are classified according to only one criterion (or) factor.
- **Explain the meaning and use of Analysis of Variance?**
  Analysis of variance to test the homogeneity several means.
  Uses:
  (i) It helps to find out the F-test
  (ii) Between the samples we can find the variances.
- **Define the term Completely Randomized Design.**
  The completely randomized design is the simplest of all the designs, based on principles of randomization and replication. In this design, treatments are allocated at random to the experimental units over the entire experimental.
- **What is a randomized block design?**
  Let us consider an agricultural experiment using which we wish to test the effect of 'k' fertilizing treatments on the yield of crops. We assume that we know some information about the soil fertility of the plots. Then we divide the plots into 'h' blocks. Thus the plots in each block will be of homogeneous fertility as far as possible within each block, the 'k' treatments are given to the 'k' plots in a perfectly random manner, such that each treatment occurs only once in any block. But the same k treatments are repeated from block to block. This design is called Randomized Block Design
- **State the differences between CRD and RBD.**

S.No.	CRD	RBD
1	This design provides a one-way classified data according to levels of a single factor namely 'treatment'	The analysis of the design is simple and straight forward as in the case of two-way classification.
2	It has a simple layout	The analysis of this decision is not as simple as a completely randomized design.
3	Grouping of the experimental site so as to allocate the treatments at random to the experimental units is not done	Treatments are allocated at random within the units of each stratum.

- **Define Latin Square Design:**
  Here for k treatments we should have k^2 experimental units arranged in a square. So that each row as well as each column contains k units. Such a layout is known as k x k latin square design. The treatments should be allocated in a random manner in such a way that each treatment occurs in each row and each column.
- **What are the basic principal of experimental design? (Apr/May 2015 (R2013 & R2008))**

Soln: There are three basic principles of experimental design. They are:

  (iv) Randomization
  (v) Replication
  (vi) Local Control

- **Is** 2×2 **Latin square is possible? Why? (Apr/May, & Nov/Dec 2015)**

Soln: No because SSE degree of freedom is (k-1)(k-2) where k is number of rows and columns
In 2X2 Latin square k=2 so SSE degree of freedom is zero, If error is zero we are not able to find calculated value of F

- **Define (a) Mean Square (b) Complete randomized design (Nov/Dec 2015)**

Soln: (a) mean squares are used to determine whether factors (treatments) are significant. The treatment mean square is obtained by dividing the treatment sum of squares by the degrees of freedom. The treatment mean square represents the variation between the sample means.

(b) The completely randomized design is the simplest of all the designs, based on principles of randomization and replication. In this design, treatments are allocated at random to the experimental units over the entire experimental.

- **What is the aim of designs of experiments? (Apr/May 2015)**

Soln: The design of experiments is the design of any task that aims to describe or explain the variation of information under conditions that are hypothesized to reflect the variation. The term is generally associated with true experiments in which the design introduces conditions that directly affect the variation, but may also refer to the design of quasi-experiments in which natural conditions that influence the variation are selected for observation.

PART-B

1) **The following are the numbers of mistakes made in 5 successive days of 4 Technicians working a photographic laboratory**

Technicians-I	**Technicians-II**	**Technicians-III**	**Technicians-IV**
6	**14**	**10**	**9**
14	**9**	**12**	**12**
10	**12**	**7**	**8**
8	**10**	**15**	**10**
11	**14**	**11**	**11**

Test the level of signification $\alpha = 0.01$ **whether the difference among the 4 sample can be attributed to chance?**

Soln: H_0: There is no significance difference betwee the Technicians

H_1: There is a significance difference betwee the Technicians

We shifted our origin to 10

X_1 (X_1-10)	X_2 (X_2-10)	X_3 (X_3-10)	X_4 (X_4-10)	Total	X_1^2	X_2^2	X_3^2	X_4^2
-4	4	0	-1	-1	16	16	0	1
4	-1	2	2	7	16	1	4	4
0	2	-3	-2	-3	0	4	9	4
-2	0	5	0	3	4	0	25	0
1	4	1	1	7	1	16	1	1
ΣX_1	ΣX_2	ΣX_3	ΣX_4	Σy(Total)	ΣX_1^2	ΣX_2^2	ΣX_3^2	ΣX_4^2
-1	**9**	**5**	**0**	**T=13**	**37**	**37**	**39**	**10**

Step-1: N→ Number of Data given in the Problem (N=20)

Step 2: T=**13** (From above Table)

Step 3: $\frac{T^2}{N} = \frac{(13)^2}{20} = 8.45$(Correction Factor)

Step4 : $TSS = \Sigma X_1^2 + \Sigma X_2^2 + \Sigma X_3^2 + \Sigma X_4^2 - \frac{T^2}{N} = (37 + 37 + 39 + 10 - 8.45) = 114.55$

Step 5: $SSC = \frac{ZX_1^2}{N_1} + \frac{ZX_2^2}{N_1} + \frac{ZX_3^2}{N_1} + \frac{ZX_4^2}{N_1} - \frac{T^2}{N} = (\frac{(-1)^2}{5} + \frac{9^2}{5} + \frac{5^2}{5} + 0 - 8.45) = 12.95$

Step 6: SSE = TSS – SSC = 114.55-12.95 = **101.6**

Source of Variance	**Sum of Squares**	**Degree of Freedom**	**Mean sum of Squares**	**Variance Ratio**	**Table Value at 1% level**
Between Columns	SSC = 12.95	C-1 = 4-1 = 3	$MSC = \frac{SSC}{C-1}$ = 4.317	$\cancel{F = \frac{MSC}{MSE} < 1}$ $F = \frac{MSE}{MSC} = 1.471$	$F_c(16,3) = 26.87$
Error	SSE = 101.6	N-C = 20-4 =16	$MSE = \frac{SSE}{N-C}$ = 6.35		
Total	TSS = 114.55	N-1= 20-1 =19	**Condition Always F >1**		

Conclusion→ Calculate F_c(1.471) < table value of F_c(26.87) so we accept H_0

(ie)There is no significance difference betwee the Technicians

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2) **There are three main brands of a certain powder, A set of 120 sample values is examined and found to be allocated among four groups (A,B,C,D) and three brands are (I,II,III) are shown under**

Brands	Groups			
I	0	4	8	15
II	5	8	13	6
III	8	19	11	13

Is there any significance difference in brands preference answer at **5%**Level?

Soln: H_0: There is no significance difference in Brands

H_1: There is a significance difference in Brands

Brand	**Groups**				Total	X_1^2	X_2^2	X_3^2	X_4^2
	X_1 (A)	X_2 (B)	X_3 (C)	X_4 (D)					
I (Y_1)	0	4	8	15	$\Sigma y_1 = 27$	0	16	64	225
II (Y_2)	5	8	13	6	$\Sigma y_2 = 32$	25	64	169	36
III (Y_3)	8	19	11	13	$\Sigma y_3 = 51$	64	361	121	169
Total	ΣX_1	ΣX_2	ΣX_3	ΣX_4	**Σy(Total)**	ΣX_1^2	ΣX_2^2	ΣX_3^2	ΣX_4^2
	13	**31**	**32**	**34**	**T=110**	**89**	**441**	**354**	**430**

Step-1: N→ Number of Data given in the Problem (N=12)

Step 2: T=**110** (From above Table)

Step 3: $\frac{T^2}{N} = \frac{(110)^2}{12} = 1008.3(\text{ Correction Factor})$

Step4 : $TSS = \Sigma X_1^2 + \Sigma X_2^2 + \Sigma X_3^2 + \Sigma X_4^2 — \frac{T^2}{N} = (89 + 441 + 354 + 430 — 1008.3) = 305.7$

Step 5: $SSR = \frac{Zy_1^2}{N_2} + \frac{Zy_2^2}{N_2} + \frac{Zy_3^2}{N_2} — \frac{T^2}{N} = (\frac{(27)^2}{4} + \frac{32^2}{4} + \frac{51^2}{4} — 1008.3) = 80.2$

Step 6: SSE = TSS – SSR = 305.7 – 80.2 =**225.5**

Source of Variance	**Sum of Squares**	**Degree of Freedom**	**Mean sum of Squares**	**Variance Ratio**	**Table Value at 5% level**
Between Rows	SSR = 80.2	R-1 = 3-1 = 2	$MSC = \frac{SSR}{R — 1} = 40.1$	$F_R = \frac{MSR}{MSE} = 1.999$ $F = \cancel{\frac{MSE}{MSR}} < 1$	$F_R(2,9) = 4.26$
Error	SSE = 225	N-C = 12-3 =9	$MSE = \frac{SSE}{N — R} = 20.06$		
Total	TSS = 305.7	N-1= 12-1 =11	**Condition Always F >1**		

Conclusion→ Calculate F_R (1.999) < table value of F_R (4.26) so we accept H_0

(ie)There is no significance difference in Brands

3) **Three different machines are used for production ,on the basis of the outputs ,setup One – Way ANOVA table and test whether the machines are equally effective.**

MACHINE I	**MACHINE II**	**MACHINE III**
10	**9**	**20**
15	**7**	**16**
11	**5**	**10**
10	**6**	**14**

Given that the value of F at 5% level of significance for (2,9) d.f is 4.26

Solution:

MACHINES		
X_1	X_2	X_3
10	9	20
15	7	16
11	5	10
10	6	14
$\Sigma X_1 = 56$	$\Sigma X_2 = 27$	$\Sigma X_3 = 60$

Null hypothesis H_0 : The machines are equally effective

Alternate Hypothesis H_1 : The machines are not equally effective

Step:1

Grand Total (G) = 56 + 27 + 60

T = 143

Step:2

Correction factor (C.F) = $\frac{T^2}{N}$

$= \frac{(143)^2}{12}$

C.F = 1704.08

Step:3

TSS = Total sum of squares.

$= 10^2 + 15^2 + 11^2 + 10^2 + 9^2 + ... - C.F$

= 1866.25 – 1704.08

TSS = 284.92

Step :4

SSC = Sum of squares between samples.

$$= \frac{(\Sigma X_1)^2}{n} + \frac{(\Sigma X_2)^2}{n} + \frac{(\Sigma X_3)^2}{n} - C.F$$

$$= \frac{(56)^2}{4} + \frac{(27)^2}{4} + \frac{(60)^2}{4} - 1704.08$$

$$= \frac{3136}{4} + \frac{729}{4} + \frac{3600}{4} - 1704.08$$

$= 784 + 182.25 + 400 - 1704.08$

$= 1866.25 - 1704.08$

SSC $= 162.17$

Step: 5

SSE = TSS – SSC

= 284.92 – 162.17

SSE = 122.75

Source of variation	Sum of squares	Degrees of freedom		
Between samples	SSC =162.17	$v_1 = C - 1$=3-1 = 2		
Within samples	SSE =122.75	$v_2 = n - C =$ 12- 3 = 9		
Total	TSS = 284.92	$n - 1$=12 -1 =11		

RESULT:

F calculated value = 5.945

T tab(2,9) df at 5% level = 4.26

Fcal > F tab

5.945 > 4.26

$\therefore H_0$ is rejected. Hence we conclude that the machines are not equally effective.

4) **Three samples below have been obtained from normal population with equal variances .test the hypothesis that the samples means are equal.**

Samples.		
8	**7**	**12**
10	**5**	**19**
7	**10**	**13**
14	**9**	**12**
11	**9**	**14**

The value of F at 5% level of significance is 3.88

Soln:

Null hypothesis H_0 :The sample means are equal

Alternate Hypothesis H_1 :The sample means are not equal

Samples.		
8	7	12
10	5	19
7	10	13
14	9	12
11	9	14
$\Sigma X_1 = 50$	$\Sigma X_2 = 40$	$\Sigma X_3 = 70$

Step: 1

Grand total (G) = 50 + 40 + 70

T = 160

Step: 2

Correction factor (C.F) = $\dfrac{T^2}{N} = \dfrac{(160)^2}{15}$

C.F = 1706.7

Step : 3

TSS = Total sum squares

$= \quad 8^2 + 10^2 + 7^2 + 14^2 + \text{..........................} C.F$

$= \quad 1880 - 1706.7$

$= \quad 173.3$

Step: 4

SSC = Sum of squares between samples

$$= \frac{(\Sigma X_1)^2}{n} + \frac{(\Sigma X_2)^2}{n} + \frac{(\Sigma X_3)^2}{n} - C.F$$

$$= \frac{(50)^2}{5} + \frac{(40)^2}{5} + \frac{(70)^2}{5} - 1706.7$$

$$= 500 + 320 + 980 \ - 1706.7$$

$$= 1800 - 1706.7$$

SSC $= 93.3$

Step : 5

SSE = TSS – SSC

$= 173.3 - 93.3$

SSE = 80

Source of variation	Sum of squares	Degrees of freedom	Mean square	F-ratio
Between samples	SSC =93.3	$v_1 = C - 1$ = 3-1 = 2	$MSC = \frac{93.3}{2} = 46.65$	$F_C = \frac{MSC}{MSE}$ (or) $= \frac{MSE}{MSC}$
Within samples	SSE =80	$v_2 = n - C$ = 15-3 = 12	$MSE = \frac{80}{12} = 6.7$	$F_C = \frac{46.65}{6.7} = 6.97$
Total	TSS = 173.3	$n - 1$=15-1 =14		

F calculated value = 6.97

T tab(2,12) df at 5% level = 3.88

Fcal > F tab

6.97 > 3.88$\therefore H_0$ is rejected. Hence we conclude that the sample means are not equal (i.e. There is a significant difference between the means of the three samples.

5) **An Experiment was designed to study the performance of 4 different detergents for cleaning fuel injectors the following cleanness reading were obtained with specially Designed equipments for 12 tanks of Gas distributed over 3 different models of engine**

Detergent	**Engine-I**	**Engine-II**	**Engine-I**	**Total**
A	45	43	51	**139**
B	47	46	52	**145**
C	48	50	55	**153**
D	42	37	49	**128**
Total	**182**	**176**	**207**	**565**

Perform the ANOVA & Test at 0.01 level of significance whether there are differences in detergent or in engines

H_0: { (i) There is no significance difference in Engines
(ii)There is no significance difference in Detergents

$$H_1: \begin{cases} \text{(i) There is a significance difference in Engines} \\ \text{(ii)There is a significance difference in Detergents} \end{cases}$$

We shifted our origin to 50

Brand	**Groups**			Total	X_1^2	X_2^2	X_3^2
	X_1 (I)	X_2 (II)	X_3 (III)				
A(Y_1)	-5	-7	1	$\Sigma y_1 = 11$	25	49	1
B(Y_2)	-3	-4	2	$\Sigma y_2 = -5$	9	16	4
C(Y_3)	-2	0	5	$\Sigma y_3 = 3$	4	0	25
D(Y_4)	-8	-13	-1	$\Sigma y_4 = -22$	64	169	1
Total	ΣX_1	ΣX_2	ΣX_3	**Σy(Total)**	ΣX_1^2	ΣX_2^2	ΣX_3^2
	-18	**-24**	**7**	**T= -35**	**102**	**234**	**31**

Step-1: N→ Number of Data given in the Problem (N=12)

Step 2: T = **-35** (From above Table)

Step 3: $\frac{T^2}{N} = \frac{(-35)^2}{12} = 102.08$(Correction Factor)

Step4 : TSS = $(\Sigma X_1^2 + \Sigma X_2^2 + \Sigma X_3^2 + \Sigma X_4^2) - \frac{T^2}{N} = (102 + 234 + 31 - 102.08) = 264.92$

Step 5: SSR= $\left(\frac{Zy_1^2}{N_2} + \frac{Zy_2^2}{N_2} + \frac{Zy_3^2}{N_2} + + \frac{Zy_4^2}{N_2}\right) - \frac{T^2}{N} = \left(\frac{-11^2}{3} + \frac{(-5)^2}{3} + \frac{3^2}{3} + \frac{(-22)^2}{3} - 102.08\right) = 110.91$

Step 6: SSC = $\left(\frac{ZX_1^2}{N_1} + \frac{ZX_2^2}{N_1} + \frac{ZX_3^2}{N_1} + \frac{ZX_4^2}{N_1}\right) - \frac{T^2}{N} = \left(\frac{18^2}{4} + \frac{(-24)^2}{4} + \frac{7^2}{4} - 102.08\right) = 135.17$

Step 7: SSE = TSS –SSC- SSR = 264.92 — 135.17 — 110.91 = 18.84

Source of Variance	Sum of Squares	Degree of Freedom	Mean sum of Squares	Variance Ratio	Table Value at 1% level
Column Treatment	SSC = 135.17	C-1=2	$MSC = \frac{SSC}{C-1} = \frac{135.17}{2} = 67.58$	$F_c = \frac{MSC}{MSE} = \frac{67.585}{3.14} = 21.52$	$F_C(2,6) = 10.92$
Row Treatment	SSR = 110.91	R-1=3	$MSR = \frac{SSR}{R-1} = \frac{110.91}{3} = 36.97$	$F_R = \frac{MSR}{MSE} = \frac{36.97}{3.14} = 11.77$	$F_R(3,6) = 9.78$
Error	SSE = 18.84	N-C-R+1= 6	$MSE = \frac{SSE}{N-C-R+1} = \frac{18.84}{6} = 3.14$		
Total	TSS = 264.92	**Condition Always F >1**			

Conclusion→ Calculate Value > table value in both the Cases so we Reject H_0

(ie)There is a significance difference between Detergents & Engines

6) Five Doctors each test five treatments for a certain disease and observe the number of days each patients recover the results are (Anna University --Dec-13)

Doctors	**Treatments**				
	I	**II**	**III**	**IV**	**V**
A	10	14	23	19	20
B	11	15	24	17	21
C	9	11	20	16	19
D	8	13	17	17	20
E	12	15	19	15	22

Discuss the Difference between Doctors and Treatments?

Soln:

$$H_0 : \begin{cases} \text{(i) There is no significance difference between Doctors} \\ \text{(ii)There is no significance difference between Treatments} \end{cases}$$

$$H_1 : \begin{cases} \text{(i) There is a significance difference between Doctors} \\ \text{(ii)There is a significance difference between Treatments} \end{cases}$$

We shifted our origin to 16

Doctors	**Treatments**					Total	X_1^2	X_2^2	X_3^2	X_4^2	X_5^2
	X_1	X_2	X_3	X_4	X_5						
A (Y_1)	-6	-2	7	3	4	$\Sigma y_1 = 6$	36	4	49	9	16
B (Y_2)	-5	-1	8	1	5	$\Sigma y_2 = 8$	25	1	64	1	25
C (Y_3)	-7	-5	4	0	3	$\Sigma y_3 = —4$	49	16	16	0	9
D (Y_4)	-8	-3	1	1	4	$\Sigma y_4 = —5$	64	9	9	1	16
E (Y_5)	-4	-1	3	-1	6	$\Sigma y_5 = 3$	16	1	1	1	36
Total	ΣX_1	ΣX_2	ΣX_3	ΣX_4	ΣX_5	**Σy(Total)**	ΣX_1^2	ΣX_2^2	ΣX_3^2	ΣX_4^2	ΣX_5^2
	-30	**-12**	**23**	**4**	**22**	**T=8**	**190**	**31**	**139**	**12**	**102**

Step-1: N→ Number of Data given in the Problem (N=25)

Step 2: T = **8** (From above Table)

Step 3: $\frac{T^2}{N} = \frac{(8)^2}{25} = 2.56$(Correction Factor)

Step4 : TSS = $(\Sigma X_1^2 + \Sigma X_2^2 + \Sigma X_3^2 + \Sigma X_4^2 — \frac{T^2}{N}) = 474 — 2.56 = 471.44$

Step 5: SSR= $(\frac{Zy_1^2}{N_2} + \frac{Zy_2^2}{N_2} + \frac{Zy_3^2}{N_2} + + \frac{Zy_4^2}{N_2} + \frac{Zy_5^2}{N_2}) — \frac{T^2}{N}$

SSR= $(\frac{(6)^2}{5} + \frac{(8)^2}{5} + \frac{(-4)^2}{5} + \frac{(-5)^2}{5} + \frac{(3)^2}{5} — 2.56) = 30 — 2.56 = 27.64$

Step 6: SSC = $(\frac{ZX_1^2}{N_1} + \frac{ZX_2^2}{N_1} + \frac{ZX_3^2}{N_1} + \frac{ZX_4^2}{N_1} + \frac{ZX_5^2}{N_1}) — \frac{T^2}{N}$

$$\text{SSC} = (\frac{(—30)^2}{5} + \frac{(—12)^2}{5} + \frac{23^2}{5} + \frac{(4)^2}{5} + \frac{(22)^2}{5} — 2.56) = 410 — 2.56 = 407.44$$

Step 7: SSE = TSS –SSC- SSR = $471.44 - 407.44 - 27.64 = 36.56$

Source of Variance	Sum of Squares	Degree of Freedom	Mean sum of Squares	Variance Ratio	Table Value at 1% level
Column Treatment	SSC = 407.44	C-1=4	$MSC = \frac{SSC}{C-1} = \frac{407.44}{4} = 101.86$	$F_c = \frac{MSC}{MSE} = \frac{101.86}{2.28} = 44.67$	$F_C(4,6) = 3.01$
Row Treatment	SSR = 27.44	R-1=4	$MSR = \frac{SSR}{R-1} = \frac{27.44}{4} = 6.86$	$F_R = \frac{MSR}{MSE} = \frac{6.86}{2.28} = 3.01$	$F_R(4,6) = 3.01$
Error	SSE = 36.56	N-C-R+1= 16	$MSE = \frac{SSE}{N-C-R+1} = \frac{36.56}{16} = 2.28$		
Total	TSS = 471.44	**Condition Always F >1**			

Calculated Value $F_R(3.01) \le$ table value $F_R(3.01)$ so we accept H_0

Calculated Value $F_C(44.67) >$ table value $F_C(3.01)$ so we Reject H_0

Conclusion:

There is no significance difference between Doctors but,

There is a significance difference between Trearments

7) **The following table gives monthly sales (in thousand rupees) of a certain firm in three states by its four salesman.**

	Salesman			
States	**I**	**II**	**III**	**IV**
A	**6**	**5**	**3**	**8**
B	**8**	**9**	**6**	**5**
C	**10**	**7**	**8**	**7**

Setup the analysis of variance table and test whether there is any significant difference (i) between the sales by the firm salesman ,(ii) between sales in the three states.

Solution:

H_0 :There is no significant difference between the sales by the firm's salesman

H_1 : There is significant difference between the sales by the firm's salesman

H_0 :There is no significant difference between the three states.

H_1 : There is significant difference between the three states

States	Saleman				Total
	I	II	III	IV	
A	6	5	3	8	22
B	8	9	6	5	28
C	10	7	8	7	32
Total	24	21	17	20	T = 82

Step : 1

Correction factor (C.F) = $\frac{T^2}{N} = \frac{(82)^2}{12}$

C.F = 560.333

Step;2

TSS = Sum of squares of each values – C.F

$= 6^2 + 8^2 + 10^2 + 5^2 + 9^2 + ...- 560.333$

= 602 – 560.333

TSS = 41.667

Step : 3

SSC = Sum of squares between columns, (salesman)

$= \frac{1}{3}\left[24^2 + 21^2 + 17^2 + 20^2\right]$ - C.F

$= \frac{1}{3}\left[24^2 + 21^2 + 17^2 + 20^2\right]$ - 560.333

= 568.667 - 560.333

SSC = 8.334

Step :4

SSR = Row sum of squares.(states)

$= \frac{1}{4}\left[22^2 + 28^2 + 32^2\right]$ - C.F

$= \frac{1}{4}\left[22^2 + 28^2 + 32^2\right]$ -560.333

= 573 - 560.333

SSR = 12.667

Step :5

SSE = Error sum of squares.

= TSS – SSC – SSR

= 41.667 – 8.334 – 12.667

SSE = 20.666

Source of variation	Sum of squares	Degrees of freedom	Mean square	F-ratio
Between columns	SSC = 8.334	k-1= 4-1=3	$MSC = \frac{8.334}{3} = 2.778$	$F_C = \frac{3.444}{2.778} = 1.239$
Between rows	SSR=12.667	r-1=3 -1 =2	$MSR = \frac{12.667}{2} = 6.334$	$F_R = \frac{6.334}{3.444} = 1.84$
Residual error	SSE=20.667	(k-1)(r-1)= (3)(2)=6	$MSE = \frac{20.667}{6} = 3.444$	
Total	TSS=41.667	rk-1		

RESULT :1

F Calculated value = 1.239

F tab(3,6) df at 5% level = 4.75

F Cal < F tab

1.239 < 4.75

H_0 is accepted.

Hence we conclude that there is no significant difference between the sales by the firm's salesman.

RESULT :2

F Calculated value = 1.84

F tab(2,6) df at 5% level = 5.14

F Cal < F tab

1.84 < 4.75

H_0 is accepted. Hence we conclude that there is no significant difference between the sales in the three states.

$\therefore$ There is no significant difference in the states as far as sales are concerned at 5% level of significance

8) A tea company appoints four saleman A,B,C,D and observes their sales in three seasons summer ,winter , monsoon. The figures (in lakhs) are given in the following table.

Seasons	**Salesman**				**Season's Total**
	A	**B**	**C**	**D**	
Summer	**36**	**36**	**21**	**35**	**128**
Winter	**28**	**29**	**31**	**32**	**120**
Monsoon	**26**	**28**	**29**	**29**	**112**
Salesman's Total	**90**	**93**	**81**	**96**	**360**

(i) Does the salesman significantly differ in performance ?

(ii) Is there significant difference between the seasons?

Solution:

H_0 :The salesman does not differ significantly differ in performance.

H_1 : The salesman differs significantly differ in their performance.

H_0 : There is no significant difference between the three seasons.

H_1 : There is significant difference between the three seasons

Step:1

Correction factor (C.F) = $\frac{T^2}{N} = \frac{(360)^2}{12}$

C.F = 10800

Step :2

TSS = Sum of squares of each values – C.F

$= 36^2 + 36^2 + 21^2 + 35^2 + 28^2 + \ldots\ldots\ldots\ldots\ldots\ldots - 10800$

= 11010 – 10800

TSS = 210

Step :3

SSC = Sum of squares between columns, (salesman)

$= \frac{1}{3}\left[90^2 + 93^2 + 81^2 + 96^2\right]$ - C.F

$= \frac{1}{3}\left[90^2 + 93^2 + 81^2 + 96^2\right]$-10800

= 10842 -10800

SSC = 42

Step :4

SSR = Sum of squares between rows.(seasons)

$= \frac{1}{4}\left[128^2 + 120^2 + 112^2\right]$ - C.F = $\frac{1}{4}\left[128^2 + 120^2 + 112^2\right]$ -10800 = 10832 – 10800 SSR = 32

SSE = Error sum of squares. = TSS – SSC – SSR = 210 – 42 -32 SSE = 136

Source of variation	Sum of squares	Degrees of freedom	Mean square	F-ratio
Between columns	SSC =42	k-1= 4 – 1 =3	$MSC = \frac{42}{3} = 14$	$F_C = \frac{22.67}{14} = 1.619$
Between rows	SSR=32	r-1 = 3 -1 =2	$MSR = \frac{32}{2} = 16$	$F_R = \frac{22.67}{16} = 1.41$
Residual error	SSE=136	(k-1)(r-1) $3 \times 2 = 6$	$MSE = \frac{136}{6} = 22.67$	
Total	TSS=210	rk-1=11		

RESULT :1

F Calculated value = 1.619

F tab(3,6) df at 5% level = 4.75

F Cal < F tab

1.619 < 4.75

H_0 is accepted.

Hence we conclude that the salesman do not differ significantly in their performance .

RESULT :2

F Calculated value = 1.41

F tab(2,6) df at 5% level = 5.14

F Cal < F tab

1.41 < 5.14

H_0 is accepted. Hence we conclude that there is no significant difference between the three seasons.

9) Preform a Two – way ANOVA on the data given below.

Plots of land	**Treatments**			
	A	**B**	**C**	**D**
I	**38**	**40**	**41**	**39**
II	**45**	**42**	**49**	**36**
III	**40**	**38**	**42**	**42**

Solution:

H_0 :There is no significant difference between Treatments

H_1 : There is significant difference between Treatments

H_0 :There is no significant difference between Plots

H_1 : There is significant difference between Plots

Plots of land	Treatments				Total
	A	B	C	D	
I	38	40	41	39	158
II	45	42	49	36	172
III	40	38	42	42	162
Total	123	120	132	117	492

Step :1

Correction factor (C.F) = $\dfrac{T^2}{N} = \dfrac{(492)^2}{12}$

C.F = 20172

Step :2

TSS = Sum of squares of each values – C.F

$= 38^2 + 45^2 + 40^2 + 40^2 + 42^2 + \ldots\ldots\ldots\ldots\ldots\ldots . - 20172$

= 20304 - 20172

TSS = 132

Step :3

SSC = Sum of squares between columns, (Treatments)

$= \frac{1}{3}\left[123^2 + 120^2 + 132^2 + 117^2\right]$ - C.F

$= \frac{1}{3}\left[123^2 + 120^2 + 132^2 + 117^2\right]$-20172

= 20214-20172

SSC = 42

Step :4

SSR = Sum of squares between rows.(plots)

$= \frac{1}{4}\left[158^2 + 172^2 + 162^2\right]$ - C.F

$= \frac{1}{4}\left[158^2 + 172^2 + 162^2\right]$ - 20172

= 20198 - 210172

SSR = 26

Step :5

SSE = Error sum of squares.

= TSS – SSC – SSR

= 132 -42 -26

SSE = 64

ANOVA TABLE

Source of variation	Sum of squares	Degrees of freedom	Mean square	F-ratio
Between columns	SSC =42	k-1= 4 -1 =3	$MSC = \frac{42}{3} = 14$	$F_C = \frac{14}{10.67} = 1.312$
Between rows	SSR=26	r-1=3 -1 =2	$MSR = \frac{26}{2} = 13$	$F_R = \frac{13}{10.67} = 1.218$
Residual error	SSE=64	(k-1)(r-1) $3 \times 2 = 6$	$MSR = \frac{64}{6} = 10.67$	
Total	TSS=132	rk-1=11		

F Calculated value = 1.312

F tab(3,6) df at 5% level = 4.75

F Cal < F tab

1.312 < 4.75

H_0 is accepted. Hence we conclude there is no significant difference between treatments

F Calculated value = 1.218

F tab(2,6) df at 5% level = 5.14

F Cal < F tab

1.218 < 5.14

H_0 is accepted. Hence we conclude that there is no significant difference between the Plots.

10) The Following Latin Square of a design when four varieties of seeds are being tested set up the analysis table and state your conclusion you may carry out suitable change of origin and scale

A 105	**B 95**	**C 125**	**D 115**
C 115	**D 125**	**A 105**	**B 105**
D 115	**C 95**	**B 105**	**A 115**
B 95	**A 135**	**D 95**	**C 115**

Soln: Let H_0 = {(i) There is no significance Difference in Seeds
(ii) There is no significance Difference in Treatments
(iii) There is no significance Difference in Lands

H_1 = {(i) There is a significance Difference in Seeds
(ii) There is a significance Difference in Treatments
(iii) There is a significance Difference in Lands

Shifted Origin to 100 and divided by 5,

Lands	**Seeds**				Total	X_1^2	X_2^2	X_3^2	X_4^2
	X_1 (A)	X_2 (B)	X_3 (C)	X_4 (D)					
I (Y_1)	1	-1	5	3	$\Sigma y_1 = 8$	1	1	25	9
II (Y_2)	3	5	1	1	$\Sigma y_2 =$ **10**	9	25	1	1
III (Y_3)	3	-1	1	3	$\Sigma y_3 =$ **6**	9	1	1	9
IV (Y_4)	-1	7	-1	3	$\Sigma y_4 =$ **8**	1	49	1	9
Total	ΣX_1	ΣX_2	ΣX_3	ΣX_4	**Σy(Total)**	ΣX_1^2	ΣX_2^2	ΣX_3^2	ΣX_4^2
	6	**10**	**6**	**10**	**T= 32**	**20**	**76**	**28**	**28**

Step-1: N→ Number of Data given in the Problem (N=16)

Step 2: T = **32** (From above Table)

Step 3: $\frac{T^2}{N} = \frac{(32)^2}{16} = 64$(Correction Factor)

Step4 : TSS = $(\Sigma X_1^2 + \Sigma X_2^2 + \Sigma X_3^2 + \Sigma X_4^2 — \frac{T^2}{N}) = (20 + 76 + 28 + 28) — 64 = 88$

Step 5: SSR= $(\frac{Zy_1^2}{N_2} + \frac{Zy_2^2}{N_2} + \frac{Zy_3^2}{N_2} + + \frac{Zy_4^2}{N_2}) — \frac{T^2}{N} =(\frac{(8)^2}{4} + \frac{(10)^2}{4} + \frac{(6)^2}{4} + \frac{(8)^2}{4} — 64) = 2$

Step 6: SSC = $(\frac{ZX_1^2}{N_1} + \frac{ZX_2^2}{N_1} + \frac{ZX_3^2}{N_1} + \frac{ZX_4^2}{N_1} +) — \frac{T^2}{N} = (\frac{(6)^2}{4} + \frac{(10)^2}{4} + \frac{(6)^2}{4} + \frac{(10)^2}{4} — 64) = 4$

SST					Total
A	1	1	3	7	**12**
B	-1	1	1	-1	**0**
C	5	3	-1	3	**10**
D	3	5	3	-1	**10**

Step 7: SST = $(\frac{(12)^2}{4} + \frac{(0)^2}{4} + \frac{(10)^2}{4} + \frac{(10)^2}{4} — 64) = 22$

Step 8: SSE = TSS –SSC- SSR- SST = $88 - 2 - 4 - 22 = 60$

Source of Variance	**Sum of Squares**	**Degree of Freedom**	**Mean sum of Squares**	**Variance Ratio**	**Table Value at 5% level**
Between Columns	SSC=4	K-1=3	$MSC = \frac{SSC}{K-1} = \frac{4}{3} = 1.33$	$F_c = \frac{MSE}{MSC} = \frac{10}{1.33} = 7.52$	$F_c(6,3) = 8.94$
Between Rows	SSR=2	K-1=3	$MSR = \frac{SSR}{K-1} = \frac{2}{3} = 0.67$		$F_R(6,3) = 8.94$
Between Treatments	SST=22	K-1=3	$MST = \frac{SST}{K-1} = \frac{22}{3} = 7.33$	$F_R = \frac{MSE}{MSR} = \frac{10}{0.67} = 14.9$	$F_T(6,3) = 8.94$
Error	SSE=60	(K-1)(K-2) =6	$MSE = \frac{SSE}{(K-1)(K-2)} = \frac{60}{6} = 10$	$F_T = \frac{MSE}{MST} = \frac{10}{7.33} = 1.36$	
Total	TSS=88	$K^2 - 1 = 15$	Condition Always F >1		

Calculated Value $F_C(7.52)$ < table value $F_C(8.94)$ so we accept H_0

Calculated Value $F_R(14.9)$ > table value $F_R(8.94)$ so we Reject H_0

Calculated Value $F_T(1.36)$ < table value $F_T(8.94)$ so we accept H_0

Conclusion:

There is no significance difference between Seeds & Treatments, But

There is a significance difference between Lands

11) Analyze the following Latin Square experiment at 1% level

A (12)	**D (20)**	**C (16)**	**B (10)**
D (18)	**A (14)**	**B (11)**	**C (14)**
B (12)	**C (15)**	**D (19)**	**A (13)**
C (16)	**B (11)**	**A (15)**	**D (20)**

The Letters (A,B,C,D) denotes the treatments & the figures in brackets denotes the observation

Soln: We Shifted our origin to 12

Let H_0 = {(i) There is no significance Difference in Seeds
(ii) There is no significance Difference in Treatments
(iii) There is no significance Difference in Lands

(i) There is a significance Difference in Seeds

H_1 = {(ii) There is a significance Difference in Treatments

(iii) There is a significance Difference in Lands

Lands	**Seeds**				Total	X_1^2	X_2^2	X_3^2	X_4^2
	X_1 (A)	X_2 (B)	X_3 (C)	X_4 (D)					
I (Y_1)	0	8	4	2	$\Sigma y_1 = 10$	0	64	16	4
II (Y_2)	6	2	-1	2	$\Sigma y_2 = 9$	36	4	1	4
III (Y_3)	0	3	7	1	$\Sigma y_3 = 11$	0	9	49	1
IV (Y_4)	4	-1	3	8	$\Sigma y_4 = 14$	16	1	9	64
Total	ΣX_1	ΣX_2	ΣX_3	ΣX_4	**Σy(Total)**	ΣX_1^2	ΣX_2^2	ΣX_3^2	ΣX_4^2
	10	**12**	**13**	**9**	**T= 44**	**52**	**78**	**75**	**73**

Step-1: N→ Number of Data given in the Problem (N=16)

Step 2: T = **44** (From above Table)

Step 3: $\frac{T^2}{N} = \frac{(44)^2}{16} = 121$(Correction Factor)

Step4 : TSS = $(\Sigma X_1^2 + \Sigma X_2^2 + \Sigma X_3^2 + \Sigma X_4^2 - \frac{T^2}{N}) = (52 + 78 + 75 + 73) - 121 = 157$

Step 5: SSR= $(\frac{Zy_1^2}{N_2} + \frac{Zy_2^2}{N_2} + \frac{Zy_3^2}{N_2} + + \frac{Zy_4^2}{N_2}) - \frac{T^2}{N} = (\frac{(10)^2}{4} + \frac{(9)^2}{4} + \frac{(11)^2}{4} + \frac{(14)^2}{4} - 121) = 3.5$

Step 6: SSC = $(\frac{ZX_1^2}{N_1} + \frac{ZX_2^2}{N_1} + \frac{ZX_3^2}{N_1} + \frac{ZX_4^2}{N_1} +) - \frac{T^2}{N} = (\frac{(10)^2}{4} + \frac{(12)^2}{4} + \frac{(13)^2}{4} + \frac{(9)^2}{4} - 121) = 2.5$

SST					Total
A	0	2	3	1	**6**
B	0	-1	-1	-2	**-4**
C	4	3	4	2	**13**
D	6	8	7	8	**29**

Step 7: SST = $(\frac{(6)^2}{4} + \frac{(-4)^2}{4} + \frac{(13)^2}{4} + \frac{(29)^2}{4} - 121) = 144.5$

Step 8: SSE = TSS –SSC- SSR- SST = $157 - 2.5 - 3.5 - 144.5 = 6.5$

Source of Variance	**Sum of Squares**	**Degree of Freedom**	**Mean sum of Squares**	**Variance Ratio**	**Table Value at 1% level**
Between Columns	SSC=2.5	K-1=3	$MSC = \frac{SSC}{K-1} = \frac{2.5}{3} = 0.83$	$F_c = \frac{MSE}{MSC} = \frac{1.08}{0.83} = 1.301$	$F_c(6,3) = 27.91$
Between Rows	SSR=3.5	K-1=3	$MSR = \frac{SSR}{K-1} = \frac{3.5}{3} = 1.17$		$F_R(3,6) = 9.78$

Between Treatments	SST=144.5	K-1=3	$MST = \frac{SST}{K-1}$ $= \frac{144.5}{3} = 48.17$	$F_R = \frac{MSR}{MSE}$ $= \frac{1.17}{1.08}$ $= 1.08$ $F_T = \frac{MST}{MSE}$ $= \frac{48.17}{1.08}$ $= 44.6$	$F_T(3,6)$ $= 9.78$
Error	SSE=6.5	(K-1)(K-2) =6	$MSE = \frac{SSE}{(K-1)(K-2)}$ $= \frac{65}{6} = 1.08$		
Total	TSS=157	$K^2 - 1$ $= 15$	Condition Always F >1		

Calculated Value $F_C(1.301)$ < table value $F_C(27.91)$ so we accept H_0

Calculated Value $F_R(1.08)$ < table value $F_R(9.78)$ so we accept H_0

Calculated Value $F_T(44.6)$ > table value $F_T(9.78)$ so we Reject H_0

Conclusion: There is no significance difference between Seeds & Lands , But

There is a significance difference between Treatments

12) Three varieties of a crop are tested in the Randomized block design with four replications, the layout being has given below: The yields are given in kilograms Analyse for significance

(Apr/May 2015 (R13 &R08))

C48	**A51**	**B52**	**A49**
A47	**B49**	**C52**	**C51**
B49	**C53**	**A49**	**B50**

Soln: H_0: $\begin{cases} \text{(i) There is no significance difference between Yields} \\ \text{(ii) There is no significance difference between Crops} \end{cases}$

H_1 : $\begin{cases} \text{(i) There is a significance difference between Yields} \\ \text{(ii) There is a significance difference between Crops} \end{cases}$

Yields	**Treatments**				Total	X_1^2	X_2^2	X_3^2	X_4^2
	X_1	X_2	X_3	X_4					
A (Y_1)	48	51	52	49	$\Sigma y_1 = 200$	2304	2601	2704	2401
B (Y_2)	47	49	52	51	$\Sigma y_2 = 199$	2209	2401	2704	2601
C (Y_3)	49	53	49	50	$\Sigma y_3 = 201$	2401	2809	2401	2500
Total	ΣX_1	ΣX_2	ΣX_3	ΣX_4	**Σy(Total)**	ΣX_1^2	ΣX_2^2	ΣX_3^2	ΣX_4^2
	144	**153**	**153**	**150**	**T= 600**	**6914**	**7811**	**7809**	**7502**

Step-1: N→ Number of Data given in the Problem (N=12)

Step 2: T = **600** (From above Table)

Step 3: $\frac{T^2}{N} = \frac{(600)^2}{12} = 30000$(Correction Factor)

Step4 : TSS = $(\Sigma X_1^2 + \Sigma X_2^2 + \Sigma X_3^2 + \Sigma X_4^2 - \frac{T^2}{N}) = 36$

Step 5: SSR= $(\frac{Zy_1^2}{N_2} + \frac{Zy_2^2}{N_2} + \frac{Zy_3^2}{N_2} + + \frac{Zy_4^2}{N_2} + \frac{Zy_5^2}{N_2}) - \frac{T^2}{N}$

$$SSR= (\frac{(200)^2}{4} + \frac{(199)^2}{4} + \frac{(201)^2}{4} - 30000) = 0.5$$

Step 6: SSC = $(\frac{ZX_1^2}{N_1} + \frac{ZX_2^2}{N_1} + \frac{ZX_3^2}{N_1} + \frac{ZX_4^2}{N_1} + \frac{ZX_5^2}{N_1}) - \frac{T^2}{N}$

$$SSC = (\frac{(144)^2}{3} + \frac{(153)^2}{3} + \frac{(153)^2}{3} + \frac{(150)^2}{3} - 30000) = 18$$

Step 7: SSE = TSS –SSC- SSR = 36 — 0.5 — 18 = 17. 5

Source of Variance	Sum of Squares	Degree of Freedom	Mean sum of Squares	Variance Ratio	Table Value at 5% level
Column Treatment	SSC = 18	C-1=3	$MSC = \frac{SSC}{C-1} = \frac{184}{3} = 6$	$F_c = \frac{MSC}{MSE} = \frac{6}{2.91} = 2.057$	$F_C(3,6) = 4.75$
Row Treatment	SSR = 0.5	R-1=2	$MSR = \frac{SSR}{R-1} = \frac{0.5}{2} = 0.25$	$F_R = \frac{MSE}{MSR} = \frac{2.91}{0.25} = 11.667$	$F_R(6,2) = 19.32$
Error	SSE = 17.5	N-C-R+1= 6	$MSE = \frac{SSE}{N-C-R+1} = \frac{17.5}{6} = 2.91$		
Total	TSS = 36	**Condition Always F >1**			

Calculated Value F_R(11.66) < table value F_R(19.32) so we Accept H_0

Calculated Value F_C(2.057) < table value F_C(4.75) so we Accept H_0

Conclusion: There is no significance difference between Yields and crops

13) **Analyse the variance in the latin square of yields in (Kgs) of paddy where A,B,C,D denote the different method of cultivation. Examine whether the different method of cultivation have given significantly different yields (Apr/May 2015(R13 &R08))**

D 122	**A 121**	**C 123**	**B 122**
B 124	**C 123**	**A 122**	**D 125**
A 120	**B 119**	**D 120**	**C 121**
C 122	**D 123**	**B 121**	**A 122**

Soln: Let H_0 = {
(i) There is no significance Difference in Seeds
(ii) There is no significance Difference in Treatments
(iii) There is no significance Difference in Lands

(i) T*h*ere is a significance Difference in Seeds

H_1 = {(ii) T*h*ere is a significance Difference in Treatments

(iii) T*h*ere is a significance Difference in Lands

Lands	Seeds				Total	X_1^2	X_2^2	X_3^2	X_4^2
	X_1 (A)	X_2 (B)	X_3 (C)	X_4 (D)					
I (Y_1)	122	121	123	122	$\Sigma y_1 = 488$	14884	14641	15129	14884
II (Y_2)	124	123	122	125	$\Sigma y_2 = 494$	15376	15129	14884	15625
III (Y_3)	120	119	120	121	$\Sigma y_3 = 480$	14400	14161	14400	14641
IV (Y_4)	122	123	121	122	$\Sigma y_4 = 488$	14884	15129	14641	14884
Total	ΣX_1	ΣX_2	ΣX_3	ΣX_4	**Σy(Total)**	ΣX_1^2	ΣX_2^2	ΣX_3^2	ΣX_4^2
	488	**486**	**486**	**490**	**T= 1950**	**59544**	**59060**	**59054**	**60034**

Step-1: N→ Number of Data given in the Problem (N=16)

Step 2: T = **1950** (From above Table)

Step 3: $\frac{T^2}{N} = \frac{(1950)^2}{16} = 237656.3$ (Correction Factor)

Step4 : TSS = $(\Sigma X_1^2 + \Sigma X_2^2 + \Sigma X_3^2 + \Sigma X_4^2 — \frac{T^2}{N}) = 35.75$

Step 5: SSR= $(\frac{Zy_1^2}{N_2} + \frac{Zy_2^2}{N_2} + \frac{Zy_3^2}{N_2} + + \frac{Zy_4^2}{N_2}) — \frac{T^2}{N} = (\frac{(488)^2}{4} + \frac{(494)^2}{4} + \frac{(480)^2}{4} + \frac{(488)^2}{4} — 237656.3) = 24.75$

Source of Variance	**Sum of Squares**	**Degree of Freedom**	**Mean sum of Squares**	**Variance Ratio**	**Table Value at 5% level**
Between Columns	SSC=2.75	K-1=3	$MSC = \frac{SSC}{K — 1} = \frac{2.75}{3} = 0.91$	$F_c = \frac{MSC}{MSE} = \frac{0.91}{0.66} = 1.375$ $F_R = \frac{MSR}{MSE} = \frac{8.25}{0.66} = 12.375$ $F_T = \frac{MST}{MSE} = \frac{1.41}{0.66} = 2.125$	$F_c(3,6) = 4.75$ $F_R(3,6) = 4.75$ $F_T(3,6) = 4.75$
Between Rows	SSR=24.75	K-1=3	$MSR = \frac{SSR}{K — 1} = \frac{24.75}{3} = 8.25$		
Between Treatments	SST=4.25	K-1=3	$MST = \frac{SST}{K — 1} = \frac{4.25}{3} = 1.41$		
Error	SSE=4	(K-1)(K-2) =6	$MSE = \frac{SSE}{(K — 1)(K — 2)} = \frac{4}{6} = 0.66$		
Total	TSS=35.75	$K^2 — 1 = 15$	Condition Always F >1		

Step 6: SSC = $\left(\frac{ZX_1^2}{N_1}+\frac{ZX_2^2}{N_1}+\frac{ZX_3^2}{N_1}+\frac{ZX_4^2}{N_1}+\right)—\frac{T^2}{N}=\left(\frac{(488)^2}{4}+\frac{(486)^2}{4}+\frac{(486)^2}{4}+\frac{(490)^2}{4}—237656.3\right)=2.75$

SST					Total
A	121	122	120	122	**485**
B	122	124	119	121	**486**
C	123	123	121	122	**489**
D	122	125	120	123	**490**

Step 7: SST = $\left(\frac{(485)^2}{4}+\frac{(486)^2}{4}+\frac{(489)^2}{4}+\frac{(490)^2}{4}—237656.3\right)=4.25$

Step 8: SSE = TSS –SSC- SSR- SST = 35.75 — 24.75 — 2 75 — 4.25 = 4

Calculated Value F_C(1.375) < table value F_C(4.75) so we Accept H_0

Calculated Value F_R(12.375) > table value F_R(4.75) so we Reject H_0

Calculated Value F_T(2.125) < table value F_T(4.75) so we Accept H_0

Conclusion: There is no significance difference between Seeds &Treatments , But

There is a significance difference between Lands

14) Four different, through supposed by equivalent, forms of a standardized reading achievements test where give to each of five students and the followings are the scores which they obtained (Nov/Dec 2015)

	Student-1	**Student-2**	**Student-3**	**Student-4**	**Student-5**
Form A	**75**	**73**	**59**	**69**	**84**
Form B	**83**	**72**	**56**	**70**	**92**
Form C	**86**	**61**	**53**	**72**	**88**
Form D	**73**	**67**	**62**	**79**	**95**

Perform two way analysis of variance to test at the level of significance α = 0.01 whether it is reasonable to treat the four form are equivalent, Are the scores of the students significantly difference α = 0.01 level?

Soln: H_0: {
(i) There is no significance difference between Students
(ii)There is no significance difference between Forms

H_1 : {
(i) There is a significance difference between Students
(ii)There is a significance difference between Forms

Forms	**Students**					Total	X_1^2	X_2^2	X_3^2	X_4^2	X_5^2
	X_1	X_2	X_3	X_4	X_5						
A (Y_1)	75	73	59	69	84	$\Sigma y_1 = 360$	5625	5329	3481	4761	7056
B (Y_2)	83	72	56	70	92	$\Sigma y_2 = 373$	6889	5184	3136	4900	8464
C (Y_3)	86	61	53	72	88	$\Sigma y_3 = 360$	7396	3721	2809	5184	7744
D (Y_4)	73	67	62	79	95	$\Sigma y_4 = 376$	5329	4489	3844	6241	9025
Total	ΣX_1	ΣX_2	ΣX_3	ΣX_4	ΣX_5	**Σy(Total)**	ΣX_1^2	ΣX_2^2	ΣX_3^2	ΣX_4^2	ΣX_5^2
	317	**273**	**230**	**290**	**359**	**T=1469**	**25239**	**18723**	**13270**	**21086**	**32289**

Step-1: N→ Number of Data given in the Problem (N=20)

Step 2: T = **1469** (From above Table)

Step 3: $\frac{T^2}{N} = \frac{(1469)^2}{20} = 107898.1$ (Correction Factor)

Step4 : TSS = $(\Sigma X_1^2 + \Sigma X_2^2 + \Sigma X_3^2 + \Sigma X_4^2 — \frac{T^2}{N}) = 2708.95$

Step 5: SSR= $(\frac{Zy_1^2}{N_2} + \frac{Zy_2^2}{N_2} + \frac{Zy_3^2}{N_2} + + \frac{Zy_4^2}{N_2} + \frac{Zy_5^2}{N_2}) — \frac{T^2}{N}$

SSR= $(\frac{(360)^2}{5} + \frac{(373)^2}{5} + \frac{(360)^2}{5} + \frac{(376)^2}{5} — 107898.1\ ) = 42.95$

Step 6: SSC = $(\frac{ZX_1^2}{N_1} + \frac{ZX_2^2}{N_1} + \frac{ZX_3^2}{N_1} + \frac{ZX_4^2}{N_1} + \frac{ZX_5^2}{N_1}) — \frac{T^2}{N}$

$$SSC = (\frac{(317)^2}{4} + \frac{(273)^2}{4} + \frac{(230)^2}{4} + \frac{(290)^2}{4} + \frac{(359)^2}{4} — 107898.1) = 2326.7$$

Step 7: SSE = TSS –SSC- SSR = 2708.95 — 42.95 — 2326.7 = 339.3

Source of Variance	**Sum of Squares**	**Degree of Freedom**	**Mean sum of Squares**	**Variance Ratio**	**Table Value at 5% level**
Column Treatment	SSC = 2326.7	C-1=4	$MSC = \frac{SSC}{C — 1} = \frac{2326.7}{4} = 581.67$	$F_c = \frac{MSC}{MSE} = \frac{581.67}{28.27} = 20.57$	Fc(4,12) = 3.25
Row Treatment	SSR = 42.95	R-1=3	$MSR = \frac{SSR}{R — 1} = \frac{42.95}{3} = 14.31$	$F_R = \frac{MSE}{\cancel{MSR}} = \frac{28.27}{14.31} = 1.975$	FR(12,3) = 8.74
Error	SSE = 339.3	N-C-R+1= 12	$MSE = \frac{SSE}{N — C — R + 1} = \frac{339.3}{12} = 28.27$		
Total	TSS = 2708.95	**Condition Always F >1**			

Calculated Value FR(1.97) < table value FR(8.74) so we Accept H0

Calculated Value Fc(20.57) > table value Fc(3.25) so we Reject H0

Conclusion:

There is a significance difference between Students, but not in Forms,

-The following data related to the Latin square experiment on four varieties of paddy A,B,C & D

(Nov/Dec 2015)

18 A	**21 C**	**25 D**	**11 B**
22 D	**12 B**	**15 A**	**19 C**
15 B	**20 A**	**23 C**	**24 D**
22 C	**21 D**	**10 B**	**17 A**

Analyse the result and offer your comments of α = 0.05 level of significance

Soln: Let H0 = {
(i) There is no significance Difference in Seeds
(ii) There is no significance Difference in Treatments
(iii) There is no significance Difference in Lands

(i) There is a significance Difference in Seeds
H_1 = {(ii) There is a significance Difference in Treatments
(iii) There is a significance Difference in Lands

Lands	Seeds				Total	X_1^2	X_2^2	X_3^2	X_4^2
	X_1 (A)	X_2 (B)	X_3 (C)	X_4 (D)					
I (Y_1)	18	21	25	11	$\Sigma y_1 = 75$	324	441	625	121
II (Y_2)	22	12	15	19	$\Sigma y_2 = \mathbf{68}$	484	144	225	361
III (Y_3)	15	20	23	24	$\Sigma y_3 = \mathbf{82}$	225	400	529	576
IV (Y_4)	22	21	10	17	$\Sigma y_4 = 70$	484	441	100	289
Total	ΣX_1	ΣX_2	ΣX_3	ΣX_4	**Σy(Total)**	ΣX_1^2	ΣX_2^2	ΣX_3^2	ΣX_4^2
	77	**74**	**73**	**71**	**T= 295**	**1517**	**1426**	**1479**	**1347**

Step-1: N→ Number of Data given in the Problem (N=16)

Step 2: T = **295** (From above Table)

Step 3: $\frac{T^2}{N} = \frac{(295)^2}{16} = 5439.06$(Correction Factor)

Step4 : TSS = $(\Sigma X_1^2 + \Sigma X_2^2 + \Sigma X_3^2 + \Sigma X_4^2 - \frac{T^2}{N}) = 329.93$

Step 5: SSR= $(\frac{Zy_1^2}{N_2} + \frac{Zy_2^2}{N_2} + \frac{Zy_3^2}{N_2} + + \frac{Zy_4^2}{N_2}) - \frac{T^2}{N} = (\frac{(75)^2}{4} + \frac{(68)^2}{4} + \frac{(82)^2}{4} + \frac{(70)^2}{4} - 5439.06) = 29.18$

Step 6: SSC = $(\frac{ZX_1^2}{N_1} + \frac{ZX_2^2}{N_1} + \frac{ZX_3^2}{N_1} + \frac{ZX_4^2}{N_1} +) - \frac{T^2}{N} = (\frac{(77)^2}{4} + \frac{(74)^2}{4} + \frac{(73)^2}{4} + \frac{(71)^2}{4} - 5439.06) = 4.68$

SST					Total
A	18	15	20	17	**70**
B	11	12	15	10	**48**
C	21	19	23	22	**85**
D	25	22	24	21	**92**

Step 7: SST = $(\frac{(70)^2}{4} + \frac{(48)^2}{4} + \frac{(85)^2}{4} + \frac{(92)^2}{4} - 5439.06) = 284.18$

Step 8: SSE = TSS –SSC- SSR- SST = $329.93 - 29.18 - 4.68 - 284.18 = 11.87$

Source of Variance	Sum of Squares	Degree of Freedom	Mean sum of Squares	Variance Ratio	Table Value at 5% level
Between Columns	SSC=4.68	K-1=3	MSC = $\frac{SSC}{K-1}$ $= \frac{4.68}{3} = 1.56$	$F_c = \frac{MSE}{MSC}$ $= \frac{1.97}{1.56}$ $= 1.26$	$F_C(6,3)$ = 8.94
Between Rows	SSR=29.18	K-1=3	MSR = $\frac{SSR}{K-1}$ $= \frac{29.19}{3} = 9.72$		$F_R(3,6)$ = 4.75

Between Treatments	SST=284.18	K-1=3	$MST = \frac{SST}{K-1}$ $= \frac{284.18}{3} = 94.72$	$F_R = \frac{MSR}{MSE}$ $= \frac{9.72}{1.97}$ $= 4.91$ $F_T = \frac{MST}{MSE}$ $= \frac{94.72}{1.97}$ $= 47.86$	$F_T(3,6)$ $= 4.75$
Error	SSE=11.875	(K-1)(K-2) =6	MSE $= \frac{SSE}{(K-1)(K-2)}$ $= \frac{11.875}{6} = 1.97$		
Total	TSS=329.93	$K^2 - 1$ $= 15$	Condition Always F >1		

Calculated Value $F_C(1.26)$ < table value $F_C(8.94)$ so we Accept H_0

Calculated Value $F_R(4.91)$ > table value $F_R(4.75)$ so we Reject H_0

Calculated Value $F_T(47.86)$ > table value $F_T(4.75)$ so we Reject H_0

Conclusion:

There is no significance difference between Seeds, But there is a significance difference between Treatments & Lands

Describe the Latin Square Layout.

Latin Square Designs are probably not used as much as they should be - they are very efficient designs. Latin square designs allow for two blocking factors. In other words, these designs are used to simultaneously control (or eliminate) **two sources of nuisance variability**. For instance, if you had a plot of land the fertility of this land might change in both directions, North -- South and East -- West due to soil or moisture gradients. So, both rows and columns can be used as blocking factors. However, you can use Latin squares in lots of other settings. As we shall see, Latin squares can be used as much as the RCBD in industrial experimentation as well as other experiments.

Whenever, you have more than one blocking factor a Latin square design will allow you to remove the variation for these two sources from the error variation. So, consider we had a plot of land, we might have blocked it in columns and rows, i.e. each row is a level of the row factor, and each column is a level of the column factor. We can remove the variation from our measured response in both directions if we consider both rows and columns as factors in our design.

The Latin Square Design gets its name from the fact that we can write it as a square with Latin letters to correspond to the treatments. The treatment factor levels are the Latin letters in the Latin square design. The number of rows and columns has to correspond to the number of treatment levels. So, if we have four treatments then we would need to have four rows and four columns in order to create a Latin square. This gives us a design where we have each of the treatments and in each row and in each column.

This is just one of many 4×4 squares that you could create. In fact, you can make any size square you want, for any number of treatments - it just needs to have the following property associated with it - that each treatment occurs only once in each row and once in each column.

Consider another example in an industrial setting: the rows are the batch of raw material, the columns are the operator of the equipment, and the treatments (A, B, C and D) are an industrial process or protocol for producing a particular product.

What is the model? We let:

$$y_{ijk} = \mu + \rho_i + \beta_j + r_k + e_{ijk}$$

$$i = 1, \dots, t$$
$$j = 1, \dots, t$$

[$k = 1, \dots, t$] where — $k = d(i, j)$ and the total number of observations

$N = t^2$ (the number of rows times the number of columns) and t is the number of treatments.

Note that a Latin Square is an incomplete design, which means that it does not include observations for all possible combinations of *i*, *j* and *k*. This is why we use notation k = d(i, j). Once we know the row and column of the design, then the treatment is specified. In other words, if we know *i* and *j*, then *k* is specified by the Latin Square design.

This property has an impact on how we calculate means and sums of squares, and for this reason we can not use the balanced ANOVA command in Minitab even though it looks perfectly balanced. We will see later that although it has the property of orthogonality, you still cannot use the balanced ANOVA command in Minitab because it is not complete.

An assumption that we make when using a Latin square design is that the three factors (treatments, and two nuisance factors) **do not interact**. If this assumption is violated, the Latin Square design error term will be inflated.

The randomization procedure for assigning treatments that you would like to use when you actually apply a Latin Square, is somewhat restricted to preserve the structure of the Latin Square. The ideal randomization would be to select a square from the set of all possible Latin squares of the specified size. However, a more practical randomization scheme would be to select a standardized Latin square at random (these are tabulated) and then:

1. randomly permute the columns,
2. randomly permute the rows, and then
3. assign the treatments to the Latin letters in a random fashion.

Consider a factory setting where you are producing a product with 4 operators and 4 machines. We call the columns the operators and the rows the machines. Then you can randomly assign the specific operators to a row and the specific machines to a column. The treatment is one of four protocols for producing the product and our interest is in the average time needed to produce each product. If both the machine and the operator have an effect on the time to produce, then by using a Latin Square Design this variation due to machine or operators will be effectively removed from the analysis.

The following table gives the degrees of freedom for the terms in the model.

AOV	***df***	***df* for the example**
Rows	t-1	3
Cols	t-1	3
Treatments	t-1	3
Error	(t-1)(t-2)	6
Total	(t^2 - 1)	15

A Latin Square design is actually easy to analyze. Because of the restricted layout, one observation per treatment in each row and column, the model is orthogonal.

If the row, ρ_i, and column, β_j, effects are random with expectations zero, the expected value of Y_{ijk} is $\mu + r_k$. In other words, the treatment effects and treatment means are orthogonal to the row and column effects. We can also write the sums of squares, as seen in Table 4.10 in the text.

We can test for row and column effects, but our focus of interest in a Latin square design is on the treatments. Just as in RCBD, the row and column factors are included to reduce the error variation but are not typically of interest. And, depending on how we've conducted the experiment they often haven't been randomized in a way that allows us to make any reliable inference from those tests.

Note: if you have missing data then you need to use the general linear model and test the effect of treatment after fitting the model that would account for the row and column effects.

In general, the General Linear Model tests the hypothesis that:

$$H_0 : r_i = 0 \text{ vs. } H_a : r_i \neq 0$$

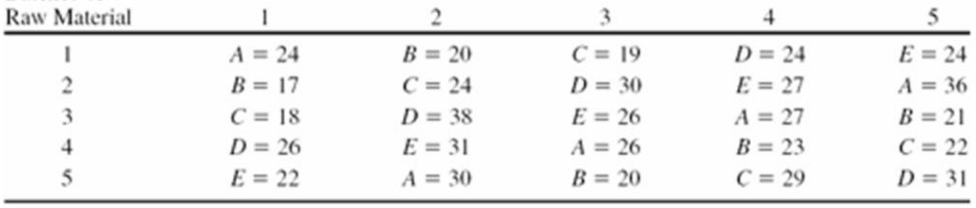
To test this hypothesis we will look at the *F*-ratio which is written as:

$$F = \frac{MS(r_k|\mu, \rho_i, \beta_j)}{MSE(\mu, \rho_i, \beta_j, r_k)} \sim F((t — 1), (t — 1)(t — 2))$$

To get this in Minitab you would use GLM and fit the three terms: rows, columns and treatments.
The *F* statistic is based on the adjusted MS for treatment.

The Rocket Propellant Problem – A Latin Square Design

Table 4-8 Latin Square Design for the Rocket Propellant Problem



Batches of Raw Material	Operators				
	1	2	3	4	5
1	$A = 24$	$B = 20$	$C = 19$	$D = 24$	$E = 24$
2	$B = 17$	$C = 24$	$D = 30$	$E = 27$	$A = 36$
3	$C = 18$	$D = 38$	$E = 26$	$A = 27$	$B = 21$
4	$D = 26$	$E = 31$	$A = 26$	$B = 23$	$C = 22$
5	$E = 22$	$A = 30$	$B = 20$	$C = 29$	$D = 31$

Table 4-13 (4-12 in 7th ed) shows some other Latin Squares from $t = 3$ to $t = 7$ and states the number of different arrangements available.

Statistical Analysis of the Latin Square Design

The statistical (effects) model is:

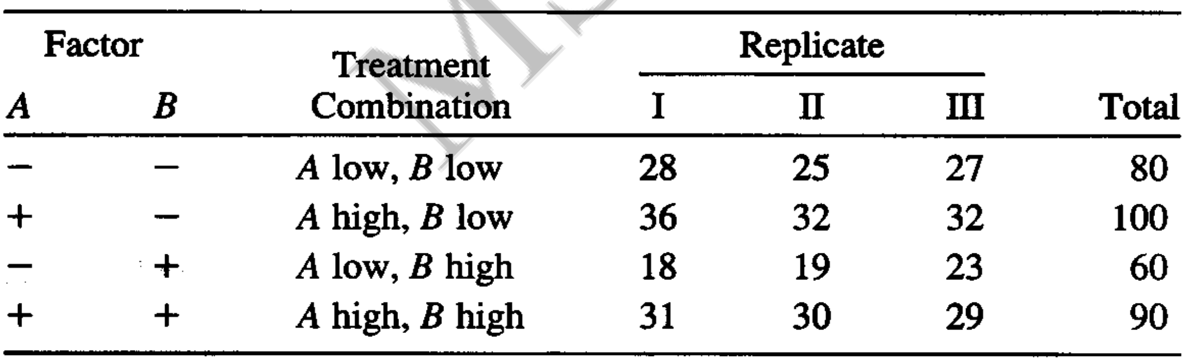
$$Y_{ijk} = \mu + p_i + \beta_j + \tau_k + \varepsilon_{ijk} \begin{cases} i = 1,2, \dots, p \\ j = 1,2, \dots, p \\ k = 1,2, \dots, p \end{cases}$$

but $k = d(i, j)$ shows the dependence of *k* in the cell *i, j* on the design layout, and $p = t$ the number of treatment levels.
The statistical analysis (ANOVA) is much like the analysis for the RCBD.

The 2^2 Factorial Design

Two factors, A and B, and each factor has two levels, low and high

The concentration of reactant v.s. the amount of the catalyst



Factor		Treatment Combination	Replicate			Total
A	*B*		I	II	III	
−	−	*A* low, *B* low	28	25	27	80
+	−	*A* high, *B* low	36	32	32	100
−	+	*A* low, *B* high	18	19	23	60
+	+	*A* high, *B* high	31	30	29	90

- "-" And "+" denote the low and high levels of a factor, respectively
- Low and high are arbitrary terms
- Geometrically, the four runs form the corners of a square
- Factors can be quantitative or qualitative, although their treatment in the final model will be different

-

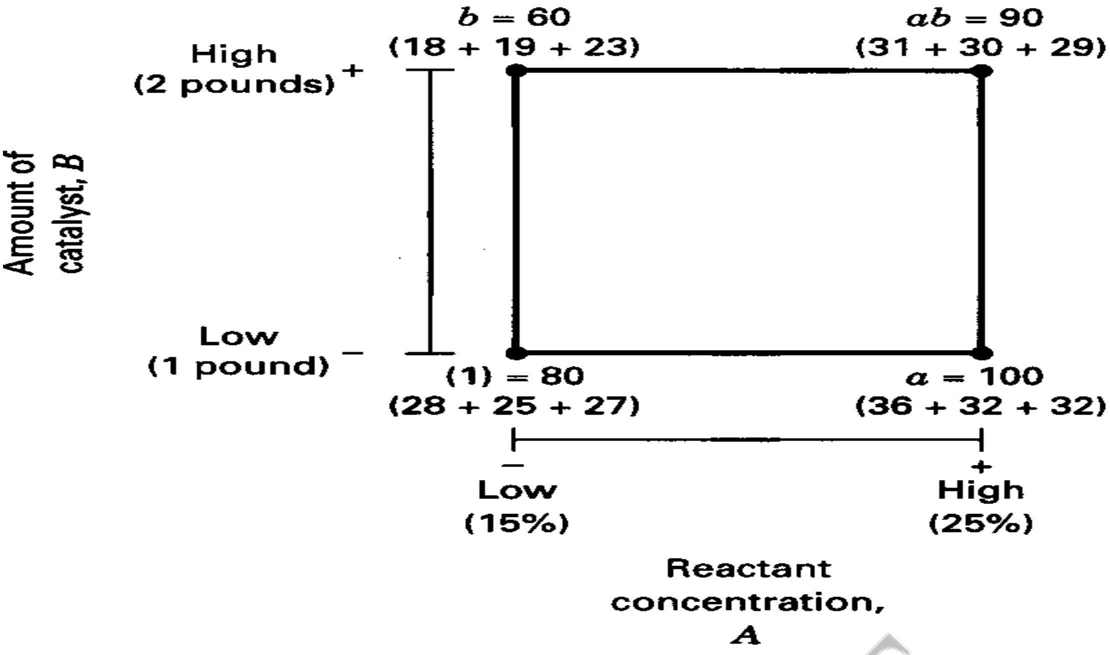


Figure 6-1 Treatment combinations in the 2^2 design.

ab: the total of *n* replicates taken at the treatment combination.

The main effects

$$A = \frac{1}{2n}\{[ab - b] + [a - (1)]\} = \frac{1}{2n}\,[ab + a - b - (1)]$$

$$= \frac{ab + a}{2n} - \frac{b + (1)}{2n} = \bar{y}_{A^+} - \bar{y}_{A^-}$$

$$B = \frac{1}{2n}\{[ab - a] + [b - (1)]\} = \frac{1}{2n}\,[ab + b - a - (1)]$$

$$= \frac{ab + b}{2n} - \frac{a + (1)}{2n} = \bar{y}_{B^+} - \bar{y}_{B^-}$$

The interaction effect

$$AB = \frac{1}{2n}\{[ab - b] - [a - (1)]\} = \frac{1}{2n}\,[ab + (1) - a - b]$$

$$= \frac{ab + (1)}{2n} - \frac{b + a}{2n}$$

The total effects

$$Contrast_A = ab + a - b - (1)$$

$$Contrast_B = ab + b - a - (1)$$

$$Contrast_{AB} = ab + (1) - a - b$$

Average effect of a factor = the change in response produced by a change in the level of that factor averaged over the levels if the other factors.

- (1), *a*, *b* and

- Sum of squares:

$$SS_A = \frac{[ab + a - b - (1)]^2}{4n}$$

$$SS_B = \frac{[ab + b - a - (1)]^2}{4n}$$

$SS_E = SS_T — SS_{Æ} — SS_B — SS_{ÆB}$

$$SS_{AB} = \frac{[ab + (1) - b - a]^2}{4n}$$

$$SS_T = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{n} y_{ijk}^2 - \frac{y_{...}^2}{4n}$$

$$SS_E = SS_T - SS_A - SS_B - SS_{AB}$$

The Analysis of Variance is completed by computing the total sum of squares SS_T with $(4n — 1)$ d.f as usual and the Error Sum of Squares SS_E with $(4(n — 1)$ d.f)

Table of plus and minus signs

	I	A	B	AB
(1)	+	–	–	+
a	+	+	–	–
b	+	–	+	–
ab	+	+	+	+

- The regression model:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$

  - x_1 and x_2 are coded variables that represent the two factors, i.e. x_1 (or x_2) only take values on –1 and 1.
  - Use least square method to get the estimations of the coefficients
  - For that example,

$$\hat{y} = 27.5 + \left(\frac{8.33}{2}\right) x_1 + \left(\frac{-5.00}{2}\right) x_2$$

  - Model adequacy: residuals and normal probability plot

Degree of Freedom as follows

RBD		LSD	
Treatments	4 — 1 = 3	Treatments	4 — 1 = 3
Blocks	b — 1	Row	4 — 1 = 3
Error	3(b — 1)	Columns	4 — 1 = 3
Total	4(b — 1)	Error	(4 — 1)(4 — 2) = 6
		Total	15

Problem-1

14.037	14.165	13.972	13.907
14.821	14.757	14.843	14.878
13.880	13.860	14.032	13.914
14.888	14.921	14.415	14.932

The above table presents the results of a 2^2 factorial design with n = 4 replicates, using the factor

A= deposition time and B= Arsenic Flow rate

The two Level of deposition time are - = short & + = long ,

The two Level of Arsenic Flow rate - = 55 % & + = 59% , The response variable epitaxial layer thickness (μ_m)

2^2 design for Epitaxial layer thickness (μ_m)

Treatment Combinations	**Design Factors**			**Thickness (μ_m)**				**Thickness**	
	A	**B**	**AB**					**Total**	**Average**
(1)	-	-	+	14.037	14.165	13.972	13.907	56.081	14.020
a	+	-	-	14.821	14.757	14.843	14.878	59.299	14.825
b	-	+	-	13.880	13.860	14.032	13.914	55.686	13.922
ab	+	+	+	14.888	14.921	14.415	14.932	59.156	14.789

$$A = \frac{1}{2n}[a + ab — b — (1)] = \frac{1}{2(4)}[59.299 + 59.156 — 55.686 — 56.081] = \frac{1}{8}[6.688] = 0.836$$

$$B = \frac{1}{2n}[b + ab — a — (1)] = \frac{1}{2(4)}[55.686 + 59.156 — 59.299 — 56.081] = \frac{1}{8}[—0.536] = —0.067$$

$$AB = \frac{1}{2n}[(1) + ab — a — b] = \frac{1}{2(4)}[56.081 + 59.156 — 59.299 — 55.686] = \frac{1}{8}[0.256] = 0.032$$

$$SS_{Æ} = \frac{[a + ab — b — (1)]^2}{4n} = \frac{[6.688]^2}{16} = 2.7956$$

$$SS_B = \frac{[b + ab — a — (1)]^2}{4n} = \frac{[—0.536]^2}{16} = 0.0181$$

$$SS_{ÆB} = \frac{[(1) + ab — a — b]^2}{4n} = \frac{[0.252]^2}{16} = 0.0040$$

Source of variations	Sum of squares	Degree of freedom	Mean Square	Variance Ratio	Table Value 5% Level	Table Value 1% Level
A	$SS_{Æ} = 2.7956$	1	$MS_{Æ} = \frac{SS_{Æ}}{d.f}$ $= 2.7956$	$F_{Æ} = \frac{MS_{Æ}}{MS_E}$ $= 134.4$	$F_{Æ}(1,12)$ $= 4.75$	$F_{Æ}(1,12)$ $= 9.33$
B	$SS_B = 0.0181$	1	$MS_B = \frac{SS_B}{d.f}$ $= 0.0181$	$F_B = \frac{MS_B}{MS_E}$ $= 0.87$	$F_B(1,12)$ $= 4.75$	$F_B(1,12)$ $= 9.33$
AB	$SS_{ÆB} = 0.0040$	1	$MS_{ÆB} = \frac{SS_{ÆB}}{d.f}$ $= 0.0040$	$F_{ÆB} = \frac{MS_{ÆB}}{MS_E}$ $= 0.19$	$F_{ÆB}(1,12)$ $= 4.75$	$F_{ÆB}(1,12)$ $= 9.33$
Error	$SS_E = 0.2495$	12	$MS_E = \frac{SS_E}{d.f}$ $= 0.0208$	Always F > 1		

$$SS_T = \{(14.037)^2 + (14.165)^2 + \cdots + (14.415)^2 + (14.932)^2\}$$

$$-\{\frac{56.081 + 59.299 + 55.686 + 59.156}{16}^{2}\}$$

$$SS_T = 3.0672$$

$$SS_E = SS_T — SS_Æ — SS_B — SS_{ÆB} = 3.0672 — 2.7956 — 0.0181 — 0.004 = 0.2495$$

Analysis of Variance Epitaxial Process Experiment

Here Cal $F_A >$ Table F_A

Cal $F_B <$ Table F_B

Cal $F_{AB} <$ Table F_{AB}

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2) Find out the main effects and interactions in the following 2^2 Factorial experiment and write down the analysis of variance Table

	(I) 00	a 10	b 01	ab 11
BLOCK-I	64	25	30	6
BLOCK-II	75	14	50	33
BLOCK-III	76	12	41	17
BLOCK-IV	75	33	25	10

Soln:

Taking Deviation y = 37 , We get

Treatment Combinations	**BLOCKS**				**TOTAL**	X_1^2	X_2^2	X_3^2	X_4^2
	I X_1	II X_2	III X_3	IV X_4					
(y_1) (1)	27	38	39	38	142	729	1444	1521	1444
(y_2) a	-12	-23	-25	-4	-64	144	529	625	16
(y_3) b	-7	13	4	-12	-2	49	169	16	144
(y_4) ab	-31	-4	-20	-27	-82	961	16	400	729
Total	**-23**	**24**	**-2**	**-5**	**-6**	**1883**	**2158**	**2562**	**2333**

Step:1 N = 16

Step:2 T = -6

Step:3 Correction Factor (CF) $= \frac{T^2}{N} = \frac{(-6)^2}{16} = \frac{36}{16} = 2.25$

Step:4 TSS $= \Sigma X_1^2 + \Sigma X_2^2 + \Sigma X_3^2 + \Sigma X_4^2 — \frac{T^2}{N} = 1883 + 2158 + 2562 + 2333 — 2.25 = 8933.75$

Step:5 SSC $= \frac{(\Sigma X_1)^2}{N_1} + \frac{(\Sigma X_2)^2}{N_1} + \frac{(\Sigma X_3)^2}{N_1} + \frac{(\Sigma X_4)^2}{N_1} — \frac{T^2}{N} = \frac{(-23)^2}{4} + \frac{(24)^2}{4} + \frac{(-2)^2}{4} + \frac{(-5)^2}{4} — 2.25$

$$SSC = 281.25$$

N_1 → Number of elements in each row

Step:6 SSR $= \frac{(\Sigma Y_1)^2}{N_2} + \frac{(\Sigma Y_2)^2}{N_2} + \frac{(\Sigma Y_3)^2}{N_2} + \frac{(\Sigma Y_4)^2}{N_2} - \frac{T^2}{N} = \frac{(142)^2}{4} + \frac{(-64)^2}{4} + \frac{(-2)^2}{4} + \frac{(-82)^2}{4} - 2.25$

$$SSR = 7745.75$$

Step:7: SSE = TSS – SSC – SSR = 8933.75 – 281.25 – 7744.75 = **907.75**

For 2^2 Experiment

Contrast A = $a + ab - b - (1) = [—64 — 82 + 2 - 142] = -286$

Contrast B = $b + ab - a - (1) = [-2 - 82 + 64 - 142] = -162$

Contrast AB = $(1) + ab - b - a = [142 - 82 + 2 + 64] = 126$

Main Effects:

$$A = \frac{1}{2}(a + ab - b - (1)) = -\frac{286}{2} = -143$$

$$B = \frac{1}{2}(b + ab - a - (1)) = -\frac{162}{2} = -81$$

$$AB = \frac{1}{2}((1) + ab - b - a) = \frac{126}{2} = 63$$

$$SS_{Æ} = \frac{(a + ab - b - (1))^2}{16} = \frac{(-286)^2}{16} = 5112.25$$

$$SS_{B} = \frac{(b + ab - a - (1))^2}{16} = \frac{(-162)^2}{16} = 1640.25$$

$$SS_{ÆB} = \frac{((1) + ab - b - a)^2}{16} = \frac{(126)^2}{16} = 992.25$$

Source of variance	Degree of freedom	Sum of Squares	Mean Square	Variance Ratio	Table Value	
					5% Level	1% Level
Blocks	3	281.5	93.83	$\frac{100.86}{93.83} = 1.075$	F(9,3) = 8.81	27.35
Treatments	3	7744.75	2581.58	$\frac{2581.88}{100.86} = 25.6$	F(3,9) = 3.86	6.99
A	1	5112.25	5112.25	$\frac{5112.25}{100.86} = 50.69$	F(1,9) = 5.12	6.99
B	1	1640.25	1640.25	$\frac{1640.25}{100.86} = 16.26$	F(1,9) = 5.12	6.99
AB	1	992.25	992.25	$\frac{992.25}{100.86} = 9.84$	F(1,9) = 5.12	6.99
Error	9	907.75	100.86	Always F > 1		

$$\text{Error (d.}f) = N - c - r + 1 = 16 - 4 - 4 + 1 = 9$$

$$\text{Cal } F_{Æ} > \text{Tab } F_{Æ}$$

$$\text{Cal } F_{B} > \text{Tab } F_{B}$$

$$\text{Cal } F_{ÆB} > \text{Tab } F_{ÆB}$$

I-YEAR B.E./B.TECH – SEMESTER II
QUESTION BANK - MA3257 STATISTICS& NUMERICAL METHODS

UNIT-III SOLUTION OF EQUATIONS AND EIGEN VALUE PROBLEMS

PART A

1.	**Write down the order of convergence and condition for convergence of fixed point iteration method $x = g(x)$**
	Solution: The order of convergence is one and condition for convergence is $\lvert g'(x)\rvert \le 1$, for $x \in I$ where I is the interval containing the root of $x = g(x)$.
2.	**State fixed point theorem.**
	Solution: Let $f(x)=0$ be the given equation whose actual root is 'a'. The equation $f(x)=0$ be written as $x=g(x)$. Let I be the interval containing the root 'a'. If $\lvert g'(x)\rvert \le 1$ for all x in I then the sequence of successive approximations $x_0, x_1, x_2 \cdots$, will converges to a, if the initial starting value x_0 is chosen in I.
3.	**Locate the positive root for the equation $xe^x = \cos x$**
	Solution: Let $f(x) = xe^x - \cos x$ When $x = 0 \Rightarrow f(0) = (0)e^0 - \cos(0) = 0 - 1 = -1$ (Negative) When $x = 1 \Rightarrow f(1) = (1)e^1 - \cos(1) = 2.718 - 0.999 = 1.719$ (Positive) The positive root in magnitude lies in the interval (0, 1).
4.	**Locate the negative root for the equation $x^3 - 2x + 5 = 0$, approximately.**
	Solution: Let $f(x) = x^3 - 2x + 5$; When $x = -1 \Rightarrow f(-1) = (-1)^3 - 2(-1) + 5 = -1 + 2 + 5 = 6$ (Positive) When $x = -2 \Rightarrow f(-2) = (-2)^3 - 2(-2) + 5 = -8 + 4 + 5 = 1$ (Positive) When $x = -3 \Rightarrow f(-3) = (-3)^3 - 2(-3) + 5 = -27 + 6 + 5 = -16$ (Negative) The negative root in magnitude lies in the interval (–3, –2).
5.	**Solve $e^x - 3x = 0$ by the method of iteration.**
	Solution: Given $f(x) = e^x - 3x$, $f(x) = 0 \Rightarrow e^x - 3x = 0 \Rightarrow 3x = e^x \Rightarrow \mathrm{x} = \frac{1}{3}\mathrm{e}^{\mathrm{x}} = \mathrm{g(x)}$, $g'(x) = \frac{1}{3}e^x$. Here $\lvert g'(x)\rvert < 1$ for all x in the interval (0,1). Hence, the iteration converges. Let $x_0 = 0.6$. We obtain the following results. $x_1 = \frac{1}{3}e^{x_0} \Rightarrow x_1 = \frac{1}{3}e^{0.6} \Rightarrow x_1 = 0.60737$ $x_2 = \frac{1}{3}e^{x_1} \Rightarrow x_2 = \frac{1}{3}e^{0.60737} \Rightarrow x_2 = 0.61187$, $x_3 = \frac{1}{3}e^{x_2} \Rightarrow x_3 = \frac{1}{3}e^{0.61187} \Rightarrow x_3 = 0.61452$ The last two iteration values are equal. The required root is $x = 0.61$
6.	**State the order and criterion of convergence of Newton-Raphson method for $f(x) = 0$**
	Solution: The order of convergence of Newton-Raphson method is 2. The criterion of convergence of Newton-Raphson Method is $\lvert f(x)\, f''(x)\rvert < \lvert f'(x)\rvert^2$

7. **Write down the Newton-Raphson method formula.**

Solution: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} \quad n = 0,1,\ldots$

8. **Derive a formula to find the value of $\sqrt{N}$, where N is a real number, by Newton's method**

Solution: $\text{Let } x = \sqrt{N} \Rightarrow x^2 = N, \quad \text{take } f(x) = x^2 - N \Rightarrow f'(x) = 2x$

$$\text{By Newton's iterative formula, } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n} \Rightarrow x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]$$

9. **Using Newton's method, find the root between 0 and 1 of $x^3 - 6x + 4$ correct to 2 decimal places.**

Solution: Given $f(x) = x^3 - 6x + 4, \; f'(x) = 3x^2 - 6$

We know that $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n^3 - 6x_n + 4}{3x_n^2 - 6} = \dfrac{3x_n^3 - 6x_n - x_n^3 + 6x_n - 4}{3x_n^2 - 6} = \dfrac{2x_n^3 - 4}{3x_n^2 - 6}$

Take $x_0 = 1, \; x_1 = \dfrac{2x_0^3 - 4}{3x_0^2 - 6} = \dfrac{2(1)^3 - 4}{3(1)^2 - 6} = \dfrac{2}{3} = 0.6666,$

$$x_2 = \frac{2x_1^3 - 4}{3x_1^2 - 6} = \frac{2(0.6666)^3 - 4}{3(0.6666)^2 - 6} = 0.7301,$$

$$x_3 = \frac{2x_2^3 - 4}{3x_2^2 - 6} = \frac{2(0.7301)^3 - 4}{3(0.7301)^2 - 6} = 0.7320.$$

The required root is $x = 0.73$

10. **What are the merits of Newton's method of iteration?**

Solution:

1. Newton's method is successfully used to improve the results obtained by other methods.
2. It is applicable to the solution of equations involving algebraic functions as well as transcendental functions.

11. **Distinguish between direct and iterative method of solving simultaneous equation.**

Solution:

S.No	**Direct Method**	**Indirect Method**
1.	We get exact solution	Approximate solution
2.	Simple, take less time	Time consuming laborious

12. **For solving a linear system of equations, compare Gauss Elimination method and Gauss Jordan method.**

Solution:

S.No.	**Gauss-Elimination method**	**Gauss – Jordan method**
1.	Coefficient matrix is transformed into upper triangular matrix.	Coefficient matrix is transformed into diagonal matrix.
2.	Solution is obtained by back substitution method.	Solution is obtained by direct method.

13. **Solve by Gauss Elimination method $2x - y = 1, x - 3y = 0$**

	Solution: Given $2x-y=1,\ x-3y=0$ The given system is equivalent to $\begin{bmatrix}2 & -1\\ 1 & -3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix} \Rightarrow$ AX = B $[A,B]=\left[\begin{array}{cc\|c}2 & -1 & 1\\ 1 & -3 & 0\end{array}\right]$ Now, we will make the matrix A as a upper triangular matrix $\sim\left[\begin{array}{cc\|c}2 & -1 & 1\\ 0 & 5 & 1\end{array}\right] R_2 \rightarrow R_1-2R_2$ $\therefore 5y=1$ and $2x-y=1 \quad \Rightarrow y=\frac{1}{5}$ and $2x-\frac{1}{5}=1$ $2x=1+\frac{1}{5}=\frac{6}{5}$, Hence $x=\frac{3}{5},\ y=\frac{1}{5}$
14.	**Explain briefly Gauss Jordan method to solve simultaneous equations.**
	Solution: In this method, the augmented matrix is reduced to a diagonal matrix (or even a unit matrix). Here we get the solutions directly, without using back substitution method.
15.	**Solve by Gauss-Jordan method, the following system** $\mathbf{5x+4y=15,\ 3x+7y=12}$
	Solution: The given system is equivalent to $\begin{bmatrix}5 & 4\\ 3 & 7\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}15\\ 12\end{bmatrix}\Rightarrow AX=B$ $[A,B]=\left[\begin{array}{cc\|c}5 & 4 & 15\\ 3 & 7 & 12\end{array}\right]$ Now, we will make the matrix A as a diagonal matrix $\sim\left[\begin{array}{cc\|c}5 & 4 & 15\\ 0 & 23 & 15\end{array}\right] R_2 \rightarrow 5R_2-3R_1$ $\sim\left[\begin{array}{cc\|c}115 & 0 & 285\\ 0 & 23 & 15\end{array}\right] R_1 \rightarrow 23R_1-4R_2$ $\therefore 115x=285 \Rightarrow x=\frac{285}{115}=2.4783, \quad 23y=15 \Rightarrow y=\frac{15}{23}=0.6522$
16.	**Which of the iteration method for solving linear system of equation converges faster? Why?**
	Solution: Gauss – Seidel method is faster than other iterative methods. In Gauss – Seidel method the latest values of unknowns at each stage of iteration are used in proceeding to the next stage of iteration. Hence the convergence in Gauss – Seidel method is more rapid than Gauss – Jacobi method.
17.	**State the condition for convergence of Jacobi's iteration method for solving a system of simultaneous algebraic equations.**
	Solution: The process of iteration by Gauss-Jacobi method will converge if in each equation of the system, the absolute value of the largest coefficient is greater than the sum of the absolute values of the remaining coefficients. The coefficient of matrix should be diagonally dominate. $a_1x+b_1y+c_1z=d_1,\ a_2x+b_2y+c_2z=d_2,\ a_3x+b_3y+c_3z=d_3$ The convergence condition is $\|a_1\|>\|b_1\|+\|c_1\|,\quad \|b_2\|>\|a_2\|+\|c_2\|,\quad \|c_3\|>\|b_3\|+\|a_3\|$

18. **Find the first iteration values of x, y, z satisfying $28x+4y-z=32$, $x+2y+10z=24$ and $2x+17y+4z=35$ by Gauss – Seidel method.**

Solution: Interchanging the equation as 1, 3, 2: $28x+4y-z=32,\ 2x+17y+4z=35$

$x+2y+10z=24$ Now it is diagonally dominant: $x=\frac{1}{28}(32-4y+z),\ y=\frac{1}{17}(35-2x-4z),$

$z=\frac{1}{10}(24-x-2y)$

First iteration: Let $y=0, z=0$

$$x=\frac{1}{28}(32-4y+z)=\frac{1}{28}(32-4(0)+0)=\frac{1}{28}(32)=1.142$$

$$y=\frac{1}{17}(35-2x-4z)=\frac{1}{17}(35-2(1.142)-4(0))=\frac{1}{17}(32.716)=1.924$$

$$z=\frac{1}{10}(24-x-2y)=\frac{1}{10}(24-1.142-2(1.924))=\frac{1}{10}(19.01)=1.901$$

19. **Find the dominant Eigenvalue of the matrix $\begin{pmatrix}1&2\\3&4\end{pmatrix}$ by power method.**

Solution: Let $X_0=\begin{bmatrix}1\\1\end{bmatrix}$ be initial Eigen vector

$$AX_0=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}3\\7\end{bmatrix}=7\begin{bmatrix}0.43\\1\end{bmatrix}=7X_1$$

$$AX_1=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}0.43\\1\end{bmatrix}=\begin{bmatrix}2.43\\5.29\end{bmatrix}=5.29\begin{bmatrix}0.46\\1\end{bmatrix}=5.29X_2$$

$$AX_2=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}0.46\\1\end{bmatrix}=\begin{bmatrix}2.46\\5.38\end{bmatrix}=5.38\begin{bmatrix}0.46\\1\end{bmatrix}=5.38X_3$$

The dominant Eigenvalue is 5.38 and the corresponding Eigenvector is $\begin{bmatrix}0.46\\1\end{bmatrix}$

20. **Find the largest Eigen value and the corresponding Eigen vector of the matrix $\begin{bmatrix}1&1\\1&1\end{bmatrix}$ correct to two decimal places using power method.**

Solution: Let $X_0=\begin{bmatrix}1\\1\end{bmatrix}$ be initial Eigenvector

$$AX_0=\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=2\begin{bmatrix}1\\1\end{bmatrix}=2X_1,\quad AX_1=\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=2\begin{bmatrix}1\\1\end{bmatrix}=2X_2.$$

The largest Eigen value is 2 & the corresponding Eigen vector is $[1,\ 1]^T$.

PART-B

1. **(i). Find the real root of the equation $x^3+x^2-1=0$ using fixed point iteration.**

Solution: Let $f(x)=x^3+x^2-1$

Now $f(0)=0+0-1=-1(-ve)$

$f(1)=1+1-1=1(+ve)$

Hence a real root lies between 0 and 1

Now $x^3+x^2-1=0$ can be written as

$$x^2\left(x+1\right)-1=0$$

$$x^2=\frac{1}{x+1}$$

$$x=\frac{1}{\sqrt{x+1}}$$

Let $\phi\left(x\right)=\dfrac{1}{\sqrt{x+1}}$

Therefore $\phi'\left(x\right)=\dfrac{\dfrac{-1}{2\sqrt{x+1}}}{x+1}=\dfrac{-1}{2\left(x+1\right)^{3/2}}$

Clearly $\left|\phi'\left(x\right)\right|=\left|\dfrac{1}{2\left(x+1\right)^{3/2}}\right|<1$ in $\left(0,1\right)$

For when $x=0.5$, $\left|\phi'\left(0.5\right)\right|=\left|\dfrac{1}{2\left(1.5\right)^{3/2}}\right|<1$

The initial value is $x_0=0.5$. Now

$$x_1=\phi\left(x_0\right)=\frac{1}{\sqrt{x_0+1}}=\frac{1}{\sqrt{0.5+1}}=0.8165,$$

$$x_2=\phi\left(x_1\right)=\frac{1}{\sqrt{x_1+1}}=\frac{1}{\sqrt{0.8165+1}}=0.7420,\ x_3=\phi\left(x_2\right)=\frac{1}{\sqrt{x_2+1}}=\frac{1}{\sqrt{0.7420+1}}=0.7577,$$

$$x_4=\phi\left(x_3\right)=\frac{1}{\sqrt{x_3+1}}=\frac{1}{\sqrt{0.7577+1}}=0.7543,\ x_5=\phi\left(x_4\right)=\frac{1}{\sqrt{x_4+1}}=\frac{1}{\sqrt{0.7543+1}}=0.7550$$

$$x_6=\phi\left(x_5\right)=\frac{1}{\sqrt{x_5+1}}=\frac{1}{\sqrt{0.7550+1}}=0.7548,x_7=\phi\left(x_6\right)=\frac{1}{\sqrt{x_6+1}}=\frac{1}{\sqrt{0.7548+1}}=0.7549$$

$$x_8=\phi\left(x_7\right)=\frac{1}{\sqrt{x_7+1}}=\frac{1}{\sqrt{0.7549+1}}=0.7549$$

Two successive iteration values are equal stop the process. Hence the root is 0.7549

(ii). Find a positive root of the equation $\cos x-3x+1=0$ by the method of fixed point iteration.

Solution: Let $f\left(x\right)=\cos x-3x+1$

$$f\left(0\right)=\cos\left(0\right)-3\left(0\right)+1=1-0+1=2\left(+ve\right)$$

$$f\left(1\right)=\cos\left(1\right)-3\left(1\right)+1=-1.000=\left(-ve\right)$$

Therefore a root lies between 0 and 1. The given equation can be written as

$$x=\frac{1}{3}\left(1+\cos x\right)$$

Let $\phi\left(x\right)=\dfrac{1}{3}\left(1+\cos x\right)$

$$\phi'\left(x\right)=\frac{-\sin x}{3}$$

Clearly, $\left|\phi'(x)\right| = \left|\dfrac{-\sin x}{3}\right| = \dfrac{1}{3}\left|\sin x\right| < 1 \ in \ (0,1).$

The initial value is $x_0 = 0.$

The successive approximation are as follows:

$$x_1 = \phi(x_0) = \frac{1}{3}(1+\cos x_0) = \frac{1}{3}(1+\cos 0) = 0.6667$$

$$x_2 = \phi(x_1) = \frac{1}{3}(1+\cos x_1) = \frac{1}{3}(1+\cos 0.6667) = 0.5953$$

$$x_3 = \frac{1}{3}(1+\cos x_2) = 0.6093, \ x_4 = \frac{1}{3}(1+\cos x_3) = 0.6067, \ x_5 = \frac{1}{3}(1+\cos x_4) = 0.6072$$

$$x_6 = \frac{1}{3}(1+\cos x_5) = 0.6071, \ x_7 = \frac{1}{3}(1+\cos x_6) = 0.6071$$

Two successive iteration values are equal stop the process. Hence the root is 0.6071

2. **(i). Compute the real root of $x\log_{10} x = 1.2$ correct to three decimal places using Newton-Raphson method.**

Solution:

Given $f(x) = x\log_{10} x - 1.2$

Now $f(2) = 2\log_{10} 2 - 1.2 = 2(0.3010) - 1.2 = -0.5980(-ve)$

$f(3) = 3\log_{10} 3 - 1.2 = 3(0.4771) - 1.2 = 0.2313(+ve)$

Here $f(2)$ and $f(3)$ are opposite in sign, therefore the root of $f(x) = 0$ lies between 2 and 3.

Here $\left|f(3)\right| < \left|f(2)\right|.$ Therefore we can take the initial approximation to the root is $x_0 = 3.$

Now $f(x) = x\log_{10} x - 1.2$

$$f'(x) = \log_{10} x + x\frac{1}{x}\log_{10} e \qquad \left[\because \frac{d}{dx}\log_a x = \frac{1}{x}\log_a e\right]$$

$$= \log_{10} x + 0.4343 \qquad \left[\because \log_{10} e = 0.4343\right]$$

We know that Newton's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The first approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0\log_{10} x_0 - 1.2}{\log_{10} x_0 + 0.4343} = 3 - \frac{3\log_{10} 3 - 1.2}{\log_{10} 3 + 0.4343} = 2.746$$

The second approximation to the root is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1\log_{10} x_1 - 1.2}{\log_{10} x_1 + 0.4343} = 2.746 - \frac{2.746\log_{10} 2.746 - 1.2}{\log_{10} 2.746 + 0.4343} = 2.741$$

The third approximation to the root is given by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{x_2\log_{10} x_2 - 1.2}{\log_{10} x_2 + 0.4343} = 2.741 - \frac{2.741\log_{10} 2.741 - 1.2}{\log_{10} 2.741 + 0.4343} = 2.741$$

Hence the real root of $f(x) = 0,$ correct to three decimal places is 2.741

(ii). Using Newton-Raphson method, find the real root of $3x + \sin x - e^x = 0$ by choosing initial approximation $x_0 = 0.5$

Solution:

Given $f(x)=3x+\sin x-e^{x}$

Therefore $f'(x)=3+\cos x-e^{x}$

Now $f(0.5)=3(0.5)+\sin(0.5)-e^{0.5}=0.3307(+ve)$

$f(2)=3(2)+\sin(2)-e^{2}=-0.4798(-ve)$

Here $f(0.5)$ is positive and $f(2)$ is negative. Therefore the root lies between 0.5 and 2. Since $\left|f(2)\right|<\left|f(0.5)\right|$ we can take the initial approximation $x_0=0.5.$

We know that Newton's formula is

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

The first approximation to the root is given by

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}=0.5-\frac{3(0.5)+\sin(0.5)-e^{0.5}}{3+\cos(0.5)-e^{0.5}}=0.3516$$

The second approximation to the root is given by

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}=0.3516-\frac{3(0.3516)+\sin(0.3516)-e^{0.3516}}{3+\cos(0.3516)-e^{0.3516}}=0.3516-\left[\frac{-0.02214}{2.5175}\right]=0.3604$$

The third approximation to the root is given by

$$x_3=x_2-\frac{f(x_2)}{f'(x_2)}=0.3604-\frac{3(0.3604)+\sin(0.3604)-e^{0.3604}}{3+\cos(0.3604)-e^{0.3604}}=0.3604-[-0.0000217]=0.3604$$

Hence the real root of $f(x)=0,$ correct to three decimal places is 0.3604

(ii). Find Newton's iterative formula to find the reciprocal of a given number *N* and hence find the value of $\frac{1}{19}$

Let $x=\frac{1}{N}\Rightarrow\frac{1}{x}=N$

Let $f(x)=\frac{1}{x}-N=0$

$f'(x)=\frac{-1}{x^2}$

We know that Newton's iterative formula is

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{\frac{1}{x_n}-N}{\frac{-1}{x^2}}=x_n+x_n^2\left[\frac{1}{x_n}-N\right]=x_n+x_n-Nx_n^2=x_n\left[2-Nx_n\right]\text{ ------ (1)}$$

To find $\frac{1}{19}$:

Substitute $N=19$ and $n=0$ in (1) we get

$x_1=x_0\left[2-19x_0\right]$

Since $\frac{1}{8}=0.06$ we take $x_0=0.06$

The first approximation is given by

$x_1 = x_0\left[2-19x_0\right] = 0.06\left[2-19\left(0.06\right)\right] = 0.0516$

The second approximation is given by

$x_2 = x_1\left[2-19x_1\right] = 0.0516\left[2-19\left(0.0516\right)\right] = 0.0526$

The third approximation is given by

$x_3 = x_2\left[2-19x_2\right] = 0.0526\left[2-19\left(0.0526\right)\right] = 0.0526$

Therefore the value of x_2 and x_3 are equal

Hence the value of $\frac{1}{19} = 0.0526$

3. **(i). Solve the following linear system of equations by Gauss elimination method**
$\mathbf{2x+3y+z=-1,\ \ 5x+y+z=9,\ 3x+2y+4z=11}$

Solution:

Write the given system of equation in augmented matrix form $\left[A,B\right] = \left(\begin{array}{ccc|c} 2 & 3 & 1 & -1 \\ 5 & 1 & 1 & 9 \\ 3 & 2 & 4 & 11 \end{array}\right)$

<table>
<tr><td>$\sim\left(\begin{array}{ccc|c} 2 & 3 & 1 & -1 \\ 0 & -13 & -3 & 23 \\ 3 & 2 & 4 & 11 \end{array}\right)$ $R_2 : 2R_2 - 5R_1$</td><td>$\begin{array}{lcccc} 2R_2: & 10 & 2 & 2 & 18 \\ -5R_1: & -10 & -15 & -5 & 5 \\ \hline R_2: & 0 & -13 & -3 & 23 \end{array}$</td></tr>
<tr><td>$\sim\left(\begin{array}{ccc|c} 2 & 3 & 1 & -1 \\ 0 & -13 & -3 & 23 \\ 0 & -5 & 5 & 25 \end{array}\right)$ $R_3 : 2R_3 - 3R_1$</td><td>$\begin{array}{lcccc} 2R_3: & 6 & 4 & 8 & 22 \\ -3R_1: & -6 & -9 & -3 & 3 \\ \hline R_3: & 0 & -5 & 5 & 25 \end{array}$</td></tr>
<tr><td>$\sim\left(\begin{array}{ccc|c} 2 & 3 & 1 & -1 \\ 0 & -13 & -3 & 23 \\ 0 & 0 & 80 & 210 \end{array}\right)$
$R_3 : 13R_3 - 5R_2$</td><td>$\begin{array}{lcccc} 13R_3: & 0 & -65 & 65 & 325 \\ -5R_2: & 0 & 65 & 15 & -115 \\ \hline R_3: & 0 & 0 & 80 & 210 \end{array}$</td></tr>
</table>

By back substitution

$80\text{z} = 210 \Rightarrow \text{z} = \frac{210}{80} \Rightarrow \text{z} = 2.625$

$-13\text{y} - 3\text{z} = 23$

$-13\text{y} - 3(2.625) = 23 \quad \Rightarrow \text{y} = -2.375$

$2\text{x} + 3\text{y} + \text{z} = -1$

$2\text{x} + 3(-2.375) + 2.625 = -1 \quad \Rightarrow \text{x} = 1.75$

(ii). Apply Gauss Jordan method to find the solution of the following system
$\mathbf{3x-y+2z=12,\ \ x+2y+3z=11,\ \ 2x-2y-z=2}$

Solution:

Write the given system of equation in augmented matrix form

$$[A,B] = \left[\begin{array}{ccc|c} 3 & -1 & 2 & 12 \\ 1 & 2 & 3 & 11 \\ 2 & -2 & -1 & 2 \end{array}\right]$$

$\sim\left[\begin{matrix}3 & -1 & 2\\ 0 & 7 & 7\\ 2 & -2 & -1\end{matrix}\left\vert\begin{matrix}12\\ 21\\ 2\end{matrix}\right.\right]$ $R_2 : 3R_2 - R_1$	$\begin{array}{lcccc} 3R_2 & :3 & 6 & 9 & 33\\ -R_1 & :-3 & 1 & -2 & -12\\ \hline R_2 & :0 & 7 & 7 & 21\end{array}$
$\sim\left[\begin{matrix}3 & -1 & 2\\ 0 & 7 & 7\\ 0 & -4 & -7\end{matrix}\left\vert\begin{matrix}12\\ 21\\ -18\end{matrix}\right.\right]$ $R_3 : 3R_3 - 2R_1$	$\begin{array}{lcccc} 3R_3 & :6 & -6 & -3 & 6\\ -2R_1 & :-6 & 3 & -4 & -24\\ \hline R_3 & :0 & -3 & -7 & -18\end{array}$
$\sim\left[\begin{matrix}3 & -1 & 2\\ 0 & 7 & 7\\ 0 & 0 & -21\end{matrix}\left\vert\begin{matrix}12\\ 21\\ -42\end{matrix}\right.\right]$ $R_3 : 7R_3 + 4R_2$	$\begin{array}{lcccc} 7R_3 & :0 & -28 & -49 & -126\\ 4R_2 & :0 & 28 & 28 & 84\\ \hline R_3 & :0 & 0 & -21 & -42\end{array}$
$\sim\left[\begin{matrix}3 & -1 & 2\\ 0 & 1 & 1\\ 0 & 0 & 1\end{matrix}\left\vert\begin{matrix}12\\ 3\\ 2\end{matrix}\right.\right]$ $\frac{R_2}{7};\ \frac{R_3}{-21}$	$\begin{array}{lcccc} \frac{R2}{7} & :0 & \frac{7}{7} & \frac{7}{7} & \frac{21}{7}\\ \hline R_2 & :0 & 1 & 1 & 3\\ \frac{R_3}{-21} & :0 & 0 & \frac{-21}{-21} & \frac{-42}{-21}\\ \hline R_3 & :0 & 0 & 1 & 2\end{array}$
$\sim\left[\begin{matrix}3 & -1 & 2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}\left\vert\begin{matrix}12\\ 1\\ 2\end{matrix}\right.\right]$ $R_2 : R_2 - R_3$	$\begin{array}{lcccc} R_2 & :0 & 1 & 1 & 3\\ -R_3 & :0 & 0 & 1 & 2\\ \hline R_2 & :0 & 1 & 0 & 1\end{array}$
$\sim\left[\begin{matrix}3 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}\left\vert\begin{matrix}8\\ 1\\ 2\end{matrix}\right.\right]$ $R_1 : R_1 - 2R_3$	$\begin{array}{lcccc} R_1 & :3 & -1 & 2 & 12\\ -2R_3 & :0 & 0 & -2 & -4\\ \hline R_3 & :3 & -1 & 0 & 8\end{array}$
$\sim\left[\begin{matrix}3 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}\left\vert\begin{matrix}9\\ 1\\ 2\end{matrix}\right.\right]$ $R_1 : R_1 + R_2$	$\begin{array}{lcccc} R_1 & :3 & -1 & 0 & 8\\ R_2 & :0 & 1 & 0 & 1\\ \hline R_1 & :3 & 0 & 0 & 9\end{array}$
$\sim\left[\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}\left\vert\begin{matrix}3\\ 1\\ 2\end{matrix}\right.\right]$ $\frac{R_1}{3}$	$\begin{array}{lcccc} \frac{R_1}{3} & :\frac{3}{3} & 0 & 0 & \frac{9}{3}\\ \hline R_1 & :1 & 0 & 0 & 3\end{array}$

$\Rightarrow x + 0y + 0z = 3$ ………..……………………….(1)

$0x + y + 0z = 1$ ……………………………..………..(2)

$0x + 0y + z = 1$ ………………………………….(3)

Therefore $x = 3,\ y = 1,\ z = 2$.

4. **i). Solve the following system of equations by Gauss-Jacobi method $8x - y + z = 18$, $2x + 5y - 2z = 3$, $x + y - 3z = -6$**

Solution:

As the coefficient matrix is diagonally dominant solving for x, y, z we get

$$8x - y + z = 18 \Rightarrow 8x = 18 + y - z \Rightarrow x = \frac{1}{8}(18 + y - z)$$

$$2x + 5y - 2z = 3 \Rightarrow 5y = 3 - 2x + 2z \Rightarrow y = \frac{1}{5}(3 - 2x + 2z)$$

$$x + y - 3z = -6 \Rightarrow 3z = 6 + x + y \Rightarrow z = \frac{1}{3}(6 + x + y)$$

Let the initial values be $x_0 = 0, y_0 = 0, z_0 = 0$

Iteration	$x = \frac{1}{8}(18 + y - z)$	$y = \frac{1}{5}(3 - 2x + 2z)$	$z = \frac{1}{3}(6 + x + y)$
I	$x_1 = \frac{1}{8}(18 + y_0 - z_0)$ $= \frac{1}{8}(18 + 0 - 0)$ $= \frac{1}{8}(18) = 2.250$	$y_1 = \frac{1}{5}(3 - 2x_0 + 2z_0)$ $= \frac{1}{5}(3 - 0 + 0)$ $= \frac{1}{5}(3) = 0.600$	$z_1 = \frac{1}{3}(6 + x_0 + y_0)$ $= \frac{1}{3}(6 + 0 + 0)$ $= \frac{1}{3}(6) = 2.000$
II	$x_2 = \frac{1}{8}(18 + y_1 - z_1)$ $= \frac{1}{8}(18 + 0.600 - 2.000)$ $= 2.075$	$y_2 = \frac{1}{5}(3 - 2x_1 + 2z_1)$ $= \frac{1}{5}(3 - 2(2.250) + 2(2.000))$ $= 0.500$	$z_2 = \frac{1}{3}(6 + x_1 + y_1)$ $= \frac{1}{3}(6 + 2.250 + 0.600)$ $= 2.950$
III	$x_3 = \frac{1}{8}(18 + y_2 - z_2)$ $= \frac{1}{8}(18 + 0.500 - 2.950)$ $= 1.944$	$y_3 = \frac{1}{5}(3 - 2x_2 + 2z_2)$ $= \frac{1}{5}(3 - 2(2.075) + 2(2.950))$ $= 0.950$	$z_3 = \frac{1}{3}(6 + x_2 + y_2)$ $= \frac{1}{3}(6 + 2.075 + 0.500)$ $= 2.858$
IV	$x_4 = \frac{1}{8}(18 + y_3 - z_3)$ $= \frac{1}{8}(18 + 0.950 - 2.858)$ $= 2.012$	$y_4 = \frac{1}{5}(3 - 2x_3 + 2z_3)$ $= \frac{1}{5}(3 - 2(1.944) + 2(2.858))$ $= 0.966$	$z_4 = \frac{1}{3}(6 + x_3 + y_3)$ $= \frac{1}{3}(6 + 1.944 + 0.950)$ $= 2.965$
V	$x_5 = \frac{1}{8}(18 + y_4 - z_4)$ $= \frac{1}{8}(18 + 0.966 - 2.965)$ $= 2.000$	$y_5 = \frac{1}{5}(3 - 2x_4 + 2z_4)$ $= \frac{1}{5}(3 - 2(2.012) + 2(2.965))$ $= 0.981$	$z_5 = \frac{1}{3}(6 + x_4 + y_4)$ $= \frac{1}{3}(6 + 2.012 + 0.966)$ $= 2.993$

VI	$x_6=\frac{1}{8}\left(18+y_5-z_5\right)$ $=\frac{1}{8}\left(18+0.981-2.993\right)$ $=1.999$	$y_6=\frac{1}{5}\left(3-2x_5+2z_5\right)$ $=\frac{1}{5}\left(3-2(2.000)+2(2.993)\right)$ $=0.997$	$z_6=\frac{1}{3}\left(6+x_5+y_5\right)$ $=\frac{1}{3}\left(6+2.000+0.981\right)$ $=2.994$
VII	$x_7=\frac{1}{8}\left(18+y_6-z_6\right)$ $=\frac{1}{8}\left(18+0.997-2.994\right)$ $=2.000$	$y_7=\frac{1}{5}\left(3-2x_6+2z_6\right)$ $=\frac{1}{5}\left(3-2(1.999)+2(2.994)\right)$ $=0.998$	$z_7=\frac{1}{3}\left(6+x_6+y_6\right)$ $=\frac{1}{3}\left(6+1.999+0.997\right)$ $=2.999$
VIII	$x_8=\frac{1}{8}\left(18+y_7-z_7\right)$ $=\frac{1}{8}\left(18+0.998-2.999\right)$ $=2.000$	$y_8=\frac{1}{5}\left(3-2x_7+2z_7\right)$ $=\frac{1}{5}\left(3-2(2.000)+2(2.999)\right)$ $=1.000$	$z_8=\frac{1}{3}\left(6+x_7+y_7\right)$ $=\frac{1}{3}\left(6+2.000+0.998\right)$ $=2.999$
IX	$x_9=\frac{1}{8}\left(18+y_8-z_8\right)$ $=\frac{1}{8}\left(18+1.000-2.999\right)$ $=2.000$	$y_9=\frac{1}{5}\left(3-2x_8+2z_8\right)$ $=\frac{1}{5}\left(3-2(2.000)+2(2.999)\right)$ $=1.000$	$z_9=\frac{1}{3}\left(6+x_8+y_8\right)$ $=\frac{1}{3}\left(6+2.000+1.000\right)$ $=3.000$
X	$x_{10}=\frac{1}{8}\left(18+y_9-z_9\right)$ $=\frac{1}{8}\left(18+1.000-3.000\right)$ $=2.000$	$y_{10}=\frac{1}{5}\left(3-2x_9+2z_9\right)$ $=\frac{1}{5}\left(3-2(2.000)+2(3.000)\right)$ $=1.000$	$z_9=\frac{1}{3}\left(6+x_8+y_8\right)$ $=\frac{1}{3}\left(6+2.000+1.000\right)$ $=3.000$

Hence $x=2,\ \ y=1,\ \ z=3$

(ii). Find the solution of the system of following equations by Gauss - Seidel method $\boldsymbol{x-2y+5z=12,\ 5x+2y-z=6,\ 2x+6y-3z=5}$

Solution:

The given system is

$x-2y+5z=12$

$5x+2y-z=6$

$2x+6y-3z=5$

Interchanging the equations

$5x+2y-z=6$ ----------------- (1)

$2x+6y-3z=5$ -------------------- (2)

$x-2y+5z=12$ -------------------- (3)

Clearly the coefficient matrix $\begin{bmatrix}5 & 2 & -1\\ 2 & 6 & -3\\ 1 & -2 & 5\end{bmatrix}$ is diagonally dominant. Hence we can apply Gauss-Seidel method without any difficulty.

From (1), (2) and (3) we get

$$x=\frac{1}{5}(6-2y+z) \qquad \text{---------------} \ (4)$$

$$y=\frac{1}{6}(5-2x+3z) \qquad \text{---------------} \ (5)$$

$$z=\frac{1}{5}(12-x+2y) \qquad \text{--------------} \ (6)$$

Iteration	$x=\frac{1}{5}(6-2y+z)$	$y=\frac{1}{6}(5-2x+3z)$	$z=\frac{1}{5}(12-x+2y)$
I	$x_1=\frac{1}{5}(6-2y_0+z_0)$ $=\frac{1}{5}(6-0+0)$ $=1.200$	$y_1=\frac{1}{6}(5-2x_1+3z_0)$ $=\frac{1}{6}(5-2(1.200)+0)$ $=0.433$	$z_1=\frac{1}{5}(12-x_1+2y_1)$ $=\frac{1}{5}(12-1.200+2(0.433))$ $=2.333$
II	$x_2=\frac{1}{5}(6-2y_1+z_1)$ $=\frac{1}{5}(6-2(0.433)+2.333)$ $=1.493$	$y_2=\frac{1}{6}(5-2x_2+3z_1)$ $=\frac{1}{6}(5-2(1.493)+3(2.333))$ $=1.502$	$z_2=\frac{1}{5}(12-x_2+2y_2)$ $=\frac{1}{5}(12-1.493+2(1.502))$ $=2.702$
III	$x_3=\frac{1}{5}(6-2y_2+z_2)$ $=\frac{1}{5}(6-2(1.502)+2.702)$ $=1.140$	$y_3=\frac{1}{6}(5-2x_3+3z_2)$ $=\frac{1}{6}(5-2(1.140)+3(2.702))$ $=1.804$	$z_3=\frac{1}{5}(12-x_3+2y_3)$ $=\frac{1}{5}(12-1.140+2(1.804))$ $=2.894$
IV	$x_4=\frac{1}{5}(6-2y_3+z_3)$ $=\frac{1}{5}(6-2(1.804)+2.894)$ $=1.057$	$y_4=\frac{1}{6}(5-2x_4+3z_3)$ $=\frac{1}{6}(5-2(1.057)+3(2.894))$ $=1.928$	$z_4=\frac{1}{5}(12-x_4+2y_4)$ $=\frac{1}{5}(12-1.057+2(1.928))$ $=2.960$
V	$x_5=\frac{1}{5}(6-2y_4+z_4)$ $=\frac{1}{5}(6-2(1.928)+2.960)$ $=1.021$	$y_5=\frac{1}{6}(5-2x_5+3z_4)$ $=\frac{1}{6}(5-2(1.021)+3(2.960))$ $=1.973$	$z_5=\frac{1}{5}(12-x_5+2y_5)$ $=\frac{1}{5}(12-1.021+2(1.973))$ $=2.985$
VI	$x_6=\frac{1}{5}(6-2y_5+z_5)$ $=\frac{1}{5}(6-2(1.973)+2.985)$ $=1.008$	$y_6=\frac{1}{6}(5-2x_6+3z_5)$ $=\frac{1}{6}(5-2(1.008)+3(2.985))$ $=1.990$	$z_6=\frac{1}{5}(12-x_6+2y_6)$ $=\frac{1}{5}(12-1.008+2(1.990))$ $=2.994$

VII	$x_7=\frac{1}{5}(6-2y_6+z_6)$ $=\frac{1}{5}(6-2(1.990)+2.994)$ $=1.003$	$y_7=\frac{1}{6}(5-2x_7+3z_6)$ $=\frac{1}{6}(5-2(1.003)+3(2.994))$ $=1.996$	$z_7=\frac{1}{5}(12-x_7+2y_7)$ $=\frac{1}{5}(12-1.003+2(1.996))$ $=2.998$
VIII	$x_8=\frac{1}{5}(6-2y_7+z_7)$ $=\frac{1}{5}(6-2(1.996)+2.998)$ $=1.001$	$y_8=\frac{1}{6}(5-2x_8+3z_7)$ $=\frac{1}{6}(5-2(1.001)+3(2.998))$ $=1.999$	$z_8=\frac{1}{5}(12-x_8+2y_8)$ $=\frac{1}{5}(12-1.001+2(1.999))$ $=2.999$
IX	$x_9=\frac{1}{5}(6-2y_8+z_8)$ $=\frac{1}{5}(6-2(1.999)+2.999)$ $=1.000$	$y_9=\frac{1}{6}(5-2x_9+3z_8)$ $=\frac{1}{6}(5-2(1.000)+3(2.999))$ $=2.000$	$z_9=\frac{1}{5}(12-x_9+2y_9)$ $=\frac{1}{5}(12-1.000+2(2.000))$ $=3.000$
X	$x_{10}=\frac{1}{5}(6-2y_9+z_9)$ $=\frac{1}{5}(6-2(2.000)+3.000)$ $=1.000$	$y_{10}=\frac{1}{6}(5-2x_{10}+3z_9)$ $=\frac{1}{6}(5-2(1.000)+3(3.000))$ $=2.000$	$z_{10}=\frac{1}{5}(12-x_{10}+2y_{10})$ $=\frac{1}{5}(12-1.000+2(2.000))$ $=3.000$

Hence $x=1, y=2, z=3$

5. **(i). Find the numerically largest Eigen value of A =** $\begin{bmatrix} \mathbf{25} & \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{3} & \mathbf{0} \\ \mathbf{2} & \mathbf{0} & \mathbf{-4} \end{bmatrix}$ **and the corresponding Eigen vector.**

Solution: Let the initial eigenvector be $X^{(0)}=\begin{pmatrix}1\\0\\0\end{pmatrix}$

$$AX^{(0)}=\begin{bmatrix}25&1&2\\1&3&0\\2&0&-4\end{bmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}25\\1\\2\end{pmatrix}=25\begin{pmatrix}1\\0.04\\0.08\end{pmatrix}=25X^{(1)}.$$

$$AX^{(1)}=\begin{bmatrix}25&1&2\\1&3&0\\2&0&-4\end{bmatrix}\begin{pmatrix}1\\0.04\\0.08\end{pmatrix}=\begin{pmatrix}25.2\\1.12\\1.68\end{pmatrix}=25.2\begin{pmatrix}1\\0.0444\\0.0667\end{pmatrix}=25.2X^{(2)}.\text{(ie) } X^{(2)}=\begin{pmatrix}1\\0.0444\\0.0667\end{pmatrix}$$

Repeating this , we get $25.1778\begin{pmatrix}1\\0.0450\\0.06888\end{pmatrix}, 25.1826\begin{pmatrix}1\\0.0451\\0.0685\end{pmatrix}, 25.1821\begin{pmatrix}1\\0.0451\\0.0685\end{pmatrix}.$

Therefore, the largest eigenvalue is 25.182

(ii). Find all the Eigen value of A= $\begin{bmatrix} \mathbf{5} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{-2} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{5} \end{bmatrix}$ **using power method**

Solution:

Let the initial eigenvector be $X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Now

$$X_1 = AX_0 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \\ 0.2 \end{bmatrix} = 5X_1$$

$$X_2 = AX_1 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 5.2 \\ 0 \\ 2 \end{bmatrix} = 5.2\begin{bmatrix} 1 \\ 0 \\ 0.38 \end{bmatrix} = 5.2X_2$$

$$X_3 = AX_2 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.38 \end{bmatrix} = \begin{bmatrix} 5.38 \\ 0 \\ 2.92 \end{bmatrix} = 5.38\begin{bmatrix} 1 \\ 0 \\ 0.54 \end{bmatrix} = 5.38X_3$$

$$X_4 = AX_3 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.54 \end{bmatrix} = \begin{bmatrix} 5.54 \\ 0 \\ 3.71 \end{bmatrix} = 5.54\begin{bmatrix} 1 \\ 0 \\ 0.67 \end{bmatrix} = 5.54X_4$$

$$X_5 = AX_4 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.67 \end{bmatrix} = \begin{bmatrix} 5.67 \\ 0 \\ 4.35 \end{bmatrix} = 5.67\begin{bmatrix} 1 \\ 0 \\ 0.77 \end{bmatrix} = 5.67X_5$$

$$X_6 = AX_5 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.77 \end{bmatrix} = \begin{bmatrix} 5.77 \\ 0 \\ 4.84 \end{bmatrix} = 5.77\begin{bmatrix} 1 \\ 0 \\ 0.84 \end{bmatrix} = 5.77X_6$$

$$X_7 = AX_6 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.84 \end{bmatrix} = \begin{bmatrix} 5.84 \\ 0 \\ 5.19 \end{bmatrix} = 5.84\begin{bmatrix} 1 \\ 0 \\ 0.89 \end{bmatrix} = 5.84X_7$$

$$X_8 = AX_7 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.89 \end{bmatrix} = \begin{bmatrix} 5.89 \\ 0 \\ 5.45 \end{bmatrix} = 5.89\begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = 5.89X_8$$

$$X_9 = AX_8 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.93 \end{bmatrix} = \begin{bmatrix} 5.93 \\ 0 \\ 5.62 \end{bmatrix} = 5.93\begin{bmatrix} 1 \\ 0 \\ 0.95 \end{bmatrix} = 5.93X_9$$

	$X_{10} = AX_9 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.95 \end{bmatrix} = \begin{bmatrix} 5.95 \\ 0 \\ 5.75 \end{bmatrix} = 5.95\begin{bmatrix} 1 \\ 0 \\ 0.97 \end{bmatrix} = 5.95X_{10}$ $X_{11} = AX_{10} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.97 \end{bmatrix} = \begin{bmatrix} 5.97 \\ 0 \\ 5.83 \end{bmatrix} = 5.97\begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix} = 5.97X_{11}$ $X_{12} = AX_{11} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix} = \begin{bmatrix} 5.98 \\ 0 \\ 5.89 \end{bmatrix} = 5.98\begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix} = 5.98X_{12}$ $X_{13} = AX_{12} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.98 \end{bmatrix} = \begin{bmatrix} 5.98 \\ 0 \\ 5.92 \end{bmatrix} = 5.98\begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix} = 5.98X_{13}$ Two successive iterations are equal that is $X_{12} = X_{13}$ Therefore, the dominant eigenvalue is 5.98 and corresponding eigenvector is $X = \begin{bmatrix} 1 \\ 0 \\ 0.99 \end{bmatrix}$
UNIT IV - INTERPOLATION AND APPROXIMATION	
PART A	
1.	**What is meant by Interpolation and Extrapolation?**
	Solution: Interpolation is the process of computing intermediate values of a function from a given set of tabular values of the function (i.e.) inside the interval (x_0, x_n)**.** Extrapolation is the process of finding the values outside the interval (x_0, x_n).
2.	**Find $f(x)$ as a polynomial through the points (0, 0) , (1, 1) and (2, 20).**
	Solution: Let $x_0 = 0, x_1 = 1, x_2 = 2$ and $y_0 = 0, y_1 = 1, y_2 = 20$ Lagrange's interpolation formula $y = \frac{(x-x_1)(x-x_2)}{(x_o-x_1)(x_o-x_2)}y_o + \frac{(x-x_o)(x-x_2)}{(x_1-x_o)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_o)(x_2-x_1)}y_2$ $y = f(x) = \frac{(x-1)(x-2)}{(0-1)(0-2)}(0) + \frac{(x-0)(x-2)}{(1-0)(1-2)}(1) + \frac{(x-0)(x-1)}{(2-0)(2-1)}(20)$ $y = 0 - x(x-2) + 10x(x-1) = 9x^2 - 8x$
3.	**Write down Lagrange's Inverse Interpolation formula.**
	Solution: $x = \frac{(y-y_1)(y-y_2)....(y-y_n)}{(y_o-y_1)(y_o-y_2)....(y_o-y_n)}x_o + \frac{(y-y_o)(y-y_2)....(y-y_n)}{(y_1-y_o)(y_1-y_2)....(y_1-y_n)}x_1 + ...$ $+ \frac{(y-y_0)(y-y_1)....(y-y_{n-1})}{(y_n-y_o)(y_n-y_1)....(y_n-y_{n-1})}x_n$
4.	**State the assumption for Lagrange's method**
	Solution: Lagrange's interpolation formula can be used whether the values of x, the independent variable are equally spaced or not whether the difference of y become smaller or not.
5.	**What are the advantages of Lagrange's Interpolation method over Newton's method?**
	Solution:

(i). Newton's forward and backward methods can be used when the arguments are equally spaced, whereas Lagrange's method can be used for equally and unequally spaced arguments.
(ii). Newton's forward method is suitable to interpolate near the beginning of the table and Newton's backward method is suitable to interpolate near the end of the table, while Lagrange's method can be used to interpolate anywhere in the range.

6. **Prove that $\Delta \log(f(x)) = \log\left[1+\frac{\Delta f(x)}{f(x)}\right]$**

Solution:

$$\Delta \log f(x) = \log f(x+h) - \log f(x) = \log \frac{f(x+h)}{f(x)}$$

$$= \log\left[\frac{f(x)+f(x+h)-f(x)}{f(x)}\right] = \log\left[\frac{f(x)}{f(x)} + \frac{\left(f(x+h)-f(x)\right)}{f(x)}\right] = \log\left[1+\frac{\Delta f(x)}{f(x)}\right]$$

7. **When we use Newton's forward method and Newton's backward method?**

Solution: Newton's forward formula is used to interpolate value of y nearer to the beginning value of the table. Newton's backward formula is used to interpolate value of y nearer to the end of set of tabular values. This may also be used to extrapolate closure to right of y_n.

8. **State Newton's forward and backward formula.**

Solution:

Newton's forward formula

$$y(x) = y_o + u\Delta y_o + \frac{u(u-1)}{2!}\Delta^2 y_o + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_o + ...+ \frac{u(u-1)....(u-(n-1))}{n!}\Delta^n y_o$$

where $u = \frac{x-x_0}{h}$

Newton's backward formula:

$$y(x) = y_n + v\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + ...+ \frac{v(v+1)....(v+(n-1))}{n!}\nabla^n y_n$$

where $v = \frac{x-x_n}{h}$

9. **State the error in Newton's Backward formula.**

Solution: Error = $f(x) - p_n(x) = \frac{p(p+1)(p+2)......(p+n)}{(n+1)!} h^{n+1} y^{n+1}(c)$ where $p = \frac{x-x_n}{h}$

10. **What is the relationship between the divided differences and forward differences?**

Solution: If the arguments are equally spaced, then $\Delta^n f(x_o) = \frac{\Delta^n f(x_o)}{n!h^n}$ where h is the interval of difference.

11. **Give $f(0) = -2$, $f(1) = 2$, $f(2) = 8$ find the root of the Newton's interpolating polynomial equation $f(x)=0$.**

Solution:

X	y=f(x)	$\Delta f(x)$	$\Delta^2 f(x)$
0	-2		
		4	

1	2		2
		6	
2	8		

There are only three data given. Hence, the polynomial is of degree 2

$$y(x) = y_o + u\,\Delta y_o + \frac{u(u-1)}{2!}\Delta^2 y_o + ...\quad where \quad u = \frac{x-x_0}{h}$$

$$\text{Here},, x_0 = 0, h = 1-0 = 1, \qquad \because u = x$$

$$y(x) = -2 + \frac{x}{1!}(4) + \frac{x(x-1)}{2!}(2)$$

$$= -2 + 4x + x(x-1) = -2 + 4x + x^2 - x = x^2 + 3x - 2$$

The roots of the equation $f(x) = 0 \Rightarrow x^2 + 3x - 2 = 0$

$$\Rightarrow x = \frac{-3 \pm \sqrt{9+8}}{2} = \frac{-3 \pm \sqrt{17}}{2}$$

12. **Prove that $\Delta^3 y_2 = \nabla^3 y_5$**

Solution:

$$\Delta^3 y_2 = (E-1)^3\, y_2 \qquad [\because \Delta = E-1]$$

$$= (E^3 - 3E^2 + 3E - 1)\, y_2 \qquad [\because \Delta y_x = (E-1)\, y_x = Ey_x - y_x = y_{x+h} - y_x]$$

$$= E^3 y_2 - 3E^2 y_2 + 3E y_2 - y_2$$

$$= y_5 - 3y_4 + y_3 - y_2 ....(1)$$

$$\nabla^3 y_5 = (1 - E^{-1})^3\, y_5 \qquad [\because \nabla = 1 - E^{-1}]$$

$$= (1 - 3E^{-1} + 3E^{-2} - E^{-3})\, y_5 \qquad [\because \nabla y_x = (1 - E^{-1})\, y_x = y_x - E^{-1} y_x = y_x - y_{x-h}]$$

$$= y_5 - 3E^{-1} y_5 + 3E^{-2} y_5 - E^{-3} y_5$$

$$= y_5 - 3y_4 + y_3 - y_2 ....(2)$$

13. **State any two properties of divided difference.**

Solution: (i) The divided differences are symmetrical in all their arguments. i.e. the value of any divided difference is independent of the order of the arguments.

(ii) The n^th^ order divided differences of a polynomial of degree 'n' are constants.

14. **If $f(x) = \frac{1}{x^2}$, find $f(a,b)$ and hence find $f(2,3)$.**

Solution:

$$f(a,b) = \frac{f(b) - f(a)}{b-a} = \frac{\frac{1}{b^2} - \frac{1}{a^2}}{b-a} = \frac{-(b^2-a^2)\big/ a^2b^2}{b-a} = \frac{-\cancel{(b-a)}\,(b+a)\big/ a^2b^2}{\cancel{b-a}} = \frac{-(a+b)}{a^2b^2}$$

$$\therefore f(2,3) = \frac{-(2+3)}{2^2 3^2} = \frac{-5}{36}$$

15. **Form the divided difference table for 1, 3, 6, 11 and f(x) = $x^3 + x + 2$**

Solution:

X	f(x)	Δ	Δ^2	Δ^3
1	4			

		$\frac{32-4}{3-1}=14$			
3	32		$\frac{64-14}{6-1}=10$		
		$\frac{224-32}{6-3}=64$		$\frac{20-10}{11-1}=1$	
6	224		$\frac{224-64}{11-3}=20$		
		$\frac{1344-224}{11-6}=224$			
11	1344				

16. **Show that the second order divided difference $[x_0, x_{1,} x_2]$ is independent of the order of the arguments.**

Solution: $f(x_0,x_1,x_2)=\frac{f(x_0)}{(x_0-x_1)(x_0-x_2)}+\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)}+\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}$

$\Rightarrow f(x_0,x_1,x_2)=f(x_1,x_2,x_0)=f(x_2,x_0,x_1)$.

This shows that $f(x_0,x_1,x_2)$ is independent of the order of the arguments.

17. **Given $y(0)=3$, $y(1)=12$, $y(2)=81$, $y(3)=200$, $y(4)=100$ .Find $\Delta^4 y_0$**

Solution:

X	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y_0$
0	3				
		9			
1	12		60		
		69		–10	
2	81		50		–259
		119		–269	
3	200		–219		
		–100			
4	100				

$\Delta^4 y_0 =-259$

18. **If $y(x_i)=y_i, i=0,1,2,3.......n$ write down the formula for the cubic spline polynomial $y(x)$, valid in $x_{i-1}\le x\le x_i$**

Solution: $M_{i-1}+4M_i+M_{i+1}=\frac{6}{h^2}[y_{i-1}-2y_i+y_{i+1}],\quad i=1,2,.....,n-1$

Here $h=1$,

$$y(x)=\frac{1}{6}\left[(x_i-x)^3 M_{i-1}+(x-x_{i-1})^3 M_i\right]+(x_i-x)\left[y_{i-1}-\frac{1}{6}M_{i-1}\right]+(x-x_{i-1})\left[y_i-\frac{1}{6}M_i\right]$$

where $i=1,2,....,n$

19. **What is a cubic spline and natural cubic spline?**

Solution: Third degree polynomials employed to connect each pair of data points are called cubic splines. If $M_0=0$ and $M_n=0$, then the cubic spline is called as natural cubic spline.

20. **State the properties of the cubic spline.**

Solution: A cubic spline $S(x)$ is defined by the following properties.

(i) $S(x_i) = y_i,\ i = 0, 1, \dots, n$

(ii) $S(x), S'(x), S''(x)$ are continuous in [a, b]

(iii) $S(x)$ is a cubic polynomial in each subinterval $(x_i, x_{i+1}),\ i = 0, 1, \dots, n-1$

PART-B

1. **(i). Find $y(10)$ given $y(5)=12,\ y(6)=13,\ y(9)=14$ and $y(11)=16$ by Lagrange's formula.**

Solution:

The Lagrange's interpolation formula is,

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$y = f(x) = \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)}(13) + \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)}(16)$$

Put $x = 10$

$$y(10) = f(10) = \frac{(4)(1)(-1)}{(-1)(-4)(-6)}(12) + \frac{(5)(1)(-1)}{(1)(-3)(-5)}(13) + \frac{(5)(4)(-1)}{(4)(3)(-2)}(14) + \frac{(5)(4)(1)}{(6)(5)(2)}(16)$$

$y(10) = 14.6666$

(ii). Using Lagrange's inverse interpolation formula, find the value of x when y = 13.5 from the given data

x	93.0	96.2	100.0	104.2	108.7
y	11.38	12.80	14.70	17.07	19.91

Solution: The Lagrange's inverse interpolation formula is,

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y_o-y_1)(y_o-y_2)(y_0-y_3)(y_0-y_4)}x_o + \frac{(y-y_0)(y-y_2)(y-y_3)(y-y_4)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)(y_1-y_4)}x_1 + \frac{(y-y_0)(y-y_1)(y-y_3)(y-y_4)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)(y_2-y_4)}x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_4)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)(y_3-y_4)}x_3 + \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_3)}{(y_4-y_0)(y_4-y_1)(y_4-y_2)(y_4-y_3)}x_4$$

Here $x_0 = 93.0;\ x_1 = 96.2;\ x_2 = 100;\ x_3 = 104.2;\ x_4 = 108.7;$

$y = 13.5;\ y_0 = 11.38;\ y_1 = 12.80;\ y_2 = 14.70;\ y_3 = 17.07;\ y_4 = 19.91$

$$x = \frac{(13.5-12.80)(13.5-14.70)(13.5-17.07)(13.5-19.91)}{(11.38-12.80)(11.38-14.70)(11.38-17.07)(11.38-19.91)}93.0 + \frac{(13.5-11.38)(13.5-14.70)(13.5-17.07)(13.5-19.91)}{(12.80-11.38)(12.80-14.70)(12.80-17.07)(12.80-19.91)}96.0 + \frac{(13.5-11.38)(13.5-12.80)(13.5-17.07)(13.5-19.91)}{(14.70-11.38)(14.70-12.80)(14.70-17.07)(14.70-19.91)}100$$

$$+\ \frac{(13.5-11.38)(13.5-12.80)(13.5-14.70)(13.5-19.91)}{(17.07-11.38)(17.07-12.80)(17.07-14.70)(17.07-19.91)}104.2$$

$$+\frac{(13.5-11.38)(13.5-12.80)(13.5-14.70)(13.5-17.07)}{(19.91-11.38)(19.91-12.80)(19.91-14.70)(19.91-17.07)}108.7$$

Therefore $x(13.5)=97.65567455$

2. **(i). Using Newton's forward interpolation formula, find the cubic polynomial which takes the following values. Hence find $f(3)$.**

x	0	2	4	6
$f(x)$	-14	6	18	118

Solution:

The difference table is

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	-14			
		20		
2	6		-8	
		12		96
4	18		88	
		100		
6	118			

$$y(x) = y_0+\frac{u}{1!}\Delta y_0+\frac{u(u-1)}{2!}\Delta^2 y_0+\frac{u(u-1)(u-2)}{3!}\Delta^3 y_0+\ldots$$

$$\text{here } x_0=0\ ;\ y_0=-14\ ;\ h=2\ ;\ u=\frac{x-x_0}{h}=\frac{x-0}{2}=\frac{x}{2}$$

$$y(x)=-14+\frac{u}{1!}(20)+\frac{u(u-1)}{2!}(-8)+\frac{u(u-1)(u-2)}{3!}96+\ldots$$

$$=-14+20u-4u^2+4u+16u^3-48u^2+32u+\ldots$$

$$=16u^3-52u^2+56u-14$$

$$\text{Put } u=\frac{x}{2}$$

$$y(x)=16\left(\frac{x}{2}\right)^3-52\left(\frac{x}{2}\right)^2+56\left(\frac{x}{2}\right)-14$$

$$y(x)=2x^3-13x^2+26x-14$$

$$\text{Put } x=3,\quad y(3)=2(3)^3-13(3)^2+26(3)-14=1$$

3. **(i). From the following table, estimate the number of students who obtained marks between 40 and 45.**

Marks	**30-40**	**40-50**	**50-60**	**60-70**	**70-80**
No.of Students	**31**	**42**	**51**	**35**	**31**

Solution:

x	Y	Δ	Δ^2	Δ^3	Δ^4
Below 40	31				
		42			
Below 50	73		9		
		51		-25	
Below 60	124		-16		37
		35		12	
Below 70	159		-4		
		31			
Below 80	190				

Here $u = \frac{x - x_0}{h} = \frac{45-40}{10} = 0.5$

By Newton's forward interpolation formula

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{34}\Delta^4 y_0$$

No.of students with the marks less than 45 = 48

No.of students with the marks less than 40 = 31

Therefore No.of students who obtained marks between 40 and 45 = 48 – 31 = 17

(ii). Using Newton's divided difference formula compute f(5) from the data

x	**1**	**2**	**4**	**7**	**12**
Y	**22**	**30**	**82**	**106**	**216**

Solution:

x	Y	Δ	Δ^2	Δ^3	Δ^4
1	22				
		8			
2	30		6		
		26		-1.6	
4	82		-3.6		0.194
		8		0.535	
7	106		1.75		
		22			
12	216				

$y=f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+$

$(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)+(x-x_0)(x-x_1)(x-x_2)(x-x_3)f(x_0,x_1,x_3,x_4)$

Here $f(x_0)=22, f(x_0,x_1)=8, f(x_0,x_1,x_2)=6, f(x_0,x_1,x_2,x_3)=-1.6, f(x_0,x_1,x_3,x_4)=0.194$

$f(5) = 102.144$

4. **(i). If $f(1)=1$, $f(2)=5$, $f(7)=5$ and $f(8)=4$, find a polynomial that satisfies this data using Newton's divided difference formula. hence, find f(6).**

Solution: The divided difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	1			
		4		
2	5		$\frac{-4}{6}$	
		0		$\frac{1}{14}$
7	5		$\frac{-1}{6}$	
		-1		
8	4			

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3) + ...$$

$$= 1 + (x-1)4 + (x-1)(x-2)\left(\frac{-4}{6}\right) + (x-1)(x-2)(x-7)\left(\frac{1}{14}\right)$$

$$= 1 + 4x - 4 - \frac{2}{3}(x-1)(x-2) + \frac{1}{14}(x-1)(x-2)(x-7)$$

$$= \frac{1}{42}\left[168x - 126 - 28x^2 + 84x - 56\right] + 3\left[x^3 - 3x^2 + 2x - 7x^2 + 21x - 14\right]$$

$$= \frac{1}{42}\left[-28x^2 + 252x - 182 + 3x^3 - 9x^2 + 6x - 21x^2 + 63x - 42\right]$$

$$= \frac{1}{42}\left[3x^3 - 58x^2 + 321x - 224\right]$$

$$f(6) = \frac{1}{42}\left[3(6)^3 - 58(6)^2 + 321(6) - 224\right] = 6.2381$$

(ii). From the following table find the value of $\tan 45^o 15'$

x^o :	45	46	47	48	49	50
$\tan x^o$:	1	1.03553	1.07237	1.11061	1.15037	1.19175

Solution:

Here $h = 1, \quad u = \frac{x - x_0}{h} = \frac{45^o 15' - 45^o}{1} = 15' = \frac{1^0}{4} = 0.25$ (u is dimensionless)

The difference table is

x	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
45	1.0000					
		0.03553				
			0.00131			
46	1.03553			0.0009		
		0.03684			0.00003	
47	1.07237		0.00140	0.00012		-0.00005
		0.03824			-0.00002	
48	1.11061			0.00010		
		0.03976	0.00152			
49	1.15037					
		0.04138				
			0.00162			
50	1.19175					

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

$$+ \frac{u(u-1)(u-2)(u-3)(u-4)}{5!}\Delta^5 y_0$$

$$=1.0000+\frac{1}{4}(0.03553)+\frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{2}(0.00131)+\frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{6}(0.00009)$$

$$+\frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)\left(-\frac{11}{4}\right)}{24}(0.00003)+\frac{\frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)\left(-\frac{11}{4}\right)\left(-\frac{15}{4}\right)}{120}(-0.00005)$$

$$=1.0000+0.0088825-0.0001228+0.0000049$$

$$y(45^o15')=1.00876$$

5. **(i). From the given data, find θ at x = 43 and x = 84**

X	40	50	60	70	80	90
θ	184	204	226	250	276	304

Solution: Since six data are given, $P(x)$ is of degree 5. To find θ at $x=84$, use backward interpolation formula where $u=\frac{x-x_0}{h}=\frac{43-40}{10}=0.3$

X	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$
40	184				
		20			
50	204		2		
		22		0	
60	216		2		0
		24		0	
70	250		2		0
		26		0	
80	276		2		
		28			
90	304				

$$\theta(x)=\theta_o+\frac{u}{1!}\Delta\theta_o+\frac{u(u-1)}{2!}\Delta^2\theta_o+\frac{u(u-1)(u-2)}{3!}\Delta^3\theta_o+\ldots$$

$$\theta(43)=184+(0.3)20+\frac{(0.3)(-0.7)}{2}(2)=189.79$$

$$\theta(x)=\theta_n+v\nabla\theta_n+\frac{v(v+1)}{2!}\nabla^2\theta_n+\frac{v(v+1)(v+2)}{3!}\nabla^3\theta_n+\cdots \text{ where } v=\frac{x-x_n}{h}=\frac{84-90}{10}=-0.6$$

$$\theta(84)=304+(-0.6)28+\frac{(-0.6)(0.4)}{2}(2)=286.96$$

(ii). If f(0)=1, f(1)=2, f(2)=33 and f(3)=244 find a cubic spline approximation. Assuming $M(0)=M(3)=0$. Also find $f'(1.5)$ and $f(0.35)$

Solution: Given

x	0 (x_0)	1 (x_1)	2 (x_2)	3 (x_3)
y	1 (y_0)	2 (y_1)	33 (y_2)	244 (y_3)

	Here $h = 1$; $n = 3$ *Let* $M_o = M_3 = 0$, We have $\boxed{M_{i-1}+4M_i+M_{i+1}=\frac{6}{h^2}[y_{i-1}-2y_i+y_{i+1}]}$ for $i=1,2$ $M_0+4M_1+M_2=6[y_0-2y_1+y_2]$ $M_1+4M_2+M_3=6[y_1-2y_2+y_3]$ This reduces to, (taking M_0 , $M_3=0$) $4M_1+M_2=180\to(1)$ $M_1+4M_2=1080\to(2)$ solving (1) & (2) we get $M_1=-24$, $M_2=276$ The cubic spline for $x_{i-1}\le x\le x_i$ is given by $\boxed{f(x)=y(x)=\frac{1}{6h}\left[(x_i-x)^3M_{i-1}+(x-x_{i-1})^3M_i\right]+\frac{1}{h}\left[(x_i-x)[y_{i-1}-\frac{h^2}{6}M_{i-1}\right]+\frac{1}{h}(x-x_{i-1})\left[y_i-\frac{h^2}{6}M_i\right]}\to(3)$ $\boxed{\text{For } i=1}$, $f(x)=-4x^3+5x+1$; $0\le x\le 1\ \to(4)$ $\boxed{\text{For } i=2}$, $f(x)=50x^3-162x^2+167x-53$; $1\le x\le 2\to(5)$ $\boxed{\text{For } i=3}$, $f(x)=-46x^3+414x^2-985x-715$; $2\le x\le 3\to(6)$ Equation(4), (5) & (6) give the cubic spline in each sub–interval Hence $f(x)=\begin{cases}-4x^3+5x+1\ ;\ 0\le x\le 1\\ 50x^3-162x^2+167x-53\ ;\ 1\le x\le 2\\ -46x^3+414x^2-985x-715\ ;\ 2\le x\le 3\end{cases}$ $f(x)=50x^3-162x^2+167x-53;\ 1\le x\le 2$ $f'(x)=150x^2-324x+167$ $f'(1.5)=150(1.5^2)-324(1.5)+167=18.5$ $f(x)=-4x^3+5x+1\ ;\ 0\le x\le 1$ $f(0.35)=-4(0.35^3)+5(0.35)+1=2.26$
UNIT III - NUMERICAL DIFFERENTIATION AND INTEGRATION	
PART – A	
1.	**Define Numerical Differentiation.**
	Solution: Numerical differentiation is the process of computing the values of $\frac{dy}{dx},\frac{d^2y}{dx^2}$…. For some particular values of x from the given data (x_i,y_i) where y is not known explicitly.
2.	**State Newton's formula to find the derivatives $\frac{dy}{dx},\frac{d^2y}{dx^2}$ at x = x₀ using forward differences.**
	Solution: $y'=f'(x)=\frac{dy}{dx}=\frac{1}{h}\left[\Delta y_o-\frac{1}{2}\Delta^2y_o+\frac{1}{3}\Delta^3y_o-........\right]$ $y''=f''(x)=\frac{d^2y}{dx^2}=\frac{1}{h^2}\left[\Delta^2y_o-\Delta^3y_o+\frac{11}{12}\Delta^4y_o-.......\right]$

3. **State Newton's formula to find the derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ at $x \neq x_0$ using forward differences.**

Solution:

$$y' = f'(x) = \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_o + \frac{(2u-1)}{2!}\Delta^2 y_o + \frac{(3u^2-6u+2)}{3!}\Delta^3 y_o + ...\right]$$

$$y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_o + (u-1)\Delta^3 y_o + \left(\frac{6u^2-18u+11}{12}\right)\Delta^4 y_o + ...\right] \text{ where } u = \frac{x-x_o}{h}$$

4. **Write the Newton's backward formula for the first and second order derivatives at the value $x = x_n$**

Solution:

$$y' = f'(x) = \frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + ...\right]$$

$$y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + ...\right]$$

5. **State Newton's formula to find the derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ at $x \neq x_n$ using backward differences.**

Solution:

$$y' = f'(x) = \frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{(2v+1)}{2!}\nabla^2 y_n + \frac{(3v^2+6v+2)}{3!}\nabla^3 y_n + ...\right]$$

$$y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (v+1)\nabla^3 y_n + \left(\frac{6v^2+18v+11}{12}\right)\nabla^4 y_n + ...\right] \text{where } v = \frac{x-x_n}{h}$$

6. **Find the first derivative of f(x) at x = 0.4 from the following table.**

X	0.1	0.2	0.3	0.4
f(x)	1.10517	1.2214	1.34986	1.49182

Solution: Since x = 0.4 is the final value of the given table, we use Newton's backward formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.1	1.10517			
		0.11623		
0.2	1.2214		0.01223	
		0.12846		0.00127
0.3	1.34986		0.0135	
		0.14196		
0.4	1.49182			

$$\left(\frac{dy}{dx}\right)_{x=0.4} = \left(\frac{dy}{dx}\right)_{v=0} = \frac{1}{h}[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + ...)$$

$$= \frac{1}{0.1}[0.14196 + \frac{1}{2}(0.0135) + \frac{1}{3}(0.00127)) = 1.4913$$

7. **Define Quadrature.**

Solution: The process of evaluating a definite integral from a set of tabulated values of function is called as quadrature.

8. **What is the Geometrical interpretation of Trapezoidal rule?**

Solution: The area of the region enclosed by the curve y = f(x), the x – axis, the ordinates x = a and x = b is approximated by the sum of the area of n trapeziums.

9. **Write the formula for trapezoidal rule.**

Solution: $I = \int_{x_0}^{x_n} f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})]$, where $h = \frac{x_n - x_0}{n}$, n is no. of sub intervals

10. **State Simpson's one-third rule.**

Solution: I= $\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}[(y_0 + y_{2n}) + 2(y_2 + y_4 + ...+ y_{2n-2}) + 4(y_1 + y_3 + ... y_{2n-1})]$

11. **Evaluate $\int_0^6 \frac{1}{1+x^2}dx$ by taking h=1 by Simpson's $\frac{3}{8}^{th}$ rule.**

Solution:

X	0	1	2	3	4	5	6
Y	1	0.5	0.2	0.1	0.058824	0.038462	0.027027

$$\int_a^b ydx = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$$

$$I = \int_0^6 \frac{1}{1+x^2}dx = \frac{3}{8}[(1 + 0.027027) + 3(0.5 + 0.2 + 0.058824 + 0.038462) + 2(0.1)] = 1.35708$$

12. **Evaluate $\int_{\frac{1}{2}}^{1} \frac{dx}{x}$ Trapezoidal rule, dividing the range into 4 equal parts.**

Solution: $h = \frac{\text{upper limit} - \text{lower limit}}{\text{no. of intervals}} = \frac{1-\frac{1}{2}}{4} = 0.125$

X	0.5	0.625	0.75	0.875	1
$y = \frac{1}{x}$	2	1.6	1.333	1.142	1

$$I = \int_{x_0}^{x_n} f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})] = \frac{0.125}{2}[(2+1) + 2(1.6 + 1.333 + 1.142)] = 0.696$$

13. **Compare Trapezoidal rule and Simpson's one-third rule.**

Solution:

Trapezoidal rule	Simpson's one-third rule
It has no specific restriction on number of segments. i.e. n = any number	It requires even number of segments and odd number of pointsi.e. n = even
Degree of interpolating polynomial y(x) is one.	Degree of interpolating polynomial y(x) is two.
It is approximated by trapezium	It is approximated by set of parabolas.

14. **What approximation is used in deriving Simpson's rule of integration?**

Solution: Simpson's one third rule approximates the area of two adjacent strips, by the area under a quadratic parabola.

15. **What is the order of error in Simpson's 1/3rd rule?**

Solution:

$$\text{Error} = \frac{-(b-a)}{180} h^4 y^{iv}(\ \xi\ )\ ,\quad a \le x \le b\ ;\quad \text{Order of the error} = h^4$$

16. **Apply Simpson's 1/3 rule to evaluate** $\int_0^2 \frac{dx}{x^2+x+1}$ **having h = 0.25**

Solution:

X	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
$y=\frac{1}{x^2+x+1}$	1.000	0.7619	0.5714	0.4324	0.3333	0.2623	0.2105	0.1720	0.1428

By Simpson's 1/3rd rule. $\int_a^b ydx = \frac{h}{3}[(y_0+y_8)+4(y_1+y_3+y_5)+2(y_2+y_4+y_6)]$

$$= \frac{0.25}{3}[(1+0.1428)+2(0.5714+0.3333+0.2105)+4(0.7619+0.4324+0.2623+0.1720)] = 0.8129$$

17. **Given the following data, evaluate** $\int_0^4 e^x dx$ **by using trapezoidal rule with h = 1.**

Solution:

X	0	1	2	3	4
e^x	1	2.72	7.39	20.09	54.60

$$I = \int_{x_0}^{x_n} f(x)dx = \frac{h}{2}[(y_0+y_4)+2(y_1+y_2+y_3)] = \frac{1}{2}\left[(1+54.60)+2(2.72+7.39+20.09)\right] = 58$$

18. **Evaluate** $\int_{-1}^{1}\frac{1}{x+3}\,dx$ **by Gaussian two-point formula**

Solution: $I = \int_{-1}^{1}\frac{1}{x+3}\,dx = f\left(-\frac{1}{\sqrt{3}}\right)+f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{\left(3-\frac{1}{\sqrt{3}}\right)}+\frac{1}{\left(3+\frac{1}{\sqrt{3}}\right)} = 0.6903$

19. **State Gaussian three-point quadrature formula.**

Solution: $\int_{-1}^{1} f(x)\,dx = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right)+f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9}f(0)$

20. **Write the Simpson's rule to evaluate $\iint f(x,y)dxdy$**

Solution: Simpson's rule for Double Integration is

I= $\frac{hk}{9}$ {sum of values of 'f' at the four corners+2(sum of the values of 'f' at odd position on the boundary except corners) + 4(sum of the values of 'f' at even position on the boundary except corners) + 4(sum of the values of 'f' at odd position on odd row) + 8(sum of the values of 'f' at even position on odd row) + 8(sum of the values of 'f' at odd position on even row) + 16(sum of the values of 'f' at even position on even row)}

PART – B

1. **(i). Find the first, second derivatives of the function f(x) at x = 1.5**

X	**1.5**	**2.0**	**2.5**	**3.0**	**3.5**	**4.0**
f(x)	**3.375**	**7.0**	**13.625**	**24.0**	**38.875**	**59.0**

Solution:

x	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.5	3.375				
		3.625			
2.0	7.0		3.0		
		6.625		0.75	
2.5	13.625		3.75		0
		10.375		0.75	
3.0	24.0		4.5		0
		14.875		0.75	
3.5	38.875		5.25		
		20.125			

Derivatives of Newton's forward formula is $x_0 = 1.5, h = 0.5$

$$f'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - ......\right]$$

$$f'(1.5) = \frac{1}{0.5}\left[3.625 - \frac{1}{2}(3.0) + \frac{1}{3}(0.75)\right] = 4.75$$

$$f''(x_0) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - ......\right]$$

$$f''(1.5) = \frac{1}{(0.5)^2}\left[3 - 0.75\right] = 9$$

$$f'''(x_0) = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 \ldots\ldots\right]$$

$$f'''(1.5) = \frac{1}{(0.5)^3}\left[0.75\right] = 6$$

(ii). Given the data below, find y'(6)

X	0	2	3	4	7	9
Y	4	26	58	112	466	922

Solution:

The divided difference table is

x	y	Δ	Δ²	Δ³	Δ⁴
0	4				
		11			
2	26		7		
		32		1	
3	58		11		0
		54		1	
4	112		16		0
		118		1	
7	446		22		
		228			
9	922				

Newton's divided difference formula.

$$y = f(x) = f(x_o) + (x - x_o) f(x_o, x_1) + (x - x_o)(x - x_1) f(x_o, x_1, x_2) + \ldots\ldots$$

$$= 4 + (x - 0)(11) + (x - 0)(x - 2)(7) + (x - 0)(x - 2)(x - 3)(1)$$

$$y = 4 + 11x + 7x^2 - 14x + x^3 - 5x^2 + 6xx^3 + 2x^2 + 3x + 4$$

$$\Rightarrow y' = 3x^2 + 4x + 3\,;$$

$$\text{put } x = 6,\ y'(6) = 3(6)^2 + 4(6) + 3 = 135$$

2. **(i). Find the first two derivatives of $(x)^{\frac{1}{3}}$ at x=50 and x= 56, for the following table.**

X	50	51	52	53	54	55	56
y = $(x)^{\frac{1}{3}}$	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

Solution:

x	y	Δy	Δ^2y	Δ^3y
50	2.6840			
		0.0244		
51	3.7084		-0.0003	
		0.0241		0
52	3.7325		-0.0003	
		0.0238		0
53	3.7563		-0.0003	
		0.0235		0
54	3.7798		-0.0003	
		0.0232		0
55	3.8030		-0.0003	
		0.0229		
56	3.8259			

By Newton forward formula,

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h}[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 + ...]$$

$$\left(\frac{dy}{dx}\right)_{x=50} = \frac{1}{1}[0.0244 - \frac{1}{2}(-0.0003) + \frac{1}{3}(0)] = 0.02455$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=50} = \frac{1}{(1)^2}[-0.0003] = -0.0003$$

Newton backward formula is

$$\left(\frac{dy}{dx}\right)_{x=56} = \left(\frac{dy}{dx}\right)_{v=0} = \frac{1}{h}[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + ...]$$

$$= \frac{1}{1}[0.0229 + \frac{1}{2}(-0.0003) + \frac{1}{3}(0)] = 0.02275$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=56} = \frac{1}{h^2}[\nabla^2 y_n + \nabla^3 y_n.........] = \frac{1}{(1)^2}[-0.0003] = -0.0003$$

(ii). Given the following values of x and y, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at x = 1.05

X	**1**	**1.05**	**1.10**	**1.15**	**1.20**	**1.25**	**1.30**
f(x)	**1**	**1.025**	**1.049**	**1.072**	**1.095**	**1.118**	**1.140**

Solution:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1	1						
		0.025					
1.05	1.025		-0.001				
		0.024		0			
1.1	1.049		-0.001		0.001		
		0.023		0.001		-0.002	
1.15	1.072		0		-0.001		0.002
		0.023		0		0	
1.2	1.095		0		-0.001		
		0.023		-0.001			
1.25	1.118		-0.001				
		0.022					
1.3	1.14						

$$y' = f'(x) = \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_o + \frac{(2u-1)}{2!}\Delta^2 y_o + \frac{(3u^2-6u+2)}{3!}\Delta^3 y_o + ...\right]$$

$$y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_o + (u-1)\Delta^3 y_o + \left(\frac{6u^2-18u+11}{12}\right)\Delta^4 y_o + ...\right] \text{ where } u = \frac{x-x_o}{h}$$

$$\text{where } u = \frac{1.05-1}{0.05} = 1$$

$$f'(x) = \frac{1}{0.05}\left[\begin{array}{l} 0.024 + \frac{(2(1)-1)}{2!}(-0.001) + \frac{(3(1)^2-6(1)+2)}{3!}(0.001) \\ + \frac{(4(1)^3-18(1)^2+22(1)-6)}{4!}(-0.001) \end{array}\right]$$

$$f'(x) = 0.4717$$

$$f''(x) = \frac{1}{(0.05)^2}\left[-0.001 + (1)(0.001) + \left(\frac{6(1)^2 - 18(1) + 11}{12}\right)(-0.001)\right]$$

$$f''(x) = 0.0333$$

3. **(i). The table given below reveals the velocity v of a body during the time 't'. Find its acceleration at t = 1.1**

T	**1.0**	**1.1**	**1.2**	**1.3**	**1.4**
V	**43.1**	**47.7**	**52.1**	**56.4**	**60.8**

Solution:

t	V	Δv	Δ^2v	Δ^3v	Δ^4v
1.0	43.1				
		4.6			
1.1	47.7		-0.2		
		4.4		0.1	
1.2	52.1		-0.1		0.1
		4.3		0.2	
1.3	56.4		-0.1		
		4.4			
1.4	60.8				

Acceleration = $\frac{dv}{dt}$

Use Newton's forward difference formula,

$$\frac{dv}{dt} = \frac{1}{h}\left[\Delta y_o + \frac{(2u-1)}{2!}\Delta^2 y_o + \frac{(3u^2-6u+2)}{3!}\Delta^3 y_o + ...\right]$$

$$u = \frac{x - x_0}{h} = \frac{1.1 - 1.0}{0.1} = 1$$

$$\frac{dv}{dt} = \frac{1}{0.1}\left[4.6 + \frac{(2(1)-1)}{2!}(-0.2) + \frac{(3(1)^2 - 6(1) + 2)}{3!}(0.1) + \frac{(4(1)^3 - 18(1)^2 + 22(1) - 6)}{4!}(0.1)\right]$$

$$\frac{dv}{dt} = 44.9167$$

Acceleration when t = 1.1 is 44.9167

(ii). Find the value of $\log 2^{1/3}$ from $\int_0^1 \frac{x^2}{1+x^3}dx$ using Simpson's $\frac{1}{3}$ rule with $h = 0.25$

Solution: Here h = 0.25 $y = \dfrac{x^2}{1+x^3}$

x	0	0.25	0.5	0.75	1
y	0	0.0615	0.2222	0.3956	0.5

By Simpson's rule,

$$I = \frac{h}{3}\left[(y_0+y_4)+2(y_2)+4(y_1+y_3)\right]$$

$$= \frac{0.25}{3}\left[(0+0.5)+2(0.2222)+4(0.0615+0.3956)\right] = 0.2311$$

Actual Integration:

$$\int_0^1 \frac{x^2}{1+x^3} = \left[\frac{1}{3}\log(1+x^3)\right]_0^1$$

$$= \left[\frac{1}{3}\left[\log(1+(1)^3)-\log(1+(0)^3)\right]\right] = \frac{1}{3}\log 2 = 0.2311$$

(iii). By dividing the range into 10 equal parts, evaluate $\int_0^{\pi} \sin x\, dx$ by using Trapezoidal rule

Verify your results by actual integration.

Solution:

Here a = 0 ; b = π ; n =10

$$h = \frac{b-a}{n} = \frac{\pi - 0}{10} = \frac{\pi}{10}$$

x	0	$\frac{\pi}{10}$	$\frac{2\pi}{10}$	$\frac{3\pi}{10}$	$\frac{4\pi}{10}$	$\frac{5\pi}{10}$	$\frac{6\pi}{10}$	$\frac{7\pi}{10}$	$\frac{8\pi}{10}$	$\frac{9\pi}{10}$	π
y = sin x	0	0.3090	0.5878	0.8090	0.9511	1	0.9511	0.8090	0.5878	0.3090	0

By Trapezoidal rule, $\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}[(y_0+y_n)+2(y_1+y_2+y_3+\dots+y_{n-1})]$

$$\int_{x_o}^{x_n} y(x)\, dx = \frac{h}{2}\left[\begin{array}{l}(0+0)+2(0.3090+0.5878+0.8090\\ +0.9511+1+0.9511+0.8090+0.5878+0.3090\end{array}\right] = 1.9843$$

Actual Integration:

$$\int_0^{\pi} \sin x\, dx = \left[-\cos x\right]_0^{\pi} = \left[-\cos\pi + \cos 0\right] = 2$$

4. **(i). Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Romberg's method. Hence obtain an approximate value for π.**

Solution:

H = 0.5 $y = \frac{1}{1+x^2}$

x	0	0.5	1
y	1	0.80	0.50

$$I = \frac{0.5}{2}\left[1.5+2(0.8)\right] = 0.775$$

h = 0.25

x	0	0.25	0.5	0.75	1
y	1	0.9412	0.8	0.64	0.5

$$I = \frac{0.25}{2}\left[1.5+2(0.9412+0.8+0.64)\right] = 0.78280$$

h = 0.125

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	1	0.9846	0.9412	0.8767	0.8	0.7191	0.64	0.5664	0.5

$$I = \frac{0.125}{2}\left[1.5+2(0.9846+0.9412+0.8767+0.8+0.7191+0.64+0.5664)\right]$$

I = 0.784750

Using Romberg's formula for I_1 *and* I_2, we have

$$I = I_2 + \left(\frac{I_2 - I_1}{3}\right) = 0.7854$$

Using Romberg's formula for I_2 *and* I_3, we have

$$I = I_3 + \left(\frac{I_3 - I_2}{3}\right) = 0.7854$$

Evaluation:

W.K.T

$$\int_0^1 \frac{dx}{1+x^2} = 0.7854$$

$$\left[\tan^{-1} x\right]_0^1 = 0.7854$$

$$\left[\tan^{-1} 1 - \tan^{-1} 0\right] = 0.7854$$

$$\frac{\pi}{4} - 0 = 0.7854$$

$$\pi = 3.1416$$

(ii). Evaluate $\int_{0.2}^{1.5} e^{-x^2}dx$ by using three-point Gaussian Quadrature.

Solution: Transform the variable from x to t by the transformation

$$x=\left(\frac{b-a}{2}\right)t+\left(\frac{b+a}{2}\right)$$

here a=0.2 , b=1.5

$x = 0.65t + 0.85 \Rightarrow dx = 0.65dt$

Limit :

x	0.2	1.5
t	-1	1

$$\int_{0.2}^{1.5} e^{-x^2}dx = \int_{-1}^{1} e^{-(0.65t+0.85)^2}(0.65)dt = (0.65)\int_{-1}^{1} e^{-(0.65t+0.85)^2}dt$$

here $f(t)=e^{-(0.65t+0.85)^2} \Rightarrow f(0)=0.4855,\ f\left(-\sqrt{\frac{3}{5}}\right)=0.8869,\ f\left(\sqrt{\frac{3}{5}}\right)=0.1601$

$$\int_{-1}^{1} e^{-(0.65t+0.85)^2}dt=\frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right)+f\left(\sqrt{\frac{3}{5}}\right)\right]+\frac{8}{9}f(0)=1.0133$$

$$\therefore \int_{0.2}^{1.5} e^{-x^2}dx=(0.65)(1.0133)=0.65865$$

5.

(i). Evaluate $\int_{1}^{1.4}\int_{2}^{2.4}\frac{dx\,dy}{xy}$ by using Trapezoidal rule, verify your results by actual integration.

Solution: By Trapezoidal rule

I = $\int_c^d\int_a^b f(x,y)\,dx\,dy=\frac{hk}{4}$ {sum of values of f at the four corners

+2(sum of the values of f at the remaining nodes on the boundary)

+4(sum of the values of f at the interior nodes)}

Where $h=\frac{b-a}{n};k=\frac{d-c}{n}$

Let $f(x,y)=\frac{1}{xy}$ $\quad h=\frac{2.4-2}{4}=0.1$ $\quad k=\frac{1.4-1}{4}=0.1$

y/x	2	2.1	2.2	2.3	2.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.1	0.4545	0.4329	0.4132	0.3953	0.3788
1.2	0.4167	0.3968	0.3788	0.3623	0.3472
1.3	0.3846	0.3663	0.3497	0.3344	0.3205
1.4	0.3571	0.3401	0.3247	0.3106	0.2976

$$\int_1^{1.4}\int_2^{2.4}\frac{dx\,dy}{xy}=\frac{0.1\times 0.1}{4}\left[\begin{array}{r}(0.5+0.4167+0.2976+0.3571)+2\begin{pmatrix}0.4762+0.4545+0.4348\\ +0.3788+0.3472+0.3205\\ +0.3106+0.3247+0.3401\\ +0.3846+0.4167+0.4545\end{pmatrix}\\ +4\begin{pmatrix}0.4329+0.4132+0.3953+0.3968+\\ 0.3788+0.3623+0.3663+0.3497+0.3344\end{pmatrix}\end{array}\right]$$

$$\int_1^{1.4}\int_2^{2.4}\frac{dx\,dy}{xy}=0.0614$$

Actual Integration:

$$\int_1^{1.4}\int_2^{2.4}\frac{dx\,dy}{xy}=\int_1^{1.4}\frac{dy}{y}\int_2^{2.4}\frac{dx}{x}$$

$$=\left[\log y\right]_1^{1.4}\left[\log x\right]_2^{2.4}=(\log 1.4-\log 1)(\log 2.4-\log 2)$$

$$=(\log 1.4)(\log 1.2)$$

$$\int_1^{1.4}\int_2^{2.4}\frac{dx\,dy}{xy}=0.0613$$

(ii). Evaluate $\int_1^2\int_3^4\frac{dxdy}{(x+y)^2}$ by using Trapezoidal and Simpson's rule with h = k = 0.5

Solution:

y\x	3	3.5	4
1	0.0625	0.0494	0.04
1.5	0.0494	0.04	0.0331
2	0.04	0.0331	0.0278

Trapezoidal rule:

$$I=\frac{hk}{4}\left\{\begin{array}{l}\text{sum of values of at the four corners}+\\ 2(\text{sum of the values ofat the remaining nodes on the boundary})\\ +4(\text{sum of the values of at the interior nodes})\end{array}\right\}$$

$$=\frac{0.5\times 0.5}{4}[(0.0625+0.04+0.04+0.0278)$$

$$+2(0.0494+0.0331+0.0331+0.0494)+4(0.04)]$$

$$=0.0413$$

	Simpson's rule: I= $\frac{hk}{9}$ {sum of values of f at the four corners+2(sum of the values of f at odd position on the boundary except corners) + 4(sum of the values of f at even position on the boundary except corners) + 4(sum of the values of f at odd position on odd row) + 8(sum of the values of f at even position on odd row) + 8(sum of the values of f at odd position on even row) + 16(sum of the values of f at even position on even row)} $= \frac{0.5\times 0.5}{9}[(0.0625+0.04+0.04+0.0278)$ $+4(0.0494+0.0331+0.0331+0.0494)+16(0.04)]$ $\mathrm{I}=0.0408$

UNIT – IV

Interpolation ,Numerical Differentiation and Numerical Integration

Interpolating Function:

Let a set of tabular values of a function y = f(x) , where the explicit nature of the function is not known, then f(x) is replaced by a similar function $\emptyset(x)$, such that f(x) and $\emptyset(x)$ agree with set of tabulated points. Any other value may be calculated from $\emptyset(x)$.This function $\emptyset(x)$ is known as an interpolating function

Inverse interpolation

It is the process of finding the values of x corresponding to a value of y, not present in the table.

Remark:

The process of computing the value of a function inside the given range is called interpolation. The process of computing the value of a function outside the given range is called extrapolation

Interpolation with unequal intervals:

Lagrange's interpolation:

Suppose $x_1, x_2, x_3,\ldots. x_n$, *and the corresponding* $y_1 , y_2 , y_3 \ldots y_n$ *given then*

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3).......(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3).......(x_0-x_n)} y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3).......(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3).......(x_1-x_n)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3).......(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3).......(x_2-x_n)} y_2$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots +$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3).......(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)(x_n-x_3).......(x_n-x_{n-1})} y_n$$

Remark:

1. Newton's formula can be used only when the values of the independent variable x are equally spaced. But Lagrange's interpolation formula can be used whether the values of the independent variable x are equally spaced or not. Lagrange's formula can be used for inverse interpolation also, while Newton's formula cannot be used.

2. *Lagrange's interpolation formula can be used for equal intervals.*

The Lagrange's formula for inverse interpolation.

Sol.

$$x = f(y) = \frac{(y-y_1)(y-y_2)(y-y_3).......(y-y_n)}{(y_0-y_1)(y_0-y_2)(y_0-y_3).......(y_0-y_n)}x_0$$

$$+\frac{(y-y_0)(y-y_2)(y-y_3).......(y-y_n)}{(y_1-y_0)(y_1-y_2)(y_1-y_3).......(y_1-y_n)}x_1$$

$$+\frac{(y-y_0)(y-y_1)(y-y_3).......(y-y_n)}{(y_2-y_0)(y_2-y_1)(y_2-y_3).......(y_2-y_n)}x_2$$

$$+.................................+$$

$$+\frac{(y-y_0)(y-y_1)(y-y_2)(y-y_3).......(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)(y_n-y_2)(y_n-y_3).......(y_n-y_{n-1})}x_n$$

Examples:

1. Find the quadratic polynomial that fits y(x) = x^4 at x = 0, 1, 2.

Sol. The following data is

x :	0	1	2
y=x^4 :	0	1	16

By Lagrange's formula

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1$$

$$+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

$$y = f(x) = \frac{(x-1)(x-2)}{(0-1)(0-2)}.0 + \frac{(x-0)(x-2)}{(1-0)(1-2)}.1 + \frac{(x-0)(x-1)}{(2-0)(2-1)}.16$$

$$y = -x(x-2) + 8x(x-1)$$

$$y(x) = 7x^2 - 6x.$$

2. Use Lagrange's formula to find the quadratic polynomial that takes these values

x	***:***	***0***	***1***	***3***
y	***:***	***0***	***1***	***0***

Then find y(2).

Sol.

By Lagrange's formula

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

$$y = f(x) = \frac{(x-1)(x-3)}{(0-1)(0-3)}.0 + \frac{(x-0)(x-3)}{(1-0)(1-3)}.1 + \frac{(x-0)(x-1)}{(3-0)(3-1)}.0$$

$$y(x) = \frac{x^2-3x}{-2}$$

Hence y(2) = 1.

3. Using Lagrange's interpolation formula calculate the profit in the year 2000 from the following data

Year	***:***	***1997***	***1999***	***2001***	***2002***
Profit in lakhs of Rs.	:	***43***	***65***	***159***	***248***

Sol. Lagrange's interpolation formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

Here x = 2000

$$\therefore\ y = f(x) = \frac{(2000-1999)(2000-2001)(2000-2002)}{(1997-1999)(1997-2001)(1997-2002)}43$$

$$+\frac{(2000-1997)(2000-2001)(2000-2002)}{(2000-1997)(2000-2001)(2000-2002)}65$$

$$+\frac{(2000-1997)(2000-1999)(2000-2002)}{(2001-1997)(2001-1999)(2001-2002)}159$$

$$+\frac{(2000-1997)(2000-1999)(2000-2001)}{(2002-1997)(2002-1999)(2002-2001)}248$$

$$y = f(x) = \frac{(1)(-1)(-2)}{(-2)(-4)(-5)}43+\frac{(3)(-1)(-2)}{(2)(-2)(-3)}65$$

$$+\frac{(3)(1)(-2)}{(4)(2)(-1)}159+\frac{(3)(1)(-1)}{(5)(3)(1)}248$$

y = – 2.15 + 32.5 + 119.25 – 49.6

y = 100.

4. Given the values x : 14 17 31 35

f(x) : 68.7 64.0 44.0 39.1

Find the value of f(x) when x = 27.

Sol.

Lagrange's interpolation formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

Given x = 27

$$\therefore\ y=f(x)=\frac{(27-17)(27-31)(27-35)}{(14-17)(14-31)(14-35)}68.7$$

$$+\frac{(27-14)(27-31)(27-35)}{(17-14)(17-31)(17-35)}64$$

$$+\frac{(27-14)(27-17)(27-35)}{(31-14)(31-17)(31-35)}44$$

$$+\frac{(27-14)(27-17)(27-31)}{(35-14)(35-17)(35-31)}39.1$$

$$y=f(x)=\frac{(10)(-4)(-8)}{(-3)(-17)(-21)}68.7+\frac{(13)(-4)(-8)}{(3)(-14)(-18)}64$$

$$+\frac{(13)(10)(-8)}{(17)(14)(-4)}44+\frac{(13)(10)(-4)}{(21)(18)(4)}39.1$$

f(x) = –20.5266 + 35.2169 + 48.0672 – 13.4471

(i.e.) f(x) = 49.3104

5. Using Lagrange's interpolation formula fit a polynomial to the following data

x :	***–1***	***0***	***2***	***3***
y :	***–8***	***3***	***1***	***12***

and hence find y at x = 1.5

Sol.

Lagrange's interpolation formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$y = f(x) = \frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)}(-8)$$

$$+ \frac{(x+1)(x-2)(x-3)}{(0+1)(0-2)(0-3)}(3)$$
$$+ \frac{(x+1)(x-0)(x-3)}{(2+1)(2-0)(2-3)}(1)$$
$$+ \frac{(x+1)(x-0)(x-2)}{(3+1)(3-0)(3-2)}(12)$$

$$y = \frac{2}{3}x(x^2-5x+6) + \frac{1}{2}(x+1)(x^2-5x+6) - \frac{1}{6}x(x^2-2x-3)$$

$$+ x(x^2-x-2)$$

$$y = \frac{1}{6}[(4x^3-20x^2+24x)+(3x^3-12x^2+3x+18)$$

$$+(-x^3+2x^2+3x)+(6x^3-6x^2-12x)]$$

$$y = \frac{1}{6}(12x^3-36x^2+18x+18)$$
$$y = 2x^3-6x^2+3x+3.$$

$y(1.5) = 2(1.5)^3 – 6(1.5)^2 + 3(1.5) + 3$

$= 0.75$

6. Given $\log_{10} 654 = 2.8156$**,** $\log_{10} 658 = 2.8182$**,** $\log_{10} 659 = 2.8189$**,** $\log_{10} 661 = 2.8202$ ***. Find*** $\log_{10} 656$ ***by using Lagrange's formula.***

Sol.

Let y = $\log_{10} x$

The following data is

x :	654	658	659	661
$\log_{10} x$:	2.8156	2.8182	2.8189	2.8202

Here x = 656.

Lagrange's interpolation formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$y = \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)} 2.8156 + \frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)} 2.8182 + \frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)} 2.8189 + \frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)} 2.8202$$

$$y = \frac{(-2)(-3)(-5)}{(-4)(-5)(-7)} 2.8156 + \frac{(2)(-3)(-5)}{(4)(-1)(-3)} 2.8182 + \frac{(2)(-2)(-5)}{(5)(1)(-2)} 2.8189 + \frac{(2)(-2)(-3)}{(7)(3)(2)} 2.8202$$

y = 0.6033 + 7.0455 – 5.6378 + 0.8058

y = 2.8168

(i.e.) $\log_{10} 656$ = 2.8168

7.Find the polynomial f(x) by using Lagrange's formula and hence find f(3) for

x :	***0***	***1***	***2***	***5***
f(x) :	***2***	***3***	***12***	***147***

Sol. Lagrange's interpolation formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$y = f(x) = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2)$$

$$+ \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3)$$

$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12)$$

$$+ \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

$$y = \frac{-1}{5}(x-1)(x^2-7x+10) + \frac{3}{4}x(x^2-7x+10) - 2x(x^2-6x+5)$$

$$+ \frac{49}{20}x(x^2-3x+2)$$

$$y = \frac{1}{20}[(-4x^3+32x^2-68x+40)+(15x^3-105x^2+150x)$$

$$+(-40x^3+240x^2-200x)+(49x^3-147x^2+98x)]$$

$$y=\frac{1}{20}(20x^3+20x^2-20x+40)$$

$$y=x^3+x^2-x+2.$$

Now, f(3) = 3^3+3^2-3+2

(i.e.) f(3) = 35.

8. Using Lagrange's formula, Prove that

$$y_0=\frac{1}{2}(y_1+y_{-1})-\frac{1}{8}\left[\frac{1}{2}\left(y_{-3}-y_{-1}\right)-\frac{1}{2}\left(y_1-y_3\right)\right]$$

Sol. The following data is

x :	1	–1	–3	3
y :	y_1	y_{-1}	y_{-3}	y_3

Here x = 0.

Lagrange's interpolation formula is

$$y_x=f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

Given x = 0

$$\therefore\ y_0=\frac{(0+1)(0+3)(0-3)}{(1+1)(1+3)(1-3)}y_1+\frac{(0-1)(0+3)(0-3)}{(-1-1)(-1+3)(-1-3)}y_{-1}$$

$$+\frac{(0-1)(0+1)(0-3)}{(-3-1)(-3+1)(-3-3)}y_{-3}+\frac{(0-1)(0+1)(0+3)}{(3-1)(3+1)(3+3)}y_3$$

$$y_0 = \frac{9}{16} y_1 + \frac{9}{16} y_{-1} - \frac{1}{16} y_{-3} - \frac{1}{16} y_3$$

$$y_0 = \left(\frac{1}{2} + \frac{1}{16}\right) y_1 + \left(\frac{1}{2} + \frac{1}{16}\right) y_{-1} - \frac{1}{16} y_{-3} - \frac{1}{16} y_3$$

$$y_0 = \frac{1}{2}\left(y_1 + y_{-1}\right) - \frac{1}{8}\left[\frac{1}{2}\left(y_{-3} - y_{-1}\right) - \frac{1}{2}\left(y_1 - y_3\right)\right]$$

9. Applying Lagrange's formula to find the roots of the equation f(x) = 0 when f(30) = –30, f(34) = –13, f(38) = 3, f(42) = 18.

Sol.

To find x [(i.e.) the roots of the equation f(x) = 0], we have to use Lagrange's inverse interpolation formula.

The following data is

x :	30	34	38	42
f(x) :	–30	–13	3	18

Lagrange's inverse interpolation formula is

$$x = f(y) = \frac{(y - y_1)(y - y_2)(y - y_3)}{(y_0 - y_1)(y_0 - y_2)(y_0 - y_3)} x_0$$

$$+ \frac{(y - y_0)(y - y_2)(y - y_3)}{(y_1 - y_0)(y_1 - y_2)(y_1 - y_3)} x_1$$

$$+ \frac{(y - y_0)(y - y_1)(y - y_3)}{(y_2 - y_0)(y_2 - y_1)(y_2 - y_3)} x_2$$

$$+ \frac{(y - y_0)(y - y_1)(y - y_2)}{(y_3 - y_0)(y_3 - y_1)(y_3 - y_2)} x_3$$

Here y = 0 [since given f(x) = 0 (i.e.) y = 0]

$$\therefore\ x = \frac{(0 + 13)(0 - 3)(0 - 18)}{(-30 + 13)(-30 - 3)(-30 - 18)} 30$$

$$+\frac{(0+30)(0-3)(0-18)}{(-13+30)(-13-3)(-13-18)}34$$

$$+\frac{(0+30)(0+13)(0-18)}{(3+30)(3+13)(3-18)}38$$

$$+\frac{(0+30)(0+13)(0-3)}{(18+30)(18+13)(18-3)}42$$

$$x=-\frac{21060}{26928}+\frac{55080}{8432}+\frac{266760}{7920}-\frac{49140}{22320}$$

$$x = -0.7821 + 6.5323 + 33.6818 - 2.2016$$

$$x = 37.2304$$

Newton’s divided difference table:

Let y=f(x) which takes the values $f(x_0), f(x_1), \dots, f(x_n)$ corresponding to the arguments $x_0, x_1, \dots x_n$

$$f(x_0,x_1)=\frac{f(x_1)-f(x_0)}{x_1-x_0}=\frac{f(x_0)-f(x_1)}{x_0-x_1}= f(x_1,x_0)=\Delta f$$

The second divided difference is

$$f(x_0,x_1,x_2)=\frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0}=\Delta^2 f$$

The third divided difference is

$$f(x_0,x_1,x_2,x_3)=\frac{f(x_1,x_2,x_3)-f(x_0,x_1,x_2)}{x_2-x_0}=\Delta^3 f$$

x	$y=f(x)$	Δ **f**	Δ^2f	Δ^3f
X_0	$f(x_0)$			
		$f(x_0,x_1)$		
x_1	$f(x_1)$		$f(x_0,x_1,x_2)$	
		$f(x_1,x_2)$		$f(x_0,x_1,x_2,x_3)$
x_2	$f(x_2)$		$f(x_1,x_2,x_3)$	
		$f(x_2,x_3)$		
x_3	$f(x_3)$			

Examples:

1. ***Given*** $u_0 = 1, u_1 = 11, u_2 = 21, u_3 = 28, u_4 = 29. Find \Delta^4 u_0.$

Sol.

x	y = u_x	Δu_x	$\Delta^2 u_x$	$\Delta^3 u_x$	$\Delta^4 u_x$
0	1				
		10			
1	11		0		
		10		–3	
2	21		–3		0
		7		–3	
3	28		–6		
		1			
4	29				

Hence $\Delta^4 u_0 = 0.$

2. ***If*** $u_1 = 1, u_3 = 17, u_4 = 43, u_5 = 89.$ ***Find the value of*** u_2.

Sol.

Let the missing term be y_1.

x	y = u_x	Δu_x	$\Delta^2 u_x$	$\Delta^3 u_x$	$\Delta^4 u_x$
1	1				
		y_1–1			
2	y_1		–2y_1+18		
		17–y_1		3y_1–61	
3	17		y_1-43		–4y_1+124
		26		–y_1+63	
4	43		20		
		46			
5	89				

By assumption, we have

$$-4y_1 + 124 = 0$$

$$y_1 = 31.$$

3.Find the second divided differences with arguments a,b,c if f(x) = 1/x.

Sol. The divided difference table is

x	y = 1/x	Δy	$\Delta^2 y$
a	1/a		
		–1/ab	
			1/abc
b	1/b		
		–1/bc	
c	1/c		

4.If f(x) = 1/x², find f(a,b) and f(a,b,c) by using divided differences.

Sol. The divided difference table is

x	y = $1/x^2$	Δy	$\Delta^2 y$
a	$1/a^2$		
		$-(a+b)/a^2b^2$	
b	$1/b^2$		$(ab + bc + ca) / a^2b^2c^2$
		$-(b+c)/b^2c^2$	
c	$1/c^2$		

Newton's Divided difference Formula for Unequal Intervals:

Suppose $y = f(x)$ takes the values $f(x_0), f(x_1), f(x_2), \ldots. f(x_n)$ for the corresponding arguments $x_0, x_1, x_2, \ldots x_n$ then

$$y = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0 + ..$$

Examples:

1.Using Newton divided difference formula find u(3) given
u(1) = –26, u(2) = 12, u(4) = 256, u(6) = 844.

Sol.

The divided difference table is

x	y = u(x)	Δy	$\Delta^2 y$	$\Delta^3 y$
1	–26			
		38		
			28	
2	12			3
		122		
			43	
4	256			
		294		
6	844			

Newton divided difference formula is

$$y = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0$$

$$+ (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0 + \ldots\ldots\ldots\ldots\ldots$$

Here x = 3

y = –26 + (3 –1).(38) + (3 –1)(3 – 2)(28) + (3 –1)(3 – 2)(3 – 4)(3)

= – 26 + 76 + 56 – 6

= 132 – 32

= 100

(i.e.) u(3) = 100.

2. ***Using Newton divided difference method find f(1.5) using the data f(1.0) = 0.7651977, f(1.3) = 0.6200860, f(1.6) = 0.4554022,***

f(1.9) = 0.2818186, f(2.2) = 0.1103623.

Sol.

The divided difference table is

x	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	0.7651977				
		–0.4837057			
1.3	0.6200860				
			–0.1087338		
		–0.548946		0.0658783	
					0.0018251
1.6	0.4554022				
			–0.0494433		
		–0.578612		0.0680684	
1.9	0.2818186				
			0.0118183		
		–0.571521			
2.2	0.1103623				

Newton divided difference formula is

$$y = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0$$

$$+ (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0$$

$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)\Delta^4 y_0 + \dots\dots\dots\dots\dots$$

Here x = 1.5

y = 0.7651977 + (1.5 – 1)(–0.4837057)

+ (1.5 – 1)(1.5 – 1.3)(–0.1087338)

+ (1.5 – 1)(1.5 – 1.3)(1.5 – 1.6)(0.0658783)

+ (1.5 – 1)(1.5 – 1.3)(1.5 – 1.6)(1.5 – 1.9)(0.0018251)

= 0.7651977 – 0.2418529 – 0.0108734 – 0.0006588 + 0.0000073

= 0.5118199

(i.e.) f(1.5) = 0.5118199

3. ***Given*** $u_0 = -4,\ u_1 = -2,\ u_4 = 220,\ u_5 = 546,\ u_6 = 1148$

Find u_2 ***and*** u_3.

Sol.

The divided difference table is

x	y = u_x	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	–4				
		2			
1	–2				
			18		
		74		9	
					1
4	220				
			63		
		326		15	
5	546				
			138		
		602			
6	1148				

Newton divided difference formula is

$$y = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0$$

$$+ (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0$$

$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)\Delta^4 y_0 + \dots\dots\dots\dots\dots$$

$$y = -4 + (x - 0)(2) + (x - 0)(x - 1)(18)$$

$$+ (x - 0)(x - 1)(x - 4)(9)$$

$$+ (x - 0)(x - 1)(x - 4)(x - 5)(1)$$

$$u_2 = -4 + (2)(2) + (2)(1)(18) + (2)(1)(-2)(9) + (2)(1)(-2)(-3)(1)$$

= – 4 + 4 + 36 – 36 + 12

= 12.

u_3 = – 4 +(3)(2) + (3)(2)(18) + (3)(2)(–1)(9) + (3)(2)(–1)(–2)(1)

$= -4 + 6 + 108 - 54 + 12$

$= 68.$

Newton’s Forward and Backward Interpolation Formula:

Forward differences:

If $y_0, y_2, y_3, \ldots y_n$ denote the set of values of $y = f(x)$ for the following $x_0, x_1, x_2, \ldots x_n$ then $\Delta y_{n-1} = y_n - y_{n-1}$ and the arguments are equally spaced then the Newton’s Forward Interpolation Formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots\ldots\ldots$$

Where $u = \dfrac{x - x_0}{h}$

Examples:

1 Obtain the interpolation quadratic polynomial for the given data by using Newton forward difference formula

X : 0 2 4 6

Y : -3 5 21 45

Sol. The difference table is

x	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$
0	-3			
		8		
			8	
2	5			0
		16		
			8	
4	21			
		24		
6	45			

Newton Forward Interpolation formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots$$

where $u = \frac{x - x_0}{h} = \frac{x-0}{2} = \frac{x}{2}$

$$y = -3 + (x/2)(8) + \frac{(x/2)(x/2-1)}{2!}(8) + 0$$

$$y = -3 + 4x + x(x-2)$$

$y = x^2 + 2x - 3.$

2.Using Newton's Forward Interpolation formula find the polynomial f(x) satisfying the following data. Hence find f(2).

x :	***0***	***5***	***10***	***15***
f(x) :	***14***	***379***	***1444***	***3584***

Sol.

Let y = f(x).

The difference table is

x	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$
0	14			
		365		
			700	
5	379			375
		1065		
			1075	
10	1444			
		2140		
15	3584			

Newton Forward Interpolation formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots$$

where $u = \frac{x - x_0}{h} = \frac{x-0}{5} = \frac{x}{5}$

$$y = 14 + \frac{x}{5}(365) + \frac{(x/5)(x/5-1)}{2!}(700) + \frac{(x/5)(x/5-1)(x/5-2)}{3!}(375)$$

$$= 14 + 73x + x(x-5)(14) + x(x-5)(x-10).\frac{1}{2}$$

$$(i.e.)\, f(x) = \frac{1}{2}\left[x^3 + 13x^2 + 56x + 28\right]$$

$$\therefore\ f(2) = \frac{1}{2}\left[2^3 + 13(2)^2 + 56(2) + 28\right] = 100.$$

3. Construct Newton's forward interpolation polynomial for the following data.

x :	***4***	***6***	***8***	***10***
y :	***1***	***3***	***8***	***16***

Use it to find the value of y for x = 5.

Sol.

The difference table is

x	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1			
		2	3	
6	3			0
		5	3	
8	8			
		8		
10	16			

Newton Forward Interpolation formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \text{............}$$

where $u = \frac{x - x_0}{h} = \frac{x-4}{2}$

$$y = 1 + \frac{x-4}{2}(2) + \frac{1}{2!}\left(\frac{x-4}{2}\right)\left(\frac{x-4}{2} - 1\right)(3) + 0$$

$$= 1 + (x-4) + \frac{3}{8}(x-4)(x-6)$$

$$= \frac{8 + 8x - 32 + 3(x^2 - 10x + 24)}{8} = \frac{1}{8}\left(3x^2 - 22x + 48\right)$$

$$\therefore\ y(5) = \frac{1}{8}\left[3(5)^2 - 22(5) + 48\right] = \frac{13}{8} = 1.625$$

4.Given $\sin 45^0 = 0.7071, \sin 50^0 = 0.7660, \sin 55^0 = 0.8192, \sin 60^0 = 0.8660$ ***Find sin 52^0 by Newton's formula.***

Sol.

To find sin 52^0 , we use Newton's forward formula. Let y = sinx^0

The difference table is

x	y = sinx^0	Δy	$\Delta^2 y$	$\Delta^3 y$
45	0.7071			
		0.0589		
			-0.0057	
50	0.7660			-0.0007
		0.0532		
			-0.0064	
55	0.8192			
		0.0468		
60	0.8660			

Newton Forward Interpolation formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + ...$$

where $u = \frac{x - x_0}{h} = \frac{52-45}{5} = \frac{7}{5} = 1.4$

$$y = 0.7071 + (1.4)(0.0589) + \frac{(1.4)(1.4-1)}{2!}(-0.0057)$$

$$+\frac{(1.4)(1.4-1)(1.4-2)}{3!}(-0.0007)$$

y = 0.7880

(i.e.) sin 52^0 = 0.7880

5. From the following data, estimate the number of persons earning weekly wages between 60 and 70 rupees.

Wage (in Rs.)	***Below 40***	***40 – 60***	***60 – 80***	***80 – 100***	***100 – 120***
No. of person (in thousands)	***250***	***120***	***100***	***70***	***50***

Sol.

The difference table is

Wage x	No. of persons y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	250				
		120			
Below 60	370		-20		
		100		-10	
Below 80	470		-30		20
		70		10	
Below 100	540		-20		
		50			
Below 120	590				

Let us calculate the number of persons whose weekly wages below 70.

So we will use Newton's forward formula.

Newton Forward Interpolation formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots\dots\dots\dots$$

where $u = \frac{x - x_0}{h} = \frac{70-40}{20} = 1.5$

$$y = 250 + (1.5)(120) + \frac{(1.5)(1.5-1)}{2!}(-20) + \frac{(1.5)(1.5-1)(1.5-2)}{3!}(-10)$$

$$+ \frac{(1.5)(1.5-1)(1.5-2)(1.5-3)}{4!}(20)$$

$y = 423.59 \approx 424.$

$\therefore$ Number of person whose weekly wages below 70 = 424

Number of person whose weekly wages below 60 = 370

$$\left.\begin{array}{l}\therefore \text{Number of persons whose weekly} \\ \text{wages between Rs.60 and Rs.70}\end{array}\right\} = 424 - 370 = 54 \text{ } \textit{thousands.}$$

6. *From the following table, find the value of tan 45 ^{0}15' by Newton's Forward Interpolation formula.*

x ° :	***45***	***46***	***47***	***48***	***49***	***50***
tan x ° :	***1***	***1.03553***	***1.07237***	***1.11061***	***1.15037***	***1.19175***

Sol.

x^o	y = tan x^o	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
45	1					
		0.03553				
46	1.03553					
			0.00131			
		0.03684		0.00009		
					0.00003	
47	1.07237					
			0.00140			
		0.03824		0.00012		-0.00005
					-0.00002	
48	1.11061					
			0.00152			
		0.03976		0.00010		
49	1.15037					
			0.00162			
		0.04138				
50	1.19175					

Newton Forward Interpolation formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots$$

where $u = \frac{x - x_0}{h} = \frac{45^\circ 15' - 45^\circ}{1^\circ} = 0.25$

$$y = 1 + (0.25)(0.03553) + \frac{(0.25)(0.25-1)}{2!}(0.00131)$$

$$+ \frac{(0.25)(0.25-1)(0.25-2)}{3!}(0.00009)$$

$$+ \frac{(0.25)(0.25-1)(0.25-2)(0.25-3)}{4!}(0.00003)$$

$$+ \frac{(0.25)(0.25-1)(0.25-2)(0.25-3)(0.25-4)}{5!}(-0.00005)$$

$$y = 1 + 0.00888 - 0.00012 + 0.0000049 - ....$$

(i.e.) tan 45 ^{0}15' = 1.00876

Backward differences:

If y_0, y_2, y_3,…y_n denote the set of values of $y = f(x)$ for the following x_0, x_1, x_2, …x_n then $\nabla y_{n-1} = y_n - y_{n-1}$ and the arguments are equally spaced then the Newton's Backward Interpolation Formula is

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + ...$$

Where $u = \frac{x - x_n}{h}$

Examples.

1. Given x : 0 0.1 0.2 0.3 0.4

e^x : 1 1.1052 1.2214 1.3499 1.4918

Find the value of y = e^x when x = 0.38.

Sol.

To find y = e^x when x = 0.38, we use Newton's Backward formula .

x	y = e^x	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0	1				
		0.1052			
0.1	1.1052				
			0.011		
		0.1162		0.0013	
					-0.0002
0.2	1.2214				
			0.0123		
		0.1285		0.0011	
0.3	1.3499				
			0.0134		
		0.1419			
0.4	1.4918				

Newton Backward Interpolation formula is

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \ldots\ldots\ldots\ldots$$

where $u = \frac{x - x_n}{h} = \frac{0.38 - 0.4}{0.1} = -0.2$

$$y = 1.4918 + (-0.2)(0.1419) + \frac{(-0.2)(-0.2+1)}{2!}(0.0134)$$

$$+ \frac{(-0.2)(-0.2+1)(-0.2+2)}{3!}(0.0011)$$

$$+ \frac{(-0.2)(-0.2+1)(-0.2+2)(-0.2+3)}{4!}(-0.0002)$$

y = 1.4623

2. The following data are taken from the steam table

Temp ^{0} c : 140 150 160 170 180

Pressure kg f/cm^2 : 3.685 4.854 6.302 8.076 10.225

Find the pressure at temperature t = 175^0.

Sol. To find the pressure f(t) at temperature t = 175^0 , we use Newton's Backward formula.

The difference table is

t	y = f(t)	$\nabla f(t)$	$\nabla^2 f(t)$	$\nabla^3 f(t)$	$\nabla^4 f(t)$
140	3.685				
		1.169			
150	4.854				
			0.279		
		1.448		0.047	
					0.002
160	6.302				
			0.326		
		1.774		0.049	
170	8.076				
			0.375		
		2.149			
180	10.225				

Newton Backward Interpolation formula is

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \text{............}$$

where $u = \dfrac{x - x_n}{h} = \dfrac{175 - 180}{10} = -0.5$

$$y = 10.225 + (-0.5)(2.149) + \frac{(-0.5)(-0.5+1)}{2!}(0.375)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}(0.049)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{4!}(0.002)$$

y = 9.1005

3.Use Newton Backward formula to construct an interpolating polynomial of degree 3 for the data : f(– 0.75) = – 0.07181250, f(– 0.5) = – 0.024750, f(– 0.25) = 0.33493750, f(0) = 1.10100 Hence find $f\left(-\frac{1}{3}\right)$.

Sol. The difference table is

x	y = f(x)	∇y	$\nabla^2 y$	$\nabla^3 y$
–0.75	–0.07181250			
		0.0470625		
			0.312625	
–0.5	–0.024750			0.09375
		0.3596875		
			0.406375	
–0.25	0.33493750			
		0.7660625		
0	1.10100			

Newton Backward Interpolation formula is

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \ldots\ldots\ldots\ldots$$

where $u = \frac{x - x_n}{h} = \frac{x-0}{0.25} = 4x$

$$y = 1.10100 + 4x(0.7660625) + \frac{(4x)(4x+1)}{2!}(0.406375)$$

$$+\frac{(4x)(4x+1)(4x+2)}{3!}(0.09375)$$

$$y = 1.10100 + 3.06425x + 3.251x^2 + 0.81275x + x^3 + 0.75x^2 + 0.125x$$

$$y = f(x) = x^3 + 4.001x^2 + 4.002x + 1.10100$$

$$\therefore\ f\left(-\frac{1}{3}\right) = (-1/3)^3 + 4.001(-1/3)^2 + 4.002(-1/3) + 1.10100$$

$$= 0.174519$$

4. From the given table, the values of y are consecutive terms of a series of which 23.6 is the sixth term. Find the first and tenth terms of the series.

x :	**3**	**4**	**5**	**6**	**7**	**8**	**9**
y :	**4.8**	**8.4**	**14.5**	**23.6**	**36.2**	**52.8**	**73.9**

Sol.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
3	4.8						
		3.6					
4	8.4						
			2.5				
		6.1		0.5			
					0		
5	14.5						
			3				
		9.1		0.5		0	
					0		
6	23.6						
			3.5				0
		12.6		0.5		0	
					0		
7	36.2						
			4				
		16.6		0.5			
8	52.8						
			4.5				
		21.1					
9	73.9						

To find y(1), we use Newton’s forward interpolation formula

To find y(10), we use Newton’s backward interpolation formula

Newton Forward Interpolation formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots$$

where $u = \frac{x - x_0}{h} = \frac{1-3}{1} = -2$

$$y = 4.8 + (-2)(3.6) + \frac{(-2)(-2-1)}{2!}(2.5) + \frac{(-2)(-2-1)(-2-2)}{3!}(0.5) + 0$$

$= 4.8 – 7.2 + 7.5 – 2$

y(1) = 3.1

Newton Backward Interpolation formula is

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \ldots\ldots\ldots\ldots$$

where $u = \frac{x - x_n}{h} = \frac{10-9}{1} = 1$

$$y = 73.9 + (1)(21.1) + \frac{(1)(1+1)}{2!}(4.5) + \frac{(1)(1+1)(1+2)}{3!}(0.5) + 0$$

$$= 73.9 + 21.1 + 4.5 + 0.5$$

$$y(10) = 100$$

Remark:

1. The nth divided difference of a polynomial of nth degree is **constant.**

2. *Forward, backward, central differences and divided difference.*

Δ f(x) = f(x+h) – f(x)

∇ f(x) = f(x) – f(x – h)

δ f(x) = f(x+h) – f(x – h)

Δ f(x) = $\frac{f(x_1) - f(x_0)}{x_1 - x_0}$

Numerical Differentiation & Integration

Numerical differentiation is used to find the derivatives of f(x) by using Newton's divided difference interpolation formula or Newton's forward interpolation formula and Newton's backward interpolation formula

Note:

1. *Numerical differentiation can be used only when the difference of some order are* constant.
2. When the function is given in the form of table of values instead of giving analytical expression we use numerical differentiation

Numerical Differentiation:

Derivatives Using Divided Differences (Unequal):

First fit a polynomial for the given data using Newton's divided difference interpolation formula and computing the derivatives with respect to given variable.

Example:

1. Find $y'(6)$ ***from the following data***

x :	***0***	***2***	***3***	***4***	***7***	***9***
y :	***4***	***26***	***58***	***112***	***466***	***922***

Sol. Since the arguments are not equally spaced, we will use Newton's divided difference formula.

The divided difference table is

x	y= f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	4					
		11				
2	26					
			7			
		32				
				1		
3	58					
			11			
					0	
		54				
				1		
4	112					0
			16			
					0	
		118				
				1		
7	466					
			22			
		228				
9	922					

Newton divided difference formula is

$$y = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0$$

$$+ (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0 + \ldots\ldots\ldots\ldots\ldots$$

$$y(x) = 4 + (x - 0)(11) + (x - 0)(x - 2)(7) + (x - 0)(x - 2)(x - 3)(1)$$

$$y(x) = x^3 + 2x^2 + 3x + 4$$

$$y'(x) = 3x^2 + 4x + 3$$

$$\therefore\ y'(6) = 3(6)^2 + 4(6) + 3$$

= **135.**

First Derivatives Using Newton forward and backward difference formula:

Newton's forward interpolation formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots\ldots\ldots$$

where $u = \frac{x - x_0}{h}$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \frac{2u^3-9u^2+11u-3}{12}\Delta^4 y_0 + ...\right]$$

Newton’s backward interpolation formula is

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + ...$$

where $u = \frac{x - x_n}{h}$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2}\nabla^2 y_n + \frac{3u^2+6u+2}{6}\nabla^3 y_n + \frac{2u^3+9u^2+11u+3}{12}\nabla^4 y_n + ...\right]$$

Derivatives Using Newton forward and backward difference formula:

First and second derivative formula at x = x₀

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + ...\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - ....\right]$$

First and second derivative formula at x = xₙ

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + ...\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + ...\right]$$

Example:

1. Find the error in the derivative of f(x) = cos x by computing directly and using the approximation $f'(x) = \frac{f(x+h) - f(x-h)}{2h}$ ***at x = 0.8 choosing h = 0.1***

Sol. $f(x) = \cos x \Rightarrow f'(x) = -\sin x \Rightarrow f'(0.8) = -0.717$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

$$f'(0.8) = \frac{f(0.8+0.1) - f(0.8-0.1)}{2(0.1)}$$

$$= \frac{f(0.9) - f(0.7)}{0.2}$$

$$= \frac{\cos(0.9) - \cos(0.7)}{0.2}$$

$$= -0.716$$

Error = – 0.001

2.. If f(x) = a^x ($a \neq 0$) is given for x = 0, 0.5, 1. Show by numerical differentiation that $f'(0) = 4\sqrt{a} - a - 3.$

Sol. For x = 0, 0.5, 1, the values of y = f(x) = a^x are a^0, $a^{0.5}$, a^1

(i.e.) 1, $\sqrt{a}$, a

x	y = a^x	Δy	$\Delta^2 y$
0	1		
		$\sqrt{a}$ -1	
0.5	$\sqrt{a}$		a-2 $\sqrt{a}$ +1
		a- $\sqrt{a}$	
1	a		

$$\left(\frac{dy}{dx}\right)_{x=0} = f'(0) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + ...\right]$$

$$= \frac{1}{0.5}\left[\sqrt{a} - 1 - \frac{1}{2}(a - 2\sqrt{a} + 1)\right]$$

$$= 2\left[2\sqrt{a} - \frac{a}{2} - \frac{3}{2}\right]$$

$$= 4\sqrt{a} - a - 3.$$

3.Find $f'(3)$ ***and*** $f''(3)$ ***for the following data:***

x :	***3.0***	***3.2***	***3.4***	***3.6***	***3.8***	***4.0***
f(x) :	***–14***	***–10.032***	***–5.296***	***–0.256***	***6.672***	***14***

Sol. The difference table is

x	y= f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
3.0	– 14					
		3.968				
3.2	–10.032					
			0.768			
		4.736				
				–0.464		
3.4	–5.296					
			0.304			
					2.048	
		5.04				
				1.584		
3.6	–0.256					–5.12
			1.888			
					–3.072	
		6.928				
				–1.488		
3.8	6.672					
			0.4			
		7.328				
4.0	14					

Newton Forward Interpolation formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots$$

where $u = \dfrac{x - x_0}{h}$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \ldots\ldots\ldots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=3} = \frac{1}{0.2}\left[3.968 - \frac{1}{2}(0.768) + \frac{1}{3}(-0.464) - \frac{1}{4}(2.048) + \frac{1}{5}(-5.12)\right]$$

(i.e.) $f'(3)$ = **9.4665**

$$\left(\frac{d^2 y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \ldots\ldots\ldots\right]$$

$$\left(\frac{d^2 y}{dx^2}\right)_{x=3} = \frac{1}{(0.2)^2}\left[0.768 - (-0.464) + \frac{11}{12}(2.048) - \frac{5}{6}(-5.12)\right]$$

(i.e.) $f''(3)$ **= 184.4**

4.The following data gives the velocity of a particle for 20 seconds at an interval of 5 seconds. Find the initial acceleration using the entire data

time (sec.) :	***0***	***5***	***10***	***15***	***20***
velocity (m/sec.) :	***0***	***3***	***14***	***69***	***228***

Sol. The difference table is

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
0	0				
		3			
5	3		8		
		11		36	
10	14		44		24
		55		60	
15	69		104		
		159			
20	228				

Initial Accleration = $\frac{dv}{dt}$ at t = 0.

$$\left(\frac{dv}{dt}\right)_{t=t_0} = \frac{1}{h}\left[\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0 - \frac{1}{4}\Delta^4 v_0 + \ldots\ldots\ldots\ldots\right]$$

$$\left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{5}\left[3 - \frac{1}{2}(8) + \frac{1}{3}(36) - \frac{1}{4}(24)\right]$$

= **1 m/sec^2 .**

5.Find $\frac{d\theta}{dt}$ ***at t = 3 and t = 8 given***

t : 1 3 5 7 9

θ **:** ***85.3 74.5 67 60.5 54.3***

Sol. The difference table is

t	θ	$\Delta\theta$	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$
1	85.3				
		–10.8			
3	74.5		3.3		
		–7.5		–2.3	
5	67		1		1.6
		–6.5		–0.7	
7	60.5		0.3		
		–6.2			
9	54.3				

$$\left(\frac{d\theta}{dt}\right)_{t=t_0} = \frac{1}{h}\left[\Delta\theta_0 - \frac{1}{2}\Delta^2\theta_0 + \frac{1}{3}\Delta^3\theta_0 - \frac{1}{4}\Delta^4\theta_0 + \ldots\ldots\ldots\ldots\right]$$

$$\left(\frac{d\theta}{dt}\right)_{t=3} = \frac{1}{2}\left[-10.8 - \frac{1}{2}(3.3) + \frac{1}{3}(-2.3) - \frac{1}{4}(1.6)\right]$$

$$= \mathbf{-\,4.1167}$$

To find $\frac{d\theta}{dt}$ at t = 8, we use Newton backward interpolation formula.

Newton Backward Interpolation formula is

$$\theta = \theta_n + u\nabla\theta_n + \frac{u(u+1)}{2!}\nabla^2\theta_n + \frac{u(u+1)(u+2)}{3!}\nabla^3\theta_n + \ldots\ldots\ldots$$

where $u = \frac{t-t_n}{h} = \frac{8-9}{2} = -0.5$

$$\frac{d\theta}{dt} = \frac{d\theta}{du}.\frac{du}{dt}$$

$$= \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2}\nabla^2 y_n + \frac{3u^2+6u+2}{6}\nabla^3 y_n + \frac{2u^3+9u^2+11u+3}{12}\nabla^4 y_n + ...\ \right]$$

$$\left(\frac{d\theta}{dt}\right)_{t=8} = \frac{1}{2}\left[-6.2 + \frac{2(-0.5)+1}{2}(0.3) + \frac{3(-0.5)^2+6(-0.5)+2}{6}(-0.7)\right.$$

$$\left. + \frac{2(-0.5)^3+9(-0.5)^2+11(-0.5)+3}{12}(1.6)\ \right]$$

= – 3.1188

6.Find the value of sec 31^0 from the following data :

θ ***(deg)***	***:***	***31^0***	***32^0***	***33^0***	***34^0***
tanθ	***:***	***0.6008***	***0.6249***	***0.6494***	***0.6745***

Sol. Let y = tanθ

The difference table is

θ	y = tanθ	Δy	$\Delta^2 y$	$\Delta^3 y$
31	0.6008			
		0.0241		
32	0.6249		0.0004	
		0.0245		0.0002
33	0.6494		0.0006	
		0.0251		
34	0.6745			

$$\left(\frac{d(\tan\theta)}{d\theta}\right)_{\theta=31^0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - ....\right]$$

$$\sec^2 31^0 = \frac{1}{1^0}\left[0.0241 - \frac{1}{2}(0.0004) + \frac{1}{3}(0.0002)\right]$$

$$= \frac{1}{1^0}\left[0.02396666\right] \qquad (\because \pi = 180^0$$

$$= \frac{0.02396666}{0.017453292} \qquad \frac{\pi}{180} = 1^0)$$

$$\sec^2 31^0 = 1.373188852$$

(i.e.) sec 31^0 = **1.1718**

7.Consider the following table of data :

x :	***0.2***	***0.4***	***0.6***	***0.8***	***1.0***
f(x) :	***0.9798652***	***0.9177710***	***0.8080348***	***0.6386093***	***0.3843735***

Find $f'(0.25)$ ***using Newton Forward interpolation formula and*** $f'(0.95)$ ***using Newton Backward interpolation formula.***

Sol. The difference table is

x	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.2	0.9798652				
		–0.0620942			
0.4	0.9177710		–0.047642		
		–0.1097362		–0.0120473	
0.6	0.8080348		–0.0596893		–0.0130737
		–0.1694255		–0.025121	
0.8	0.6386093		–0.0848103		
		–0.2542358			
1.0	0.3843735				

Newton Forward Interpolation formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots$$

$$\text{where } u = \frac{x - x_0}{h} = \frac{0.25 - 0.2}{0.2} = \frac{0.05}{0.2} = 0.25$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{2u - 1}{2}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{6}\Delta^3 y_0 \right.$$

$$\left. + \frac{2u^3 - 9u^2 + 11u - 3}{12}\Delta^4 y_0 + \ldots\ldots \right]$$

$$\left(\frac{dy}{dx}\right)_{x=0.25} = \frac{1}{0.2}\left[-0.0620942 + \frac{2(0.25) - 1}{2}(-0.047642) \right.$$

$$+ \frac{3(0.25)^2 - 6(0.25) + 2}{6}(-0.0120473)$$

$$\left. + \frac{2(0.25)^3 - 9(0.25)^2 + 11(0.25) - 3}{12}(-0.0130737) \right]$$

$$= \frac{1}{0.2}\,[\,-0.0620942 + 0.0119105 - 0.0013804 + 0.0008512]$$

(i.e.) $f'(0.25) =$ **– 0.2535645**

Newton Backward Interpolation formula is

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \ldots\ldots\ldots\ldots$$

$$\text{where } u = \frac{x - x_n}{h} = \frac{0.95 - 1.0}{0.2} = \frac{-0.05}{0.2} = -0.25$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{h}\left[\nabla y_n + \frac{2u + 1}{2}\nabla^2 y_n + \frac{3u^2 + 6u + 2}{6}\nabla^3 y_n \right.$$

$$\left. + \frac{2u^3 + 9u^2 + 11u + 3}{12}\nabla^4 y_n + \ldots\ldots \right]$$

$$\left(\frac{dy}{dx}\right)_{x=0.95} = \frac{1}{0.2}\left[-0.2542358 + \frac{2(-0.25) + 1}{2}(-0.0848103) \right.$$

$$+\frac{3(-0.25)^2+6(-0.25)+2}{6}(-0.025121)$$

$$+\frac{2(-0.25)^3+9(-0.25)^2+11(-0.25)+3}{12}(-0.0130737)\ \Big]$$

$$=\frac{1}{0.2}[-0.2542358-0.0212026-0.0028784-0.0008512]$$

(i.e.) $f'(0.95)$ = – **1.39584**

8. *Find the maximum and minimum value of y tabulated below*

x : –2 –1 0 1 2 3 4

y : 2 –0.25 0 –0.25 2 15.75 56

Sol. The difference table is

x	y= f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
–2	2						
		–2.25					
–1	–0.25						
			2.5				
		0.25		–3			
0	0						
			–0.5				
					6		
		–0.25		3			
1	–0.25					0	
			2.5				
					6		
							0
		2.25		9			
2	2					0	
			11.5				
					6		
		13.75		15			
3	15.75						
			26.5				
		40.25					
4	56						

Newton Forward Interpolation formula is

$$y=y_0+u\Delta y_0+\frac{u(u-1)}{2!}\Delta^2 y_0+\frac{u(u-1)(u-2)}{3!}\Delta^3 y_0+\text{...........}$$

where $u=\frac{x-x_0}{h}$

$$\frac{dy}{du} = \left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 \right.$$

$$\left. + \frac{2u^3-9u^2+11u-3}{12}\Delta^4 y_0 + ... \right]$$

$$= -2.25 + \frac{2u-1}{2}(2.5) + \frac{3u^2-6u+2}{6}(-3)$$

$$+ \frac{2u^3-9u^2+11u-3}{12}(6)$$

$$= \frac{-4.5+5u-2.5-3u^2+6u-2+2u^3-9u^2+11u-3}{2}$$

$$= \frac{2u^3-12u^2+22u-12}{2}$$

$$\frac{dy}{du} = u^3-6u^2+11u-6$$

For maximum or minimum , $\frac{dy}{dx} = 0$

$$\Rightarrow \frac{1}{h}.\frac{dy}{du} = 0$$

$$\Rightarrow \frac{dy}{du} = 0$$

$$(i.e.) u^3-6u^2+11u-6 = 0$$

Solving, we get u = 1,2,3.

Now, $u = \frac{x-x_0}{h} \Rightarrow x = x_0 + uh$

$\Rightarrow x = -1,0,1$ corresponding to u = 1,2,3.

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\frac{d^2y}{du^2}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{6u^2-18u+11}{12}\Delta^4 y_0 + ...\right]$$

At u = 1 and 3, $\frac{d^2y}{dx^2} > 0$ and at u = 2, $\frac{d^2y}{dx^2} < 0$

$\therefore$ y has maximum at x = 0 and has minimum at x = –1, 1.

Hence **maximum value of y = 0 at x = 0**

and minimum value of y = – 0.25 at x = –1, 1.

9.Find $f'(4)$ *and* $f''(4)$ **from the following data**

x : 0 2 3 5

y : 8 6 20 108

Sol. Since the arguments are not equally spaced, we will use Newton’s divided difference formula.

The divided difference table is

x	y= f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$
0	8			
		–1		
2	6			
			5	
		14		
				1
3	20			
			10	
		44		
5	108			

Newton divided difference formula is

$$y = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0$$

$$+ (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0 + \ldots\ldots\ldots\ldots\ldots$$

$$y(x) = 8 + (x - 0)(-1) + (x - 0)(x - 2)(5) + (x - 0)(x - 2)(x - 3)(1)$$

$$y(x) = 8 - x + 5x^2 - 10x + x^3 - 5x^2 + 6x$$

$$y(x) = x^3 - 5x + 8$$

$$y'(x) = 3x^2 - 5,\ \ y''(x) = 6x$$

$$\therefore\ f'(4) = 3(4)^2 - 5 = 43$$

$$f''(4) = 6(4) = 24.$$

Numerical Integration:

Numerical integration is used to find the integration value of function f(x) from the tabulated values

Trapezoidal rule:

Suppose the arguments x_0, x_1, x_2,…,x_n equally spaced and also the corresponding y_0,y_1,y_2,..,y_n has given

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + ...y_{n-1}\right]$$

or

$$\int_{a}^{b} f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + ...y_{n-1}\right]$$

Where $h = \frac{b-a}{n}$ n = number of intervals

Remark:

1. The Trapezoidal rule is so called, because it approximates the integral by the sum of n trapezoids.
2. *in deriving the Trapezoidal formula, the arc of the curve y = f(x) over each sub interval is replaced by its* chord.

Examples:

1. ***Using Trapezoidal rule evaluate*** $\int_0^{\pi} \sin x dx$ ***by dividing the range into 6 equal parts.***

Sol. $h = \frac{\pi - 0}{6} = \frac{\pi}{6}$

When h = $\frac{\pi}{6}$, the values of y = sinx are

x :	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	π
y=sinx:	0	0.5	.8660	1	.8660	0.5	0

Trapezoidal rule is

$$\int_0^{\pi} \sin\, x dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + ...y_{n-1}\right]$$

$$= \frac{\pi}{6(2)}\left[(0+0) + 2(0.5 + 0.8660 + 1 + 0.8660 + 0.5)\right]$$

$$= 0.9770$$

2. Write down the Trapezoidal rule to evaluate $\int_1^6 f(x)dx$ ***with h = 0.5***

Sol. Trapezoidal rule is

$$\int_1^6 f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + ...y_{n-1}\right]$$

$$= \frac{0.5}{2}\left[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + ...y_9)\right]$$

3.Using Trapezoidal rule, evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ ***taking 8 intervals.***

Sol. $h = \frac{1-(-1)}{8} = \frac{2}{8} = 0.25$

When h = 0.25, the values of y = $\frac{1}{1+x^2}$ are

x : –1	–0.75	–0.50	–0.25	0	0.25	0.50	0.75	1
y : 0.5	0.64	0.8	0.9412	1	0.9412	0.8	0.64	0.5

Trapezoidal rule is

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + ...y_{n-1}\right]$$

$$\int_{-1}^{1} \frac{dx}{1+x^2} = \frac{0.25}{2}[(0.5 + 0.5) + 2(0.64 + 0.8 + 0.9412 + 1 + 0.9412$$

$$+ 0.8 + 0.64)]$$

= 0.125[1 + 11.5248]

= 0.125 (12.5248)

= **1.5656**

4. Evaluate $\int_0^1 e^{-x^2} dx$ by dividing the range of integration into four equal parts using Trapezoidal rule.

Sol. $h = \frac{1-0}{4} = 0.25$

When h = 0.25, the values of $y = e^{-x^2}$ are

x :	0	0.25	0.50	0.75	1
y :	1	0.9394	0.7788	0.5698	0.3679

Trapezoidal rule is

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + ...y_{n-1}\right]$$

$$\int_0^1 e^{-x^2} dx \ = \frac{0.25}{2}[(1 + 0.3679) + 2(0.9394 + 0.7788 + 0.5698)]$$

= 0.125 [1.3679 + 4.576]

= 0.125 (5.9439)

= **0.7430**

Simpson's 1/3 rd Rule:

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + ... + y_{n-1}.)$$
$$+ 2(y_2 + y_4 + y_6 + ... + y_{n-2}) \ ]$$

Simpson's 3/8 th rule:

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + ...)$$
$$+ 2(y_3 + y_6 + y_9 + ...) \ ]$$

Remark:

1. The condition for Simpson's 3/8 rule is the number of sub-intervals should be a multiple of 3.
2. In Trapezoidal rule, there is no restriction on the number of intervals whereas in Simpson's 1/3rd rule, the number of intervals should be even.
3. *by Simpson's1/3rd rule as well as by Simpson's 3/8th rule, the* number of intervals should be a multiple of 6.

Examples:

1. ***Can you use Simpson's rule for the following data:***

x :	***7.47***	***7.48***	***7.49***	***7.50***	***7.51***	***7.52***
f(x) :	***1.93***	***1.95***	***1.98***	***2.01***	***2.03***	***2.06***

Why?

Sol. We cannot use Simpson's rule, since the number of ordinates is 6 (even).

2. ***Using Simpson's rule find*** $\int_0^4 e^x\,dx$ ***given $e^0 = 1$, $e^1 = 2.72$, $e^2 = 7.39$, $e^3 = 20.09$, $e^4 = 54.6$***

Sol. The following data is

x :	0	1	2	3	4
y :	1	2.72	7.39	20.09	54.6

Simpson's 1/3rd rule is

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}\Big[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \dots\dots\dots)$$

$$+ 2(y_2 + y_4 + y_6 + \dots\dots\dots\dots\dots\dots\dots)\ \Big]$$

$$\int_0^4 e^x\,dx = \frac{1}{3}\big[(1 + 54.6) + 4(2.72 + 20.09) + 2(7.39)\big]$$

$$= 53.8733$$

3.Find an approximate value of log $_e$5 by calculating to four decimal places by Simpson's rule the integral $\int_0^5 \frac{dx}{4x+5}$ dividing the range into 10 equal parts.

Sol. $h = \frac{5-0}{10} = 0.5$

When h = 0.5, the values of y = $\frac{1}{4x+5}$ are

x :	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
y :	0.2	0.1429	0.1111	0.0909	0.0769	0.0667	0.0588	0.0526	0.0476	0.0435	0.04

Simpson's 1/3rd rule is

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + ...)\right.$$

$$\left. + 2(y_2 + y_4 + y_6 + ...)\ \right]$$

$$\int_0^5 \frac{dx}{4x+5} = \frac{0.5}{3}\left[(0.2 + 0.04) + 4(0.1429 + 0.0909 + 0.0667\right.$$

$$\left.+ 0.0526 + 0.0435) + 2(0.1111 + 0.0769 + 0.0588 + 0.0476)\right]$$

$$\int_0^5 \frac{dx}{4x+5} = \frac{0.5}{3}[0.24 + 1.5864 + 0.5888]$$

$$= \frac{0.5}{3}(2.4152)$$

= **0.4025**

<u>To find log $_e$5</u>

We have $\int_0^5 \frac{dx}{4x+5}$ = 0.4025

Integrating we get

$$\left[\frac{\log(4x+5)}{4}\right]_0^5 = 0.4025$$

$$\frac{1}{4}\left[\log 25 - \log 5\right] = 0.4025$$

$$\log 5 = 4(0.4025)$$

(i.e.) log 5 = **1.6100**

4.Evaluate $\int_0^1 \frac{dx}{1+x^2}$ ***take h = 0.125. Hence find*** π ***using Simpson's rule.***

Sol. When h = 0.125, the values of y = $\frac{1}{1+x^2}$ are

x :	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y :	1	0.9846	0.9412	0.8767	0.8	0.7191	0.64	0.5664	0.5

Simpson's 1/3rd rule is

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \dots\dots\dots)\right.$$

$$\left. + 2(y_2 + y_4 + y_6 + \dots\dots\dots\dots\dots\dots\dots)\ \right]$$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{0.125}{3}\left[(1+0.5) + 4(0.9846 + 0.8767 + 0.7191\right.$$

$$\left. + 0.5664) + 2(0.9412 + 0.8 + 0.64)\right]$$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{0.125}{3}[1.5 + 12.5872 + 4.7624]$$

$$= \frac{0.125}{3}[18.8496]$$

= **0.7854**

To find π

We have $\int_0^1 \frac{dx}{1+x^2}$ = 0.7854

$$\left[\frac{1}{1}\tan^{-1}\left(\frac{x}{1}\right)\right]_0^1 = 0.7854$$

$$\frac{\pi}{4} - 0 = 0.7854$$

(i.e.) π = **3.1416**

5.Evaluate $\int_0^6 \frac{dx}{1+x^2}$ ***by dividing the range into 6 equal parts using Simpson's rule.***

Sol. $h = \frac{6-0}{6} = 1$

When h = 1, the values of $y = \frac{1}{1+x^2}$ are

x :	0	1	2	3	4	5	6
y :	1	0.5	0.2	0.1	0.0588	0.0385	0.0270

Since we are dividing the range into 6 equal parts, we use Simpson's 3/8th rule.

Simpson's 3/8 rule is

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}\Big[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + ...)$$

$$+ 2(y_3 + y_6 + y_9 + ...)\ \Big]$$

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3(1)}{8}[(1+0.0270)+3(0.5+0.2+0.0588+0.0385)+2(0.1)]$$

$$= \frac{3}{8}[1.0270 + 2.3919 + 0.2]$$

= **1.3571**

6. Evaluate $\int_4^{5.2} \log_e x dx$ ***using Simpson's rule.***

Sol. We can divide the range into 6 equal parts and use Simpson's 3/8th rule.

$$h = \frac{5.2-4}{6} = 0.2$$

When h = 0.2, the values of y = log $_e$x are

x :	4	4.2	4.4	4.6	4.8	5	5.2
y :	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

Simpson's 3/8 rule is

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + ...)$$

$$+ 2(y_3 + y_6 + y_9 + ...)\ ]$$

$$\int_{4}^{5.2} \log_e x dx = \frac{3(0.2)}{8}[(1.3863 + 1.6487) + 3(1.4351 + 1.4816$$

$$+ 1.5686 + 1.6094) + 2(1.5261)]$$

$$= \frac{3(0.2)}{8} [3.035 + 18.2841 + 3.0522]$$

$$= \mathbf{1.8278}$$

7. By dividing the range into 10 equal parts, evaluate $\int_0^{\pi} \sin x\ dx$ by using Simpson's 1/3rd rule. It is possible to evaluate the same by Simpson's 3/8th rule. Justify your answer.

Sol. $h = \frac{\pi - 0}{10} = \frac{\pi}{10}$

When h = $\frac{\pi}{10}$, the values of y = sin x are

x :	0	π/10	2 π/10	3 π/10	4 π/10	5 π/10	6 π/10	7 π/10	8 π/10	9 π/10	π
y :	0	0.3090	0.5878	0.8090	0.9511	1	0.9511	0.8090	0.5878	0.3090	0

Simpson's 1/3rd rule is

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + ...)$$

$$+ 2(y_2 + y_4 + y_6 + ...)\ ]$$

$$\int_0^{\pi} \sin x\ dx = \frac{\pi/10}{3}[(0 + 0) + 4(0.3090 + 0.8090 + 1 + 0.8090 + 0.3090)$$

$$+ 2(0.5878 + 0.9511 + 0.9511 + 0.5878)]$$

$$= \frac{\pi}{30}[0 + 12.944 + 6.1556]$$

$$= \mathbf{2.0001}$$

8. A river is 80 meters wide. The depth 'd' in meters at a distance x meters from one bank is given by the following table. Calculate the area of cross-section of the river using Simpson's one third rule.

x :	**0**	**10**	**20**	**30**	**40**	**50**	**60**	**70**	**80**
d:	**0**	**4**	**7**	**9**	**12**	**15**	**14**	**8**	**3**

Sol. Simpson's 1/3rd rule is

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + ...)$$

$$+ 2(y_2 + y_4 + y_6 + ...) \;]$$

$$\int_0^{80} y \; dx = \frac{10}{3}[(0+3) + 4(4+9+15+8) + 2(7+12+14)]$$

$$= 710 \text{ sq. meters.}$$

UNIT V

Numerical Solution of Ordinary Differential Equations

Single step method:

In one-step methods, we use the data of just one preceding step..

Suppose the ordinary differential equation has given. we can find the numerical solution by using

1. Taylors series method
2. Euler method
3. Euler modified method
4. Runge – Kutta 4th order method

These methods are called single steps method.

Taylor series method:

Consider the ordinary differential equation $\frac{dy}{dx} = f(x, y)$ With $y(x_0) = y_0$ then

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + \frac{(x - x_0)^3}{3!}y_0''' + \frac{(x - x_0)^4}{4!}y^{iv} + ...$$

Examples

1. Solve the differential equation $\frac{dy}{dx} = x + y + xy$***, y (0) = 1 by Taylor series method to get the value of y at x = h.***

Sol. Given

$$y' = x + y + xy$$
$$x_0 = 0, y_0 = 1$$

$$\begin{array}{ll}
y' = x + y + xy & y_0' = 0 + 1 + 0 = 1 \\
y'' = 1 + y' + xy' + y & y_0'' = 1 + 1 + 0 + 1 = 3 \\
y''' = y'' + xy'' + y' + y' & y_0''' = 3 + 1 + 0 + 1 = 5 \\
y'''' = y''' + 2y'' + xy''' + y'' & y_0'''' = 5 + 6 + 0 + 3 = 14
\end{array}$$

Taylor’s series is

$$y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \frac{(x-x_0)^4}{4!}y^{iv} + ...$$

$$y(h) = 1 + (h-0)(1) + \frac{(h-0)^2}{2}(3) + \frac{(h-0)^3}{6}(5) + \frac{(h-0)^4}{24}(14) + ...$$

$$y(h) = 1 + h + \frac{3}{2}h^2 + \frac{5}{6}h^3 + \frac{7}{12}h^4 + ...$$

2.Using Taylor's series find y at x = 0.1 if $\frac{dy}{dx} = x^2y - 1,$ ***y(0) = 1.***

Sol. Given $x_0 = 0, y_0 = 1$

$$y' = x^2y - 1 \Rightarrow y_0' = x_0{}^2 y_0 - 1 = 0 - 1 = -1$$

$$y'' = x^2y' + y.2x \Rightarrow y_0'' = x_0{}^2 y_0' + 2x_0y_0 = 0 + 0 = 0$$

$$y''' = x^2y'' + y'.2x + 2x.y' + 2y.1$$

$$\Rightarrow y_0''' = x_0{}^2 y_0'' + 4x_0y_0' + 2y_0 = 0 + 0 + 2 = 2$$

$$y^{iv} = x^2y''' + y''.2x + 4x.y'' + 4y'.1 + 2y'$$

$$\Rightarrow y_0{}^{iv} = x_0{}^2 y_0''' + 2x_0y_0'' + 4x_0y_0'' + 6y_0'$$

$$\Rightarrow y_0{}^{iv} = 0 + 0 + 0 + (-6) = -6$$

Taylor's series about $x = x_0$ is given by

$$y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \frac{(x-x_0)^4}{4!}y_0{}^{iv} + ...$$

$$y(x) = 1 + (x-0)(-1) + \frac{(x-0)^2}{2!}(0) + \frac{(x-0)^3}{3!}(2) + \frac{(x-0)^4}{4!}(-6) + ...$$

$$y(x) = 1 - x + 0 + \frac{x^3}{6}(2) + \frac{x^4}{24}(-6) + ...$$

$$y(x) = 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + ...$$

$$y(0.1) = 1 - (0.1) + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + ...$$

= 1 – 0.1 + 0.00033 – 0.000025

= **0.9003**

3.By means of Taylor series expansion, find y at x = 0.1 and x = 0.2 correct to three decimal places, given $\frac{dy}{dx} - 2y = 3e^x, y(0) = 0.$

Sol. Given $x_0 = 0, y_0 = 0$

$$y' = 2y + 3e^x \Rightarrow y_0' = 2y_0 + 3e^{x_0} = 0 + 3(1) = 3$$

$$y'' = 2y' + 3e^x \Rightarrow y_0'' = 2y_0' + 3e^{x_0} = 2(3) + 3(1) = 9$$

$$y''' = 2y'' + 3e^x \Rightarrow y_0''' = 2y_0'' + 3e^{x_0} = 2(9) + 3(1) = 21$$

$$y^{iv} = 2y''' + 3e^x \Rightarrow y_0^{iv} = 2y_0''' + 3e^{x_0} = 2(21) + 3(1) = 45$$

Taylor's series about $x = x_0$ is given by

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + \frac{(x - x_0)^3}{3!}y_0''' + \frac{(x - x_0)^4}{4!}y_0^{iv} + ...$$

$$y(x) = 0 + (x - 0)(3) + \frac{(x - 0)^2}{2!}(9) + \frac{(x - 0)^3}{3!}(21) + \frac{(x - 0)^4}{4!}(45) + ...$$

$$y(x) = 3x + \frac{9x^2}{2} + \frac{7x^3}{2} + \frac{15x^4}{8} + ...$$

$$y(0.1) = 3(0.1) + \frac{9(0.1)^2}{2} + \frac{7(0.1)^3}{2} + \frac{15(0.1)^4}{8} + ...$$

= 0.3 + 0.045 + 0.0035 + 0.0001875

= **0.3487**

$$y(0.2) = 3(0.2) + \frac{9(0.2)^2}{2} + \frac{7(0.2)^3}{2} + \frac{15(0.2)^4}{8} + ...$$

= 0.6 + 0.18 + 0.028 + 0.003

= **0.811**

4. Use Taylor series solution to solve numerically $\frac{dy}{dx} = xy^{\frac{1}{3}}$, y(1) = 1. Tabulate y for x = 1.1, 1.2

Sol. Given $x_0 = 1, y_0 = 1$

$$y' = xy^{\frac{1}{3}} \Rightarrow y_0' = x_0 y_0^{\frac{1}{3}} = 1(1) = 1$$

$$y'' = x\frac{1}{3}y^{-\frac{2}{3}}.y' + y^{\frac{1}{3}} \Rightarrow y_0'' = x_0\frac{1}{3}y_0^{-\frac{2}{3}}.y_0' + y_0^{\frac{1}{3}} = \frac{1}{3} + 1 = \frac{4}{3}$$

$$y''' = xy'\left(\frac{-2}{9}\right)y^{-5/3}y' + \frac{1}{3}xy^{-2/3}y'' + \frac{1}{3}y^{-2/3}y'.1 + \frac{1}{3}y^{-2/3}y'$$

$$\Rightarrow y_0''' = \frac{-2}{9} + \frac{4}{9} + \frac{1}{3} + \frac{1}{3} = \frac{8}{9}$$

Taylor's series about $x = x_0$ is given by

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + \frac{(x - x_0)^3}{3!}y_0''' + \frac{(x - x_0)^4}{4!}y_0^{iv} + ...$$

$$y(x) = 1 + (x - 1)(1) + \frac{(x - 1)^2}{2!}\left(\frac{4}{3}\right) + \frac{(x - 1)^3}{3!}\left(\frac{8}{9}\right) + ...$$

$$y(1.1) = 1 + (1.1 - 1)(1) + \frac{(1.1 - 1)^2}{2!}\left(\frac{4}{3}\right) + \frac{(1.1 - 1)^3}{3!}\left(\frac{8}{9}\right) + ...$$

$$= 1 + 0.1 + \frac{2(0.1)^2}{3} + \frac{4(0.1)^3}{27} + ...$$

= 1 + 0.1 + 0.0067 + 0.00014

= **1.1068**

$$y(1.2) = 1 + (1.2 - 1)(1) + \frac{(1.2 - 1)^2}{2!}\left(\frac{4}{3}\right) + \frac{(1.2 - 1)^3}{3!}\left(\frac{8}{9}\right) + ...$$

$$= 1 + 0.2 + \frac{2(0.2)^2}{3} + \frac{4(0.2)^3}{27} + ...$$

= 1 + 0.2 + 0.0267 + 0.0012

= **1.2279**

Taylor's series method for simultaneous first order differential equation:

Suppose the differential equation is of the form

$\frac{dy}{dx} = f(x,y,z)$; $\frac{dz}{dx} = g(x,y,z)$ with y(x$_0$)=y$_0$, z(x$_0$)=z$_0$ then

$$y = y_0 + \frac{h}{1!} y_0^1 + \frac{h^2}{2!} y_0^{11} + \frac{h^3}{3!} y_0^{111} + ...$$

$$z = z_0 + \frac{h}{1!} z_0^1 + \frac{h^2}{2!} z_0^{11} + \frac{h^3}{3!} z_0^{111} + ...$$

Taylor’s series for second order differential equation:

Consider differential equation is $\frac{d^2y}{dx^2} = f(x,y,\frac{dy}{dx}), y(x_0) = y_0, y'(x_0) = y_0'$ which are known values put $y' = p \ , \ y'' = p' = f(x, y, p)$ and also

$$y_1 = y_0 + \frac{h}{1} y_0' + \frac{h^2}{2} y_0'' + ...$$

$$y_2 = y_1 + \frac{h}{1} y_1' + \frac{h^2}{2} y_1'' + ...$$

Examples:

1. Find the value of y(1.1) and y(1.2) from $\frac{d^2y}{dx^2} + y^2 \frac{dy}{dx} = x^3, y(1) = 1, y'(1) = 1$ ***by using Taylor’s series method.***

Sol. Given $y'' + y^2 y' = x^3$ ---------(1)

Put $y' = z$ --------(2) then $y'' = z'$ --------(3)

Sub (2) and (3) in (1), we get

$$z' + y^2 z = x^3$$

$$z' = x^3 - y^2 z \quad \text{---------(4)}$$

The initial conditions are y(1) = 1, $y'(1) = 1$

(i.e.) y(1) = 1, z(1) = 1 (since $y' = z$)

(i.e.) $x_0 = 1, y_0 = 1, z_0 = 1$

Now to solve (1), it is enough if we solve the two first order differential equations (2) and (4).

$$y' = z \qquad\qquad z' = x^3 - y^2 z$$

$$\Rightarrow y_0' = z_0 = 1 \qquad\qquad \Rightarrow z_0' = x_0{}^3 - y_0{}^2 z_0 = 1 - 1 = 0$$

$$y'' = z' \qquad\qquad z'' = 3x^2 - y^2 z' - z.2y.y'$$

$$\Rightarrow y_0'' = z_0' = 0 \qquad\qquad \Rightarrow z_0'' = 3(1) - 0 - 2(1)(1)(1) = 1$$

$$y''' = z'' \qquad\qquad z''' = 6x - y^2 z'' - z'.2y.y' - 2[y\,z.y'' + y\,y'.z' + y'z.y']$$

$$\Rightarrow y_0''' = z_0'' = 1 \qquad\qquad \Rightarrow z_0''' = 6(1) - (1)(1) - 0 - 2[0 + 0 + 1] = 6 - 1 - 2 = 3$$

$$y^{iv} = z'''$$

$$\Rightarrow y_0{}^{iv} = z_0''' = 3$$

Taylor’s series about $x = x_0$ is given by

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \frac{(x - x_0)^3}{3!} y_0''' + \frac{(x - x_0)^4}{4!} y_0{}^{iv} + ...$$

$$y(x) = 1 + (x - 1)(1) + \frac{(x - 1)^2}{2!}(0) + \frac{(x - 1)^3}{3!}(1) + \frac{(x - 1)^4}{4!}(3) + ...$$

$$y(1.1) = 1 + (1.1 - 1)(1) + \frac{(1.1 - 1)^3}{6} + \frac{(1.1 - 1)^4}{24}(3) + ...$$

$$= 1 + 0.1 + \frac{(0.1)^3}{6} + \frac{(0.1)^4}{8} + ...$$

$= 1 + 0.1 + 0.00017 + 0.0000125$

= **1.1002**

$$y(1.2) = 1 + (1.2 - 1)(1) + \frac{(1.2 - 1)^3}{6} + \frac{(1.2 - 1)^4}{24}(3) + ...$$

$$= 1 + 0.2 + \frac{(0.2)^3}{6} + \frac{(0.2)^4}{8} + ...$$

$= 1 + 0.2 + 0.0013 + 0.0002$

= **1.2015**

2.Using Taylor series method find correct to four decimal places, the value of y(0.1) given $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$.

Sol. **<u>Hint</u>**:

$$y_0' = 1,\ y_0'' = 2,\ y_0''' = 8,\ y'^{v}{}_{0} = 28$$

$$y(x) = 1 + x + x^2 + \frac{4}{3}x^3 + \frac{7}{6}x^4 + ...$$

y(0.1) = 1.11145

3. Find by Taylor series method, the values of y at x = 0.1 and x = 0.2 to four decimal places from $\frac{dy}{dx} = x^2 y - 1,\ y(0) = 1.$

Sol. **Hint**:

$$y_0' = -1,\ y_0'' = 0,\ y_0''' = 2,\ y'^{v}{}_{0} = -6$$

$$y(x) = 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + ...$$

y(0.1) = 1.8344, y(0.2) = 0.8023

Euler's method:

Suppose f(x,y) and also the initial condition $y(x_0)=y_0$ has given then $y_{n+1} = y_n + hf(x_n,y_n)$ where n=0,1,2,3....and also f(x,y) = dy/dx

Examples:

1.Use Euler's method to approximate y when x = 0.1 given that $\frac{dy}{dx} = \frac{y-x}{y+x}$ ***with y = 1 for x = 0.***

Sol. We break up the interval 0.1 into five subintervals, we get the answer in more accurate form. So take h = 0.02

Given $f(x,y) = \frac{y-x}{y+x}$.

Also given $x_0 = 0, y_0 = 1$ and h = 0.02

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_1 = y_0 + h\left[\frac{y_0 - x_0}{y_0 + x_0}\right] = 1 + (0.02)\left[\frac{1-0}{1+0}\right]$$

$$= 1.02$$

(i.e.) **y(0.02) = 1.02**

$x_1 = x_0 + h$

= 0 + 0.02

= 0.02

$y_2 = y_1 + hf(x_1, y_1)$

$y_2 = y_1 + h\left[\dfrac{y_1 - x_1}{y_1 + x_1}\right]$ = 1.02 + (0.02) $\left[\dfrac{1.02 - 0.02}{1.02 + 0.02}\right]$

= 1.0392

(i.e.) **y(0.04) = 1.0392**

$x_2 = x_1 + h$

= 0.02 + 0.02

= 0.04

$y_3 = y_2 + hf(x_2, y_2)$

$y_3 = y_2 + h\left[\dfrac{y_2 - x_2}{y_2 + x_2}\right]$ = 1.0392+ (0.02) $\left[\dfrac{1.0392 - 0.04}{1.0392 + 0.04}\right]$

= 1.0577

(i.e.) **y(0.06) = 1.0577**

$x_3 = x_2 + h$

= 0.04+ 0.02

= 0.06

$y_4 = y_3 + hf(x_3, y_3)$

$y_4 = y_3 + h\left[\dfrac{y_3 - x_3}{y_3 + x_3}\right]$ = 1.0577 + (0.02) $\left[\dfrac{1.0577 - 0.06}{1.0577 + 0.06}\right]$

= 1.0756

(i.e.) **y(0.08) = 1.0756**

$$x_4 = x_3 + h$$

= 0.06 + 0.02

= 0.08

$$y_5 = y_4 + hf(x_4, y_4)$$

$$y_5 = y_4 + h\left[\frac{y_4 - x_4}{y_4 + x_4}\right] = 1.0756 + (0.02)\left[\frac{1.0756 - 0.08}{1.0756 + 0.08}\right]$$

= 1.0928

(i.e.) **y(0.1) = 1.0928**

modified Euler's method:

suppose f(x,y) = dy/dx, y(x$_0$)=y$_0$

$y_{n+1} = y_n + hf(x_n + h/2, y_n + h/2 f(x_n, y_n))$

Examples:

1.Using modified Euler's method, find y(0.1) if $\frac{dy}{dx} = x^2 + y^2, y(0) = 1.$

Sol. Given $f(x,y) = x^2 + y^2$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$$y_1 = y_0 + \frac{h}{2}\{f(x_0, y_0) + f[x_0 + h, y_0 + hf(x_0, y_0)]\}$$

$$f(x_0, y_0) = x_0^{\ 2} + y_0^{\ 2} = 0 + 1 = 1$$

$$y_1 = 1 + \frac{0.1}{2}\{1 + f[0 + 0.1, 1 + 0.1(1)]\}$$

$$y_1 = 1 + \frac{0.1}{2}\{1 + f[0.1, 1.1]\}$$

$$y_1 = 1 + \frac{0.1}{2}\{1 + 1.22\}$$

$$y_1 = 1.111$$

2.Solve $\frac{dy}{dx} = 1 - y$ ***with the initial condition x = 0, y = 0. Using Euler's algorithm, tabulate the solutions at x = 0.1, 0.2, 0.3, 0.4. Get the solutions by Euler's modified method also.***

Sol. Given $f(x, y) = 1 - y$.

Also given $x_0 = 0, y_0 = 0$ and h = 0.1

Euler's method

$$y_1 = y_0 + hf(x_0, y_0)$$

$y_1 = y_0 + h[1 - y_0]$ = 0 + (0.1)[1 – 0]

= 0.1

(i.e.) **y(0.1) = 0.1**

$x_1 = x_0 + h$

= 0 + 0.1

= 0.1

$$y_2 = y_1 + hf(x_1, y_1)$$

$y_2 = y_1 + h[1 - y_1]$ = 0.1+ (0.1)[1 – 0.1

]

= 0.19

(i.e.) **y(0.2) = 0.19**

$x_2 = x_1 + h$

= 0.1 + 0.1

= 0.2

$$y_3 = y_2 + hf(x_2, y_2)$$

$y_3 = y_2 + h[1 - y_2]$ = 0.19 + (0.1)[1 – 0.19]

= 0.271

(i.e.) **y(0.3) = 0.271**

$x_3 = x_2 + h$

= 0.2 + 0.1

= 0.3

$$y_4 = y_3 + hf(x_3, y_3)$$

$y_4 = y_3 + h[1 - y_3]$ = 0.271 + (0.1)[1 – 0.271]

$$= 0.3439$$

(i.e.) **y(0.4) = 0.3439**

Euler's modified method

$$y_1 = y_0 + \frac{h}{2}\{f(x_0, y_0) + f[x_0 + h, y_0 + hf(x_0, y_0)]\}$$

$$f(x_0, y_0) = 1 - y_0$$

$$= 1 - 0$$

$$= 1$$

$$y_1 = 0 + \frac{0.1}{2}\{1 + f[0 + 0.1, 0 + (0.1)(1)]\} = 0 + \frac{0.1}{2}\{1 + f[0.1, 0.1]\}$$

$$= 0 + \frac{0.1}{2}\{1 + (1 - 0.1)\}$$

$$= 0.095$$

(i.e.) **y(0.1) = 0.095**

$$x_1 = x_0 + h$$

$$= 0 + 0.1$$

$$= 0.1$$

$$y_2 = y_1 + \frac{h}{2}\{f(x_1, y_1) + f[x_1 + h, y_1 + hf(x_1, y_1)]\}$$

$$f(x_1, y_1) = 1 - y_1$$

$$= 1 - 0.095$$

$$= 0.905$$

$$y_2 = 0.095 + \frac{0.1}{2}\{0.905 + f[0.1 + 0.1, 0.095 + (0.1)(0.905)]\}$$

$$y_2 = 0.095 + \frac{0.1}{2}\{0.905 + f[0.2, 0.1855]\} = 0.095 + \frac{0.1}{2}\{0.905 + (1 - 0.1855)\}$$

$$= 0.095 + \frac{0.1}{2}\{0.905 + 0.8145\}$$

$= 0.18098$

(i.e.) **y(0.2) = 0.18098**

$$x_2 = x_1 + h$$

$= 0.1 + 0.1$

$= 0.2$

$$y_3 = y_2 + \frac{h}{2}\{f(x_2, y_2) + f[x_2 + h, y_2 + hf(x_2, y_2)]\}$$

$$f(x_2, y_2) = 1 - y_2$$

$= 1 – 0.18098$

$= 0.81902$

$$y_3 = 0.18098 + \frac{0.1}{2}\{0.81902 + f[0.2 + 0.1, 0.18098 + (0.1)(0.81902)]\}$$

$$y_3 = 0.18098 + \frac{0.1}{2}\{0.81902 + f[0.3, 0.2629]\}$$

$$= 0.18098 + \frac{0.1}{2}\{0.81902 + (1 - 0.2629)\} = 0.18098 + \frac{0.1}{2}\{0.81902 + 0.7371\}$$

$= 0.2588$

(i.e.) **y(0.3) = 0.2588**

$$x_3 = x_2 + h$$

$= 0.2 + 0.1$

$= 0.3$

$$y_4 = y_3 + \frac{h}{2}\{f(x_3, y_3) + f[x_3 + h, y_3 + hf(x_3, y_3)]\}$$

$$f(x_3, y_3) = 1 - y_3$$

$= 1 – 0.2588$

$= 0.7412$

$$y_4 = 0.2588 + \frac{0.1}{2}\{0.7412 + f[0.3 + 0.1, 0.2588 + (0.1)(0.2588)]\}$$

$$y_4 = 0.2588 + \frac{0.1}{2}\{0.7412 + f[0.4, 0.3329]\}$$

$$y_4 = 0.2588 + \frac{0.1}{2}\{0.7412 + (1 - 0.3329)\}$$

$$y_4 = 0.2588 + \frac{0.1}{2}\{0.7412 + 0.6671\}$$

= 0.3292

(i.e.) **y(0.4) = 0.3292**

3.Given that $\frac{dy}{dx} = \log_{10}(x+y)$ with the initial condition that y = 1 when x = 0, use Euler's modified method to find y for x = 0.2 and x = 0.5 in more accurate form.

Sol. Given $f(x,y) = \log_{10}(x+y)$.

Also given $x_0 = 0, y_0 = 1$. Take h = 0.1

By Euler modified method,

$$y_1 = y_0 + \frac{h}{2}\{f(x_0, y_0) + f[x_0 + h, y_0 + hf(x_0, y_0)]\}$$

$$f(x_0, y_0) = \log_{10}(x_0 + y_0)$$

$$= \log_{10}(0+1)$$

= 0

$$y_1 = 1 + \frac{0.1}{2}\{0 + f[0 + 0.1, 1 + (0.1)(0)]\} = 1 + \frac{0.1}{2}\{0 + f[0.1, 1]\}$$

$$= 1 + \frac{0.1}{2}\{0 + \log_{10}(0.1 + 1)\}$$

= 1.0021

(i.e.) **y(0.1) = 1.0021**

$$x_1 = x_0 + h$$

= 0 + 0.1

$= 0.1$

$$y_2 = y_1 + \frac{h}{2}\{f(x_1, y_1) + f[x_1 + h, y_1 + hf(x_1, y_1)]\}$$

$$f(x_1, y_1) = \log_{10}(x_1 + y_1)$$

$$= \log_{10}(0.1 + 1.0021)$$

$$= 0.0422$$

$$y_2 = 1.0021 + \frac{0.1}{2}\{0.0422 + f[0.1 + 0.1, 1.0021 + (0.1)(0.0422)]\}$$

$$y_2 = 1.0021 + \frac{0.1}{2}\{0.0422 + f[0.2, 1.0063]\}$$

$$y_2 = 1.0021 + \frac{0.1}{2}\{0.0422 + \log_{10}(0.2 + 1.0063)\}$$

$= 1.0083$

(i.e.) **y(0.2) = 1.0083**

$$x_2 = x_1 + h$$

$= 0.1 + 0.1$

$= 0.2$

$$y_3 = y_2 + \frac{h}{2}\{f(x_2, y_2) + f[x_2 + h, y_2 + hf(x_2, y_2)]\}$$

$$f(x_2, y_2) = \log_{10}(x_2 + y_2)$$

$$= \log_{10}(0.2 + 1.0083)$$

$$= 0.0822$$

$$y_3 = 1.0083 + \frac{0.1}{2}\{0.0822 + f[0.2 + 0.1, 1.0083 + (0.1)(0.0822)]\}$$

$$y_3 = 1.0083 + \frac{0.1}{2}\{0.0822 + f[0.3, 1.0165]\}$$

$$y_3 = 1.0083 + \frac{0.1}{2}\{0.0822 + \log_{10}(0.3 + 1.0165)\}$$

$= 1.0184$

(i.e.) **y(0.3) = 1.0184**

$$x_3 = x_2 + h$$

$$= 0.2 + 0.1$$

$$= 0.3$$

$$y_4 = y_3 + \frac{h}{2}\left\{f(x_3, y_3) + f[x_3 + h, y_3 + hf(x_3, y_3)]\right\}$$

$$f(x_3, y_3) = \log_{10}(x_3 + y_3)$$

$$= \log_{10}(0.3 + 1.0184)$$

$$= 0.12005$$

$$y_4 = 1.0184 + \frac{0.1}{2}\left\{0.12005 + f[0.3 + 0.1, 1.0184 + (0.1)(0.12005)]\right\}$$

$$y_4 = 1.0184 + \frac{0.1}{2}\left\{0.12005 + f[0.4, 1.0304]\right\}$$

$$y_4 = 1.0184 + \frac{0.1}{2}\left\{0.12005 + \log_{10}(0.4 + 1.0304)\right\}$$

$$= 1.0322$$

(i.e.) **y(0.4) = 1.0322**

$$x_4 = x_3 + h$$

$$= 0.3 + 0.1$$

$$= 0.4$$

$$y_5 = y_4 + \frac{h}{2}\left\{f(x_4, y_4) + f[x_4 + h, y_4 + hf(x_4, y_4)]\right\}$$

$$f(x_4, y_4) = \log_{10}(x_4 + y_4)$$

$$= \log_{10}(0.4 + 1.0322)$$

$$= 0.1560$$

$$y_5 = 1.0322 + \frac{0.1}{2}\left\{0.1560 + f[0.4 + 0.1, 1.0322 + (0.1)(0.1560)]\right\}$$

$$y_5 = 1.0322 + \frac{0.1}{2}\left\{0.1560 + f[0.5, 1.0478]\right\}$$

$$y_5 = 1.0322 + \frac{0.1}{2}\left\{0.1560 + \log_{10}(0.5 + 1.0478)\right\}$$

= 1.0495

(i.e.) **y(0.5) = 1.0495**

Runge kutta method of 4th order for first order Equations :

Suppose

$\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ *has given then* $y_{n+1} = y_n + \Delta y$

Where $\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$
$$k_4 = hf(x_n + h, y_n + k_3)$$

Examples:

1. ***Using R-K method of fourth order, solve*** $y' = 3x + \frac{1}{2}y$ ***with y(0) = 1 at x = 0.2 taking h = 0.1***

Sol. Given $f(x,y) = 3x + \frac{1}{2}y$

Also given $x_0 = 0, y_0 = 1$. Take h = 0.1

<u>To find y(0.1)</u>

$$k_1 = hf(x_0, y_0) = (0.1)\left(3x_0 + \frac{y_0}{2}\right) = (0.1)\left(3(0) + \frac{1}{2}\right)$$

= 0.05

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)\ f\left(0 + \frac{0.1}{2}, 1 + \frac{0.05}{2}\right)$$

$$= (0.1)\ f(0.05, 1.025) = 0.1\left(3(0.05) + \frac{1.025}{2}\right)$$

$$= 0.0663$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)\ f\left(0 + \frac{0.1}{2}, 1 + \frac{0.0663}{2}\right)$$

$$= (0.1)\ f(0.05, 1.0332) = 0.1\left(3(0.05) + \frac{1.0332}{2}\right)$$

$$= 0.0667$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3\right) = (0.1)\ f(0 + 0.1, 1 + 0.0667)$$

$$= (0.1)\ \left(3(0.1) + \frac{1.0667}{2}\right)$$

$$= 0.0833$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.05 + 2(0.0663) + 2(0.0667) + 0.0833]$$

$$= 0.0666$$

$$y_1 = y_0 + \Delta y$$

$$= 1 + 0.0666$$

$$= 1.0666$$

(i.e.) y(0.1) = **1.0666**

$$x_1 = x_0 + h$$

$$= 0 + 0.1$$

$$= 0.1$$

<u>To find y(0.2)</u>

$$k_1 = hf(x_1, y_1) = (0.1)\ \left(3x_1 + \frac{y_1}{2}\right) = (0.1)\ \left(3(0.1) + \frac{1.0666}{2}\right)$$

$$= 0.0833$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1)\ f\left(0.1 + \frac{0.1}{2},\ 1.0666 + \frac{0.0833}{2}\right)$$

$$= (0.1)\ f(0.15, 1.1083) = 0.1\left(3(0.15) + \frac{1.1083}{2}\right)$$

$$= 0.1004$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.1)\ f\left(0.1 + \frac{0.1}{2},\ 1.0666 + \frac{0.1004}{2}\right)$$

$$= (0.1)\ f(0.15, 1.1168) = 0.1\left(3(0.15) + \frac{1.1168}{2}\right)$$

$$= 0.1008$$

$$k_4 = hf\left(x_1 + h, y_1 + k_3\right) = (0.1)\ f(0.1 + 0.1, 1.0666 + 0.1008)$$

$$= (0.1)\ \left(3(0.2) + \frac{1.1674}{2}\right)$$

$$= 0.1184$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.0833 + 2(0.1004) + 2(0.1008) + 0.1184]$$

$$= 0.1007$$

$$y_2 = y_1 + \Delta y$$

= 1.0666 + 0.1007

= 1.1673

(i.e.) y(0.2) = **1.1673**

2. ***Use 4th order R-K method to solve*** $y' = xy$ ***for x = 1.2, 1.4, 1.6 Initially x = 1, y = 2 (take h = 0.2)***

Sol. Given $f(x, y) = xy$

Also given $x_0 = 1, y_0 = 2$. Take h = 0.2

To find y(1.2)

$$k_1 = hf(x_0, y_0) = (0.2)\ (x_0 y_0) = (0.2)\ [(1)(2)]$$

$$= 0.4$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.2)\ f\left(1 + \frac{0.2}{2}, 2 + \frac{0.4}{2}\right)$$

$$= (0.2)\ f(1.1, 2.2) = (0.2)[(1.1)(2.2)]$$

$$= 0.484$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.2)\ f\left(1 + \frac{0.2}{2}, 2 + \frac{0.484}{2}\right)$$

$$= (0.2)\ f(1.1, 2.242) = (0.2)[(1.1)(2.242)]$$

$$= 0.4932$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.2)\ f(1 + 0.2, 2 + 0.4932)$$

$$= (0.2)\ f(1.2, 2.4932) = (0.2)[(1.2)(2.4932)]$$

$$= 0.5984$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.4 + 2(0.484) + 2(0.4932) + 0.5984]$$

$$= 0.4921$$

$$y_1 = y_0 + \Delta y$$

$$= 2 + 0.4921$$

$$= 2.4921$$

(i.e.) **y(1.2) = 2.4921**

$$x_1 = x_0 + h$$

$$= 1 + 0.2$$

$$= 1.2$$

<u>To find y(1.4)</u>

$$k_1 = hf(x_1, y_1) = (0.2)\ (x_1 y_1) = (0.2)\ [(1.2)(2.4921)]$$

$$= 0.5981$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.2)\ f\left(1.2 + \frac{0.2}{2}, 2.4921 + \frac{0.5981}{2}\right)$$

$$= (0.2)\ f(1.3, 2.7912) = (0.2)[(1.3)(2.7912)]$$

$$= 0.7257$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.2)\ f\left(1.2 + \frac{0.2}{2}, 2.4921 + \frac{0.7257}{2}\right)$$

$$= (0.2)\ f(1.3, 2.8550) = (0.2)[(1.3)(2.8550)]$$

$$= 0.7423$$

$$k_4 = hf\left(x_1 + h, y_1 + k_3\right) = (0.2)\ f(1.2 + 0.2, 2.4921 + 0.7423)$$

$$= (0.2)\ f(1.4, 3.2344) = (0.2)[(1.4)(3.2344)]$$

$$= 0.9056$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.5981 + 2(0.7257) + 2(0.7423) + 0.9056]$$

$$= 0.74$$

$$y_2 = y_1 + \Delta y$$

$$= 2.4921 + 0.74$$

$$= 3.2321$$

(i.e.) **y(1.4) = 3.2321**

$$x_2 = x_1 + h$$

$$= 1.2 + 0.2$$

$$= 1.4$$

<u>To find y(1.6)</u>

$$k_1 = hf(x_2, y_2) = (0.2)\ (x_2 y_2) = (0.2)\ [(1.4)(3.2321)]$$

$$= 0.9050$$

$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = (0.2)\ f\left(1.4 + \frac{0.2}{2}, 3.2321 + \frac{0.9050}{2}\right)$$

$$= (0.2)\ f(1.5, 3.6846) = (0.2)[(1.5)(3.6846)]$$

$$= 1.1054$$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = (0.2)\ f\left(1.4 + \frac{0.2}{2}, 3.2321 + \frac{1.1054}{2}\right)$$

$$= (0.2)\ f(1.5, 3.7848) = (0.2)[(1.5)(3.7848)]$$

$$= 1.1354$$

$$k_4 = hf\left(x_2 + h, y_2 + k_3\right) = (0.2)\ f(1.4 + 0.2, 3.2321 + 1.1354)$$

$$= (0.2)\ f(1.6, 4.3675) = (0.2)[(1.6)(4.3675)]$$

$$= 1.3976$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.9050 + 2(1.1054) + 2(1.1354) + 1.3976]$$

$$= 1.1307$$

$$y_3 = y_2 + \Delta y$$

$$= 3.2321 + 1.1307$$

$$= 4.3628$$

(i.e.) **y(1.6) = 4.3628**

Multistep Method:

In multi step methods, where in each step, we use data from more than one of the preceding steps

multistep methods available for solving ordinary differential equation
i) Milne's predictor - corrector method

ii) Adam's Bashforth predictor – corrector method.

Milne's predictor and corrector Method:

Consider $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ *has given*

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2y_{n-2}{}' - y_{n-1}{}' + 2y_n{}'\right]$$

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y_{n-1}{}' + 4y_n{}' + y_{n+1}{}'\right]$$

Examples:

1. ***Solve*** $y' = x - y^2, 0 \le x \le 1$***, y(0) = 0, y(0.2) = 0.02, y(0.4) = 0.0795, y(0.6) = 0.1762 by Milne's method to find y(0.8) and y(1).***

Sol. Given $y' = x - y^2$ and h = 0.2

$x_0 = 0 \qquad y_0 = 0$

$x_1 = 0.2 \qquad y_1 = 0.02$

$x_2 = 0.4 \qquad y_2 = 0.0795$

$x_3 = 0.6 \qquad y_3 = 0.1762$

$x_4 = 0.8 \qquad y_4 = ?$

$x_5 = 1 \qquad y_5 = ?$

By **Milne's predictor formula**, we have

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

To get y_4, put n = 3 we get

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y'_1 - y'_2 + 2y'_3\right]$$

Now, $y_1' = x_1 - {y_1}^2$

$= 0.2 - (0.02)^2$

$= 0.1996$

$y_2' = x_2 - {y_2}^2$

$= 0.4 - (0.0795)^2$

$= 0.3937$

$y_3' = x_3 - {y_3}^2$

$= 0.6 - (0.1762)^2$

$= 0.5690$

$$y_{4,p} = 0 + \frac{4(0.2)}{3}\left[2(0.1996) - (0.3937) + 2(0.5690)\right]$$

$y(0.8)_p =$ **0.3049**

By **Milne's corrector formula**, we have

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

To get y_4, put n = 3 we get

$$y_{4,c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

Now, $y_4' = x_4 - {y_4}^2$

$= 0.8 - (0.3049)^2$

$= 0.7070$

$$y_{4,c} = 0.0795 + \frac{0.2}{3}\left[0.3937 + 4(0.5690) + 0.7070\right]$$

$y(0.8)_c = 0.3046$

Again, $y_4' = x_4 - {y_4}^2$ $= 0.8 - (0.3046)^2$

$= 0.7072$

$$y_{4,c} = 0.0795 + \frac{0.2}{3}\left[0.3937 + 4(0.5690) + 0.7072\right]$$

$y(0.8)_c^{(2)} =$ **0.3046**

To find y_5 (or) y(1), put n = 4 in the Milne's formula.

To get y_5, put n = 4 in Milne's predictor formula, we get

$$y_{5,p} = y_1 + \frac{4h}{3}\left[2y_2' - y_3' + 2y_4'\right]$$

$$= 0.02 + \frac{4(0.2)}{3}\left[2(0.3937) - 0.5690 + 2(0.7070)\right]$$

$y(1)_p$ = **0.4553**

Now put n = 4 in Milne's corrector formula, we get

$$y_{5,c} = y_3 + \frac{h}{3}\left[y_3' + 4y_4' + y_5'\right]$$

$y_5' = x_5 - y_5^2$ = 1 – $(0.4553)^2$ = 0.7927

$$y_{5,c} = 0.1762 + \frac{0.2}{3}\left[0.5690 + 4(0.7070) + 0.7927\right]$$

$y(1)_c$ = 0.4555

Again, $y_5' = x_5 - y_5^2$ = 1 – $(0.4555)^2$ = 0.7925

$$y_{5,c} = 0.1762 + \frac{0.2}{3}\left[0.5690 + 4(0.7070) + 0.7925\right]$$

$y(1)_c^{(2)}$ = **0.4555**

2. Given $\frac{dy}{dx} = x^3 + y, y(0) = 2$

i) Compute y(0.2), y(0.4) and y(0.6) by R-K method of 4th order.

ii) Hence find y(0.8) by Milne's predictor corrector method taking h = 0.2

Sol. Given $f(x, y) = x^3 + y$

Also given $x_0 = 0, y_0 = 2$. Take h = 0.2

To find y(0.2)

$k_1 = hf(x_0, y_0)$ = $(0.2)(x_0{}^3 + y_0)$ = (0.2) [0 + 2]

= 0.4

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$ = (0.2) $\left[\left(x_0 + \frac{h}{2}\right)^3 + \left(y_0 + \frac{k_1}{2}\right)\right]$

= (0.2) $\left[\left(0 + \frac{0.2}{2}\right)^3 + \left(2 + \frac{0.4}{2}\right)\right]$

= 0.4402

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$ = (0.2) $\left[\left(x_0 + \frac{h}{2}\right)^3 + \left(y_0 + \frac{k_2}{2}\right)\right]$

= (0.2) $\left[\left(0 + \frac{0.2}{2}\right)^3 + \left(2 + \frac{0.4402}{2}\right)\right]$

= 0.4442

$k_4 = hf(x_0 + h, y_0 + k_3)$ = (0.2) $[(x_0 + h)^3 + (y_0 + k_3)]$

= (0.2) $[(0 + 0.2)^3 + (2 + 0.4442)]$

= 0.4904

$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

= $\frac{1}{6}[0.4 + 2(0.4402) + 2(0.4442) + 0.4904]$

= 0.4432

$y_1 = y_0 + \Delta y$ = 2 + 0.4432

= 2.4432

(i.e.) **y(0.2) = 2.4432**

$x_1 = x_0 + h$

= 0 + 0.2

= 0.2

To find y(0.4)

$k_1 = hf(x_1, y_1)$ = $(0.2)(x_1^{\ 3} + y_1)$

= $(0.2)[(0.2)^3 + 2.4432]$

= 0.4902

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$ = $(0.2)\left[\left(x_1 + \frac{h}{2}\right)^3 + \left(y_1 + \frac{k_1}{2}\right)\right]$

= $(0.2)\left[\left(0.2 + \frac{0.2}{2}\right)^3 + \left(2.4432 + \frac{0.4902}{2}\right)\right]$

= 0.5431

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$ = $(0.2)\left[\left(x_1 + \frac{h}{2}\right)^3 + \left(y_1 + \frac{k_2}{2}\right)\right]$

= $(0.2)\left[\left(0.2 + \frac{0.2}{2}\right)^3 + \left(2.4432 + \frac{0.5431}{2}\right)\right]$

= 0.5484

$k_4 = hf\left(x_1 + h, y_1 + k_3\right)$ = $(0.2)[(x_1 + h)^3 + (y_1 + k_3)]$

= $(0.2)[(0.2 + 0.2)^3 + (2.4432 + 0.5484)]$

= 0.6111

$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

= $\frac{1}{6}[0.4902 + 2(0.5431) + 2(0.5484) + 0.6111]$

= 0.5474

$y_2 = y_1 + \Delta y$ = 2.4432 + 0.5474

= 2.9906

(i.e.) **y(0.4) = 2.9906**

$x_2 = x_1 + h$ = 0.2 + 0.2

= 0.4

To find y(0.6)

$k_1 = hf(x_2, y_2)$ = $(0.2)(x_2{}^3 + y_2)$ = (0.2) [(0.4)^3 + 2.9906]

= 0.6109

$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = (0.2)\left[\left(x_2 + \frac{h}{2}\right)^3 + \left(y_2 + \frac{k_1}{2}\right)\right]$$

$$= (0.2)\left[\left(0.4 + \frac{0.2}{2}\right)^3 + \left(2.9906 + \frac{0.6109}{2}\right)\right]$$

= 0.6842

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = (0.2)\left[\left(x_2 + \frac{h}{2}\right)^3 + \left(y_2 + \frac{k_2}{2}\right)\right]$$

$$= (0.2)\left[\left(0.4 + \frac{0.2}{2}\right)^3 + \left(2.9906 + \frac{0.6842}{2}\right)\right]$$

= 0.6915

$$k_4 = hf(x_2 + h, y_2 + k_3) = (0.2)[(x_2 + h)^3 + (y_2 + k_3)]$$

$$= (0.2)[(0.4 + 0.2)^3 + (2.9906 + 0.6915)]$$

= (0.2)[3.8981]

= 0.7796

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.6109 + 2(0.6842) + 2(0.6915) + 0.7796]$$

= 0.6903

$y_3 = y_2 + \Delta y$ = 2.9906 + 0.6903

= 3.6809

(i.e.) **y(0.6) = 3.6809**

$x_3 = x_2 + h$

= 0.4 + 0.2

= 0.6

To find y(0.8)

Given $y' = x^3 + y$ and h = 0.2

$x_0 = 0$	$y_0 = 2$
$x_1 = 0.2$	$y_1 = 2.4432$
$x_2 = 0.4$	$y_2 = 2.9906$
$x_3 = 0.6$	$y_3 = 3.6809$
$x_4 = 0.8$	$y_4 = ?$

By **Milne's predictor formula**, we have

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

To get y_4, put n = 3 we get

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y'_1 - y'_2 + 2y'_3\right]$$

Now, $y'_1 = x_1{}^3 + y_1$ = $(0.2)^3$ + 2.4432

= 2.4512

$y'_2 = x_2{}^3 + y_2$ = $(0.4)^3$ + 2.9906

= 3.0546

$y'_3 = x_3{}^3 + y_3$ = $(0.6)^3$ + 3.6809

= 3.8969

$$y_{4,p} = 2 + \frac{4(0.2)}{3}\left[2(2.4512) - (3.0546) + 2(3.8969)\right]$$

$y(0.8)_p =$ **4.5711**

By **Milne's corrector formula**, we have

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

To get y_4, put n = 3 we get

$$y_{4,c} = y_2 + \frac{h}{3}\left[y'_2 + 4y'_3 + y'_4\right]$$

Now, $y'_4 = x_4^{\ 3} + y_4$ = $(0.8)^3$ + 4.5711

= 5.0831

$$y_{4,c} = 2.9906 + \frac{0.2}{3}\left[3.0546 + 4(3.8969) + 5.0831\right]$$

$$y(0.8)_c = 4.5723$$

Again, $y'_4 = x_4^{\ 3} + y_4$ = $(0.8)^3$ + 4.5723

= 5.0843

$$y_{4,c} = 2.9906 + \frac{0.2}{3}\left[3.0546 + 4(3.8969) + 5.0843\right]$$

$y(0.8)_c^{(2)} =$ **4.5724**

3. Given that $y'' + xy' + y = 0, y(0) = 1, y'(0) = 0$. ***Obtain for x = 0.1, 0.2, 0.3 by Taylor's series method and find the solution for y(0.4) by Milne's method.***

Sol. Given $y'' + xy' + y = 0$ ---------(1)

Put $y' = z$ ---------(2) then $y'' = z'$ ---------(3)

Sub (2) and (3) in (1), we get

$z' + xz + y = 0 \Rightarrow$ $z' = -xz - y$ ---------(4)

The initial conditions are y(0) = 1, $y'(0) = 0$

(i.e.) y(0) = 1, z(0) = 0 (since $y' = z$)

(i.e.) $x_0 = 0, y_0 = 1, z_0 = 0$, h = 0.1

Now to solve (1), it is enough if we solve the two first order differential equations (2) and (4).

$$y' = z$$
$$\Rightarrow y_0' = z_0 = 0$$

$$z' = -xz - y$$
$$\Rightarrow z_0' = -x_0 z_0 - y_0 = 0 - 1 = -1$$

$$y'' = z'$$
$$\Rightarrow y_0'' = z_0' = -1$$

$$z'' = -x.z' - z.1 - y'$$
$$\Rightarrow z_0'' = 0 - 0 - 0 = 0$$

$$y''' = z''$$
$$\Rightarrow y_0''' = z_0'' = 0$$

$$z''' = -xz'' - z'.1 - z' - y''$$
$$\Rightarrow z_0''' = 0 - (-1) - (-1) - (-1) = 3$$

$$y^{iv} = z'''$$
$$\Rightarrow y_0^{iv} = z_0''' = 3$$

Taylor's series about $x = x_0$ is given by

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \frac{(x-x_0)^4}{4!}y_0^{iv} + ...$$

$$y(x) = 1 + (x-0)(0) + \frac{(x-0)^2}{2!}(-1) + \frac{(x-0)^3}{3!}(0) + \frac{(x-0)^4}{4!}(3) + ...$$

$$y(x) = 1 - \frac{x^2}{2} + \frac{x^4}{8} + ...$$

$$y(0.1) = 1 - \frac{(0.1)^2}{2} + \frac{(0.1)^4}{8} + ...$$

= **0.9950**

$$y(0.2) = 1 - \frac{(0.2)^2}{2} + \frac{(0.2)^4}{8} + ...$$

= **0.9802**

$$y(0.3) = 1 - \frac{(0.3)^2}{2} + \frac{(0.3)^4}{8} + ...$$

= **0.9560**

Now Taylor's series for z(x) is

$$z(x) = z_0 + (x - x_0)z_0' + \frac{(x-x_0)^2}{2!} z_0'' + \frac{(x-x_0)^3}{3!} z_0''' + ...$$

$$z(x) = 0 + (x-0)(-1) + \frac{(x-0)^2}{2!}(0) + \frac{(x-0)^3}{3!}(3) + ...$$

$$z(x) = -x + \frac{x^3}{2} + ...$$

$$z_1 = z(0.1) = -(0.1) + \frac{(0.1)^3}{2} + ...$$

$$= -\ 0.0995$$

$$z_2 = z(0.2) = -(0.2) + \frac{(0.2)^3}{2} + ...$$

$$= -\ 0.1960$$

$$z_3 = z(0.3) = -(0.3) + \frac{(0.3)^3}{2} + ...$$

$$= -\ 0.2865$$

$$z_4 = z(0.4) = -(0.4) + \frac{(0.4)^3}{2} + ...$$

$$= -\ 0.3680$$

Hence $y_1' = z_1 = -0.0995$

$y_2' = z_2 = -0.1960$

$y_3' = z_3 = -0.2865$

$y_4' = z_4 = -0.3680$

By **Milne’s predictor formula**, we have

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2y_{n-2}' - y_{n-1}' + 2y_n'\right]$$

To get y_4, put n = 3 we get

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$y_{4,p} = 1 + \frac{4(0.1)}{3}\left[2(-0.0995) - (-0.1960) + 2(-0.2865)\right]$$

$y(0.4)_p =$ **0.9232**

By **Milne's corrector formula**, we have

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

To get y_4, put n = 3 we get

$$y_{4,c} = y_2 + \frac{h}{3}\left[y'_2 + 4y'_3 + y'_4\right]$$

$$y_{4,c} = 0.9802 + \frac{0.2}{3}\left[-0.1960 + 4(-0.2865) - 0.3680\right]$$

$y(0.4)_c =$ **0.9232**

5. Consider the initial value problem $\frac{dy}{dx} = y - x^2 + 1, y(0) = 0.5$

i) Using the modified Euler method find y(0.2)

ii) Using 4th order R-K method, find y(0.4) and y(0.6)

Sol.

Given $f(x, y) = y - x^2 + 1$.

Also given $x_0 = 0, y_0 = 0.5$. Take h = 0.2

By Euler modified method,

$$y_1 = y_0 + \frac{h}{2}\left\{f(x_0, y_0) + f[x_0 + h, y_0 + hf(x_0, y_0)]\right\}$$

$$f(x_0, y_0) = y_0 - x_0^{\ 2} + 1 = 0.5 - 0 + 1$$

$$= 1.5$$

$$y_1 = 0.5 + \frac{0.2}{2}\left\{1.5 + f[0 + 0.2, 0.5 + (0.2)(1.5)]\right\}$$

$$= 0.5 + \frac{0.2}{2}\left\{1.5 + f[0.2, 0.8]\right\}$$

$$= 0.5 + \frac{0.2}{2}\left\{1.5 + [0.8 - (0.2)^2 + 1]\right\}$$

= 0.826

(i.e.) **y(0.2) = 0.826**

$$x_1 = x_0 + h$$

= 0 + 0.2

= 0.2

To find y(0.4)

Given $f(x, y) = y - x^2 + 1$

Also we have $x_1 = 0.2, y_1 = 0.826$ Take h = 0.2

$$k_1 = hf(x_1, y_1) = (0.2)(y_1 - {x_1}^2 + 1)$$

$$= (0.2)[(0.826 - (0.2)^2 + 1]$$

= 0.3572

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.2)\left[\left(y_1 + \frac{k_1}{2}\right) - \left(x_1 + \frac{h}{2}\right)^2 + 1\right]$$

$$= (0.2)\left[\left(0.826 + \frac{0.3572}{2}\right) - \left(0.2 + \frac{0.2}{2}\right)^2 + 1\right]$$

= 0.3829

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.2)\left[\left(y_1 + \frac{k_2}{2}\right) - \left(x_1 + \frac{h}{2}\right)^2 + 1\right]$$

$$= (0.2)\left[\left(0.826 + \frac{0.3829}{2}\right) - \left(0.2 + \frac{0.2}{2}\right)^2 + 1\right]$$

= 0.3855

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.2)[(y_1 + k_3) - (x_1 + h)^2 + 1]$$

$$= (0.2)[(0.826 + 0.3855) - (0.2 + 0.2)^2 + 1]$$

= 0.4103

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.3572 + 2(0.3829) + 2(0.3855) + 0.4103]$$

$$= 0.3841$$

$$y_2 = y_1 + \Delta y = 0.8260 + 0.3841$$

$$= 1.2101$$

(i.e.) **y(0.4) = 1.2101**

$$x_2 = x_1 + h$$

$$= 0.2 + 0.2$$

$$= 0.4$$

<u>To find y(0.6)</u>

$$k_1 = hf(x_2, y_2) = (0.2)(y_2 - x_2{}^2 + 1)$$

$$= (0.2)[(1.2101 - (0.4)^2 + 1]$$

$$= 0.41002$$

$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = (0.2)\left[\left(y_2 + \frac{k_1}{2}\right) - \left(x_2 + \frac{h}{2}\right)^2 + 1\right]$$

$$= (0.2)\left[\left(1.2101 + \frac{0.41002}{2}\right) - \left(0.4 + \frac{0.2}{2}\right)^2 + 1\right]$$

$$= 0.43302$$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = (0.2)\left[\left(y_2 + \frac{k_2}{2}\right) - \left(x_2 + \frac{h}{2}\right)^2 + 1\right]$$

$$= (0.2)\left[\left(1.2101 + \frac{0.43302}{2}\right) - \left(0.4 + \frac{0.2}{2}\right)^2 + 1\right]$$

$$= 0.43532$$

$$k_4 = hf\left(x_2 + h, y_2 + k_3\right) = (0.2)[(y_2 + k_3) - (x_2 + h)^2 + 1]$$

$$= (0.2)[(1.2101 + 0.43532) - (0.4 + 0.2)^2 + 1]$$

= 0.4571

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.41002 + 2(0.43302) + 2(0.43532) + 0.4571]$$

= 0.4340

$y_3 = y_2 + \Delta y$ = 1.2101 + 0.4340

= 1.6441

(i.e.) **y(0.6) = 1.6441**

$x_3 = x_2 + h$

= 0.4 + 0.2

= 0.6

6.Given $y' = x(x^2 + y^2)e^{-x}, \quad y(0) = 1$ ***find y at x = 0.1, 0.2 and 0.3 by Taylor's series method and compute y(0.4) by Milne's method.***

Sol. **Hint**: $y_0' = 0, \quad y_0'' = 1, \quad y_0''' = -1$

Taylor's series for y(x) is

$$y(x) = 1 + \frac{x^2}{2} - \frac{x^3}{6} + ...$$

y(0.1) = 1.0048, y(0.2) = 1.0187, y(0.3) = 1.0405

To find y(0.4)

$y_1' = 0.0923, \quad y_2' = 0.1765, \quad y_3' = 0.2606$

$y(0.4)_p =$ **1.0706**

$y_4' = 0.3502$. $\quad y(0.4)_c =$ **1.0710**

Revision Problems:

1.Using fourth order Runge kutta method, solve the following equation taking each step of h = 0.1 Given y(0) = 3, $\frac{dy}{dt} = \frac{4t}{y} - t\,y$ ***Calculate y for x = 0.1 and 0.2***

Sol. **Hint**:

<u>To find y(0.1)</u>

$k_1 = 0$, $k_2 = -0.0083$, $k_3 = -0.0083$, $k_4 = -0.0165$, $\Delta y = -0.0083$

y(0.1) = 2.9917

<u>To find y(0.2)</u>

$k_1 = -0.0165$, $k_2 = -0.0246$, $k_3 = -0.0246$, $k_4 = -0.0324$, $\Delta y = -0.0246$

y(0.2) = 2.9671

2.Given $y' + x y^2 + y = 0, \quad y(0) = 1,$ find the value of y(0.2) by using Runge-kutta method of 4th order.

Sol. **<u>Hint</u>**:

<u>To find y(0.1)</u>

$k_1 = -0.1$, $k_2 = -0.0995$, $k_3 = -0.0995$, $k_4 = -0.0982$, $\Delta y = -0.0994$

y(0.1) = 0.9006

<u>To find y(0.2)</u>

$k_1 = -0.0982$, $k_2 = -0.0960$, $k_3 = -0.0962$, $k_4 = -0.0934$, $\Delta y = -0.0960$

y(0.2) = 0.8046

3.Apply Runge-kutta method to find approximate value of y for x = 0.2 in steps of 0.1 if $\frac{dy}{dx} = x + y^2$ given that y = 1 when x = 0.

Sol. **<u>Hint</u>**:

<u>To find y(0.1)</u>

$k_1 = 0.1$, $k_2 = 0.1153$, $k_3 = 0.1169$, $k_4 = 0.1347$, $\Delta y = 0.1165$

y(0.1) = 1.1165

<u>To find y(0.2)</u>

$k_1 = 0.1347$, $k_2 = 0.1552$, $k_3 = 0.1576$, $k_4 = 0.1823$, $\Delta y = 0.1571$

y(0.2) = 1.2736

4. Using Euler's method, solve numerically the equation $y' = x + y, \quad y(0) = 1,$

for x = 0.0 (0.2) (1.0). Check your answer with the exact solution.

Sol. Using Euler's method, to solve y for x = 0.0 (0.2) (1.0), we take h = 0.2

$$y_1 = y(0.2) = 1.2, \quad y_2 = y(0.4) = 1.48, \quad y_3 = y(0.6) = 1.856,$$
$$y_4 = y(0.8) = 2.3472, \quad y_5 = y(1.0) = 2.9766$$

Now, $y' = x + y$

$$y' - y = x \ (or) \frac{dy}{dx} - y = x$$

$$The \quad solution \quad is \quad y e^{\int P dx} = \int Q e^{\int P dx} dx + c$$

$$Here \quad P = -1, \ Q = x$$

$$e^{\int P dx} = e^{\int -dx} = e^{-x}$$

$$\therefore y e^{-x} = \int x e^{-x} dx + c = \left[x\left(\frac{e^{-x}}{-1}\right) - (1)\left(\frac{e^{-x}}{1}\right) \right] + c$$

$$y e^{-x} = -x e^{-x} - e^{-x} + c$$

$$y = -x - 1 + c e^{x}$$

$$Given \quad y(0) = 1$$

$$1 = 0 - 1 + c \quad \Rightarrow c = 2$$

$$\therefore y = 2e^{x} - x - 1$$

x:	0	0.2	0.4	0.6	0.8	1.0
Euler y:	1	1.2	1.48	1.856	2.3472	2.9766
Exact y:	1	1.2428	1.5836	2.0442	2.6511	3.4366

Boundary Conditions:

The conditions on y or y' or their combinations are prescribed at 2 different values of x are called Boundary Conditions.

Boundary value problem:

the differential equation together with boundary conditions are called boundary value problem.

Example: dy/dx + xy = sinx with y(0) =1

Finite Difference Solution of second order ordinary differential equation

Suppose a boundary value problem $y''+a(x)y'+b(x)y(x)=c(x)---(1)$ together with the boundary conditions $y(x_0)=\alpha,\ y(x_n)=\beta$ *is given when* $x\in(x_0,x_n)$.

We replace $y'(x)$ *and* $y''(x)$ by the difference formula given by

$$y_i'=\frac{y_{i+1}-y_{i-1}}{2h},\quad y_i''=\frac{y_{i+1}+y_{i-1}-2y_i}{h^2}$$

Substituting $y'(x)$ *and* $y''(x)$ in (1) and simplifying, we get

$$y_{i+1}\left(1+\frac{h}{2}a_i\right)+y_i(h^2b_i-2)+y_{i-1}\left(1-\frac{h}{2}a_i\right)=c_ih^2 -------(2)$$

where i = 1,2,……n-1 and $y_0=\alpha,\ y_n=\beta,\ a_i=a(x_i),\ b_i=b(x_i),\ c_i=c(x_i)$.

Equation (2) will give (n-1) equations for i = 1,2,…..n-1 which is a tridiagonal system and together with $y_0=\alpha,\ y_n=\beta,$ we get (n+1) equations in the (n+1) unknowns $y_0,y_1,y_2,........y_n$. Solving from these (n+1) equations, we get $y_0,y_1,y_2,........y_n$ values. (i.e.) the value of y at x = $x_0,x_1,x_2,........x_n$.

Problems

1. ***Using the finite difference method, find y(0.25), y(0.5) and y(0.75) satisfying the differential equation*** $\frac{d^2y}{dx^2}+y=x$ ***subject to the boundary conditions y(0) = 0, y(1) = 2.***

Sol. Given $x\in(0,1)$

Here $h=0.25=\frac{1}{4}=\frac{1-0}{4}$, then we have n = 4. $\left[\because h=\frac{b-a}{n}\right]$

Also $\frac{d^2y}{dx^2}+y=x$

$$\Rightarrow \frac{y_{i+1}+y_{i-1}-2y_i}{h^2}+y_i=x_i$$

$$\Rightarrow 16(y_{i+1}+y_{i-1}-2y_i)+y_i = x_i$$

$$\Rightarrow 16y_{i+1}-31y_i+16y_{i-1} = x_i, i=1,2,3.$$

(i.e.) When i = 1,2,3 we have

$$16y_2-31y_1+16y_0 = x_1$$
$$16y_3-31y_2+16y_1 = x_2$$
$$16y_4-31y_3+16y_2 = x_3$$

Given $y_0 = y(0) = 0, y_4 = y(1) = 2.$ Also $x_1 = 0.25, x_2 = 0.5, x_3 = 0.75$

(i.e.) $16y_2 - 31y_1 + 16(0) = 0.25$

$$16y_3-31y_2+16y_1 = 0.5$$
$$16(2)-31y_3+16y_2 = 0.75$$

(i.e.) $-31y_1 + 16y_2 = 0.25 ------(1)$

$$16y_1-31y_2+16y_3 = 0.5------(2)$$
$$16y_2-31y_3 = 0.75-32 = -31.25----(3)$$

Solving (1), (2) and (3), we get

$y_1 = 0.5443, y_2 = 1.0702, y_3 = 1.5604$

Tabulating the values, we have

x	**0**	**0.25**	**0.5**	**0.75**	**1**
y	**0**	**0.5443**	**1.0702**	**1.5604**	**2**

2. Solve $xy''+y=0$, y(1) = 1, y(2) = 2 with h = 0.25 by finite difference method.

Sol. Given $x \in (1,2)$

Here $h = 0.25 = \frac{1}{4} = \frac{2-1}{4}$, then we have n = 4. $\left[\because h = \frac{b-a}{n}\right]$

Also $xy''+y=0$

$$y''+\frac{y}{x}=0 \quad \Rightarrow \frac{y_{i+1}+y_{i-1}-2y_i}{h^2}+\frac{y_i}{x_i}=0$$

$$\Rightarrow 16(y_{i+1}+y_{i-1}-2y_i)+\frac{y_i}{x_i}=0$$

$$\Rightarrow 16y_{i+1}+\left(\frac{1}{x_i}-32\right)y_i+16y_{i-1}=0, i=1,2,3.$$

(i.e.) When i = 1,2,3 we have

$$16y_2+\left(\frac{1}{x_1}-32\right)y_1+16y_0=0$$

$$16y_3+\left(\frac{1}{x_2}-32\right)y_2+16y_1=0$$

$$16y_4+\left(\frac{1}{x_3}-32\right)y_3+16y_2=0$$

Given $y_0=y(1)=1, y_4=y(2)=2.$ Also $x_1=1.25, x_2=1.5, x_3=1.75$

(i.e.) $16y_2+\left(\frac{1}{1.25}-32\right)y_1+16(1)=0$

$$16y_3+\left(\frac{1}{1.5}-32\right)y_2+16y_1=0$$

$$16(2)+\left(\frac{1}{1.75}-32\right)y_3+16y_2=0$$

(i.e.) $-31.2y_1+16y_2=-16----------(1)$

$$16y_1-31.3333y_2+16y_3=0------(2)$$

$$16y_2-31.4286y_3+16y_2=-32----(3)$$

Solving (1), (2) and (3), we get

$y_1=1.3513, y_2=1.6350, y_3=1.8505$

Tabulating the values, we have

x	**1**	**1.25**	**1.5**	**1.75**	**2**
y	**1**	**1.3513**	**1.6350**	**1.8505**	**2**

3. Solve $y'' - xy = 0$ given y(0) = –1,y(1) = 2 by finite difference method taking n = 2.

Sol. Given $x \in (0,1)$

Also given n = 2.

So, $h = \frac{1-0}{2} = \frac{1}{2}.$

Also $y'' - xy = 0 \Rightarrow \frac{y_{i+1} + y_{i-1} - 2y_i}{h^2} - x_i y_i = 0$

$$\Rightarrow 4(y_{i+1} + y_{i-1} - 2y_i) - x_i y_i = 0$$

$$\Rightarrow 4y_{i+1} - (8 + x_i)y_i + 4y_{i-1} = 0, i = 1.$$

(i.e.) When i = 1 we have

$$4y_2 - (8 + x_1)y_1 + 4y_0 = 0$$

Given $y_0 = y(0) = -1, y_2 = y(1) = 2.$ Also $x_1 = 0.5$

(i.e.) $4(2) - (8 + 0.5)y_1 + 4(-1) = 0$

$$-8.5y_1 = -4$$

$$y_1 = 0.4706$$

Tabulating the values, we have

x	**0**	**0.5**	**1**
y	**–1**	**0.4706**	**2**

4. Obtain the finite difference scheme for the differential equation $2\frac{d^2y}{dx^2} + y = 5.$

Sol. Given $2\frac{d^2y}{dx^2} + y = 5.$

$$2\left(\frac{y_{i+1} + y_{i-1} - 2y_i}{h^2}\right) + y_i = 5$$

$$2y_{i+1} + (h^2 - 4)y_i + 2y_{i-1} = 5h^2$$

5. Using finite difference method, solve for y given the differential equation

$\frac{d^2y}{dx^2} + y + 1 = 0$, $x \in (0,1)$ and the boundary conditions y(0) = y(1) = 0, taking

i) h = ½ ii) h = ¼

Sol. Hint:

i) $y_{i+1} - \frac{7}{4} y_i + y_{i-1} = -\frac{1}{4}, \ i = 1.$

$y_1 = 0.1428$

ii) $y_{i+1} - \frac{31}{16} y_i + y_{i-1} = -\frac{1}{16}, \ i = 1,2,3.$

$y_1 = y_3 = 0.1047, y_2 = 0.1403.$

6. Using finite difference method solve $\frac{d^2y}{dx^2} = y$ in (0,2) given y(0) = 0, y(2) = 3.63 subdividing the range of x into 4 equal parts.

Sol. $y_{i+1} - \frac{9}{4} y_i + y_{i-1} = 0, \ i = 1,2,3.$

$y_1 = 0.5268, \ y_2 = 1.1853, \ y_3 = 2.1401$

7. Using finite difference method, solve for y given the differential equation

$y'' - 64y + 10 = 0$, $x \in (0,1)$ given y(0) = y(1) = 0, subdividing the interval into

i) 4 equal parts ii) 2 equal parts.

Sol. i) $y_{i+1} - 6y_i + y_{i-1} = -\frac{5}{8}, \ i = 1,2,3.$

$y_1 = y_3 = 0.1287, y_2 = 0.1471.$

ii) $y_{i+1} - 18y_i + y_{i-1} = -\frac{5}{2}, \ i = 1.$

$y_1 = 0.1389$

8. Solve $y'' - y = x$, $x \in (0,1)$ given y(0) = y(1) = 0 using finite differences dividing the interval into 4 equal parts.

Sol. $16y_{i+1} - 33y_i + 16y_{i-1} = x_i$, $i = 1,2,3.$

$$y_1 = -0.0349,\ y_2 = -0.0563,\ y_3 = -0.05004$$