UNIT – I

PARTIAL DIFFERENTIAL EQUATION

A partial Differential Equation is an equation involving a function of two or more variables and some of its partial derivatives. Therefore a PDE contains one dependent variable and more than one independent variable. Hence the main difference between partial and ordinary differential equation is the number of independent variables involved in the equation.

The order of a PDE is the order of the highest partial derivatives occurring in the equation.

FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS:

There are two methods to form a PDE.

1. By Elimination of arbitrary constants.
2. By Elimination of arbitrary functions.

1. Formation of PDE by elimination of arbitrary constants:

Let $f(x, y, z, a, b) = 0 ------- (1)$ be an eqn which contains two arbitrary constants a and b. To eliminate this two constant we need atleast three eqn. Therefore partially differentiating eqn (1) w.r.to x and y, we get two more eqn.

From these three eqn we can eliminate a and b. Similarly , for eliminating three constants we need four equations and so on.

We use the following notations

$$p = \frac{\partial z}{\partial x}\ , q = \frac{\partial z}{\partial y}\ , r = \frac{\partial^2 z}{\partial x^2}\ , s = \frac{\partial^2 z}{\partial x \partial y}\ , t = \frac{\partial^2 z}{\partial y^2}$$

PROBLEMS:

1. Form the PDE by eliminating the arbitrary constants from $z = (x^2 + a)(y^2 + b)$.

Sol:

Given $z = (x^2 + a)(y^2 + b) \quad ....(1)$

Diff (1) partially w.r.to x and y

From (2) and (3) we get,

$$p = 2x(y^2 + b) \quad .....(2)$$
$$q = 2y(x^2 + a) \quad .....(3)$$
$$\frac{p}{2x} = y^2 + b \quad .....(4)$$
$$\frac{q}{2y} = x^2 + a \quad ......(5)$$

Substituting (4) and (5) in (1) , we get,

$$z = \frac{p}{2x}\frac{q}{2y}$$

$$\Rightarrow \; pq = 4xyz$$

2. From a PDE by eliminating the arbitrary constants a and b from $\log(az-1) = x + ay + b$.

   Sol:

   Given $\log(az-1) = x + ay + b \quad ......(1)$

   Diff (1) partially w.r.to x and y

   From (2)and (3),

$$\frac{ap}{az-1} = 1 \quad .....(2)$$
$$\frac{aq}{az-1} = a \quad .....(3)$$
$$ap = (az-1)$$
$$\Rightarrow a = \frac{1}{z-p} \quad .....(4)$$
$$q = (az-1) \quad .....(5)$$

   Substituting (4) and (5)

$$q = \frac{z}{z-p} - 1$$
$$\Rightarrow q(z-p) = p$$
$$\Rightarrow p(q+1) = zq$$

3. Form a PDE by eliminating arbitrary constants a,b,and c from $z = ax + by + cxy.$

Sol:

Given $z = ax + by + cxy$ .....(1)

Diff (1) partially w.r.to x and y,

$$p = a + cy \quad .....(2)$$

$$q = b + cx \quad .....(3)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = c \quad .....(4)$$

From (1) , (2) ,(3)we cannot eliminate the arbitrary constants,
So diff (2) and (3) partially w.r.to x and y

$$r = \frac{\partial^2 z}{\partial x^2} = 0 \quad .....(5)$$

$$t = \frac{\partial^2 z}{\partial y^2} = 0 \quad .....(6)$$

Sub (4) in (3) , we get

$$q = b + sx$$

$$\Rightarrow b = q - sx \quad .....(7)$$

Sub (4) in (2) , we get

$$p = a + sy$$

$$\Rightarrow a = p - sy \quad .....(8)$$

Sub (7), (8), and (4) in (1) , we get

$$z = (p - sy)x + (q - sx) + sxy.$$

4. Form a PDE by eliminating the arbitrary constants a and b from $(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha.$

Sol:

Given $(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha.$ ......(1)

Diff (1) partially w.r.to x and y

$$2(x-a)=2zp\cot^2\alpha$$

$$\Rightarrow (x-a)=zp\cot^2\alpha \quad .....(2)$$

$$2(y-b)=2zp\cot^2\alpha$$

$$\Rightarrow (y-b)=zp\cot^2\alpha .....(3)$$

Sub (2) and (3) in (1) , we get

$$z^2p^2\cot^4\alpha+z^2q^2\cot^4\alpha=z^2\cot^2\alpha$$

$$\Rightarrow p^2+q^2=\tan^2\alpha$$

5. Find the PDE of all planes having equal intercepts on the x and y axis.

   Sol:

Equation of such plane is $\frac{x}{a}+\frac{y}{a}+\frac{z}{b}=1$ .....(1)

Diff (1) partially w.r.to x and y

$$\frac{1}{a}+\frac{p}{b}=0$$

$$\Rightarrow p=\frac{-b}{a} \quad .....(2)$$

$$\frac{1}{a}+\frac{q}{b}=0$$

$$\Rightarrow q=\frac{-b}{a} \quad .....(3)$$

From (2) and (3) , we get

$$p = q \ .$$

6. Form the PDE from $z=ax^3+by^3$

Sol:

Given $z=ax^3+by^3$ .....(1)

Diff partially w.r.to x and y

$$p=3ax^2 \Rightarrow a=\frac{p}{3x^2} \quad .....(2)$$

$$q=3by^2 \Rightarrow b=\frac{q}{3y^2} \quad .....(3)$$

sub (2) and (3) in (1)

$$px + qy = 3z$$

7. Form the PDE from $z = ax^n + by^n$

Sol:

Diff partially w.r.to x and y

$$p = 3ax^{n-1} \Rightarrow a = \frac{p}{3x^{n-1}} \qquad .....(2)$$

$$q = 3by^{n-1} \Rightarrow b = \frac{q}{3y^{n-1}} \qquad .....(3)$$

sub (2) and (3) in (1)

$$px + qy = zn$$

8. Form the PDE from $z = (2x^2 + a)(3y - b)$

Sol:

Given $z = (2x^2 + a)(3y - b) \qquad ....(1)$

Diff partially w.r.to x and y

$$p = (3y - b)4x \Rightarrow (3y - b) = \frac{p}{4x} \qquad .....(2)$$

$$q = (2x^2 + a)(3) \Rightarrow (2x^2 + a) = \frac{q}{3y^2} \qquad .....(3)$$

Sub (2) and (3) in (1)

$$12\ xz = pq$$

9. Find the PDE of all planes having equal intercepts on the x and y axis.

Sol:

The Plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad .....(1)$

Diff partially w.r.to x and y

$$\frac{1}{a} + \frac{p}{c} = 0 \qquad .....(2)$$

$$\frac{1}{b} + \frac{q}{c} = 0 \qquad .....(3)$$

From (2) and (3) ,we get

$$\frac{p}{q}=1$$

$$\Rightarrow p=q$$

Formation of PDE by elimination of arbitrary functions:

The elimination of one arbitrary function from a given relation gives a PDE of first order while elimination of two arbitrary function from a given relation gives a second or higher order PDE.

10. Form the PDE by eliminating the arbitrary function from $z=f\left(x^2-y^2\right)$.

Sol:

Given $z=f\left(x^2-y^2\right)$ .....(1)

Diff (1) partially w.r.to x and y

$$p=f'\left(x^2-y^2\right)(2x)$$

$$\Rightarrow f'\left(x^2-y^2\right)=\frac{p}{2x} \quad .....(2)$$

$$q=f'\left(x^2-y^2\right)(-2y)$$

$$\Rightarrow f'\left(x^2-y^2\right)=\frac{-q}{2y} \quad .....(3)$$

From (2) and (3) , we get

$$\frac{p}{2x}=\frac{-q}{2y}$$

$$\Rightarrow py=-qx$$

$$py+qx=0.$$

11. Eliminate the arbitrary function f from $z=f\left(\frac{xy}{z}\right)$ and form the PDE.

Sol:

Given $z=f\left(\frac{xy}{z}\right)$

$$p = f'\left(\frac{xy}{z}\right).\frac{zy - xy.p}{z^2} \qquad .....(1)$$

$$q = f'\left(\frac{xy}{z}\right).\frac{zx - xy.q}{z^2} \qquad .....(2)$$

$$(1) \Rightarrow f'\left(\frac{xy}{z}\right) = \frac{pz^2}{zy - xyp} \qquad .....(3)$$

Sub (3) in (2) , we get

$$q = \frac{pz^2}{zy - xyp}.\frac{zx - xyq}{z^2} = \frac{px(z - yq)}{zy - xyp}$$

$$qzy - xypq = pxz - xypq$$

$$\Rightarrow px = qy$$

12.Form the PDE by eliminating the arbitrary functions f and g from $z = f(2x + y) + g(3x - y)$

Sol:Given $z = f(2x + y) + g(3x - y)$ ……(1)

$$p = 2f'(2x + y) + 3g(3x - y)$$

$$p = 2f' + 3g' \qquad .....(2)$$

$$q = f'(2x + y) - g(3x - y)$$

$$q = f' - g' \qquad .....(3)$$

$$r = 4f'' + 9g'' \qquad .....(4)$$

$$s = 2f'' - 3g'' \qquad .....(5)$$

$$t = f'' + g'' \qquad .....(6)$$

Eliminating f" and g" from (4) , (5)and (6) , we get

$$\begin{vmatrix} 4 & 9 & r \\ 2 & -3 & s \\ 1 & 1 & t \end{vmatrix} = 0$$

$$4(-3t - s) - 9(2t - s) + r(2 + 3) = 0$$

$$-12t - 4s - 18t + 9s + 5r = 0$$

$$5r + 5s - 30t = 0$$

$$\Rightarrow r + s - 6t = 0.$$

13. Form the PDE by eliminating arbitrary function f from $z=e^{ay} f(ax+by)$.

Sol:

Given $z=e^{ay} f(ax+by)$ .....(1)

$$p=e^{ay} f'(ax+by).a \quad .....(2)$$

$$q=e^{ay} f'(ax+by).b+f(ax+by)e^{ay}.a \quad .....(3)$$

$$(1)\Rightarrow f(ax+by)=\frac{z}{e^{ay}} \quad .....(4)$$

$$(2)\Rightarrow f'(ax+by)=\frac{p}{ae^{ay}} \quad .....(5)$$

Sub (4) and (5) in (3) , we get

$$q=be^{ay}\frac{p}{ae^{ay}}+\frac{z}{e^{ay}}.e^{ay}.a$$

$$p=\frac{bp}{a}+az$$

14. Form the PDE from $z=f(2x-6y)$

Sol:

Given $z=f(2x-6y)$

Diff partially w.r.to x and y

$$p=f'(2x-6y).2 \quad ....(2)$$

$$q=f'(2x-6y).(-6) \quad .....(3)$$

From (2) and (3) , we get

$$\frac{p}{q}=\frac{2}{-6}$$

$$\Rightarrow 3p=-q$$

15.Form the PDE from $z=f(x^2+y^2)$

Sol:

Given $z=f(x^2+y^2)$ ....(1)

Diff partially w.r.to x and y

$$p = f'(x^2 + y^2).2x \qquad .....(2)$$

$$q = f'(x^2 + y^2).2y \qquad .....(3)$$

From (2) and (3)

$$py - qx = 0$$

16. Form the PDE from $z = f\left(\frac{y}{x}\right)$

Sol:

Given $z = f\left(\frac{y}{x}\right) \qquad .....(1)$

Diff partially w.r.to x and y

$$p = f'.\left(\frac{-y}{x^2}\right) \qquad .....(2)$$

$$q = f'.\left(\frac{1}{x}\right) \qquad .....(3)$$

From (2) and (3) we get

$$\frac{p}{q} = \frac{-y}{x}$$

$$\Rightarrow px + qy = 0$$

17. Form the PDE from $x + y + z = f(x^2 + y^2 + z^2)$

Sol:

Given $x + y + z = f(x^2 + y^2 + z^2) \qquad ....(1)$

Diff partially w.r.to x and y

$$1 + p = f'(x^2 + y^2 + z^2)(2x + 2zp) \qquad .....(2)$$

$$1 + q = f'(x^2 + y^2 + z^2)(2y + 2zq) \qquad .....(3)$$

From (2) and (3)

$$\frac{1 + p}{1 + q} = \frac{x + zp}{y + zq}$$

$$(y - z)p + (z - x)q = x - y$$

18. Form the PDE from $ax+by+cz=\phi(x^2+y^2+z^2)$

Sol:

Given $ax+by+cz=\phi(x^2+y^2+z^2)$ .....(1)

Diff partially w.r.to x and y

$$a+cp=\phi'(x^2+y^2+z^2)(2x+2zp) \quad .....(2)$$

$$b+cq=\phi'(x^2+y^2+z^2)(2y+2zq) \quad .....(3)$$

From (2) and (3)

$$\frac{a+cp}{b+cq}=\frac{x+zp}{y+zq}$$

$$(a+cp)y+(aq+bp)z=(b+cq)x$$

19. Form the PDE by eliminating the arbitrary functions from $z=f(2x+3y)+g(2x+y)$

Sol:

Given $z=f(2x+3y)+g(2x+y)$ .....(1)

Diff partially w.r.to x and y

$$p=f'(2x+3y).2+g'(2x+y).2$$

$$q=f'(2x+3y).3+g'(2x+y).1$$

$$r=f''(2x+3y).4+g''(2x+y).4 \quad ....(2)$$

$$s=f''(2x+3y).6+g''(2x+y).2 \quad ....(3)$$

$$t=f''(2x+3y).9+g''(2x+y) \quad .....(4)$$

Eliminating f and g from (2) , (3),(4)

$$3r-8s+4t=0$$

20. Form the PDE from $z=x^2 f(y)+y^2 g(x)$

Sol:

Given $z=x^2 f(y)+y^2 g(x)$ ....(1)

Diff partially w.r.to x and y

$$p=2x\,f(y)+y^2\,g'(x)$$

$$q=x^2\,f'(y)+2y\,g(x)$$

$$r = 2f(y) + y^2 g''(x) \qquad .....(2)$$

$$s = 2x f'(y) + 2y g'(x) \qquad .....(3)$$

$$t = x^2 f''(y) + 2g(x) \qquad ......(4)$$

Eliminating f and g from (2), (3) and (4)

$$s = \frac{2q}{x} + \frac{2p}{y} - \frac{4z}{xy}$$

$$4z = 2yq + 2px - sxy$$

21. Form the PDE from $z = f(y) + \phi(x + y + z)$

Sol:

Given $z = f(y) + \phi(x + y + z) \qquad ....(1)$

Diff partially w.r.to x and y

$$p = (1+p)\,\phi' \qquad ....(2)$$

$$q = (1+q)\,\phi' \qquad ....(3)$$

$$r = \phi'.(r) + (1+p)^2\,\phi'' \qquad .....(4)$$

$$s = \phi'.(s) + (1+p)(1+q)\,\phi'' \qquad .....(5)$$

$$t = f'' + t\,.\,\phi' + (1+q)^2\,\phi'' \qquad .....(6)$$

$$\frac{r}{s} = \frac{\phi'.(r) + (1+p)^2\,\phi''}{\phi'.(s) + (1+p)(1+q)\,\phi''}$$

$$(1+q)\,.r = (1+p)\,.s$$

22. Form the PDE from $z = f(x+t) + g(x-t)$

Sol:

Given $z = f(x+t) + g(x-t) \qquad ...(1)$

Diff partially w.r.to x and y

$$p = f' + g'$$

$$q = f' - g'$$

$$r = f'' + g'' \qquad .....(2)$$

$$s = f'' - g'' \qquad .....(3)$$

$$t = f'' + g'' \qquad .....(4)$$

From (2),(3) and (4)

$$r - t = f'' + g'' - f'' - g''$$
$$r - t = 0$$

Formation of PDE by eliminating the function ϕ from $\phi(u,v)=0$ where u and v are functions of x , y and z.

Let $\phi(u,v)=0$ .....(1) be a given function of u and v.

Diff (1) partially w.r.to x and y we get,

$$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x}=0 \qquad .....(2)$$
$$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y}=0 \qquad .....(3)$$

To eliminate ϕ, it is enough to eliminate $\frac{\partial \phi}{\partial u}$ *and* $\frac{\partial \phi}{\partial v}$

Eliminating ϕ from (2) and (3) , we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}=0 \qquad .....(4)$$

The above four partial derivatives can be derived from u and v.

23. Form the PDE by eliminating f, from $f(xy+z^2, x+y+z)=0$.

Sol:

Given $f(xy+z^2, x+y+z)=0$ .....(1)

$\therefore$ (1) is of the form $\phi(u,v)=0$

Let $u=xy+z^2$ *and* $v=x+y+z$

$$\frac{\partial u}{\partial x}=y+2zp \qquad \frac{\partial v}{\partial x}=1+p$$
$$\frac{\partial u}{\partial y}=x+2zq \qquad \frac{\partial v}{\partial y}=1+q$$

Sub the above derivatives in $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}=0$

$$\begin{vmatrix} y+2zp & 1+p \\ y+2zq & 1+q \end{vmatrix}=0$$

$$(1+q)(y+2zp)-(1+p)(y+2zq)=0$$

$$(2z-x)p+(y-2z)q=x-y.$$

24. Form the PDE by eliminating f, from $f(z^2-xy, \frac{x}{z})=0.$

Sol:

Given $f(z^2-xy, \frac{x}{z})=0.$

$\therefore$ (1) is of the form $\phi(u,v)=0$

Let $u=z^2-xy \ and \ v=\frac{x}{z}$

$$\frac{\partial u}{\partial x}=2zp-y \qquad \frac{\partial v}{\partial x}=\frac{z-px}{z^2}$$

$$\frac{\partial u}{\partial y}=2zq-x \qquad \frac{\partial v}{\partial y}=\frac{-xq}{z^2}$$

Sub the above derivatives in $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}=0$

$$\begin{vmatrix} 2zp-y & \frac{z-px}{z^2} \\ 2zq-x & \frac{-xq}{z^2} \end{vmatrix}=0$$

$$(2zp-y)\left(\frac{-xq}{z^2}\right)-\left(\frac{z-px}{z^2}\right)(2zq-x)=0$$

$$multiplying \ by \ z^2$$

$$(2zp-y)(-xq)-(z-px)(2zq-x)=0$$

$$px^2-q(xy-2z^2)=zx.$$

25. Form the PDE by eliminating f, from $f(x^2+y^2+z^2, xyz)=0.$

Sol:

Given $f\left(x^2+y^2+z^2,xyz\right)=0.$

$\therefore$ (1) is of the form $\phi(u,v)=0$

Let $u=x^2+y^2+z^2 \quad and \quad v=xyz$

$$\frac{\partial u}{\partial x}=2x+2zp \qquad \frac{\partial v}{\partial x}=y(xp+z)$$

$$\frac{\partial u}{\partial y}=2y+2zq \qquad \frac{\partial v}{\partial y}=x(yq+z)$$

Sub the above derivatives in $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}=0$

$$\begin{vmatrix} 2x+2zp & y(xp+z) \\ 2y+2zq & x(yq+z) \end{vmatrix}=0$$

$$(2x+2zp)x(yq+z)-(2y+2zq)y(xp+z)=0$$

$$px(z^2-y^2)+qy(x^2-z^2)=z(y^2-x^2)$$

26. Form the PDE from $\phi\left(x^2+y^2+z^2 \ , lx+my+nz\right)=0$

Sol:

Given eqn is of the form $\phi(u,v)=0$

$$u=x^2+y^2+z^2 \qquad , v=lx+my+nz$$

$$u_x=2x \qquad , v_x=l$$

$$u_y=2y \qquad , v_y=m$$

$$u_z=2z \qquad , v_z=n$$

$$P=\begin{vmatrix} 2y & m \\ 2z & n \end{vmatrix}=2ny-2mz=2(ny-mz)$$

$$Q=\begin{vmatrix} 2z & n \\ 2x & l \end{vmatrix}=2zl-2xn=2(zl-xn)$$

$$R=\begin{vmatrix} 2x & l \\ 2y & m \end{vmatrix}=2xm-2yl=2(xm-yl)$$

The solution is $Pp+Qq=R$

$$(ny-mz)p+(zl-xn)q=(xm-yl)$$

27. Form the PDE from $\phi\left(x+y+z, x^2+y^2-z^2\right)=0$

Sol:

Given eqn is of the form $\phi(u,v)=0$

$$u=x+y+z \qquad ,v=x^2+y^2+z^2$$

$$u_x=1 \qquad ,v_x=2x$$

$$u_y=1 \qquad ,v_y=2y$$

$$u_z=1 \qquad ,v_z=2z$$

$$P=\begin{vmatrix}1 & 2y\\ 1 & -2z\end{vmatrix}=-2z-2y=-2(z+y)$$

$$Q=\begin{vmatrix}1 & -2z\\ 1 & 2x\end{vmatrix}=2x+2z=2(x+z)$$

$$R=\begin{vmatrix}1 & 2x\\ 1 & 2y\end{vmatrix}=2y-2x=2(y-x)$$

The solution is $Pp+Qq=R$

$$(y+z)p-(x+z)q=(y-x)$$

28. Form the PDE from $f(x^2+y^2+z^2, xyz)$

Sol:

Given eqn is of the form $\phi(u,v)=0$

$$u=x^2+y^2+z^2 \qquad , v=xyz$$

$$u_x=2x \qquad , v_x=yz$$

$$u_y=2y \qquad , v_y=xz$$

$$u_z=2z \qquad ,v_z=xy$$

$$P=\begin{vmatrix}2y & xz\\ 2z & xy\end{vmatrix}=2xy^2-2xz^2=2x(y^2-z^2)$$

$$Q=\begin{vmatrix}2z & xy\\ 2x & yz\end{vmatrix}=2yz^2-2yx^2=2y(z^2-x^2)$$

$$R=\begin{vmatrix}2x & yz\\ 2y & xz\end{vmatrix}=2zx^2-2zy^2=2z(x^2-y^2)$$

The solution is $Pp+Qq=R$

$$x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$$

29. Form the PDE from $\phi(x-y, x+y+z)=0$

Sol:

Given eqn is of the form $\phi(u,v)=0$

$u = x - y \qquad , v = x + y + z$

$u_x = 1 \qquad , v_x = 1$

$u_y = -1 \qquad , v_y = 1$

$u_z = 0 \qquad , v_z = 1$

$$P = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1 - 0 = -1$$

$$Q = \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} = 0 + 1 = 1$$

$$R = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 1 + 1 = 2$$

The solution is $Pp + Qq = R$

$$-p - q = 2$$

$$p + q + 2 = 0$$

TYPES OF SOLUTION

COMPLETE INTEGRAL (OR) COMPLETE SOLUTION:

A solution in which the number of arbitrary constants is equal to the number of independent variables is called Complete Integral (or) Complete Solution.

PARTICULAR SOLUTION:

In Complete integral if we give particular values to the arbitrary constants we get Particular Solution.

SINGULAR SOLUTION:

Let $f(x,y,z,p,q)=0$ be a PDE whose complete integral is

$\phi(x,y,z,a,b)=0$ .....(1)

Diff (1) partially w.r.to a and band then equate to zero, we get

$$\frac{\partial\phi}{\partial a}=0 \qquad .....(2)$$

$$\frac{\partial\phi}{\partial a}=0 \qquad .....(3)$$

Eliminate a and b from (1), (2)and (3). Eliminant of a and b is called Singular Solution.

TYPE 1:

EQUATION OF THE FORM $F(p,q)=0$:

In this type we are having only p and q and there is no x, y, and z. To solve this type of problems , let us assume that $z=ax+by+c$ be the solution of the given differential equation.

30. Solve pq = 1.

Sol:

Given pq = 1 .....(1)

This is of the form $F(p,q)=0$

Let $z=ax+by+c$ .....(2) be the solution of (1)

Diff (1) partially w.r.to x and y

$$\Rightarrow p=a, q=b \quad .....(3)$$

Sub (3) in (1) , we get $\Rightarrow ab=1$ .....(4)

From (4), we get $a=\frac{1}{b}$ .....(5)

Eqn (2) $z=\frac{1}{b}x+by+c$ is the complete solution of (1).

31 . Solve $p^2+q^2=npq$.

Sol:

Given $p^2+q^2=npq$ .....(1)

This is of the form $F(p,q)=0$

Let $z=ax+by+c$ .....(2) be the solution of (1)

Diff (1) partially w.r.to x and y

$$\Rightarrow p=a, q=b \quad .....(3)$$

Sub (3) in (1) , we get

$$\Rightarrow a^2 + b^2 = nab \quad .....(4)$$

$$\Rightarrow a^2 - nab + b^2 = 0$$

From (4), we get

$$a = \frac{nb \pm \sqrt{n^2b^2 - 4b^2}}{2}$$

$$\Rightarrow \quad a = \frac{b}{2}\left[n \pm \sqrt{n^2 - 4}\right] \quad .....(5)$$

Sub (5) in (4) , we get

$$z = \frac{b}{2}\left[n \pm \sqrt{n^2 - 4}\right]x + by + c$$ is the complete solution.

32. Solve the equation $pq + p + q = 0.$

Sol:

Given $pq + p + q = 0 \quad .....(1)$

This is of the form $F(p, q) = 0$

Let $z = ax + by + c \quad .....(2)$ be the solution of (1)

Diff (1) partially w.r.to x and y

$$\Rightarrow p = a, q = b \quad .....(3)$$

Sub (3) in (1), we get

$$\Rightarrow ab + a + b = 0 \quad .....(4)$$

$$ab + b + a = 0$$

$$b(a + 1) + a = 0$$

$$b(a + 1) = -a$$

$$b = \frac{-a}{a + 1} \quad .....(5)$$

Sub (5) in (1), we get

$$z = ax - \frac{a}{a+1}y + c$$ is the complete solution.

33. Solve $p^2+q^2=4$.

Sol:

Given $p^2+q^2=4$ .....(1)

This is of the form $F(p,q)=0$

Let $z=ax+by+c$ .....(2) be the solution of (1)

Diff (1) partially w.r.to x and y

$$\Rightarrow p=a, q=b \quad .....(3)$$

Sub (3) in (1) , we get $\Rightarrow a^2+b^2=4$ .....(4)

From (4) , $b=\pm\sqrt{4-a^2}$ .....(5)

Sub (5) in (1), we get $z=ax\pm y\sqrt{4-a^2}+c$ .....(6)

To find Singular integral:

Diff (6) partially w.r.to a and c and equating to zero, we get

$$\frac{\partial z}{\partial a}=x\pm\frac{1}{2\sqrt{4-a^2}}(-2a)=0$$

$$\frac{\partial z}{\partial c}=1=0$$

Here 1 = 0 is not possible. Hence there is no Singular solution.

34. Solve $\sqrt{p}+\sqrt{q}=1$

Sol:

Given $\sqrt{p}+\sqrt{q}=1$

Let $z=ax+by+c$ .....(1) be the solution.

$$f(a,b)=\sqrt{a}+\sqrt{b}-1=0$$

$$\sqrt{b}=1-\sqrt{a}$$

$$b=\left(1-\sqrt{a}\right)^2 \quad .....(2)$$

Sub (2) in (1)

$$z=ax+\left(1-\sqrt{a}\right)^2 y+c \quad .....(3)$$

This is the complete solution
To get singular solution:
Diff w.r.to c, both sides

0=1 (impossible)

$\therefore$ No singular solution.
To get general solution:

Put $c=\phi(a)$, $\Rightarrow$ $z=ax+\left(1-\sqrt{a}\right)^2 y+\phi(a)$ .....(4)

Diff w.r.to a , we get

$$0=x+2(1-\sqrt{a})\left(\frac{-1}{2\sqrt{a}}\right)y+\phi'(a) \quad .....(5)$$

Eliminating a from (4) and (5) , we get the general solution

35. Solve $q+\sin\ p=0$

Sol:

Given $f(p,q)=q+\sin p=0$ ......(1)

Let $z=ax+by+c$ .....(2) be the solution

$$f(a,b)=b+\sin a=0$$

$$b=-\sin a$$

$$z=ax-(\sin a)y+c \quad .....(3)$$

This is the complete solution.

Diff (3) w.r.to c

0=1 (impossible)

$\therefore$ No singular solution.

Put $c=\phi(a)$ in (3)

$$z=ax-(\sin a)y+\phi(a) \quad .....(4)$$

Diff w.r.to a

$$0=x-(\cos a)y+\phi'(a) \quad .....(5)$$

Eliminating a from (4) and (5), we get the general solution.

TYPE 2: CLAIRAUT'S FORM

EQUATION OF THE TYPE $z=px+qy+f(p,q)$

Assume that $z=ax+by+f(a,b)$ be the solution. This is solution is obtained by putting p=a and q=b.

TO FIND SINGULAR INTEGRAL:

The Complete integral is $z = ax + by + f(a,b)$ .....(1)

Diff (1) partially w.r.to a and b, we get

$$0 = x + \frac{\partial f(a,b)}{\partial a} \quad .....(2)$$

$$0 = y + \frac{\partial f(a,b)}{\partial b} \quad .....(3)$$

Eliminating a and b from (1) , (2) and (3) gives the Singular Solution.

NOTE:

The above complete solution gives a family of planes. The singular solution (if it exist) is a surface having the complete solution as its tangent planes.

36. Find the singular integral of $z = px + qy + pq$.

Given $z = px + qy + pq$ .....(1)

It is of the form $z = px + qy + f(p,q)$

The complete solution is $z = ax + by + ab$ .....(2) $[put \ \ p = a, q = b]$

Diff (2) partially w.r.to a and b, we get

$$0 = x + b \Rightarrow b = -x \quad .....(3)$$

$$0 = y + a \Rightarrow \ a = -y \quad .....(4)$$

Sub (3) and (4) in (2) , we get

$$z = -xy - xy + xy$$

$$z = -xy$$

Is the singular solution of (1).

37. Solve $z = px + qy + \left(\frac{q}{p} - p\right).$

Sol:

Given $z = px + qy + \left(\frac{q}{p} - p\right).$

This is of Clairaut's type.

The complete solution is $z = ax + by + \left(\frac{b}{a} - a\right).$ .....(1)

To find the singular solution:

Diff (1) partially w.r.to a and b

$$0 = x - \frac{b}{a^2} - 1 \quad .....(2)$$

$$0 = y + \frac{1}{a} \quad .....(3)$$

$$(3) \Rightarrow a = \frac{-1}{y} \quad .....(4)$$

Sub (4) in (2), we get

$$0 = x - \frac{b}{\frac{1}{y^2}} - 1$$

$$0 = x - by^2 - 1$$

$$by^2 = x - 1$$

$$b = \frac{x-1}{y^2} \quad .....(5)$$

Sub (4) and (5) in (1), we get

$$z = \frac{-x}{y} + \frac{x-1}{y} - \frac{x-1}{y} + \frac{1}{y}$$

$$= \frac{-x + x - 1 - x + 1 + 1}{y}$$

$$zy = 1 - x$$

38. Solve $z = px + qy + 2\sqrt{pq}$.

Given $z = px + qy + 2\sqrt{pq}$.

This is of Clairaut's form.

The complete solution is $z = ax + by + 2\sqrt{ab} \quad .....(1)$

To find the singular integral:

Diff (1) partially w.r.to a and b, and equating to zero ,we get

$$0 = x + \sqrt{\frac{b}{a}} \Rightarrow x = -\sqrt{\frac{b}{a}} \quad .....(2)$$

$$0 = y + \sqrt{\frac{a}{b}} \Rightarrow y = -\sqrt{\frac{a}{b}} \quad .....(3)$$

Eliminating a and b from (2) and (3)

$$xy = \left(-\sqrt{\frac{b}{a}}\right)\left(-\sqrt{\frac{a}{b}}\right)$$

$$xy = 1$$

39. Solve $(1-x)p+(2-y)q=3-z$.

Sol:

Given $(1-x)p+(2-y)q=3-z$

$$p - px + 2q - qy = 3 - z$$

$$z = px + qy - p - 2q + 3$$

The complete solution is $z = ax + by - a - 2b + 3 \quad .....(1)$

To find singular solution:

Diff (1) partially w.r.to a and b, we get

$$x - 1 = 0 \Rightarrow x = 1 \quad .....(2)$$

$$y - 2 = 0 \Rightarrow y = 2 \quad .....(3)$$

To find the singular solution we have to eliminate a and b from (1), (2) and (3).

Sub (2) and (3) in (1) we get z = 3.

TYPE 3:

EQUATIONS OF THE FORM $F(z, p, q) = 0$

Given $F(z, p, q) = 0 \quad .....(1)$ [In this type x and y do not appear explicitly]

Let $z = f(x+ay)$ be the solution of (1).

Put $x + ay = u \quad ....(2)$

Then $z = f(u) \quad .....(3)$

Diff (3) partially w.r.to x and y we get

$$p=\frac{\partial z}{\partial x}=\frac{dz}{du}.\frac{\partial u}{\partial x}=\frac{dz}{du}.1 \quad \left(\because \frac{\partial u}{\partial x}=1\right)$$

$$q=\frac{\partial z}{\partial y}=\frac{dz}{du}.\frac{\partial u}{\partial y}=\frac{dz}{du}.a \quad \left(\because \frac{\partial u}{\partial y}=a\right)$$

i.e., $p=\frac{dz}{du}$, $q=a\frac{dz}{du}$

If we substitute $p=\frac{dz}{du}$, $q=a\frac{dz}{du}$ in (1), then we get an ordinary diff eqn of the form

$f\left[z,\frac{dz}{du},a\frac{dz}{du}\right]=0$, which can be solved by method of variable seperable.

The solution of this diff eqn gives the complete integral of (1). General and singular solution can be obtained as usual.

40. Solve $p^3=qz.$

Sol:

Given $p^3=qz$ .....(1)

This is of the type $F(z,p,q)=0$

Let $z=f(x+ay)$ be the solution of (1).

Put $x+ay=u$

$$p=\frac{dz}{du} \text{ , } q=a\frac{dz}{du} \quad .....(2)$$

Sub (2) in (1) , we get

$$\left(\frac{dz}{du}\right)^3 = a\frac{dz}{du}z$$

$$\left(\frac{dz}{du}\right)^2 = az$$

$$\frac{dz}{du}=\pm\sqrt{a}\sqrt{z}$$

$$\frac{dz}{\sqrt{z}} = \pm\sqrt{a}\,du$$

Integrating , we get

$$\int \frac{dz}{\sqrt{z}} = \pm \sqrt{a} \int du$$

$$2\sqrt{z} = \pm \sqrt{a}(u) + b$$

Squaring on both sides

$$4z = a(u+b)^2 \quad where \ \ b = \frac{c}{\sqrt{a}}$$

The general solution is $4z = a(x+ay+b)^2$.

41. Solve $z^2 = p^2 + q^2 + 1$

Sol:

Given $z^2 = p^2 + q^2 + 1 \quad .....(1)$

This is of the type $F(z, p, q) = 0$

Let $z = f(x+ay)$ be the solution of (1).

Put $x+ay=u$

$$p = \frac{dz}{du} \ , \ q = a\frac{dz}{du} \quad .....(2)$$

Sub (2) in (1), we get

$$z^2 = \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 + 1$$

$$\left(\frac{dz}{du}\right)^2 (1+a^2) = z^2 - 1$$

$$\left(\frac{dz}{du}\right)^2 = \frac{z^2-1}{1+a^2}$$

$$\frac{dz}{du} = \frac{1}{\sqrt{1+a^2}}\sqrt{z^2-1}$$

$$\int \frac{dz}{\sqrt{z^2-1}} = \int \frac{du}{\sqrt{1+a^2}}$$

$$\cosh^{-1} z = \frac{1}{\sqrt{1+a^2}}.u + b$$

$$\cosh^{-1} z = \frac{1}{\sqrt{1+a^2}}(x+ay) + b$$

TYPE 4:

EQUATION OF THE FORM $F_1(x,p) = F_2(y,q)$

Let $F_1(x,p) = F_2(y,q) = k\ (say)$ .....(1)

From (1) we get $p = f_1(x,k)$, $q = f_2(y,k)$ .....(2)

We know that

$$dz = pdx + qdy \quad .....(3)$$

Sub (2) in (3) we get

$$dz = f_1(x,k)dx + f_2(y,k)dy$$

Integrating we get $z = \int f_1(x,k)dx + \int f_2(y,k)dy + b$ is the complete integral.

42. Solve $p - q = x^2 + y^2$.

Sol:Given $p - q = x^2 + y^2$.

$$p - x^2 = q - y^2 = k(say)$$

$$p - x^2 = k \ , \ q - y^2 = k$$

$$p = k + x^2 \ , \ q = k - y^2$$

$$z = \int pdx + \int qdy$$

$$= \int (x^2 + k)dx + \int (k - y^2)dy$$

$$= \frac{x^3}{3} + kx + ky - \frac{y^3}{3} + a$$

$$z = \frac{1}{3}(x^3 - y^3) + k(x+y) + a$$

43. Solve $\sqrt{p}+\sqrt{q}=x+y$.

Sol:

Given $\sqrt{p}+\sqrt{q}=x+y$

$$\sqrt{p}-x=y-\sqrt{q}=k\,(say)$$

$$\sqrt{p}-x=k \quad , y-\sqrt{q}=k$$

$$\sqrt{p}=k+x \quad , \sqrt{q}=y-k$$

$$p=(k+x)^2 \;\; , q=(y-k)^2$$

We know that

$$z=\int p\,dx+\int q\,dy$$

$$=\int (k+x)^2\,dx+\int (y-k)^2\,dy$$

$$=\frac{(k+x)^3}{3}+\frac{(y-k)^3}{3}+b$$

44. Solve $pe^y=qe^x$.

Sol:

Given $pe^y=qe^x$

$$\frac{p}{e^x}=\frac{q}{e^y}=k\,(say)$$

$$p=ke^x \;\; , q=ke^y$$

We know that

$$z=\int p\,dx+\int q\,dy$$

$$=\int ke^x\,dx+\int ke^y\,dy$$

$$=k(e^x+e^y)+b$$

45. Solve $p^2+q^2=x+y$.

Sol:

Given $p^2+q^2=x+y$

$$p^2-x=y-q^2=k$$

$$p^2=k+x \ , q^2=y-k$$

$$p=\sqrt{k+x} \, , q=\sqrt{y-k}$$

We know that

$$z=\int p\,dx+\int q\,dy$$

$$=\int \sqrt{k+x}\ dx+\int \sqrt{y-k}\ dy$$

$$=\frac{2}{3}(k+x)^{\frac{3}{2}}+\frac{2}{3}(y-k)^{\frac{3}{2}}$$

TYPE 5:

EQUATION OF THE TYPE $F(x^m p, y^n q)=0 \ and F(z, x^m p, y^n q)=0.$

CASE 1: If $m\neq 1 \ and \ n\neq 1$, then put $x^{1-m}=X$, $y^{1-n}=Y$ .....(1)

Now $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}.\frac{\partial X}{\partial x}=P(1-m).x^{-m}$ $\left[\because \frac{\partial X}{\partial x}=(1-m)\,x^{-m} \ and \ \frac{\partial z}{\partial X}=P\right]$

$$p=P(1-m)x^{-m}$$

$$x^m p=(1-m)P \quad .....(2)$$

Similarly $\frac{\partial z}{\partial y}=\frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y}=Q(1-n).y^{-n}$ $\left[\because \frac{\partial Y}{\partial y}=(1-n)\,y^{-n} \ and \ \frac{\partial z}{\partial Y}=Q\right]$

$$q=Q(1-n)y^{-n}$$

$$y^n q=(1-n)Q \quad .....(3)$$

where $P=\frac{\partial z}{\partial X}$, $Q=\frac{\partial z}{\partial Y}$

Sub (2) and (3) in the given eqn we get a diff eqn of the form $F(P,Q)=0$ (Type 1) and $F(z,P,Q)=0$ (Type 3) which can be easily solved.

CASE 2: If m=n=1 , then

Put $\log x=X$, $\log y=Y$

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x}$$

$$p=P.\frac{1}{x} \qquad \left[\because P=\frac{\partial z}{\partial X},\ \ \frac{\partial X}{\partial x}=\frac{1}{x}\right]$$

$$xp=P \qquad .....(4)$$

$$similarly\ \ yq=Q \quad \left[q=\frac{\partial z}{\partial Y}\right] \qquad .....(5)$$

Sub (4) and (5) in the given eqn we get a diff eqn of the form $F(P,Q)=0$ and $F(z,P,Q)=0$ (Type 3) which can be easily solved.

46. Solve $\frac{x^2}{p}+\frac{y^2}{q}=z$

Sol:

Given equation can be written as $\frac{1}{x^{-2}p}+\frac{1}{y^{-2}q}=z \qquad .....(1)$

This is of type $F(z,x^m p,y^n q)=0.$

Here m =-2 and n = -2

$$x^{1-(-2)}=X \Rightarrow X=x^3$$

$$y^{1-(-2)}=Y \Rightarrow Y=y^3$$

$$x^3p=3P$$

$$y^3q=3Q \qquad .....(2)$$

Sub (2) in (1) , we get

$$\frac{1}{P}+\frac{1}{Q}=3z$$

$$P+Q=3zPQ \qquad .....(3)$$

This is of $F(z,P,Q)=0$ (type 3)

Let $z=f(X+aY)$ be the solution of (3)

$$X + aY = u$$

$$z = f(u)$$

$$P = \frac{dz}{du}, Q = a\frac{dz}{du} \qquad .....(4)$$

Sub (4) in (3) we get

$$\frac{dz}{du} + a\frac{dz}{du} = 3z\left(\frac{dz}{du}\right)\left(a\frac{dz}{du}\right)$$

$$\frac{dz}{du}(1+a) = 3az\left(\frac{dz}{du}\right)^2$$

$$\frac{dz}{du} = \frac{1+a}{3az}$$

$$zdz = \frac{1+a}{3a}du$$

$$\int zdz = \int \frac{1+a}{3a}du$$

$$\frac{z^2}{2} = \frac{1+a}{3a}.u + b$$

$$\frac{z^2}{2} = \frac{1+a}{3a}(X + aY) + b$$

$$\frac{z^2}{2} = \frac{1+a}{3a}(x^3 + ay^3) + b$$

47. Solve $p^2x + q^2y = z.$

Sol:

Given $\left(px^{\frac{1}{2}}\right)^2 + \left(qy^{\frac{1}{2}}\right)^2 = z \qquad .....(1)$

This is of the type (5) , $F(z, x^m p, y^n q) = 0.$
Here

$$m=\frac{1}{2}, n=\frac{1}{2}$$

$$put\ x^{1-\frac{1}{2}}=X\ ,\ y^{1-\frac{1}{2}}=Y$$

$$x^{\frac{1}{2}}=X\ , y^{\frac{1}{2}}=Y$$

$$x^{\frac{1}{2}}p=\frac{1}{2}P$$

$$y^{\frac{1}{2}}p=\frac{1}{2}Q \qquad .....(2)$$

Sub (2) in (1), we get

$$\left(\frac{1}{2}P\right)^2+\left(\frac{1}{2}Q\right)^2=z \quad \Rightarrow P^2+Q^2=4z \qquad .....(3)$$

Let $z=f(X+aY)$ be the solution of (3)

Put

$$X+aY=u$$

$$z=f(u)$$

$$P=\frac{dz}{du}, Q=a\frac{dz}{du} \qquad .....(4)$$

Sub (4) in (3) , we get

$$\left(\frac{dz}{du}\right)^2+\left(a\frac{dz}{du}\right)^2=4z$$

$$\left(\frac{dz}{du}\right)^2(1+a^2)=4z$$

$$\left(\frac{dz}{du}\right)^2=\frac{4z}{1+a^2}$$

$$\frac{dz}{du}=\frac{2\sqrt{z}}{\sqrt{1+a^2}}$$

$$\frac{dz}{\sqrt{z}}=\frac{2\,du}{\sqrt{1+a^2}}$$

$$\int \frac{dz}{\sqrt{z}} = \int \frac{2du}{\sqrt{1+a^2}}$$

$$2\sqrt{z} = \frac{2}{\sqrt{1+a^2}}(u) + b$$

$$2\sqrt{z} = \frac{2}{\sqrt{1+a^2}}(X + aY) + b$$

$$2\sqrt{z} = \frac{2}{\sqrt{1+a^2}}\left(x^{\frac{1}{2}} + ay^{\frac{1}{2}}\right) + b$$

Take the transformation

$$X = \log x \qquad Y = \log y \qquad Z = z$$

$$\frac{\partial X}{\partial x} = \frac{1}{x} \qquad \frac{\partial Y}{\partial y} = \frac{1}{y} \qquad \frac{\partial Z}{\partial z} = 1$$

$$\frac{x}{\partial x} = \frac{1}{\partial X} \qquad \frac{y}{\partial y} = \frac{1}{\partial Y} \qquad \partial z = \partial Z$$

$$\frac{\partial Z}{\partial X} + \frac{\partial Z}{\partial Y} = 1 \qquad .....(1)$$

$$P + Q = 1$$

Let $z = aX + bY + c$ $.....(2)$ be the solution.

$$f(a,b) = a + b - 1 = 0 \Rightarrow b = 1 - a \qquad .....(3)$$

Sub (3) in (2)

$$z = aX + (1-a)Y + c$$

$$z = a\log x + (1-a)\log y + c$$

TYPE 6:

EQUATION OF THE FORM $F(z^m p, z^m q) = 0 \; and \; F(x, z^m p) = F(y, z^m q)$

CASE 1: If $m \neq -1$, $Put\, Z = z^{m+1}$

$$P = \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} = (m+1)p$$

$$\frac{P}{m+1} = z^m p \qquad .....(1)$$

$$\frac{Q}{m+1} = z^m q \qquad .....(2)$$

Sub (1) and (2) in the given equation we get a PDE of the form

$F(P,Q)=0\,(or)\,F_1(x,P)=F_2(y,Q)$ which can be easily solved .

CASE 2: If $m=-1,\ Put\ Z=\log z$

$$\frac{\partial Z}{\partial x}=\frac{\partial Z}{\partial z}.\frac{\partial z}{\partial x}=\frac{1}{z}.p=z^{-1}p$$

$$\frac{\partial Z}{\partial y}=\frac{\partial Z}{\partial z}.\frac{\partial z}{\partial y}=\frac{1}{z}.q=z^{-1}q$$

$$P=z^{-1}p\quad ,\ Q=z^{-1}q \qquad .....(1)$$

Sub (1) in the given equation we get a PDE of the form

$F(P,Q)=0\,(or)\,F_1(x,P)=F_2(y,Q)$ which can be solved easily by using type (1) or tyoe (4).

40. Solve $p^2+q^2=z(x^2+y^2)$.

Sol:

Given $p^2+q^2=z(x^2+y^2) \qquad .....(1)$

$$\frac{p^2}{z^2}+\frac{q^2}{z^2}=x^2+y^2$$

$$\left(z^{-1}p\right)^2+\left(z^{-1}q\right)^2=x^2+y^2 \qquad .....(2)$$

This is of type (6).

$$put\,Z=\log z$$

$$P=\frac{\partial Z}{\partial z}.\frac{\partial z}{\partial x}=\frac{p}{z}\quad and\ Q=\frac{\partial Z}{\partial z}.\frac{\partial z}{\partial y}=\frac{q}{z} \qquad .....(3)$$

Sub (3) in (2) , we get

$$P^2+Q^2=x^2+y^2$$

$$P^2-x^2=y^2-Q^2=a^2$$

$$P^2-x^2=a^2\ ,\ y^2-Q^2=a^2$$

$$P^2=a^2+x^2\ ,\ Q^2=y^2-a^2$$

$$P=\sqrt{a^2+x^2}\ ,Q=\sqrt{y^2-a^2} \qquad .....(4)$$

We know that

$$dZ = Pdx + Qdy$$
$$dZ = \sqrt{a^2 + x^2}\ dx + \sqrt{y^2 - a^2}\ dy$$
$$On \text{ integrating}$$
$$Z = \int \sqrt{a^2 + x^2}\ dx + \int \sqrt{y^2 - a^2}\ dy$$
$$\log z = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right) + b$$

41. Solve $z^2(p^2 + q^2) = x + y.$

Sol:

Given $z^2p^2 + z^2q^2 = x + y$

$$(zp)^2 + (zq)^2 = x + y \qquad .....(1)$$

This is of the type (6)

$$Z = z^2$$
$$P = \frac{\partial Z}{\partial x} = 2zp \quad , \ Q = \frac{\partial Z}{\partial y} = 2zq \qquad .....(2)$$

Sub (2) in (1), we get

$$P^2 + Q^2 = 4(x + y) \qquad .....(3)$$

This is of type (4).

$$P^2 + Q^2 = 4(x + y) = 4k$$
$$P^2 - 4x = 4y - Q^2 = 4k$$
$$P^2 - 4x = 4k \quad , \ 4y - Q^2 = 4k$$
$$P^2 = 4x + 4k \quad , \ Q^2 = 4y - 4k$$
$$P = 2\sqrt{x + k} \quad , \ Q = 2\sqrt{y - k} \qquad .....(4)$$

We know that

$$dZ = Pdx + Qdy \qquad .....(5)$$

Sub (4) In (5) ,and integrating we get

$$\int dZ = 2\int \sqrt{x + k}\, dx + 2\int \sqrt{y - k}\, dy$$

$$Z=\frac{4}{3}(x+k)^{\frac{3}{2}}+\frac{4}{3}(y-k)^{\frac{3}{2}}+b$$

$$z^2=\frac{4}{3}(x+k)^{\frac{3}{2}}+\frac{4}{3}(y-k)^{\frac{3}{2}}+b$$

LAGRANGE'S LINEAR EQUATIONS :

The equation of the form $Pp+Qq=R$ .....(1) is known as Lagrange's linear equation, where P,Q and R are the functions x,y and z. To solve this equation to solve the subsidiary equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \qquad .....(2)$$

If the solution of the subsidiary equation is of the form $u(x,y)=c_1 \; and \; v(x,y)=c_2$ then the solution of the lagrange's equation is $\phi(u,v)=0$.

To solve the subsidiary equation we have two methods

1. Method of grouping , 2. Method of multipliers

1. **Method of grouping:**

Consider the subsidiary eqn $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$. Take any two members say first two members or last two members or first and last members.

Now consider $\frac{dx}{P}=\frac{dy}{Q}$.

If P and Q contains z, try to eliminate it and on direct integrating we get $u(x,y)=c_1$.

Similarly take $\frac{dy}{Q}=\frac{dz}{R}$. If Q and R contains x, try to eliminate it and on direct integrating we get $v(x,y)=c_2$. $\therefore$ The solution of Lagrange's eqn is given by $\phi(u,v)=0$.

2. **Method of multipliers :**

Choose any three multipliers l,m,n which may be constants or functions of x ,y , z.

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{ldx+mdy+ndz}{lP+mQ+nR}$$

The expression $lP+mQ+nR=0$. Hence $ldx+mdy+ndz=0$. Now on direct integration we get $u(x,y,z)=c_1$. Similarly choose another set of multipliers l',m',n' such that in

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{l'dx+m'dy+n'dz}{l'P+m'Q+n'R}$$

The expression $l'P+m'Q+n'R=0$. $\therefore l'dx+m'dy+n'dz=0$ on direct integrating we get $v(x,y)=c_2$. $\therefore$ The solution of Lagrange's eqn is given by $\phi(u,v)=0$.

NOTE :

For the same problems we can apply both method of grouping and method of multipliers.

42. Solve $px^2+qy^2=z^2$

Sol:

The subsidiary equation are $\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{z^2}$

Taking 1st two members , we get

$$\frac{dx}{x^2}=\frac{dy}{y^2}$$

Integrating we get

$$\frac{-1}{x}=\frac{-1}{y}+c_1$$

$$u=\left(\frac{1}{y}-\frac{1}{x}\right)=c_1$$

Taking last two members , we get

$$\frac{dy}{y^2}=\frac{dz}{z^2}$$

Integrating we get

$$\frac{-1}{y} = \frac{-1}{z} + c_2$$

$$v = \left(\frac{1}{z} - \frac{1}{y}\right) = c_2$$

The complete solution is

$$\phi(u, v) = 0$$

$$\phi\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{z} - \frac{1}{y}\right) = 0$$

43. Solve $px + qy = z$

Sol:

Given $px + qy = z$

The subsidiary equation are $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$

Taking 1st two members , we get

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating we get

$$\log x = \log y + \log c_1$$

$$\log x - \log y = \log c_1$$

$$\log\left(\frac{x}{y}\right) = \log c_1$$

$$\left(\frac{x}{y}\right) = c_1$$

$$u = \left(\frac{x}{y}\right)$$

Taking other two members , we get

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating we get

$$\log y = \log z + \log c_2$$
$$\log y - \log z = \log c_2$$
$$\log\left(\frac{y}{z}\right) = \log c_2$$
$$\left(\frac{y}{z}\right) = c_2$$
$$v = \left(\frac{y}{z}\right)$$

The complete solution is

$$\phi(u, v) = 0$$
$$\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$$

44. Solve $xp - yq = y^2 - x^2$

Sol:

Given $xp - yq = y^2 - x^2$

The subsidiary equation are $\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{y^2 - x^2}$

Taking 1st two members , we get

$$\frac{dx}{x} = \frac{dy}{-y}$$

Integrating we get

$$\log x = -\log y + \log c_1$$
$$\log x + \log y = \log c_1$$
$$\log(xy) = \log c_1$$
$$xy = c_1$$
$$u = xy = c_1$$

Taking x , y ,1 as multipliers, we get

$$\frac{dx}{x}=\frac{dy}{-y}=\frac{dz}{y^2-x^2}=\frac{xdx+ydy+dz}{x^2-y^2+y^2-x^2}$$

$$=\frac{xdx+ydy+dz}{0}$$

$$xdx+ydy+dz=0$$

Integrating we get,

$$\frac{x^2}{2}+\frac{y^2}{2}+z=c_2$$

$$v=x^2+y^2+2z=c_2$$

The complete solution is

$$\phi(u,v)=0$$

$$\phi(xy,x^2+y^2+2z)=0$$

45. Solve $x(y-z)p+y(z-x)q=z(x-y)$

Sol:

Given $x(y-z)p+y(z-x)q=z(x-y)$

The subsidiary equation are $\frac{dx}{xy-xz}=\frac{dy}{yz-yx}=\frac{dz}{xz-zy}$

Choosing 1 ,1, 1as multipliers, we get

$$\frac{dx+dy+dz}{xy-xz+yz-yx+xz-zy}=\frac{dx+dy+dz}{0}$$

$$d(x+y+z)=0$$

Integrating we get

$$u=x+y+z=c_1$$

Choosing $\frac{1}{x},\frac{1}{y},\frac{1}{z}$ as multipliers, we get

$$\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{y-z+z-x+x-y}=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$$

Integrating we get

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2$$

$$xyz = c_2$$

The complete solution is

$$\phi(u, v) = 0$$

$$\phi(x + y + z, xyz) = 0$$

46. Solve $(y+z)p+(z+x)q=x+y$.

Sol:

Given $(y+z)p+(z+x)q=x+y$

The subsidiary equation are $\dfrac{dx}{y+z} = \dfrac{dy}{z+x} = \dfrac{dz}{x+y}$

$$\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y} = \frac{dx+dy+dz}{2(x+y+z)}$$

$$\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y}$$

$$\frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)}$$

Integrating we get

$$p^2 + q^2 = z(x^2 + y^2) \qquad .....(1)$$

$$\frac{d(x-y)}{-(x-y)} = \frac{d(x+y+z)}{2(x+y+z)}$$

Integrating we get

$$-\log(x-y) = \frac{1}{2}\log(x+y+z) - \log c_2$$

$$\frac{1}{2}\log(x+y+z) + \log(x-y) = \log c_2$$

$$\log(x-y)\sqrt{x+y+z} = \log c_2$$

$$v = (x-y)\sqrt{x+y+z} = c_2$$

The complete solution is

$$\phi(u,v)=0$$

$$\phi\left(\frac{y-z}{x-y},(x-y)\sqrt{x+y+z}\right)=0$$

47. Solve ptanx+qtany=tanz.

Sol:

Given p tanx +q tany = tanz.

Auxillary equation is

$$\frac{dx}{\tan x}=\frac{dy}{\tan y}=\frac{dz}{\tan z}$$

Taking 1^{st} and 2^{nd} eqn

$$\frac{dx}{\tan x}=\frac{dy}{\tan y}$$

$$\cot x\,dx=\cot y\,dy$$

$$\int\cot x\,dx=\int\cot y\,dy$$

$$\log\sin x=\log\sin y+\log c_1$$

$$\log\sin x-\log\sin y=\log c_1$$

$$\log\left(\frac{\sin x}{\sin y}\right)=\log c_1$$

$$u=\frac{\sin x}{\sin y}=c_1$$

Taking 2^{nd} and 3^{rd} eqns

$$\frac{dy}{\tan y}=\frac{dz}{\tan z}$$

$$\cot y\,dy=\cot z\,dz$$

$$\int\cot y\,dy=\int\cot z\,dz$$

$$\log\sin y=\log\sin z+\log c_2$$

$$\log\sin y-\log\sin z=\log c_2$$

$$\log\left(\frac{\sin y}{\sin z}\right)=\log c_2$$

$$v=\frac{\sin y}{\sin z}=c_2$$

The solution is

$$\phi(u,v)=0$$

$$\phi\left(\frac{\sin x}{\sin y},\frac{\sin y}{\sin z}\right)=0$$

48. Solve $\left(y^2+z^2-x^2\right)p-2xyq+2zx=0$

Sol:

The auxillary eqn is

$$\frac{dx}{y^2+z^2-x^2}=\frac{dy}{-2xy}=\frac{dz}{-2zx}$$

Consider first set of multipliers as (x,y,z)

$$\frac{dx}{y^2+z^2-x^2}=\frac{dy}{-2xy}=\frac{dz}{-2zx}=\frac{xdx+ydy+zdz}{-x(x^2+y^2+z^2)}$$

$$\frac{dy}{-2xy}=\frac{xdx+ydy+zdz}{-x(x^2+y^2+z^2)}$$

$$\frac{dy}{2y}=\frac{xdx+ydy+zdz}{(x^2+y^2+z^2)}$$

$$\text{int}\,egrating$$

$$\log(x^2+y^2+z^2)=\log y+\log c_1$$

$$\log(x^2+y^2+z^2)-\log y=\log c_1$$

$$\log\left(\frac{x^2+y^2+z^2}{y}\right)=\log c_1$$

$$u=\frac{x^2+y^2+z^2}{y}=c_1$$

similarly take

$$\frac{dz}{-2zx} = \frac{xdx + ydy + zdz}{-x(x^2 + y^2 + z^2)}$$

$$\frac{dz}{z} = \frac{2xdx + 2ydy + 2zdz}{(x^2 + y^2 + z^2)}$$

integrating

$$\log z = \log(x^2 + y^2 + z^2) + \log c_2$$

$$\log(x^2 + y^2 + z^2) - \log z = \log c_2$$

$$\log\left(\frac{x^2 + y^2 + z^2}{z}\right) = \log c_2$$

$$v = \frac{x^2 + y^2 + z^2}{z} = c_2$$

The solution is

$$\phi(u, v) = 0$$

$$\phi\left(\frac{x^2 + y^2 + z^2}{y}, \frac{x^2 + y^2 + z^2}{z}\right) = 0$$

HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATION:

A homogeneous linear PDE is of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + ... + a_n \frac{\partial^n z}{\partial y^n} = F(x, y) \qquad .....(1)$$

$$where\ a_0, a_1, a_2, \ ...\ a_n\ are\ constants$$

Eqn (1) can be written as

$$\left(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \ ..... + a_n D'^n\right) z = F(x, y) \qquad .....(2)$$

Where $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$.

The complete solution of homogeneous PDE consists of

1. Complementary Function
2. Particular integral

1. To find the complementary function(C.F):

The C.F is the solution of the eqn

$$\left(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \ ..... + a_n D'^n\right) z = 0 \quad .....(3)$$

In this eqn , put $D = m\ and\ D' = 1$, then we get the auxillary eqn.

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \;.....+ a_n = 0$$

Let the roots of this eqn be $m_1, m_2, m_3, \dots m_n$

Case 1 : If the roots are real (imaginary) and different then

$$z = f_1(y+m_1x) + f_2(y+m_2x) + ... + f_n(y+m_nx)$$

Case 2: If the roots are equal $m_1 = m_2 = m$

$$z = f_1(y+mx) + x f_2(y+mx) + f_3(y+m_3x) + ... + f_n(y+m_nx)$$

Case 3: If the roots are equal $m_1 = m_2 = m_3 = m$

$$z = f_1(y+mx) + x f_2(y+mx) + x^2 f_3(y+mx) + ... + f_n(y+m_nx)$$

NOTE :

If in a homogeneous PDE the RHS is zero then the C.F. gives the complete solution, and we need not to find particular integral.

49. Solve $\left(2D^2 + 5DD' + 2D'^2\right)z = 0$

Sol:

Auxiliary eqn is

$$2m^2 + 5m + 2 = 0 \qquad\qquad put\ D = m, D' = 1$$

$$(2m+1)(m+2) = 0$$

$$m = \frac{-1}{2}, -2$$

The roots are different.

The solutions is

$$z = f_1\left(y - \frac{1}{2}x\right) + f_2(y - 2x).$$

50. Solve $\left(D^2 - 6DD' + 9D'^2\right)z = 0$

Sol:

Auxiliary eqn is

$$m^2 - 6m + 9 = 0 \qquad put\ \ D = m\ , D' = 1$$

$$(m-3)^2 = 0$$

$$m = 3, 3$$

The roots are equal.

The solutions is

$$z = f_1(y + 3x) + x f_2(y + 3x).$$

51. Solve $\left(D^4 - D'^4\right)z = 0.$

Sol:

Auxiliary eqn is

$$m^4 - 1 = 0$$

$$\left(m^2 - 1\right)\left(m^2 + 1\right) = 0$$

$$m^2 - 1 = 0 \quad , \ m^2 + 1 = 0$$

$$m = \pm 1 \quad , \ m = \pm i$$

The roots are different.

The solutions is

$$z = f_1(y + x) + f_2(y - x) + f_3(y + ix) + f_4(y - ix).$$

52. Solve $\left(D^4 - 2D^3\,D' + 2DD'^3 - D'^4\right)z = 0$.

Sol:

Auxillary eqn is

$$m^4 - 2m^3 + 2m - 1 = 0$$

$$(m + 1)(m - 1)^3 = 0$$

$$m = -1, 1, 1, 1$$

The solutions is

$$z = f_1(y - x) + f_2(y + x) + x f_3(y + x) + x^2 f_4(y + x).$$

To find the Particular Integral:

Case 1 : If the RHS of a given PDE is $F(x, y) = e^{ax+by}$, then

$$P.I. = \frac{1}{\phi(D,D')} . e^{ax+by}$$

$$= \frac{1}{\phi(a,b)} . e^{ax+by} \qquad provided \phi(a,b) \neq 0$$

If $\phi(a,b)=0$, then multiply the Nr by x and differentiate the Dr w.r.to D, then apply the

Above procedure.

53. Solve $\left(D^2 - 5DD' + 6D'^2\right)z = e^{x+y}$.

Sol:

The auxiliary eqn is

$$m^2 - 5m + 6 = 0$$

$$m = 2, 3$$

$$C.F. = f_1(y+2x) + f_2(y+3x)$$

$$P.I. = \frac{1}{D^2 - 5DD + 6} e^{x+y}$$

$$= \frac{1}{1-5+6} e^{x+y} \qquad (D=1, D'=1)$$

$$= \frac{1}{2} e^{x+y}$$

The complete solution is $z = f_1(y+2x) + f_2(y+3x) + \frac{1}{2}e^{x+y}$

54. Solve $\left(D^3 - 3D^2D' + 4D'^3\right)z = e^{x+2y}$.

Sol:

The auxiliary eqn is

$$m^3-3m^2+4=0 \quad \Rightarrow m=-1,2,2$$

$$C.F.=f_1(y-x)+f_2(y+2x)+x\,f_3(y+2x)$$

$$P.I.=\frac{1}{D^3-3D^2D'+4D'^3}\ e^{x+2y} \qquad (D=1\ ,\ D'=2)$$

$$=\frac{1}{1-6+32}\ e^{x+2y} \quad =\frac{1}{27}\ e^{x+2y}$$

The complete solution is z = C.F. +P.I

$$z=f_1(y-x)+f_2(y+2x)+x\,f_3(y+2x)+\frac{1}{27}\,e^{x+2y}$$

55. Solve $\left(D^2+2DD'+D'^2\right)z=e^{x-y}$.

Sol:

The auxiliary eqn is

$$m^2+2m+1=0$$

$$m=-1,-1$$

$$C.F.=f_1(y-x)+x\,f_2(y-x)$$

$$P.I.=\frac{1}{D^2-2DD'+D'^2}\ e^{x-y} \qquad (D=1\ ,\ D'=-1)$$

$$=\frac{1}{1-2+1}\ e^{x-y}$$

$$=\frac{1}{0}\ e^{x-y}$$

$$=\frac{x}{2D+2D'}\ e^{x-y}$$

$$=\frac{x}{2-2}\ e^{x-y}$$

$$=\frac{x}{0}\ e^{x-y}$$

$$=\frac{x^2}{2}\ e^{x-y}$$

$$z=f_1(y-x)+x\,f_2(y-x)+\frac{x^2}{2}\ e^{x-y}$$

Case 2 : If given RHS is of the form $F(x,y)=\sin(mx+ny)\ or\ \cos(mx+ny)$

$$P.I.=\frac{1}{\phi(D,D')}\sin(mx+ny)\ or\ \cos(mx+ny)$$

Replace $D^2\ by -m^2$, $D'^2\ by -n^2\ and\ DD'=-mn\ \ in\ \phi(D,D')$ provided the denominator is not equal to zero. If $\phi(a,b)=0$, then multiply the Nr by x and differentiate the Dr w.r.to D, then apply the above procedure.

56. Solve $\left(D^2-2DD'+D'^2\right)z=\cos(x-3y)$.

Sol:

The auxiliary eqn is

$$m^2-2m+1=0$$
$$m=1,1$$
$$C.F.=f_1(y+x)+x\,f_2(y+x)$$

$$P.I.=\frac{1}{D^2-2DD'+D'^2}\cos(x-3y) \qquad \left(D^2=-1,D'^2=-9,DD'=-3\right)$$
$$=\frac{1}{-1-6-9}\cos(x-3y)$$
$$=\frac{-1}{16}\cos(x-3y)$$

The complete solution is

$$z=C.F.+P.I$$
$$=f_1(y+x)+x\,f_2(y+x)\frac{-1}{16}\cos(x-3y)$$

57. Solve $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\cos 2x\cos y$

Sol:

The given eqn can be written as

$$\left(D^2+DD'\right)z=\frac{1}{2}\left[\cos(2x+y)+\cos(2x-y)\right]$$

Auxiliary eqn is

$$m^2 + m = 0$$
$$m(m+1) = 0$$
$$m = 0\ ,\ m = -1$$
$$C.F. = f_1(y + 0x) + f_2(y - x)$$
$$P.I. = \frac{1}{2}\left[P.I_1 + P.I_2\right]$$
$$P.I_1 = \frac{1}{D^2 + DD'}\cos(2x+y) \quad \left(D^2 = -4\ ,\ DD' = -2\right)$$
$$= \frac{1}{-4-2}\cos(2x+y) = \frac{-1}{6}\cos(2x+y)$$
$$P.I_2 = \frac{1}{D^2 + DD'}\cos(2x-y) \quad \left(D^2 = -4\ ,\ DD' = 2\right)$$
$$= \frac{1}{-4+2}\cos(2x-y) = \frac{-1}{2}\cos(2x-y)$$
$$z = f_1(y+0x) + f_2(y-x) - \frac{1}{6}\cos(2x+y) - \frac{1}{2}\cos(2x-y)$$

Case 3:

If the RHS of the form $F(x,y) = x^m y^n$ then

$$P.I. = \frac{1}{\phi(D,D')}x^m y^n$$
$$= \left[\phi(D,D')\right]^{-1} x^m y^n$$

Expand $\left[\phi(D,D')\right]^{-1}$ by using binomial theorem and then operate on $x^m\ y^n$.

Here $\frac{1}{D}$ denotes the integration w.r.to x and $\frac{1}{D'}$ denotes the integration w.r.to y.

58. Solve $\left(D^2 - DD' - 30D'^2\right)z = xy + e^{6x+y}$.

Sol:

The auxiliary eqn is

$$m^2 - m - 30 = 0$$

$$m = 6, -5$$

$$C.F. = f_1(y+6x) + f_2(y-5x)$$

$$P.I. = P.I_1 + P.I_2$$

$$P.I_1 = \frac{1}{D^2 - DD' - 30D'^2}\, xy$$

$$= \frac{1}{D^2\left[1 - \frac{D'}{D} - \frac{30D'^2}{D^2}\right]}\, xy$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{D'}{D} + \frac{30D'^2}{D^2}\right)\right]^{-1} (xy)$$

$$= \frac{1}{D^2}\left[1 + \frac{D'}{D} + \frac{30D'^2}{D^2} + \left(\frac{D'}{D} + \frac{30D'^2}{D^2}\right)^2 + \ldots\ldots\right]$$

$$= \frac{1}{D^2}\left[1 + \frac{D'}{D}\right] xy$$

$$= \frac{1}{D^2}\left[xy + \frac{x}{D}\right]$$

$$= \frac{1}{D^2}(xy) + \frac{1}{D^3}(x)$$

$$= \frac{x^3 y}{6} + \frac{x^4}{24}$$

$$P.I_2 = \frac{1}{D^2 - DD' - 30D'^2}\, e^{6x+y}$$

$$= \frac{1}{36 - 6 - 30}\, e^{6x+y}$$

$$= \frac{1}{0} e^{6x+y}$$

$$=\frac{x}{2D-D'}e^{6x+y}$$

$$=\frac{x}{12-1}e^{6x+y}$$

$$=\frac{x}{11}e^{6x+y}$$

$$z=f_1(y+6x)+f_2(y-5x)+\frac{x^3y}{6}+\frac{x^4}{24}+\frac{x}{11}e^{6x+y}$$

59. Solve $(D^2-2DD')z=e^{2x}+x^3y$.

Sol:

The auxiliary eqn is

$$m^2-2m=0$$

$$m1=0,2$$

$$C.F.=f_1(y+0x)+f_2(y+2x)$$

$$P.I.=P.I_1+P.I_2$$

$$P.I_1=\frac{1}{D^2-2DD'}e^{2x}$$

$$=\frac{1}{4-0}e^{2x}$$

$$=\frac{e^{2x}}{4}$$

$$P.I_2=\frac{1}{D^2-2DD'}x^3y$$

$$=\frac{1}{D^2\left(1-\frac{2D'}{D}\right)}x^3y$$

$$=\frac{1}{D^2}\left(1-\frac{2D'}{D}\right)^{-1}x^3y$$

$$=\frac{1}{D^2}\left(x^3y-\frac{2}{D}(x^3)\right)$$

$$=\frac{x^5y}{20}+\frac{x^6}{60}$$

$$z=f_1(y+0x)+f_2(y+2x)+\frac{e^{2x}}{4}+\frac{x^5y}{20}+\frac{x^6}{60}$$

60. Solve $\left(D^2+4DD'-5D'^2\right)z=3e^{2x-y}+\sin(x-2y)$

Sol:

Auxillary eqn is

$$m^2+4m-5=0$$
$$m=1,-5$$
$$C.F=f_1(x+y)+f_2(x-5y)$$
$$P.I=\frac{1}{D^2+4DD'+D'^2}(3e^{2x-y})$$
$$=\frac{1}{4-8+1}(3e^{2x-y})$$
$$=\frac{-1}{3}(3e^{2x-y})$$
$$P.I=\frac{1}{D^2+4DD'-5D'^2}\sin(x-2y)$$
$$=\frac{1}{-1+8+20}\sin(x-2y)$$
$$=\frac{1}{27}\sin(x-2y)$$

The complete solution is

$$z=f_1(x+y)+f_2(x-5y)-\frac{1}{3}(3e^{2x-y})+\frac{1}{27}\sin(x-2y)$$

61. Solve $\left(D^2-2DD'+D'^2\right)z=8e^{x+2y}$

Sol:

Auxillary eqn is

$$m^2-2m+1=0$$
$$m=1,1$$
$$C.F.=f_1(x+y)+xf_2(x+y)$$
$$P.I.=\frac{1}{D^2-2DD'+D'^2}8e^{x+2y}$$
$$=\frac{1}{1-4+4}8e^{x+2y}$$
$$=8e^{x+2y}$$

The complete solution is

$$z=f_1(x+y)+xf_2(x+y)+8e^{x+2y}$$

62.Solve $\left(D^2 - DD' - 2D'^2\right)z = (y-1)e^x$.

Sol:

The auxiliary eqn is

$$m^2 - m - 2 = 0$$

$$m = -1, 2$$

$$C.F. = f_1(y-x) + f_2(y+2x)$$

$$P.I. = \frac{1}{D^2 - DD' - D'^2}(y-1)e^x$$

$$=$$

$$\frac{1}{(D-2D')(D+D')}(y-1)e^x$$

$$= \frac{1}{(D-2D')}\int (c+x-1)e^x\,dx$$

$$= \frac{1}{(D-2D')}\int ce^x\,dx + \int (x-1)e^x\,dx$$

$$= \frac{1}{(D-2D')}\left[ce^x + xe^x - e^x - e^x\right]$$

$$= \frac{1}{(D-2D')}\left[ce^x + xe^x - 2e^x\right]$$

$$= \frac{1}{(D-2D')}\left[(y-x)e^x + xe^x - 2e^x\right] \qquad (\because c = y-x)$$

$$= \frac{1}{(D-2D')}\left[ye^x - xe^x + xe^x - 2e^x\right]$$

$$= \frac{1}{(D-2D')}\left[(y-2)e^x\right]$$

$$= \int (c_1 - 2x - 2)e^x\,dx \qquad (\because y = c_1 - 2x)$$

$$= c_1\int e^x\,dx - 2\int xe^x\,dx - 2\int e^x dx$$

$$= c_1 e^x - 2xe^x + 2e^x - 2e^x$$

$$= (y+2x)e^x - 2xe^x + 2e^x - 2e^x$$

$$= ye^x$$

$$z = f_1(y-x) + f_2(y+2x) + ye^x$$

UNIT-II
FOURIER SERIES

Periodic Function:

A function f(x) is said to be periodic function if f(x+T) = f(x) where T is the positive constant for all x which is also called period.

Example:

$$f(x+2\prod)=\sin(x+2\prod)=Sinx$$
$$f(x+4\prod)=\sin(x+4\prod)=Sinx$$
$$\therefore\ Period\ T = 2\prod > 0$$

Drichlet's conditions :

A function f(x) can be expanded as a fourier series in an interval $c \le x \le c+2l$. If the following conditions are satisfied

(i) f(x) is periodic with period 2l in (c, c + 2l) and f(x) is bounded.

(ii)The function f(x) must have finite number of maxima and minima.

(iii)The function f(x) must be piecewise continuous and has a finite number of finite discontinuities.

Fourier Series:

A periodic function f(x) which satisfies Drichlet's conditions can be expanded , as cosine and sine series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Fourier coefficients:

$$a_0 = \frac{1}{l} \int_{c}^{c+2l} f(x)\, dx$$

$$a_n = \frac{1}{l} \int_{c}^{c+2l} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{l} \int_{c}^{c+2l} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Complex Form of Fourier series:

$$f(x) = \sum_{n=1}^{\infty} c_n e^{i\frac{n\pi x}{l}}$$

$$c_n = \frac{1}{2l} \int_{c}^{c+2l} f(x) e^{-i\frac{n\pi x}{l}}$$

Root mean square value :

$$\bar{y} = \sqrt{\frac{1}{b-a} \int_a^b f(x)^2 dx}$$ **Root mean square value**

$$\bar{y}^2 = \frac{1}{b-a} \int_a^b y^2 dx$$ **Effective value**

Parseval's Identity:

$$\bar{y}^2 = \frac{{a_0}^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} \left({a_n}^2 + {b_n}^2\right)$$

Problems on Fourier Series of Functions With Period 2π

1. Find the Fourier series Expansion of

$$f(x) = x^2,\ 0 < x < 2\pi\ .$$

Hence deduce that, (i) $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + ...$

(ii) $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + ...$

(iii) $\dfrac{\pi^2}{8} = \left(\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + ...\right)$

Solution :

We need to find a_0, a_n, b_n where the formulas are given by

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\ dx,\ \ a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\ dx,\ \ b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\ dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\ dx = \frac{1}{\pi}\int_0^{2\pi} x^2\ dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx$$

$$\textit{applying Bernoulli's formula}$$

$$= \frac{1}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{\left[(4\pi^2)\left(\frac{\sin n2\pi}{n}\right) - (4\pi)\left(\frac{-\cos n2\pi}{n^2}\right) + (2)\left(\frac{-\sin n2\pi}{n^3}\right)\right]_0^{2\pi} - \frac{1}{\pi}\left[(0^2)\left(\frac{\sin n0}{n}\right) - (0)\left(\frac{-\cos n0}{n^2}\right) + (2)\left(\frac{-\sin n0}{n^3}\right)\right]_0^{2\pi}\right\}$$

$$= \frac{1}{\pi}4\pi\frac{1}{n^2}$$

$$= \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$

$$\textit{applying Bernoulli's formula}$$

$$= \frac{1}{\pi}\left[(x^2)\left(\frac{-\cos nx}{n}\right) - (2x)\left(\frac{-\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{\left[(4\pi^2)\left(\frac{-\cos n2\pi}{n}\right) - (4\pi)\left(\frac{-\sin n2\pi}{n^2}\right) + (2)\left(\frac{\cos n2\pi}{n^3}\right)\right] - \left[(0^2)\left(\frac{-\cos n0}{n}\right) - (0)\left(\frac{-\sin n0}{n^2}\right) + (2)\left(\frac{\cos n0}{n^3}\right)\right]\right\}$$

$$= \frac{1}{\pi}\left[\left(\frac{-4\pi^2}{n} + \frac{2}{n^3}\right) - \left(\frac{2}{n^3}\right)\right] \qquad = -\frac{4\pi}{n}$$

Hence the Fourier series expansion is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx + \sum_{n=1}^{\infty} -\frac{4\pi}{n} \sin nx$$

Deduction 1:

Put $x = 0$ (is a point of discontinuity at end point) in the above Fourier series

$$\frac{f(0) + f(2\pi)}{2} = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos n0 + \sum_{n=1}^{\infty} -\frac{4\pi}{n} \sin n0$$

$$\Rightarrow \frac{0 + 4\pi^2}{2} = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\Rightarrow 2\pi^2 - \frac{4\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\Rightarrow \frac{2\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

i.e., $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots.$

Deduction 2:

Put $x = \pi$ (is a point of continuity) in the above Fourier series

$$\pi^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos n\pi + \sum_{n=1}^{\infty} -\frac{4\pi}{n} \sin n\pi$$

$$\Rightarrow \pi^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n$$

$$\Rightarrow \pi^2 - \frac{4\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n$$

$$\Rightarrow \frac{-\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n$$

$$\Rightarrow \frac{-\pi^2}{12} = \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^n$$

$$\Rightarrow \frac{-\pi^2}{12} = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \ldots.$$

$$\Rightarrow \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots.$$

Deduction 3:

x value substitution (such as 0, $\frac{\pi}{2}$, π, 2π) will not give the deduction.

So, let us add the above two series (given in the above two deductions),

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots.\right) + \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots.\right)$$

$$\frac{\pi^2}{4} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.\right)$$

$$\Rightarrow \frac{\pi^2}{8} = \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.\right)$$

Hence $\frac{\pi^2}{8} = \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right)$

2. If $f(x) = \frac{1}{2}(\pi - x)$ find the fourier series of period 2π in the interval $(0, 2\pi)$.Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$

Solution:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx + \sum_{n=1}^{\infty} b_n sinnx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)dx$$

$$= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(2\pi^2 - \frac{4\pi^2}{2}\right) - (0 - 0)\right]$$

$$= \frac{1}{2\pi}[2\pi^2 - 2\pi^2]$$

$=0$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)cosnxdx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)cosnxdx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\frac{sinnx}{n} - \frac{cosnx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[(-\pi)\frac{sin2n\pi}{n} - \frac{cos2n\pi}{n^2} + \frac{1}{n^2}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{1}{n^2} + \frac{1}{n^2}\right]$$

=0

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)sinnxdx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi - x)sinnxdx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\frac{-co}{n} + \frac{sinnx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\pi\frac{cos2n\pi}{n} - \frac{sin2n\pi}{n^2} + \frac{\pi}{n^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi}{n} + \frac{\pi}{n}\right]$$

$$= \frac{1}{n}$$

$$a_0 = 0\,, a_n = 0, \qquad b_n = \frac{1}{n}$$

$$f(x) = 0 + 0 + \sum_{n=1}^{\infty}\frac{1}{n}sinnx$$

$$f(x) = sinx + \frac{sin2x}{2} + \frac{sin3x}{3} + \cdots.$$

Let $x = \frac{\pi}{2}$ this point is continuous point

$$\frac{\pi}{4} = sin\frac{\pi}{2} + \frac{sin\pi}{2} + \frac{sin\frac{3\pi}{2}}{3} + \cdots.$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

3. Expand the Fourier series for $f(x) = \left(\frac{\pi - x}{2}\right)^2$, $0 < x < 2$

Hence deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \ldots \infty$

Solution:

The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx + \sum_{n=1}^{\infty} b_n sinnx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi - x)^2}{4} dx$$

$$= \frac{1}{4\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi}$$

$$= -\frac{1}{12\pi}[-\pi^3 - \pi^3]$$

$$a_0 = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)cosnxdx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}cosnxdx$$

$$= \frac{1}{4\pi}\Big[(\pi - x)^2\frac{sinnx}{n} - 2(\pi - x)(-1)\left(-\frac{cosnx}{n^2}\right) +$$
$$(2)(1)\left(\frac{-sinnx}{n^3}\right)\Big]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\frac{sinnx}{n} - 2(\pi - x)\left(\frac{cosnx}{n^2}\right) - 2\left(\frac{sinnx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\frac{2\pi}{n^2}cos2n\pi + \frac{2\pi}{n^2}cos0\right]$$

$$= \frac{1}{4\pi}\left[\frac{4\pi}{n^2}\right] = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)sinnxdx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}sinnxdx$$

$$= \frac{1}{4\pi}\Big[(\pi - x)^2\left(-\frac{cosnx}{n}\right) - 2(\pi - x)(-1)\left(-\frac{sinnx}{n^2}\right) +$$
$$(2)(1)\left(\frac{cosnx}{n^3}\right)\Big]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[-(\pi - x)^2 \frac{cos}{n} - 2(\pi - x)\left(\frac{sinnx}{n^2}\right) + 2\left(\frac{cosnx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[-\pi^2 \frac{cos2n\pi}{n} + 2\frac{cos2n\pi}{n^3} + \pi^2 \frac{cos0}{n} - 2\frac{cos0}{n^3}\right]$$

$$= \frac{1}{4\pi}\left[-\frac{\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right]$$

$$= 0$$

$$a_0 = \frac{\pi^2}{6}, a_n = \frac{1}{n^2}, \qquad b_n = 0$$

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{cosnx}{n^2}$$

$$f(x) = \frac{\pi^2}{12} + \frac{cosx}{1^2} + \frac{cos2x}{2^2} + \frac{cos3x}{3^2} + \cdots . - - - - - (1)$$

Let $x = 0$ this point is discontinuous point

$$f(x)_{x=0} = \frac{f(0)+f(2\pi)}{2}$$

$$f(x)_{x=0} = \frac{\frac{\pi^2}{4}+\frac{\pi^2}{4}}{2} = \frac{\pi^2}{4}$$

$$\therefore (1) \Longrightarrow \frac{\pi^2}{4} = \frac{\pi^2}{12} + \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\Longrightarrow \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

4. If $f(x) = x(2\pi - x) \quad for \quad 0 < x < 2\pi,$ **prove that**

$$f(x) = \frac{2\pi^2}{3} - 4\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots\right]$$

Solution:

The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx + \sum_{n=1}^{\infty} b_n sinnx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2)dx$$

$$= \frac{1}{\pi}\left[\pi x^2 - \frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[4\pi^3 - \frac{8\pi^3}{3}\right]$$

$$= \frac{12\pi^2 - 8\pi^2}{3} = \frac{4\pi^2}{3}$$

$$a_0 = \frac{4\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2)\cos nx dx$$

$$= \frac{1}{\pi}\left[(2\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (2\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[(2\pi x - x^2)\left(\frac{\sin nx}{n}\right) + (2\pi - 2x)\left(\frac{\cos nx}{n^2}\right) + 2\left(\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-2\pi}{n^2}\cos 2n\pi - \frac{2\pi}{n^2}\right]$$

$$= -\frac{2}{n^2}[1+1] = -\frac{4}{n^2}$$

$$a_n = -\frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2)\sin nx dx$$

$$= \frac{1}{\pi}\left[(2\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (2\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-(2\pi x - x^2)\left(\frac{\sin nx}{n}\right) + (2\pi - 2x)\left(\frac{\sin nx}{n^2}\right) - 2\left(\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n^3}\cos 2n\pi + \frac{2\pi}{n^3}\cos 0\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n^3} + \frac{2\pi}{n^3}\right] = 0$$

$$a_n = 0$$

$$\therefore f(x) = \frac{2\pi^2}{3} - \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx$$

$$f(x) = \frac{2\pi^2}{3} - 4\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots\right]$$

5. Find the Fourier series expansion of *f(x)* = *x sin x* in (0, 2π).

Solution:

Since the interval is 0 to 2π we need to find all the three fourier constants.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx,\quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,\quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx = \frac{1}{\pi}\Big[(x)(-\cos x) - 1(-\sin x)\Big]_0^{2\pi}$$

$$= \frac{1}{\pi}\big[2\pi(-1) - 0 - 0 + 0\big]$$

$$a_0 = -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\frac{\sin(n+1)x - \sin(n-1)x}{2}\,dx,\ \text{Provided}\ n\neq 1$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\sin(n+1)x\,dx - \int_0^{2\pi} x\sin(n-1)x\,dx\right]$$

$$= \frac{1}{2\pi}\left\{\left[(x)\left(-\frac{\cos(n+1)x}{n+1}\right) - 1\left(-\frac{\sin(n+1)x}{(n+1)^2}\right)\right] - \left[(x)\left(-\frac{\cos(n-1)x}{n-1}\right) - 1\left(-\frac{\sin(n-1)x}{(n-1)^2}\right)\right]\right\}_0^{2\pi}$$

$$\text{provided}\ n\neq 1$$

$$= \frac{1}{2\pi}\left[-\frac{2\pi}{n+1} - \left(-\frac{2\pi}{n-1}\right)\right],\ \text{provided}\ n\neq 1$$

$$= \frac{1}{2\pi}2\pi\left(-\frac{1}{n+1} - \frac{1}{n-1}\right),\ \text{provided}\ n\neq 1$$

$$a_n = \frac{2}{n^2-1},\ \text{provided}\ n\neq 1$$

When $n=1$, *we have*

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} (x\sin x)\cos 1x\,dx = \frac{1}{2\pi}\int_0^{2\pi} (x\sin 2x)\,dx$$

$$= \frac{1}{2\pi}\left[(x)\left(-\frac{\cos 2x}{2}\right) - 1\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left(-\frac{2\pi}{2}\right)$$

$$a_1 = -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\,\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\,\frac{\cos(n-1)x - \cos(n+1)x}{2}\,dx$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\cos(n-1)x\,dx - \int_0^{2\pi} x\cos(n+1)x\,dx\right]$$

$$= \frac{1}{2\pi}\left\{\left[(x)\left(\frac{\sin(n-1)x}{n-1}\right) - 1\left(-\frac{\cos(n-1)x}{(n-1)^2}\right)\right] - \left[(x)\left(\frac{\sin(n-+1)x}{n+1}\right) - 1\left(-\frac{\cos(n+1)x}{(n+1)^2}\right)\right]\right\}_0^{2\pi},$$

provided $n \neq 1$

$$= \frac{1}{2\pi}\left\{\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2}\right] - \left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2}\right]\right\},\ \textit{provided } n \neq 1$$

$= 0$, *provided* $n \neq 1$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\ \sin\ nx\ \ dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin\ x\ \sin\ 1x\ \ dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\ \frac{1-\cos\ 2x}{2} dx$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\ \ dx\ -\ \int_0^{2\pi} x\ \cos\ \ 2x\ \ dx\right]$$

$$= \frac{1}{2\pi}\left\{\left[\frac{x^2}{2}\right]_0^{2\pi} - \left[(x)\left(\frac{\sin\ 2x}{2}\right) - 1\left(-\frac{\cos\ 2x}{4}\right)\right]_0^{2\pi}\right\}$$

$$= \frac{1}{2\pi}\left[2\pi^2 - 0 + \frac{1}{4} - 0 + 0 - \frac{1}{4}\right] \qquad = \pi$$

Hence the Fourier series is

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n\ \cos\ nx\ +\ b_1 \sin x + \sum_{n=2}^{\infty} b_n\ \sin\ nx$$

$$x \sin x = -1 - \frac{1}{2}\ \cos x + \sum_{n=2}^{\infty} \frac{2}{n^2-1} \cos\ nx + \pi \sin x$$

---

6. Find the Fourier series of $f(x) = \begin{cases} 1, & 0<x<\pi \\ 2, & \pi < x < 2\pi \end{cases}$**. Hence evaluate the value of the series** $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots.$

Solution:

Here the interval given is 0 to 2π. So, let us find all the three fourier constants.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx,\; a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,\; b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} 1\,dx + \int_{\pi}^{2\pi} 2\,dx\right]$$

$$= \frac{1}{\pi}\left[(x)_0^{\pi} + (2x)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi + 2\pi\right]$$

$$= 3$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \cos nx\,dx + \int_{\pi}^{2\pi} 2\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\sin nx}{n}\right)_0^{\pi} + \left(2\frac{\sin nx}{n}\right)_0^{2\pi}\right]$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \sin nx\,dx + \int_{\pi}^{2\pi} 2\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{\cos nx}{n}\right)_0^{\pi} + \left(-2\frac{\cos nx}{n}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{(-1)^n}{n} + \frac{1}{n}\right) + \left(-\frac{2}{n} + 2\frac{(-1)^n}{n}\right)\right]$$

$$=\frac{1}{\pi}\left[\frac{(-1)^n}{n}-\frac{1}{n}\right]$$

$$b_n = \frac{1}{\pi}\begin{cases}\dfrac{-2}{n}, & if\ n\ is\ odd\\[2ex] 0, & if\ n\ is\ even\end{cases}$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

Hence the Fourier series is

$$f(x)=\frac{3}{2}+\sum_{n=1,3,5,....}^{\infty}-\frac{2}{n\pi}\sin nx$$

Deduction:

When we put x = 0, $\frac{\pi}{2}$, π, 2 π we don't get the series (As the denominator of the series is n^2 and the denominator of the Fourier series is only n). so, let us apply Parseval's identity for full range series.

$$\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)=\frac{1}{\pi}\int_0^{2\pi}[f(x)]^2\,dx$$

$$\Rightarrow \frac{9}{2}+\sum_{n=1,3,5,..}^{\infty}\frac{4}{n^2\pi^2} = \frac{1}{\pi}\left[\int_0^{\pi}1^2\,dx+\int_{\pi}^{2\pi}2^2\,dx\right]$$

$$\Rightarrow \frac{9}{2}+\frac{4}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+....\right)=\frac{1}{\pi}\left[(x)_0^{\pi}+(4x)_{\pi}^{2\pi}\right]$$

$$\Rightarrow \frac{9}{2}+\frac{4}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+....\right)=5$$

$$\Rightarrow \frac{4}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+....\right)=5-\frac{9}{2}=\frac{1}{2}$$

$$\Rightarrow \left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+....\right)=\frac{\pi^2}{8}$$

7.Find the Fourier series expansion of $f(x)=\begin{cases}1 & for\ 0<x<\pi \\ 2 & for\ \pi<x<2\pi\end{cases}$

Solution:

Here the interval given is 0 to 2π. So, let us find all the three fourier constants.

$$a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)\,dx,\ a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,\ b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$=\frac{1}{\pi}\left[\int_0^{\pi} 1\,dx+\int_{\pi}^{2\pi} 2\,dx\right]$$

$$=\frac{1}{\pi}\left[(x)_0^{\pi}+(2x)_{\pi}^{2\pi}\right]$$

$$=\frac{1}{\pi}[\pi+2\pi]$$

$$=3$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\left[\int_0^{\pi}\cos nx\,dx+\int_{\pi}^{2\pi} 2\cos nx\,dx\right]$$

$$=\frac{1}{\pi}\left[\left(\frac{\sin nx}{n}\right)_0^{\pi}+\left(2\frac{\sin nx}{n}\right)_0^{2\pi}\right]$$

$$=0$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$=\frac{1}{\pi}\left[\int_0^{\pi}\sin nx\,dx+\int_{\pi}^{2\pi} 2\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{\cos nx}{n}\right)_0^{\pi} + \left(-2\frac{\cos nx}{n}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{(-1)^n}{n} + \frac{1}{n}\right) + \left(-\frac{2}{n} + 2\frac{(-1)^n}{n}\right)\right]$$

Hence the Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \frac{3}{2} + \sum_{n=1,3,5,....}^{\infty} -\frac{2}{n\pi} \sin nx$$

Fourier Series of *f(x)* in the interval $(-\pi, \pi)$

1. In the Fourier series expansion of $f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi < x < 0 \\ 1 - \frac{2x}{\pi}, & 0 < x < \pi \end{cases}$

in $(-\pi,\pi)$, find the coefficient of sin nx.

Solution:

Since the interval is $(-\pi,\pi)$, let us verify whether the function is odd or even

$$f(-x) = \begin{cases} 1 + \frac{2(-x)}{\pi}, & -\pi < -x < 0 \\ 1 - \frac{2(-x)}{\pi}, & 0 < -x < \pi \end{cases}$$

$$= \begin{cases} 1 - \frac{2(x)}{\pi}, & 0 < x < \pi \\ 1 + \frac{2(x)}{\pi}, & -\pi < x < 0 \end{cases}$$

$$= f(x)$$

Hence the function is even. So, the coefficient of sin nx that is $b_n = 0$.

2. Find a_n in expanding e^{-ax} as a Fourier series in (-π,π).

Solution:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\left(\frac{e^{-ax}}{(a^2+b^2)}\right)(-a\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{e^{-a\pi}}{a^2+b^2}\right)(-a\cos n\pi + 0) - \left(\frac{e^{a\pi}}{a^2+b^2}\right)(-a\cos n\pi + 0)\right]$$

$$= \frac{1}{\pi}\frac{a}{a^2+b^2}\cos n\pi\,(e^{a\pi} - e^{-a\pi})$$

$$a_n = \frac{2}{\pi}\frac{a}{a^2+b^2}(-1)^n \sinh a\pi$$

3. What is the constant term a_0 and the coefficient a_n in the Fourier series expansion of $f(x) = x - x^3$ in (-π,π).

Solution:

Since the interval is (-π,π), let us verify whether the function is odd or even

$f\text{-}(x) = -x - (-x^3) = -x + x^3 = -(x - x^3) = -f(x)$. The given function is odd.

Hence, the coefficients a_0 and a_n are zero.

4. Find the constant term in the Fourier series corresponding to $f(x) = \cos^2 x$ expanded in the interval (-π,π).

Solution:

Since the interval is (-π,π), let us verify whether the function is odd or even.

$f(-x) = \cos^2(-x) = \cos^2 x = f(x)$.

Hence the function is even.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \cos^2 x \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1+\cos 2x}{2} \, dx$$

$$= \frac{1}{\pi}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}[\pi]$$

$$a_0 = 1$$

Hence the constant term in the Fourier expansion is $\frac{a_0}{2} = \frac{1}{2}$

---

5. If the Fourier series of the function f(x) = x + x², in the interval (-π,π) is $\frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{n^2}\cos nx - \frac{2}{n}\sin nx\right]$**, then find the value of the infinite series** $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots.$

Solution: Put $x = \pi$, which is an end point discontinuity. So,

$$\frac{f(-\pi)+f(\pi)}{2} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{n^2}\cos n\pi - \frac{2}{n}\sin n\pi\right]$$

$$\Rightarrow \frac{-\pi+\pi^2+\pi+\pi^2}{2} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{n^2}(-1)^n\right]$$

$$\Rightarrow \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}\right]$$

$$\Rightarrow 4\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\right] = -\frac{\pi^2}{3} + \pi^2$$

$$\Rightarrow 4\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\right] = \frac{2\pi^2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty}\left[\frac{1}{n^2}\right] = \frac{\pi^2}{6}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots. = \frac{\pi^2}{6}$$

---

6. Find a_0 if $f(x) = |x|$, expanded as a Fourier series in (-π,π).

Solution: Since $f(-x) = |-x| = |x| = f(x)$, the function is even.

$$a_0 = \frac{2}{\pi}\int_0^{\pi}|x|\,dx = \frac{2}{\pi}\int_0^{\pi}x\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{2}\right]$$

$$a_0 = \pi$$

---

7. Find the Fourier constant b_n for $f(x) = x \sin x$ in (-π,π).

Solution: $f(-x) = (-x)\sin(-x) = x \sin x = f(x)$ the function is even. Therefore, the coefficient $b_n = 0$.

---

8. Find the Fourier constant b_n for f(x) = x^2 in (-π,π).

Solution: f(- x) = $(-x)^2 = x^2$ = f(x) the function is even. Therefore, the coefficient $b_n = 0$.

---

9. Find the Fourier series expansion of $f(x) = |\sin x|$ in $-\pi < x < \pi$.

Solution: Here the interval given is $-\pi$ to π. So, let us verify whether the given function is odd or even.

$f(-x) = |\sin(-x)| = |-\sin x| = \sin x = |\sin x| = f(x)$. Hence the function is even. So, let us find the fourier constants a_0, a_n . ($b_n = 0$).

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\sin x|\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x\,dx$$

$$= \frac{2}{\pi}\left[-\cos x\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[1+1\right]$$

$$= \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\sin x|\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\sin(n+1)x - \sin(n-1)x}{2}\,dx$$

$$= \frac{1}{\pi}\left\{\left[\left(-\frac{\cos(n+1)x}{n+1}\right) - \left(-\frac{\cos(n-1)x}{n-1}\right)\right]_0^{\pi}\right\}, \ provided \ n \neq 1$$

$$= \frac{1}{\pi}\left\{\left[-\frac{(-1)^{n+1}}{n+1} + \frac{1}{n+1}\right] - \left[-\frac{(-1)^{n-1}}{n-1} + \frac{1}{n-1}\right]\right\}, provided \ n \neq 1$$

$$= \frac{1}{\pi}\left\{\left[-\frac{-(-1)^{n}}{n+1} + \frac{1}{n+1}\right] - \left[-\frac{-(-1)^{n}}{n-1} + \frac{1}{n-1}\right]\right\}, provided \ n \neq 1$$

$$= \frac{1}{\pi}\left\{\left[(-1)^n\left(\frac{1}{n+1} - \frac{1}{n-1}\right) + \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right]\right\}, provided \ n \neq 1$$

$$= \frac{1}{\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right)\left((-1)^n + 1\right), \ provided \ n \neq 1$$

$$= \frac{1}{\pi}\left(-\frac{2}{n^2-1}\right)\begin{cases} 2, \ if \ n \ is \ even \\ 0, \ if \ n \ is \ odd \end{cases}$$

When n = 1, we have

$$a_1 = \frac{2}{\pi}\int_0^{\pi} f(x) \cos 1x \ dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\sin x| \cos 1x \ dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \ \cos 1x \ dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\sin 2x}{2} \ dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}[-1 + 1]$$

$$= 0$$

Hence the Fourier series is

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\left[\sum_{n=2,4,6,..}^{\infty} \frac{1}{n^2-1} \cos nx\right].$$

10. Find the Fourier series expansion for the function $f(x) = x + x^2$ in $-\pi < x < \pi$

Solution: Here the interval is $-\pi < x < \pi$. So, let us verify whether the function is odd or even.

$$f(-x) = (-x) + (-x)^2 = -x + x^2 \neq \begin{cases} f(x) = x + x^2 \\ -f(x) = -(x + x^2) \end{cases}$$

which is neither odd nor even. So, Let us find all the three constants.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\, dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} + \frac{\pi^3}{3}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^3}{3}\right)\right]$$

$$= \frac{1}{\pi}\frac{2\pi^3}{3}$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n} - (1+2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[0 - (1+2\pi)\left(-\frac{\cos n\pi}{n^2}\right) + 0\right] - \left[0 - (1)\left(-\frac{\cos n\pi}{n^2}\right) + 0\right]\right\}$$

$$= \frac{1}{\pi}\left(-\frac{\cos n\pi}{n^2}\right)\left(-1-2\pi+1-2\pi\right)$$

$$= \frac{1}{\pi}\left(\frac{(-1)^n}{n^2}\right)4\pi$$

$$a_n = 4\frac{(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(-\frac{\cos nx}{n}\right) - (1+2x)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\begin{array}{l}\left[(\pi+\pi^2)\left(-\frac{\cos n\pi}{n}\right) - (1+2\pi)\left(-\frac{\sin n\pi}{n^2}\right) + 2\left(\frac{\cos n\pi}{n^3}\right)\right] - \\ \left[(-\pi+\pi^2)\left(-\frac{\cos n\pi}{n}\right) - (1-2\pi)\left(+\frac{\sin n\pi}{n^2}\right) + 2\left(\frac{\cos n\pi}{n^3}\right)\right]\end{array}\right\}$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n}\left(-\pi-\pi^2-\pi+\pi^2\right)\right]$$

$$= \frac{1}{\pi}\frac{(-1)^n}{n}(-2\pi)$$

$$b_n = -2\frac{(-1)^n}{n}$$

Hence the Fourier series is of the form

$$x+x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

11.Obtain the fourier series for the function $f(x)=|x| \quad in \quad -\pi<x<\pi$ **also deduce that** $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}$**.**

Sol :

$f(x)=|x|$

$f(-x)=|-x|$

$\therefore f(x)=f(-x) \quad it \; is\, an\, even\, function$

Hence $b_n=0$

Fourier series becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |x|dx \qquad \left(\because |x| = \begin{cases} -x & : -\pi < x < 0 \\ x & : 0 < x < \pi \end{cases}\right)$$

$$= \frac{2}{\pi}\int_0^{\pi} xdx = \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{\pi^2}{2} - 0\right) = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nxdx$$

$$= \frac{2}{\pi}\int_0^{\pi} |x|\cos nxdx = \frac{2}{\pi}\int_0^{\pi} x\cos nxdx$$

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$\therefore a_n = \begin{cases} -\frac{4}{\pi n^2} & : n \text{ is odd} \\ 0 & : n \text{ is even} \end{cases}$$

Hence Fourier series becomes

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,...}^{\infty} \frac{\cos nx}{n^2}$$

$Put\ x = 0\ (continuous\ \ po\ \text{int})$

$\therefore\ f(0) = 0$

$$\Rightarrow f(0) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,...}^{\infty} \frac{1}{n^2}$$

$$0 - \frac{\pi}{2} = -\frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right)$$

$$\Rightarrow \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right) = \frac{\pi^2}{8}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

12.Find the Fourier series of $f(x) = |\cos x|\, in\ (-\pi, \pi)$

Sol:

$Here\ f(x) = f(-x)$

$\therefore f(x) = |\cos x|\, is\, even\ \ function.\ Hence b_n = 0$

Fourier series is $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\cos x|\, dx$$

$$= \frac{2}{\pi}\left[\int_{0}^{\frac{\pi}{2}} \cos x dx + \int_{\frac{\pi}{2}}^{\pi} - \cos x dx\right]$$

$$= \frac{2}{\pi}\left[\left(\sin x\right)_{0}^{\pi/2} - \left(\sin x\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[(1-0)-(0-1)\right]$$

$$a_0 = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} |\cos x| \cos nx dx$$

$$= \frac{2}{\pi}\left[\int_{0}^{\pi/2} \cos nx.\cos x dx - \int_{\pi/2}^{\pi} \cos nx \cos x dx\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{2}\left(\int_{0}^{\pi/2} \cos(n+1)x + \cos(n-1)dx - \int_{\pi/2}^{\pi} \cos(n+1)x + \cos(n-1)xdx\right)\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right)_{0}^{\pi/2} - \left(\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1} + \frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos\frac{n\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}}{n-1} + \frac{\cos\frac{n\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}}{n-1}\right]$$

$$= \frac{2}{\pi}\cos\frac{n\pi}{2}\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$a_n = \frac{2}{\pi}\cos\frac{n\pi}{2}\left[\frac{-2}{n^2-1}\right] \quad n \neq 1$$

$$= \frac{-4}{\pi(n^2-1)}\cos\frac{n\pi}{2}, \quad n \neq 1$$

When n = 1 we have

$$a_1 = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cos x dx - \int_{\pi/2}^{\pi} \cos x \cos x dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x dx - \int_{\pi/2}^{\pi} \cos^2 x dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \left(\frac{1+\cos 2x}{2}\right) dx - \int_{\pi/2}^{\pi} \left(\frac{1+\cos 2x}{2}\right) dx\right]$$

$$= \frac{1}{\pi}\left[\left(x + \frac{\sin 2x}{2}\right)_0^{\pi/2} - \left(x + \frac{\sin 2x}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \pi + \frac{\pi}{2}\right] = 0$$

$$f(x) = \frac{2}{\pi} + \sum_{n=2,4,\ldots}^{\infty} \frac{-4}{\pi(n^2-1)} \cos\frac{n\pi}{2} \cos nx$$

13. Find the Fourier series expansion of $(\pi - x)^2$,

in $-\pi < x < \pi$. Hence deduce that

(i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - .... = \frac{\pi^2}{12}$,

(ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{\pi^2}{6}$

Solution:

Since the range is $-\pi < x < \pi$.

So, let us first verify whether the function is odd or even.

$f(-x) = (\pi - (-x))^2 = (\pi + x)^2$ which is entirely different function.

So, we have to find all the three fourier coefficients.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)^2 \, dx$$

$$= \frac{1}{\pi}\left[\frac{(\pi - x)^3}{-3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[0 - \frac{8\pi^3}{-3}\right]$$

$$= \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)^2 \cos nx\, dx$$

$$= \frac{1}{\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right) - 2(\pi - x)(-1)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{[0 - 0 + 0] - \left[(4\pi^2)(0) + 4\pi\left(\frac{-(-1)^n}{n^2}\right) + 2(0)\right]\right\}$$

$$= \frac{1}{\pi} 4\pi \frac{(-1)^n}{n^2}$$

$$a_n = 4\frac{(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)^2 \sin nx\, dx$$

$$= \frac{1}{\pi}\left[(\pi - x)^2\left(\frac{-\cos nx}{n}\right) - 2(\pi - x)(-1)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[0 - 0 - 2\frac{\cos n\pi}{n^3}\right] - \left[4\pi^2\left(\frac{-\cos n\pi}{n}\right) - 2(2\pi)\frac{\sin n\pi}{n^2} + 2\frac{\cos n\pi}{n^3}\right]\right\}$$

$$b_n = 4\pi\frac{(-1)^n}{n}$$

Hence the Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$(\pi - x)^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} 4\frac{(-1)^n}{n^2}\cos nx - \sum_{n=1}^{\infty} 4\pi\frac{(-1)^n}{n}\sin nx$$

Deduction (i):

Put $x = 0$ (which is a continuous point)

$$\pi^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} 4\frac{(-1)^n}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} 4\frac{(-1)^n}{n^2} = -\frac{4\pi^2}{3} + \pi^2$$

$$\Rightarrow \sum_{n=1}^{\infty} 4\frac{(-1)^n}{n^2} = -\frac{\pi^2}{3}$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots = \frac{\pi^2}{12}$$

Deduction (ii):

Put $x = \pi$ (which is a discontinuous point)

$$\frac{(\pi-\pi)^2+((\pi-(-\pi))^2}{2} = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} 4\frac{(-1)^n}{n^2}(-1)^n$$

$$\Rightarrow 2\pi^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} 4\frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} 4\frac{1}{n^2} = 2\pi^2 - \frac{4\pi^2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty} 4\frac{1}{n^2} = \frac{2\pi^2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

14.Find the Fourier series expansion of $f(x)=\begin{cases} x-1 & for -\pi<x<0 \\ x+1 & for\ 0<x<\pi \end{cases}$

Solution:

$$f(x)=\begin{cases} x-1 & for -\pi<x<0 \\ x+1 & for\ 0<x<\pi \end{cases}$$

$$f(-x)=\begin{cases} -(x+1) & for\ 0<x<\pi \\ -(x-1) & for -\pi<x<0 \end{cases}$$

$$f(-x)=-\begin{cases} x-1 & for -\pi<x<0 \\ x+1 & for\ 0<x<\pi \end{cases}$$

$$=-f(x)$$

$f(x)$ *is an odd function*

$$\therefore a_0 = a_n = 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (x+1)\sin nx dx$$

$$= \frac{2}{\pi}\left[(x+1)\left(-\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-(1+\pi)\frac{(-1)^n}{n} + \frac{1}{n}\right]$$

$$b_n = \frac{2}{\pi}\left[\frac{1-(1+\pi)(-1)^n}{n}\right]$$

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(1+\pi)(-1)^n}{n}\right]\sin nx$$

11. Obtain the fourier series for $f(x) = 1 + x + x^2$ in $(-\pi, \pi)$
Deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \cdots . = \frac{\pi^2}{6}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx + \sum_{n=1}^{\infty} b_n sinnx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (1 + x + x^2)dx$$

$$= \frac{1}{\pi}\left[x + \frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\pi + \frac{\pi^2}{2} + \frac{\pi^3}{3} + \pi - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right]$$

$$= \frac{1}{\pi}\left[2\pi + 2\frac{\pi^3}{3}\right]$$

$$= 2\frac{\pi^2}{3} + 2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) cosnxdx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (1 + x + x^2) cosnxdx$$

$$= \frac{1}{\pi}\left\{\left[(1 + x + x^2)\frac{sinnx}{n} + (1 + 2x)\frac{cosnx}{n^2} - 2\frac{sinnx}{n^3}\right]_{-\pi}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{4\pi(-1)^n}{n^2}\right\}$$

$$= \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) sinnxdx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(1 + x + x^2)sinnxdx$$

$$= \frac{1}{\pi}\left\{\left[-(1 + x + x^2)\frac{cosnx}{n} + (1 + 2x)\frac{sinnx}{n^2} - 2\frac{cosnx}{n^3}\right]_{-\pi}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\begin{matrix} -(1 + \pi + \pi^2)\frac{(-1)^n}{n} - 2\frac{(-1)^n}{n^3} + \\ (1 - \pi + \pi^2)\frac{(-1)^n}{n} + 2\frac{(-1)^n}{n^3} \end{matrix}\right]_{-\pi}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{-2\pi\frac{(-1)^n}{n}\right\}$$

$$= \left\{\frac{-2(-1)^n}{n}\right\}$$

$$= \frac{4(-1)^n}{n^2}$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}cosnx + \sum_{n=1}^{\infty}\frac{-2(-1)^n}{n}sinnx$$

Put $x = \pi$ this point is a discontinuous point

$$f(x) = \frac{\pi + \pi^2 + \pi^2 - \pi}{2} = \pi^2$$

$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^{2n}}{n^2}$

$$\pi^2 - \frac{\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4(-1)^{2n}}{n^2}$$

$$\frac{2\pi^2}{3 \times 4} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$$

$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots.$

15.Find the Fourier series of $y = x^2$ $in - \pi < x < \pi$
Hence show that $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{2^4} + \cdots$
Solution:

$y = x^2$ is an even function

Then the fourier series is

$y = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx$

Here $b_n = 0$

$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)dx$

$= \frac{2}{\pi} \int_0^{\pi} x^2 dx$

$= \frac{2}{\pi} \left[\frac{x^3}{3}\right]_0^{\pi}$

$$= \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)cosnxdx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 cosnxdx$$

$$= \frac{2}{\pi}\left[x^2 \frac{sinnx}{n} - 2x\frac{-cos}{n^2} + 2\frac{sinnx}{n^3}\right]_0^{\pi}$$

$$= \frac{4co}{n^2}$$

$$= \frac{4(-1)^n}{n^2}$$

The fourier series of f(x) is given be

$$y = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} cosnx$$

By Parseval's identity

$$\bar{y}^2 = \frac{a_0^2}{4} + \frac{1}{2}[\sum a_n^2 + b_n^2]$$

$$a_0 = \frac{2\pi^2}{3} \ , \ a_n = \frac{4(-1)^n}{n^2} \ , b_n = 0$$

$$\bar{y}^2 = \frac{1}{\pi - 0}\int_{-\pi}^{\pi} y^2 dx$$

$$\bar{y}^2 = \frac{1}{\pi + \pi}\int_{\pi}^{\pi} x^4 dx$$

16.Find b_n for xsinx in (-1,1)

Solution:

f(x) = xsinx is even function

Then b_n=0

17.If $x^2 = \frac{\pi^2}{3} - 4\left[\frac{cosx}{1^2} - \frac{cos2x}{2^2} + \frac{cos3x}{3^2} + \cdots..\right]$ $in -\pi \leq x \leq \pi$

Find $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots..$

Solution:

$$x^2 = \frac{\pi^2}{3} - 4\left[\frac{cosx}{1^2} - \frac{cos2x}{2^2} + \frac{cos3x}{3^2} + \cdots..\right]$$

Put $x = \pi$

$$4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots..\right] = \frac{2\pi^2}{3}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots.. = \frac{\pi^2}{6}$$

18.Find a_n in expanding e^{-x} $-\pi < x < \pi$ as a fourier series

Solution:

$$f(x) = e^{-x}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}cosnxdx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-cosnx + nsinnx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)}[e^{-\pi}(-cosn\pi) + e^{\pi}(cosn\pi)]$$

$$= \frac{1}{\pi(1+n^2)}[cosn\pi(e^{\pi} - e^{-\pi})]$$

$$a_n = \frac{(-1)^n}{\pi(1+n^2)} 2sinh\pi$$

Fourier Series in the Interval *(0,2l)*

1.Find the mean square value of the function *f(x) = x* in the interval *(0,l)*.

Solution:

$$Mean\ \ Square\ \ value = \frac{\int_a^b [f(x)]^2\ dx}{(b-a)}$$

$$= \frac{1}{l}\int_0^l x^2\ dx$$

$$= \frac{1}{l}\left[\frac{x^3}{3}\right]_0^l$$

$$= \frac{l^2}{3}$$

2.Find the Fourier series expansion of $f(x)=\begin{cases} x, & 0<x<l/2 \\ l-x, & l/2<x<l \end{cases}$.

Hence deduce, the value of $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$

Solution

Since the interval for fourier series is 2L

$$put\ 2L = l \Rightarrow L = \frac{l}{2}$$

$$Hence\ f(x)=\begin{cases} x, & 0<x<L \\ 2L-x, & L<x<2L \end{cases}$$

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx$$

$$= \frac{1}{L}\int_0^{2L} f(x)\,dx$$

$$= \frac{1}{L}\left\{\int_0^{L} x\,dx + \int_L^{2L}(2L-x)\,dx\right\}$$

$$= \frac{1}{L}\left\{\left[\frac{x^2}{2}\right]_0^L + \left[\frac{(2L-x)^2}{2(-1)}\right]_L^{2L}\right\}$$

$$= \frac{1}{L}\left[\frac{L^2}{2} - 0 - 0 + \frac{L^2}{2}\right]$$

$$a_0 = \frac{1}{L}\left(L^2\right) = L$$

$$L = \frac{l}{2} \Rightarrow a_0 = \frac{l}{2}$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\int_0^{2L} f(x)\ \cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\left\{\int_0^{L} x\ \cos\left(\frac{n\pi x}{L}\right)dx + \int_L^{2L}(2L-x)\ \cos\left(\frac{n\pi x}{L}\right)dx\right\}$$

$$= \frac{1}{L}\left\{\begin{aligned}&\left[(x)\left(\sin\left(\frac{n\pi x}{L}\right)\left(\frac{L}{n\pi}\right)\right)-1\left(-\cos\left(\frac{n\pi x}{L}\right)\left(\frac{L}{n\pi}\right)^2\right)\right]_0^L\\&+\left[(2L-x)\left(\sin\left(\frac{n\pi x}{L}\right)\left(\frac{L}{n\pi}\right)\right)-(-1)\left(-\cos\left(\frac{n\pi x}{L}\right)\left(\frac{L}{n\pi}\right)^2\right)\right]_L^{2L}\end{aligned}\right\}$$

$$= \frac{1}{L}\left\{\begin{aligned}&\left[L\sin n\pi\frac{L}{n\pi}+\cos n\pi\left(\frac{L}{n\pi}\right)^2-0-\left(\frac{L}{n\pi}\right)^2\right]+\\&\left[0-\left(\frac{L}{n\pi}\right)^2-L\sin n\pi\frac{L}{n\pi}+\cos n\pi\left(\frac{L}{n\pi}\right)^2\right]\end{aligned}\right\}$$

$$= \frac{1}{L}\left(\frac{L}{n\pi}\right)^2\left[2\cos n\pi - 2\right]$$

$$= \frac{2}{L}\left(\frac{L}{n\pi}\right)^2\left[(-1)^n - 1\right]$$

$$= \frac{2}{L}\frac{L^2}{n^2\pi^2}\begin{cases}-2, & if\ n\ is\ odd\\ 0, & if\ n\ is\ even\end{cases}$$

$$= \frac{2L}{n^2\pi^2}\begin{cases}-2, & if\ n\ is\ odd\\ 0, & if\ n\ is\ even\end{cases}$$

$$L = \frac{l}{2} \Rightarrow a_n = \frac{l}{n^2\pi^2}\begin{cases}-2, & if\ n\ is\ odd\\ 0, & if\ n\ is\ even\end{cases}$$

$$\Rightarrow a_n = \begin{cases}\dfrac{-2l}{n^2\pi^2} & if\ n\, is\, odd\\ 0, & if\ n\, is\, even\end{cases}$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\,\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\int_0^{2L} f(x)\;\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\left\{\int_0^{L} x\sin\left(\frac{n\pi x}{L}\right)\,dx + \int_L^{2L}(2L-x)\;\sin\left(\frac{n\pi x}{L}\right)dx\right\}$$

$$= \frac{1}{L}\left\{\begin{array}{l}\left[(x)\left(-\cos\left(\frac{n\pi x}{L}\right)\left(\frac{L}{n\pi}\right)\right)-1\left(-\sin\left(\frac{n\pi x}{L}\right)\left(\frac{L}{n\pi}\right)^2\right)\right]_0^L \\ +\left[(2L-x)\left(-\cos\left(\frac{n\pi x}{L}\right)\left(\frac{L}{n\pi}\right)\right)-(-1)\left(-\sin\left(\frac{n\pi x}{L}\right)\left(\frac{L}{n\pi}\right)^2\right)\right]_L^{2L}\end{array}\right\}$$

$$= \frac{1}{L}\left\{\begin{array}{l}\left[L(-\cos n\pi)\left(\frac{L}{n\pi}\right)+\sin n\pi\left(\frac{L}{n\pi}\right)^2-0+0\right] \\ +\left[0-\sin n\pi\left(\frac{L}{n\pi}\right)^2+L(\cos n\pi)\left(\frac{L}{n\pi}\right)+\sin n\pi\left(\frac{L}{n\pi}\right)^2\right]\end{array}\right\}$$

$$b_n = 0$$

Hence the Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad where\ L = \frac{l}{2}$$

$$f(x) = \frac{l}{4} + \sum_{n=1,3,5,..}^{\infty} \frac{-2l}{n^2\pi^2}\cos\left(\frac{2n\pi x}{l}\right)$$

Since the denominator of the series is of the form $\frac{1}{n^4}$ and the denominator in the Fourier series is of the form $\frac{1}{n^2}$ let us use the Parseval's identity.

$$\frac{a_0^2}{2} + \left(a_n^2 + b_n^2\right) = \frac{1}{L}\int_0^{2L} \left[f(x)\right]^2 dx$$

$$\Rightarrow \frac{l^2}{8} + \frac{4l^2}{\pi^4} \sum_{n=1,3,5,..}^{\infty} \frac{1}{n^4} = \frac{1}{L}\int_0^{2L} \left[f(x)\right]^2 dx$$

$$\Rightarrow \frac{l^2}{8} + \frac{4l^2}{\pi^4} \sum_{n=1,3,5,..}^{\infty} \frac{1}{n^4} = \frac{1}{L}\left\{\int_0^{L} x^2 \; dx + \int_L^{2L} (l-x)^2 dx\right\}$$

$$\Rightarrow \frac{l^2}{8} + \frac{4l^2}{\pi^4} \sum_{n=1,3,5,..}^{\infty} \frac{1}{n^4} = \frac{1}{L}\left\{\left[\frac{x^3}{3}\right]_0^L + \left[\frac{(2L-x)^3}{3(-1)}\right]_L^{2L}\right\}$$

$$\Rightarrow \frac{l^2}{8} + \frac{4l^2}{\pi^4} \sum_{n=1,3,5,..}^{\infty} \frac{1}{n^4} = \frac{1}{L}\left[\frac{L^3}{3} - 0 + \frac{L^3}{3}\right]$$

$$\Rightarrow \frac{l^2}{8} + \frac{4l^2}{\pi^4} \sum_{n=1,3,5,..}^{\infty} \frac{1}{n^4} = \frac{2}{l}\left[\frac{l^3}{24} - 0 - 0 + \frac{l^3}{24}\right] (\because L = \frac{l}{2})$$

$$\Rightarrow \frac{l^2}{8} + \frac{4l^2}{\pi^4} \sum_{n=1,3,5,..}^{\infty} \frac{1}{n^4} = \frac{2}{l}\frac{l^3}{12}$$

$$\Rightarrow \frac{l^2}{8} + \frac{4l^2}{\pi^4} \sum_{n=1,3,5,..}^{\infty} \frac{1}{n^4} = \frac{l^2}{6}$$

$$\Rightarrow \frac{4l^2}{\pi^4} \sum_{n=1,3,5,..}^{\infty} \frac{1}{n^4} = \frac{l^2}{6} - \frac{l^2}{8}$$

$$\Rightarrow \frac{4l^2}{\pi^4} \sum_{n=1,3,5,..}^{\infty} \frac{1}{n^4} = \frac{l^2}{24}$$

$$\Rightarrow \sum_{n=1,3,5,..}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{4l^2}\frac{l^2}{24}$$

$$\Rightarrow \sum_{n=1,3,5,..}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96}$$

3. Find the constant term in the Fourier expansion of $f(x) = x^2 - 2$ in $-2 < x < 2$

Solution: $f(-x) = (-x)^2 - 2 = x^2 - 2 = f(x)$ the function is even. So,

$$a_0 = \frac{2}{2}\int_0^2 (x^2 - 2)\, dx$$

$$= \left[\frac{x^3}{3} - 2x\right]_0^2 = \frac{8}{3} - 4$$

$$a_0 = -\frac{4}{3}$$

Hence the constant term in the Fourier expansion is $\frac{a_0}{2} = -\frac{2}{3}$

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4. If $f(x)$ is an odd function in the interval $(-l, l)$, write the formula to find the Fourier coefficients.

Solution: $a_0 = a_n = 0$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right) dx$$

---

5. If $f(x)$ is an even function in the interval $(-l, l)$, write the formula to find the Fourier coefficients.

Solution:

$$a_0 = \frac{2}{l}\int_0^l f(x)\, dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = 0$$

6.Find the Fourier series expansion of the function

$$f(x) = \begin{cases} l+x, & -l \le x \le 0 \\ l-x, & 0 \le x \le l \end{cases}$$ **. Hence deduce that** $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

Solution:

Here the interval is – l to l let us first check whether the function is odd or even.

$$f(-x) = \begin{cases} l + (-x), & -l \le -x \le 0 \\ l - (-x), & 0 \le -x \le l \end{cases}$$

$$= \begin{cases} l - x, & 0 \le x \le l \\ l + x, & -l \le x \le l \end{cases}$$

So, the function is even.

$$= f(x)$$

(Note that when an inequality is multiplied by -1 the inequality is reversed.)

So, let us find only the two Fourier coefficients a_0, , a_n.

$$a_0 = \frac{2}{l}\int_0^l (l - x)\, dx$$

$$= \frac{2}{l}\left[\frac{(l-x)^2}{2(-1)}\right]_0^l$$

$$= \frac{2}{l}\left[0 + \frac{l^2}{2}\right]$$

$$= l$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l (l-x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[(l-x)\left(\sin\left(\frac{n\pi x}{l}\right)\left(\frac{l}{n\pi}\right)\right) - (-1)\left(-\cos\left(\frac{n\pi x}{l}\right)\left(\frac{l}{n\pi}\right)^2\right)\right]_0^l$$

$$= \frac{2}{l}\left[0 - \cos n\pi\left(\frac{l}{n\pi}\right)^2 - 0 + \left(\frac{l}{n\pi}\right)^2\right]$$

$$= \frac{2}{l}\left(\frac{l}{n\pi}\right)^2 [1-\cos n\pi]$$

$$= \frac{2l}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$= \frac{2l}{n^2\pi^2}\begin{cases} 2, \text{ if } n \text{ is odd} \\ 0, \text{ if } n \text{ is even}\end{cases}$$

$$a_n = \begin{cases} \frac{4l}{n^2\pi^2}, \text{ if } n \text{ is odd} \\ 0, \text{ if } n \text{ is even}\end{cases}$$

Hence the Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$f(x) = \frac{l}{2} + \frac{4l}{\pi^2}\sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2}\cos\left(\frac{n\pi x}{l}\right)$$

<u>Deduction:</u>

To get the deduction put $x = 0$(which is a continuous point)

$$l = \frac{l}{2} + \frac{4l}{\pi^2} \sum_{n=1,3,5,...}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow l - \frac{l}{2} = \frac{4l}{\pi^2} \sum_{n=1,3,5,...}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \frac{l}{2} = \frac{4l}{\pi^2} \sum_{n=1,3,5,...}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1,3,5,...}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

7.Obtain the sine series for the function $f(x) = \begin{cases} x, & 0 < x < l/2 \\ l - x, & l/2 < x < l \end{cases}$

Solution:

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l} \left\{ \int_0^{l/2} x \sin\left(\frac{n\pi x}{l}\right) dx + \int_{l/2}^{l} (l - x) \sin\left(\frac{n\pi x}{l}\right) dx \right\}$$

$$= \frac{2}{l} \left\{ \begin{array}{l} \left[(x)\left(-\cos\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi} \right) - 1\left(-\sin\left(\frac{n\pi x}{l}\right)\right)\left(\frac{l}{n\pi}\right)^2 \right]_0^{l/2} \\ + \left[(l - x)\left(-\cos\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi} \right) - (-1)\left(-\sin\left(\frac{n\pi x}{l}\right)\right)\left(\frac{l}{n\pi}\right)^2 \right]_{l/2}^{l} \end{array} \right\}$$

$$= \frac{2}{l}\left\{ \begin{array}{l} \left[-\frac{l}{2}\frac{l}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)\frac{l^2}{n^2\pi^2} - 0 + 0 \right] \\ + \left[0 - 0 + \frac{l}{2}\frac{l}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)\frac{l^2}{n^2\pi^2} \right] \end{array} \right\}$$

$$= \frac{2}{l}\left[2\sin\left(\frac{n\pi}{2}\right)\frac{l^2}{n^2\pi^2} \right]$$

$$= \frac{4l}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

Hence the half range sine series is $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

$$f(x) = \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{l}\right)$$

8.Find the Fourier series expansion of $f(x) = \begin{cases} l - x, & 0 < x < l \\ 0, & l < x < 2l \end{cases}$**.**

Hence deduce the value of the series (i) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$

(ii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.$

Solution: We need to find all the three fourier constants.

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\ dx$$

$$= \frac{1}{l}\left\{\int_0^{l} l - x\ dx + \int_l^{2l} 0\ dx\right\}$$

$$= \frac{1}{l}\left[\frac{(l-x)^2}{2(-1)}\right]_0^l$$

$$= \frac{1}{l}\frac{l^2}{2}$$

$$= \frac{l}{2}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\ \cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left\{\int_0^{l}(l-x)\cos\left(\frac{n\pi x}{l}\right)dx + \int_l^{2l} 0\cos\left(\frac{n\pi x}{l}\right)dx\right\}$$

$$= \frac{1}{l}\left[(l-x)\sin\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi} - (-1)\left(-\cos\left(\frac{n\pi x}{l}\right)\right)\left(\frac{l}{n\pi}\right)^2\right]_0^l$$

$$= \frac{1}{l}\left[0 - \cos n\pi\left(\frac{l}{n\pi}\right)^2 - 0 + \left(\frac{l}{n\pi}\right)^2\right]$$

$$= \frac{1}{l}\left(\frac{l}{n\pi}\right)^2\left[1 - \cos n\pi\right]$$

$$= \frac{l}{n^2\pi^2}\left[1 - (-1)^n\right]$$

$$a_n = \frac{l}{n^2\pi^2}\begin{cases} 2, \text{ if } n \text{ is odd} \\ 0, \text{ if } n \text{ is even}\end{cases}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\ \sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left\{\int_0^{l}(l-x)\sin\left(\frac{n\pi x}{l}\right)dx + \int_l^{2l} 0 \sin\left(\frac{n\pi x}{l}\right)dx\right\}$$

$$= \frac{1}{l}\left[(l-x)\left(-\cos\left(\frac{n\pi x}{l}\right)\right)\frac{l}{n\pi} - (-1)\left(-\sin\left(\frac{n\pi x}{l}\right)\left(\frac{l}{n\pi}\right)^2\right)\right]_0^l$$

$$= \frac{1}{l}\left[0 - \sin n\pi\left(\frac{l}{n\pi}\right)^2 + l\frac{l}{n\pi} - 0\right]$$

$$b_n = \frac{l}{n\pi}$$

Hence the Fourier series is

$$f(x) = \frac{l}{4} + \frac{2l}{\pi^2}\sum_{n=1,3,5,..}^{\infty}\frac{1}{n^2}\cos\left(\frac{n\pi x}{l}\right) + \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{l}\right)$$

<u>Deduction(i)</u> The denominator of cosine terms are in the form $\frac{1}{n^2}$ and the denominator of sine terms are in the form $\frac{1}{n}$. So, to get deduction (i) , let us make all the cosine terms vanish. This can be done by taking $x = \frac{\pi}{2}, 3\frac{\pi}{2},....$ Let us put $x = \frac{l}{2}$ (which is a continuous point)

$$l-\frac{l}{2}=\frac{l}{4}+0+\frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow\frac{l}{2}=\frac{l}{4}+\frac{l}{\pi}\left[\frac{1}{1}\sin\left(\frac{\pi}{2}\right)+\frac{1}{2}\sin\pi+\frac{1}{3}\sin\left(3\frac{\pi}{2}\right)+\frac{1}{4}\sin2\pi+\frac{1}{5}\sin\left(5\frac{\pi}{2}\right)+\dots\right]$$

$$\Rightarrow\frac{l}{2}=\frac{l}{4}+\frac{l}{\pi}\left[\frac{1}{1}-\frac{1}{3}+\frac{1}{5}\dots\right]$$

$$\Rightarrow\frac{l}{2}-\frac{l}{4}=\frac{l}{\pi}\left[\frac{1}{1}-\frac{1}{3}+\frac{1}{5}\dots\right]$$

$$\Rightarrow\frac{l}{4}=\frac{l}{\pi}\left[\frac{1}{1}-\frac{1}{3}+\frac{1}{5}\dots\right]$$

$$\Rightarrow\frac{\pi}{4}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}\dots$$

Deduction(ii):

To get the deduction all the sine terms must vanish and this can be done by taking $x=0,2\pi,\dots$. So, let us put $x=0$(which is a discontinuous point).

$$\frac{f(0)+f(2l)}{2}=\frac{l}{4}+\frac{2l}{\pi^2}\sum_{n=1,3,5..}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow\frac{l}{2}=\frac{l}{4}+\frac{2l}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right]$$

$$\Rightarrow\frac{l}{2}=\frac{l}{4}+\frac{2l}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right]$$

$$\Rightarrow\frac{l}{2}-\frac{l}{4}=\frac{2l}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right]$$

$$\Rightarrow\frac{l}{4}=\frac{2l}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right]$$

$$\Rightarrow\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots$$

9. Find the Fourier series expansion of $f(x)=\begin{cases}x, & 0\le x\le 1\\ 2-x, & 1\le x\le 2\end{cases}$

Also deduce $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+.....=\frac{\pi^2}{8}$

Solution:

$$F.S\quad f(x)=\frac{a_0}{2}+\sum_1^{\infty}(a_n\cos n\pi x+b_n\sin n\pi x)$$

$$a_0=1\int_0^2 f(x)dx=\int_0^1 xdx+\int_1^2(2-x)dx=1$$

$$a_n=1\int_0^2 f(x)\cos n\pi x dx=\int_0^1 x\cos n\pi x dx+\int_1^2(2-x)\cos n\pi x dx$$

$$=\frac{2}{n^2\pi^2}\left[(-1)^n-1\right]$$

$$b_n=1\int_0^2 f(x)\sin n\pi x dx=\int_0^1 x\sin n\pi x dx+\int_1^2(2-x)\sin n\pi x dx=0$$

$$f(x)=\frac{1}{2}-\frac{4}{\pi^2}\sum_{n=odd}\frac{1}{n^2}\cos n\pi x$$

$$At\ x=1.(continuous\ pt.)\ \ F.S\ \ value=f(1)=1$$

$$\therefore 1=\frac{1}{2}-\frac{4}{\pi^2}\sum_{n=odd}\frac{1}{n^2}\cos n\pi$$

$$(For\ n=odd\ ,\ \cos n\pi=-1\ always)$$

$$1=\frac{1}{2}-\frac{4}{\pi^2}\sum_{n=odd}\frac{1}{n^2}(-1)$$

$$\frac{4}{\pi^2}\sum_{n=odd}\frac{1}{n^2}=\frac{1}{2}\Rightarrow\sum_{n=odd}\frac{1}{n^2}=\frac{\pi^2}{8}$$

Half Range Fourier Series

1.Obtain the half range cosine series for $(x-1)^2 \quad in\ 0 < x < 1$
Solution:

The half range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \qquad where\ \ l = 1$$

$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$= \frac{2}{1}\int_0^1 (x-1)^2\, dx$$

$$= 2\left[\frac{(x-1)^3}{3}\right]_0^1$$

$$= 2\left[\frac{1}{3}\right]$$

$$a_0 = \frac{2}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{1}\int_0^1 (x-1)^2 \cos n\pi x dx$$

$$= 2\left[(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right) - 2(x-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + 2\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$a_n = \frac{4}{n^2\pi^2}$$

$$\therefore\ f(x) = \frac{1}{3} + \sum_{\dots} \frac{-4}{n^2\pi^2}\cos n\pi x$$

2.Prove that $1=\frac{4}{\pi}\left[\sin\frac{\pi x}{l}+\frac{1}{3}\sin\frac{3\pi x}{l}+\frac{1}{5}\sin\frac{5\pi x}{l}+....\right]$ **in the interval**

0< x< *l*

Solution:

As per the RHS we need to find the Fourier sine series expansion of the function in 0 < x < l Here*f(x) = 1* and the interval shall be taken as *(0,l)*.

$$b_n = \frac{2}{l}\int_0^l 1 \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l}\left(-\cos\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi}\right)_0^l$$

$$= \frac{2}{l}\frac{l}{n\pi}\left[-\cos n\pi + 1\right]$$

$$= \frac{2}{n\pi}\left[-(-1)^n + 1\right]$$

$$= \frac{2}{n\pi}\begin{cases} 2, \text{ if } n \text{ is odd} \\ 0, \text{ if } n \text{ is even} \end{cases}$$

$$b_n = \begin{cases} \frac{4}{n\pi}, \text{ if } n \text{ is odd} \\ 0, \text{ if } n \text{ is even} \end{cases}$$

Hence the Fourie siner series is

$$1 = \sum_{n=1,3,5,....}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow 1 = \frac{4}{\pi}\left[\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + ...\right]$$

3.Find the half range cosine series for the function $f(x) = x, \; in \; 0 < x < l$. **Hence deduce the value of the series** $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$.

Solution:

$$a_0 = \frac{2}{l}\int_0^l x\ dx$$

$$= \frac{2}{l}\left[\frac{x^2}{2}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l^2}{2}\right]$$

$$a_0 = l$$

$$a_n = \frac{2}{l}\int_0^l x\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[x\ \sin\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi} - 1\left(-\cos\left(\frac{n\pi x}{l}\right)\left(\frac{l}{n\pi}\right)^2\right)\right]_0^l$$

$$= \frac{2}{l}\left[l\ \sin\left(\frac{n\pi l}{l}\right)\frac{l}{n\pi} - 1\left(-\cos\left(\frac{n\pi l}{l}\right)\left(\frac{l}{n\pi}\right)^2\right) - 0 - \left(\frac{l}{n\pi}\right)^2\right]$$

$$= \frac{2}{l}\left(\frac{l}{n\pi}\right)^2\left[\cos n\pi - 1\right]$$

$$= \frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$= \frac{2l}{n^2\pi^2}\begin{cases}(-2), & \text{if } n \text{ is odd}\\ 0, & \text{if } n \text{ is even}\end{cases}$$

$$a_n = \begin{cases}-\frac{4l}{n^2\pi^2}, & \text{if } n \text{ is odd}\\ 0, & \text{if } n \text{ is even}\end{cases}$$

Hence the half range series is $x = \frac{l}{2} + \sum_{n=1,3,5,\ldots}^{\infty} -\frac{4l}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right)$

Deduction:

Since the denominator of the series is n^4 and that of the cosine series is only n^2 let us apply Parseval's identity

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) = \frac{2}{L}\int_0^L [f(x)]^2\,dx$$

$$\Rightarrow \frac{l^2}{2} + \sum_{n=1,3,5,...}^{\infty} \frac{16l^2}{n^4\pi^4} = \frac{2}{l}\int_0^l x^2\,dx$$

$$\Rightarrow \frac{l^2}{2} + \sum_{n=1,3,5,...}^{\infty} \frac{16l^2}{n^4\pi^4} = \frac{2}{l}\left[\frac{x^3}{3}\right]_0^l$$

$$\Rightarrow \frac{l^2}{2} + \sum_{n=1,3,5,...}^{\infty} \frac{16l^2}{n^4\pi^4} = \frac{2l^2}{3}$$

$$\Rightarrow \sum_{n=,3,5,...1}^{\infty} \frac{16l^2}{n^4\pi^4} = \frac{2l^2}{3} - \frac{l^2}{2}$$

$$\Rightarrow \sum_{n=1,3,5,..}^{\infty} \frac{16l^2}{n^4\pi^4} = \frac{l^2}{6}$$

$$\Rightarrow \sum_{n=1,3,5,...}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96}$$

4.Find the half range sine series of $f(x) = lx - x^2 \;\; in\,(0,l)$.

Solution:

$$Let \;\; f(x) = \sum_{n-1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(lx - x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l - 2x)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[-\frac{2l^3}{n^3\pi^3}(\cos n\pi - 1)\right] = -\frac{4l^2}{n^3\pi^3}[(-1)^n - 1]$$

$$\therefore b_n = \frac{8l^2}{n^3\pi^3} \quad when \;\; n \;\; is \;\; odd$$

$$= 0 \quad when \; n \;\; is \;\; even$$

$\therefore$ The required sine series is

$$f(x) = \frac{8l^2}{\pi^3} \sum_{n=1,3,5}^{\infty} \frac{1}{n^3} \sin\frac{n\pi x}{l}$$

5.Find a sine series for *f(x)* = *x*, in (0,π).

Solution:

$$b_n \qquad = \frac{2}{\pi} \int_0^{\pi} x \; \sin \; nx \; dx$$

$$= \frac{2}{\pi} \left\{ (x) \left(-\frac{\cos \; nx}{n} \right) - 1 \left(-\frac{\sin \; nx}{n^2} \right) \right\}_0^{\pi}$$

$$= \frac{2}{\pi} \left[-\pi \frac{\cos n\pi}{n} + 0 + 0 - 0 \right]$$

$$= -2 \frac{(-1)^n}{n}$$

$$b_n \qquad = 2 \frac{(-1)^{n+1}}{n}$$

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \; nx$$

is the half range sine series .

6.Find the half range sine series for *f(x)* = 2 in 0 < *x* < π.

Solution:

$$b_n \qquad = \frac{2}{\pi} \int_0^{\pi} 2 \sin \; nx \; dx$$

$$= \frac{2}{\pi} \left\{ (2) \left(-\frac{\cos \; nx}{n} \right) \right\}_0^{\pi}$$

$$= -\frac{4}{\pi}\left[\frac{\cos n\pi}{n} - \frac{1}{n}\right] = -\frac{4}{\pi}\left[\frac{(-1)^n}{n} - \frac{1}{n}\right]$$

$$= -\frac{4}{\pi}\begin{cases} -\frac{2}{n}, & if\ n\ is\ odd \\ 0, & if\ n\ is\ even \end{cases}$$

$$b_n = \begin{cases} \frac{8}{n\pi}, & if\ n\ is\ odd \\ 0, & if\ n\ is\ even \end{cases}$$

Hence the half range sine series is $f(x) = \frac{8}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n}\sin nx$

7.The cosine series for *f(x)* = *x sin x* in 0 < *x* < π is given as

$$x\sin x = 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos nx.$$

Deduce that $1 + 2\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\right] = \frac{\pi}{2}$.

<u>Solution:</u>

As $n^2 - 1 = (n-1)(n+1)$

$$x\sin x = 1 - \frac{1}{2}\cos x - 2\left[\frac{1}{1.3}\cos 2x - \frac{1}{2.4}\cos 3x + \frac{1}{3.5}\cos 4x - \frac{1}{4.6}\cos 5x + \frac{1}{5.7}\cos 6x - \ldots\right]$$

Put $x = \pi/2$ in the above series we get

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1 - \frac{1}{2}\cos\frac{\pi}{2} - 2\left[\frac{1}{1.3}\cos 2\frac{\pi}{2} - \frac{1}{2.4}\cos 3\frac{\pi}{2} + \frac{1}{3.5}\cos 4\frac{\pi}{2} - \frac{1}{4.6}\cos 5\frac{\pi}{2} + \frac{1}{5.7}\cos 6\frac{\pi}{2} - \ldots\right]$$

$$\Rightarrow \frac{\pi}{2} = 1 - 2\left[-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \ldots\ldots\right]$$

$$\Rightarrow \frac{\pi}{2} = 1 + 2\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\ldots\right]$$

8. Find the half range cosine series of

$f(x) = x^2,\ in\ 0 < x < \pi$**. Hence deduce the value of** $\sum_{n=1}^{\infty} \frac{1}{n^4}$**.**

Solution:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

applying Bernoulli's formula

$$= \frac{2}{\pi}\left[\left(x^2\right)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[\left(\pi^2\right)\left(\frac{\sin n\pi}{n}\right) - (2\pi)\left(\frac{-\cos n\pi}{n^2}\right) + (2)\left(\frac{-\sin n\pi}{n^3}\right)\right]_0^{2\pi} - \frac{1}{\pi}\left[\left(0^2\right)\left(\frac{\sin n0}{n}\right) - (0)\left(\frac{-\cos n0}{n^2}\right) + (2)\left(\frac{-\sin n0}{n^3}\right)\right]_0^{2\pi}\right\}$$

$$= \frac{1}{\pi} 4\pi \frac{(-1)^n}{n^2}$$

$$a_n = 4\frac{(-1)^n}{n^2}$$

Hence the half range cosine series is $x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} 4\frac{(-1)^n}{n^2}$

<u>Deduction:</u>

Since the denominator of the series is n^4 and that of the cosine series is only n^2 let us apply Parseval's identity for fourier cosine series is

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) = \frac{2}{L}\int_0^L [f(x)]^2\, dx$$

$$\Rightarrow \frac{\left(\frac{4\pi^4}{9}\right)}{2} + \sum_{n=1}^{\infty}\frac{16}{n^4} = \frac{2}{\pi}\int_0^{\pi} x^4\, dx$$

$$\Rightarrow \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{2}{\pi}\left[\frac{x^5}{5}\right]_0^{\pi}$$

$$\Rightarrow \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{2}{\pi}\left[\frac{\pi^5}{5}\right]$$

$$\Rightarrow \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{2\pi^4}{5}$$

$$\Rightarrow 16\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{2\pi^4}{5} - \frac{2\pi^4}{9}$$

$$\Rightarrow 16\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{8\pi^4}{45}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

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9. Obtain the half range cosine series for $f(x)=x \quad in\ (0,\pi)$

Solution:

The half range cosine series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx dx dx$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$a_n = \begin{cases} 0 \quad if \quad n\ is\ even \\ \frac{-4}{n^2\pi} \quad if\ n\ is\ odd \end{cases}$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1,3,5...}\frac{-4}{n^2\pi}\cos nx$$

10. Find the half range sine series for

$\boldsymbol{f(x) = 3x(\pi^2 - x^2) 0 < x < \pi}$

Solution:

Let $f(x) = \sum_{n=1}^{\infty} b_n sinnx$ be the half range sine series

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) sinnx\, dx$$

$$= \frac{6}{\pi}\int_0^{\pi} x(\pi^2 - x^2) sinnx\, dx$$

$$= \frac{6}{\pi}\int_0^{\pi} (x\pi^2 - x^3) sinnx\, dx$$

$$= \frac{6}{\pi}\left[\begin{array}{c}(x\pi^2 - x^3)\left(-\frac{cosnx}{n}\right) - (\pi^2 - 3x^2)\left(-\frac{sinnx}{n^2}\right) + \\ (-6x)\left(\frac{cosnx}{n^3}\right) + 6\left(\frac{sinnx}{n^4}\right)\end{array}\right]_0^{\pi}$$

$$= \frac{6}{\pi}\left[-6\pi\frac{cosn\pi}{n^3}\right]$$

$$b_n = \frac{36.(-1)^{n+1}}{n^3}$$

The required half range series is given by

$$f(x) = \sum_{n=1}^{\infty} \frac{36.(-1)^{n+1}}{n^3} sinnx$$

$$\bar{y}^2 = \frac{2}{2\pi}\int_0^{\pi} x^4 dx$$

$$\bar{y}^2 = \frac{1}{\pi}\left(\frac{x^5}{5}\right)_0^{\pi}$$

$$\bar{y}^2 = \frac{\pi^4}{5}$$

$$\frac{\pi^4}{5} = \frac{\pi^4}{9} + \frac{16}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{\pi^4}{5} - \frac{\pi^4}{9} = 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{\pi^4(9-5)}{45} = \frac{4\pi^4}{45}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{2^4} + \cdots$$

11.Obtain cosine series $f(x) = (x-2)^2$, $0 \le x \le 2$ and deduce that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

Solution:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\left(\frac{n\pi x}{l}\right)$$

L=2

$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$= \frac{2}{2}\int_0^2 (x-2)^2 dx$$

$$a_0 = \left[\frac{(x-2)^3}{3}\right]_0^2$$

$$a_0 = 8/3$$

$$a_n = \frac{2}{l}\int_0^l f(x)cos\left(\frac{n\pi x}{l}\right)dx$$

$$a_n = \frac{2}{2}\int_0^2 (x-2)^2 cos\left(\frac{n\pi x}{l}\right)dx$$

$$a_n = \left[(x-2)^2 \frac{sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} + 2(x-2)\frac{cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2} - 2\frac{sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^3}\right]_0^2$$

$$a_n = \frac{16}{n^2\pi^2}$$

$$f(x) = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}cos\left(\frac{n\pi x}{2}\right)$$

Put x =0 is continuous point f(0)=4

$$4 = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$4-\frac{4}{3}=\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{8}{3}\left(\frac{\pi^2}{16}\right)=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots.$ ....(1)

Put x=2 is a continuous point f(2)=0

$$0=\frac{4}{3}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{-4}{3}\left(\frac{\pi^2}{16}\right)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\frac{-\pi^2}{12}=\frac{-1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\cdots.$$

$\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}+\cdots$ ....(2)

(1)+(2) $3\frac{\pi^2}{12}=2(\frac{1}{1^2}+\frac{1}{3^2}+\cdots)$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

12.Obtain xsinx as sine series in $0 < x < \pi$

Solution:

Given f(x) = xsinx

$$f(x) = \sum_{n=1}^{\infty} b_n sinnx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)sinnxdx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} xsinxsinnxdx$$

$$= -\frac{2}{\pi}\frac{1}{2}\int_0^{\pi} x[cos(n-1)x - cos(n+1)x]dx$$

$$= \frac{-1}{\pi}\int_0^{\pi} xcos(n-1)xdx - \int_0^{\pi} xcos(n+1)xdx$$

13.To which value the half range sine series corresponding to *f(x)* = *x*^{2} expressed in the interval (0,2) converges at *x* = 2?

Solution:

In order to expand in a sine series the function must be defined as an odd function in the interval (-2,2). Hence in the interval (-2,0) it should be defined in the form of $-f(-x) = -(-x)^2 = -x^2$.

$$\begin{array}{ccc} -x^2 & & x^2 \\ (------------)(-----------) & & \\ -2 & 0 & 2 \end{array}$$

Since at $x = 2$ the function is discontinuous (end point discontinuity) the Fourier (sine) series converges to

$$\frac{f(2-) + f(2+)}{2} = \frac{f(2) - f(-2)}{2}$$

$$= \frac{(2)^2 + (-(-2)^2)}{2}$$

$$= 0$$

14.Find the value of a_n in the cosine series expansion of $f(x) = 10$ in the interval (0,10).

Solution:

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{10}\int_0^{10} 10\cos\left(\frac{n\pi x}{10}\right)dx$$

$$= \frac{2}{10}10\left[\sin\left(\frac{n\pi x}{10}\right)\frac{10}{n\pi}\right]_0^{10}$$

$$= 2\frac{10}{n\pi}\left[\sin n\pi - \sin 0\right]$$

$$a_n = 0$$

Harmonic Analysis

The process of finding the fourier series for a unction y=f(x) from the tabulated values of x and y at equal intervals of x is called harmonic analysis

1.Find the Fourier series expansion upto third harmonic from the following data:

x:	**0**	**1**	**2**	**3**	**4**	**5**
f(x):	**9**	**18**	**24**	**28**	**26**	**20**

Solution:

x	Y	θ =πx/3	θ	Cosθ	y Cosθ	Cos 2θ	yCos 2θ	Cos 3θ	yCos 3θ
0	9	0	0	1	9	1	9	1	9
1	18	π/3	60	0.5	9	-0.5	-9	-1	-18
2	24	2π/3	120	-0.5	-12	-0.5	-12	1	24
3	28	3π/3	180	-1	-28	1	28	-1	-28
4	26	4π/3	240	-0.5	-13	-0.5	-13	1	26
5	20	5π/3	300	0.5	10	-0.5	-10	-1	-20
	$\sum y=$ 125				$\sum y\cos\theta$ = - 25		$\sum y\cos 2\theta$ = - 7		$\sum y\cos 3\theta$ = - 7

sinθ	ysinθ	sin 2θ	y sin 2θ	sin 3θ	y sin 3θ
0	0	0	0	0	0
0.866	15.588	0.866	15.588	0	0

0.866	20.784	-0.866	-20.784	0	0
0	0	0	0	0	0
-0.866	-22.516	0.866	22.516	0	0
-0.866	-17.32	-0.866	-17.32	0	0
	$\sum y \sin\theta$ = -3.464		$\sum y \sin 2\theta$ = 0.0		$\sum y \sin 3\theta$ = 0

$$a_0 = \frac{2}{q}\sum y = \frac{2}{6} * 125 = 41.6667$$

$$a_1 = \frac{2}{q}\sum y \cos\theta = \frac{2}{6} * (-25) = -8.3333$$

$$a_2 = \frac{2}{q}\sum y \cos 2\theta = \frac{2}{6} * (-7) = -2.3333$$

$$a_3 = \frac{2}{q}\sum y \cos 3\theta = \frac{2}{6} * (-7) = -2.3333$$

$$b_1 = \frac{2}{q}\sum y \sin\theta = \frac{2}{6} * (-3.464) = -1.15$$

$$b_2 = \frac{2}{q}\sum y \sin 2\theta = \frac{2}{6} * (0) = 0$$

$$b_3 = \frac{2}{q}\sum y \sin 3\theta = \frac{2}{6} * (0) = 0$$

Hence the Fourier series expansion

$$f(x) = \frac{a_0}{2} + (a_1 \cos\theta + b_1 \sin\theta) + (a_2 \cos 2\theta + b_2 \sin 2\theta) + (a_3 \cos 3\theta + b_3 \sin 3\theta)$$

$$\Rightarrow f(x) = 20.8334 + (-8.3333 \cos\theta - 1.15 \sin\theta) + (-2.3333 \cos 2\theta + 0 \sin 2\theta) + (-2.3333 \cos 3\theta)$$

$$\Rightarrow f(x) = 20.8334 + \left(-8.3333 \cos\frac{\pi x}{3} - 1.15 \sin\frac{\pi x}{3}\right) + \left(-2.3333 \cos 2\frac{\pi x}{3}\right) + \left(-2.3333 \cos 3\frac{\pi x}{3}\right)$$

2. Find the Fourier series upto 2nd harmonic function

X	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
F(x)	**1.0**	**1.4**	**1.9**	**1.7**	**1.5**	**1.2**	**1.0**

Solution:

$$f(x) = \frac{a_0}{2} + (a_1 cosx + b_1 sinx) + (a_2 cos2x + b_2 sin2x)$$

X	Y=f(x)	ycosx	ysinx	Ycos2x	Ysin2x
0	1.0	1.0	0	1.0	0
$\pi/3$	1.4	0.7	1.2	-0.7	1.21
$2\pi/3$	1.9	-0.95	1.64	-0.95	-1.64
π	1.7	-1.7	0	1.7	0
$4\pi/3$	1.5	-0.75	-1.29	0.75	1.29
$5\pi/3$	1.2	0.6	-1.03	-0.6	-1.03

Total	8.7	-1.1	0.5	-0.3	-1.17

$$a_0 = \frac{2}{N}\sum y = 2.9$$
$$a_1 = \frac{2}{N}\sum ycosx = -0.3$$
$$a_2 = \frac{2}{N}\sum ycos2x = -0.1$$
$$b_1 = \frac{2}{N}\sum ysin \quad 0.16$$
$$b_2 = \frac{2}{N}\sum ysin2x = -0.05$$
$$f(x) = \frac{2.9}{2} + (-0.3cosx \quad 0.16sinx) + (-0.1cos2x - 0.05sin2x)$$

3. Find the Fourier series upto 2nd harmonic function

X	**0**	**$\pi/6$**	**$2\pi/6$**	**$3\pi/6$**	**$4\pi/6$**	**$5\pi/6$**
F(x)	**0**	**9.2**	**14.4**	**17.8**	**17.3**	**11.7**

Solution:

$2l = \pi$

$l = \pi/2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\frac{\pi}{2}}\right) + \sum_{n=1}^{\infty} b_n sin\left(\frac{n\pi x}{\frac{\pi}{2}}\right)$$

f(x) $= \frac{a_0}{2} + (a_1cos2x + b_1sin2x) + (a_2cos4x + b_2sin4x)$

X	Y=f(x)	Ycos2x	Ysin2x	Ycos4x	Ysin4x
0	0	0	0	0	0
$\pi/6$	9.2	4.6	-4.6	7.967	7.967

$2\pi/6$	14.4	-7.2	-7.2	12.470	-12.470
$3\pi/6$	17.8	-17.8	17.8	0	0
$4\pi/6$	17.3	-8.65	-14.982	-8.65	-14.982
$5\pi/6$	11.7	5.85	-5.85	-10.132	-10132
Total	70.4	-23.2	-8.5	-4.677	0.347

$$a_0 = \frac{2}{6}\sum y = 23.47$$

$$a_1 = \frac{2}{6}\sum y \quad 2x = -7.73$$

$$a_2 = \frac{2}{6}\sum ycos4x = -2.833$$

$$b_1 = \frac{2}{6}\sum ysi \;\; 2x = -1.56$$

$$b_2 = \frac{2}{6}\sum ysin4x = 0.116$$

$$f(x) = 11.74 + (-7.73cos2x - 1.56sin2x) + (-2.83cos4x + 0.116sin4x)$$

4. Find the Fourier series upto 2nd harmonic function

X	**0**	**T/6**	**T/3**	**T/2**	**2T/3**	**5T/6**	**T**
F(x)	**1.98**	**1.30**	**1.060**	**1.30**	**-0.88**	**-0.5**	**1.98**

Solution:

2l=T

L=T/2

Put $\theta = \frac{2\pi}{T}$

A $= \frac{a_0}{2} + (a_1 cos\theta + b_1 sin\theta) + (a_2 cos2\theta + b_2 sin2\theta)$

T	θ	A	A$Cos\theta$	A$Cos2\theta$	A$sin\theta$	A$sin2\theta$
0	0	1.98	1.98	1.98	0	0
T/6	60	1.30	0.65	-0.65	1.13	1.13

T/3	120	1.06	-0.53	-0.53	0.92	-0.92
T/2	180	1.30	-1.30	1.30	0	0
2T/3	240	-0.88	0.44	0.44	0.76	-0.76
5T/6	300	-0,50	-0.25	0.25	0.43	0.43
Total		4.26	0.99	2.79	3.24	-0.12

$$a_0 = \frac{2}{6}\sum A = 1.42$$

$$a_1 = \frac{2}{6}\sum \mathrm{A}Cos\theta = 0.33$$

$$a_2 = \frac{2}{6}\sum \mathrm{A}Cos2\theta = 0.93$$

$$b_1 = \frac{2}{6}\sum \mathrm{A}sin\theta = 1.08$$

$$b_2 = \frac{2}{6}\sum \mathrm{A}sin2\theta = -0.04$$

$$A = 0.71 + \left(0.33cos\left(\frac{2\pi t}{T}\right) + 1.08sin\left(\frac{2\pi t}{T}\right)\right) + \left(0.93cos\left(\frac{2\pi t}{T}\right) - 0.04sin\left(\frac{2\pi t}{T}\right)\right)$$

Complex Fourier Series

1.Express the function $f(x) = e^{-x}, in -1 < x < 1$ **in the complex form of the Fourier series.**

Solution:

The Complex form of the Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \, e^{-in\pi x},$$

where $C_n = \frac{1}{2L} \int_c^{c+2L} f(x) \, e^{in\pi x} \, dx$

$$C_n = \frac{1}{2} \int_{-1}^{1} e^{-x} \, e^{in\pi x} \, dx$$

$$= \frac{1}{2} \int_{-1}^{1} e^{-(1-in\pi)x} \, dx$$

$$= \frac{1}{2} \left[\frac{e^{-(1-in\pi)x}}{-(1-in\pi)} \right]_{-1}^{1}$$

$$= \frac{1}{2} \left[\frac{e^{-(1-in\pi)}}{-(1-in\pi)} - \frac{e^{(1-in\pi)}}{-(1-in\pi)} \right]$$

$$= \frac{1}{2} \frac{1}{1-in\pi} \left[-e^{-1} \, e^{in\pi} + e \, e^{-in\pi} \right]$$

$$= \frac{1}{2} \frac{1}{1-in\pi} \frac{1+in\pi}{1+in\pi} \left[-\frac{1}{e} (\cos n\pi + i \sin n\pi) + e \, (\cos n\pi - i \sin n\pi) \right]$$

$$= \frac{1}{2} \frac{1+in\pi}{1+n^2\pi^2} (-1)^n \left(e - \frac{1}{e} \right)$$

$$C_n = \frac{e^2 - 1}{2e} \, \frac{1+in\pi}{1+n^2\pi^2} (-1)^n$$

Hence the Complex form of the Fourier series of the given function is

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \, e^{-\frac{in\pi x}{L}}$$

$$e^{-x} = \sum_{n=-\infty}^{\infty} \frac{e^2 - 1}{2e} \frac{1 + in\pi}{1 + n^2\pi^2} (-1)^n e^{-in\pi x}$$

2.Find the complex form of the F.S of the function

$f(x) = \cos ax, \ in \ -\pi < x < \pi$

Solution:

The Complex form of the Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \, e^{in\pi x},$$

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos ax e^{-inx} dx$$

$$= \frac{1}{2\pi} \left[\frac{e^{-inx}}{a^2 - n^2} (-in \cos ax + a \sin ax) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \left[\frac{e^{-in\pi}}{a^2 - n^2} (-in \cos a\pi + a \sin a\pi) - \frac{e^{in\pi}}{a^2 - n^2} (-in \cos a\pi - a \sin a\pi) \right]$$

$$= \frac{(-1)^n}{2\pi(a^2 - n^2)} \left(-in \cos a\pi + a \sin a\pi + in \cos a\pi + a \sin a\pi \right)$$

$$= \frac{(-1)^n}{\pi(a^2 - n^2)} \left(a \sin a\pi \right)$$

$$\therefore f(x) = \frac{a \sin a\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{inx}}{(a^2 - n^2)}$$

MA 3351 TRANSFORMS & PARTIAL DIFFERENTIAL EQUATIONS
UNIT IV – FOURIER TRANSFORM

Fourier integral theorem.

If f(x) is a given function defined in ($-l$, l) and satisfies Dirichlet's conditions then

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i\lambda(x-t)}\, dt\, d\lambda \quad (or) \quad f(x)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos[\lambda(x-t)]\, dt\, d\lambda$$

Definition:

Fourier transform pair.

Fourier transform of f(x) is defined as $F(s)=F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

Its Inverse Fourier transform is $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\, ds = F^{-1}[F(s)]$

Fourier cosine transform pair.

Fourier cosine transform of f(x) is $F_c(s)=F_c[f(x)]=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\cos sx\, dx$

Its Inverse Fourier cosine transform is $f(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} F_c[f(x)]\cos sx\, ds$

Fourier sine transform pair.

Fourier sine transform of f(x) is $F_s(s)=F_s[f(x)]=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\sin sx\, dx$

Its Inverse Fourier sine transform is $f(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} F_s[f(x)]\sin sx\, ds$

Parseval's identity for Fourier transform.

If F(s) is the Fourier transform of f(x) then $\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$

Parseval's identity for Fourier sine and cosine transform.

i) If $F_s(s)$ and $F_c(s)$ are the Fourier sine and Fourier cosine transform of f(x) respectively then

$$\int_{0}^{\infty}[F_s(s)]^2 ds=\int_{0}^{\infty}[f(x)]^2\, dx \quad and \quad \int_{0}^{\infty}[F_c(s)]^2 ds=\int_{0}^{\infty}[f(x)]^2\, dx$$

ii) If $F_s(s)$ and $F_c(s)$ are the Fourier sine and Fourier cosine transform of f(x) and g(x) respectively then

$$\int_{0}^{\infty} F_s(s)\, G_s(s)\, ds=\int_{0}^{\infty} f(x)\, g(x)\, dx \quad and \quad \int_{0}^{\infty} F_c(s)\, G_c(s)\, ds=\int_{0}^{\infty} f(x)\, g(x)\, dx$$

Note:

i) $F_s[x\, f(x)] = -\frac{d}{ds} F_c[f(x)]$ ii) $F_c[x\, f(x)] = \frac{d}{ds} F_s[f(x)]$

iii) $F[x^n\, f(x)] = (-i)^n \frac{d^n}{ds^n} F[f(x)]$

Problems

1. Find the Fourier transform of $f(x)=\begin{cases} a^2-x^2, & |x|<a \\ 0, & |x|\ge a \end{cases}$

Hence deduce that $(i)\ \int_0^\infty \frac{\sin s - s\cos s}{s^3}\,ds = \frac{\pi}{4}$ $\quad (ii)\ \int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos\frac{s}{2}\,ds = \frac{3\pi}{16}$

$(iii)\ \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{15}$

Sol.

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} 0.e^{isx}\,dx + \int_{-a}^{a}(a^2-x^2)\,e^{isx}\,dx + \int_{a}^{\infty} 0.e^{isx}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2-x^2)(\cos sx + i\sin sx)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2-x^2)\cos sx\,dx + i\,\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2-x^2)\sin sx\,dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{a}(a^2-x^2)\cos sx\,dx + 0$$

$$= \sqrt{\frac{2}{\pi}}\left[(a^2-x^2)\left(\frac{\sin sx}{s}\right) - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[\left\{0 - \frac{2a\cos as}{s^2} + \frac{2\sin as}{s^3}\right\} - \{0-0+0\}\right]$$

$$(i.e.)\quad F[f(x)] = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin as - as\cos as}{s^3}\right]$$

When a = 1, we have $F[f(x)] = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]$

Using ***inverse Fourier transform***, we have

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\,ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)(\cos sx - i\sin sx)\,ds$$

$$= \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\,ds - i\,\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\sin sx\,ds$$

$$= \frac{4}{\pi}\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\,ds - 0$$

$$\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\,ds = \frac{\pi}{4}f(x) \text{ -------- } (1)$$

Put $x=0$ *in equation* (1) *we get*

$$\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)ds = \frac{\pi}{4}f(0)$$

$$= \frac{\pi}{4}(1) = \frac{\pi}{4} \quad \textit{This proves } (i)$$

$f(x) = a^2 - x^2$
$f(x) = 1 - x^2$
$f(0) = 1 - 0 = 1$

$$Put \;\; x=\frac{1}{2} \;\; in \;\; equation \;\; (1) \;\; we \;\; get$$

$f(x) = a^2 - x^2$ $f(x) = 1 - x^2$ $f(\frac{1}{2}) = 1 - \frac{1}{4} = \frac{3}{4}$

$$\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos\frac{s}{2}ds = \frac{\pi}{4}f\left(\frac{1}{2}\right)$$

$$=\frac{\pi}{4}\left(1-\frac{1}{4}\right)=\frac{\pi}{4}\left(\frac{3}{4}\right)=\frac{3\pi}{16} \qquad This \;\; proves \;\; (ii)$$

Using **Parseval's identity,** we have

$$\int_{-\infty}^{\infty}|F(s)|^2\,ds = \int_{-\infty}^{\infty}|f(x)|^2\,dx$$

$$\int_{-\infty}^{\infty}\left(2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]\right)^2 ds = \int_{-\infty}^{-1}0.dx+\int_{-1}^{1}(1-x^2)^2\,dx+\int_{1}^{\infty}0.dx$$

$$\frac{8}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \int_{-1}^{1}(1-x^2)^2\,dx$$

$$\frac{16}{\pi}\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\int_{0}^{1}(1-x^2)^2\,dx$$

$$=2\int_0^1(1+x^4-2x^2)\,dx$$

$$=2\left[x+\frac{x^5}{5}-\frac{2x^3}{3}\right]_0^1$$

$$=2\left[\left\{1+\frac{1}{5}-\frac{2}{3}\right\}-\{0+0-0\}\right]$$

$$=2\left[\frac{8}{15}\right]$$

$$\frac{16}{\pi}\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{16}{15}$$

$$\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{15} \qquad This \;\; proves \;\; (iii)$$

2. Find the Fourier transform of $f(x)=\begin{cases}1-|x|, & |x|<1\\ 0\;\;\;\;, & |x|\geq 1\end{cases}$

Hence deduce that $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$

Sol. $$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,e^{isx}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-1}0.e^{isx}\,dx + \int_{-1}^{1}(1-|x|)\,e^{isx}\,dx + \int_{1}^{\infty}0.e^{isx}\,dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)(\cos sx + i\sin sx)\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)\cos sx\,dx + i\,\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)\sin sx\,dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1-|x|)\cos sx\, dx + 0$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 (1-x)\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{-\cos sx}{s^2}\right) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\left\{ 0 - \frac{\cos s}{s^2} \right\} - \left\{ 0 - \frac{1}{s^2} \right\} \right]$$

$$(i.e.) \quad F[f(x)] = \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s^2} \right]$$

Using **Parseval's identity,** we have

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

$$\int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s^2} \right] \right)^2 ds = \int_{-\infty}^{-1} 0.dx + \int_{-1}^{1} (1-|x|)^2\, dx + \int_{1}^{\infty} 0.dx$$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{1-\cos s}{s^2} \right)^2 ds = \int_{-1}^{1} (1-|x|)^2\, dx$$

$$\frac{4}{\pi} \int_0^{\infty} \left(\frac{1-\cos s}{s^2} \right)^2 ds = 2\int_0^1 (1-x)^2\, dx$$

$$\frac{4}{\pi} \int_0^{\infty} \left(\frac{1-\cos 2t}{4t^2} \right)^2 2dt = 2\left[\frac{(1-x)^3}{-3} \right]_0^1$$

Put $s = 2t$
$ds = 2dt$

$$\frac{8}{16\pi} \int_0^{\infty} \left(\frac{1-\cos 2t}{t^2} \right)^2 dt = 2\left[\{0\} - \left\{ -\frac{1}{3} \right\} \right]$$

$$\frac{1}{2\pi} \int_0^{\infty} \left(\frac{2\sin^2 t}{t^2} \right)^2 dt = \frac{2}{3}$$

$$\frac{4}{2\pi} \int_0^{\infty} \left(\frac{\sin^2 t}{t^2} \right)^2 dt = \frac{2}{3}$$

$$(i.e.) \quad \int_0^{\infty} \left(\frac{\sin t}{t} \right)^4 dt = \frac{\pi}{3}$$

3. Find the Fourier transform of $f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

Hence deduce that $(i) \int_0^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}$ $\qquad (ii) \int_0^{\infty} \left(\frac{\sin t}{t} \right)^2 dt = \frac{\pi}{2}$

Sol.

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-1} 0.e^{isx}\, dx + \int_{-1}^{1} (1)\, e^{isx}\, dx + \int_{1}^{\infty} 0.e^{isx}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \cos sx \, dx + i \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \sin sx \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{1} \cos sx \, dx + 0$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s} \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin s}{s} - 0 \right]$$

$$(i.e.) \quad F[f(x)] = \sqrt{\frac{2}{\pi}} \frac{\sin s}{s}$$

Using ***inverse Fourier transform***, we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f(x)] \, e^{-isx} \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{\sin s}{s} \right) (\cos sx - i \sin sx) \, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s}{s} \right) \cos sx \, ds - i \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s}{s} \right) \sin sx \, ds$$

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \left(\frac{\sin s}{s} \right) \cos sx \, ds - 0$$

$$\int_{0}^{\infty} \left(\frac{\sin s}{s} \right) \cos sx \, ds = \frac{\pi}{2} f(x)$$

Put $x = 0$ *we get*

$$\int_{0}^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2} f(0)$$

$$= \frac{\pi}{2}(1)$$

$f(x) = 1$
$f(0) = 1$

$$(i.e.) \quad \int_{0}^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}$$

Using **Parseval's identity,** we have

$$\int_{-\infty}^{\infty} |F(s)|^2 \, ds = \int_{-\infty}^{\infty} |f(x)|^2 \, dx$$

$$\int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} \frac{\sin s}{s} \right)^2 ds = \int_{-\infty}^{-1} 0.dx + \int_{-1}^{1} (1)^2 \, dx + \int_{1}^{\infty} 0.dx$$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s}{s} \right)^2 ds = \int_{-1}^{1} dx$$

$$\frac{4}{\pi} \int_{0}^{\infty} \left(\frac{\sin s}{s} \right)^2 ds = [x]_{-1}^{1}$$

$$= 1 - (-1)$$

$$\frac{4}{\pi} \int_{0}^{\infty} \left(\frac{\sin s}{s} \right)^2 ds = 2 \quad \Rightarrow \quad \int_{0}^{\infty} \left(\frac{\sin t}{t} \right)^2 dt = \frac{\pi}{2}$$

4. Find the sine transform of $\frac{1}{x}$

Sol. $F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$

$$F_s\left[\frac{1}{x}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{s}{t}\sin t\,\frac{dt}{s}$$

Put $sx = t$
$s\,dx = dt$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin t}{t}\,dt$$

$$\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{\pi}{2}$$

$$= \sqrt{\frac{\pi}{2}}$$

5. Find f(x) if its sine transform is e^{-as}

Sol. The inverse Fourier sine transform is given by

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s[f(x)]\sin sx\,ds$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-as}\sin sx\,ds$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-as}}{a^2+x^2}(-a\sin sx - x\cos sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[\{0\} - \left\{\frac{1}{a^2+x^2}(0-x)\right\}\right]$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{x}{x^2+a^2}$$

6. Find f(x) if its cosine transform is $f_c(p) = \begin{cases} \frac{1}{\sqrt{2\pi}}\left(a-\frac{s}{2}\right), & s < 2a \\ 0, & s \ge 2a \end{cases}$

Sol. The inverse Fourier cosine transform is given by

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c[f(x)]\cos sx\,ds$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^{2a}\frac{1}{\sqrt{2\pi}}\left(a-\frac{s}{2}\right)\cos sx ds + \int_{2a}^\infty 0 ds\right]$$

$$= \frac{1}{\pi}\left[\left(a-\frac{s}{2}\right)\left(\frac{\sin sx}{x}\right) - \left(-\frac{1}{2}\right)\left(\frac{-\cos sx}{x^2}\right)\right]_0^{2a}$$

$$= \frac{1}{\pi}\left[\left\{0 - \frac{\cos 2ax}{2x^2}\right\} - \left\{0 - \frac{1}{2x^2}\right\}\right]$$

$$= \frac{1}{\pi x^2}\,\frac{1-\cos 2ax}{2}$$

$$= \frac{\sin^2 ax}{\pi x^2}$$

7. Find the Fourier sine and cosine transform of e^{-ax}

Sol. $$F_s[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$F_s[e^{-ax}]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\sin sx-s\cos sx)\right]_0^\infty$$

$$=\sqrt{\frac{2}{\pi}}\left[\{0\}-\left\{\frac{1}{a^2+s^2}(0-s)\right\}\right]$$

$$=\sqrt{\frac{2}{\pi}}\,\frac{s}{s^2+a^2}$$

$$F_c[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$F_c[e^{-ax}]=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx+s\sin sx)\right]_0^\infty$$

$$=\sqrt{\frac{2}{\pi}}\left[\{0\}-\left\{\frac{1}{a^2+s^2}(-a+0)\right\}\right]$$

$$=\sqrt{\frac{2}{\pi}}\,\frac{a}{s^2+a^2}$$

8. Find the Fourier sine transform of $\frac{e^{-ax}}{x}$

Sol. $$F_s[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,\sin sx\,dx$$

$$F_s\left[\frac{e^{-ax}}{x}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\,\sin sx\,dx$$

Diff. w.r.t.'s' on both sides we get

$$\frac{d}{ds}F_s\left[\frac{e^{-ax}}{x}\right]=\frac{d}{ds}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\,\sin sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial}{\partial s}\left(\frac{e^{-ax}}{x}\,\sin sx\right)dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\,\cos sx.x\;dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\,\cos sx\;dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx+s\sin sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\{0\} - \left\{ \frac{1}{a^2+s^2} (-a+0) \right\} \right]$$

$$\frac{d}{ds} F_s \left[\frac{e^{-ax}}{x} \right] = \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$$

Integrating w.r.t. 's' we get

$$F_s \left[\frac{e^{-ax}}{x} \right] = \sqrt{\frac{2}{\pi}} \int \frac{a}{s^2+a^2} ds$$

$$= a\sqrt{\frac{2}{\pi}} \left[\frac{1}{a} \tan^{-1}\left(\frac{s}{a}\right) \right]$$

$$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)}$$

$$= \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{s}{a}\right)$$

9. Find the Fourier cosine transform of $\frac{e^{-ax}}{x}$

Sol. $$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx$$

$$F_c \left[\frac{e^{-ax}}{x} \right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cos sx\, dx$$

Diff. w.r.t. 's' on both sides we get

$$\frac{d}{ds} F_c \left[\frac{e^{-ax}}{x} \right] = \frac{d}{ds}\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial}{\partial s}\left(\frac{e^{-ax}}{x} \cos sx \right) dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} (-\sin sx . x)\, dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx\, dx$$

$$= -\sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2} (-a\sin sx - s\cos sx) \right]_0^\infty$$

$$= -\sqrt{\frac{2}{\pi}} \left[\{0\} - \left\{ \frac{1}{a^2+s^2} (0-s) \right\} \right]$$

$$\frac{d}{ds} F_c \left[\frac{e^{-ax}}{x} \right] = -\sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}$$

Integrating w.r.t. 's' we get

$$F_c \left[\frac{e^{-ax}}{x} \right] = -\sqrt{\frac{2}{\pi}} \int \frac{s}{s^2+a^2} ds$$

$$\boxed{\int \frac{x\,dx}{x^2+a^2} = \frac{1}{2}\log(x^2+a^2)}$$

$$= -\sqrt{\frac{2}{\pi}} \frac{1}{2} \log(s^2+a^2)$$

$$= -\frac{1}{\sqrt{2\pi}} \log(s^2+a^2)$$

10. Find the Fourier sine and cosine transform of $x e^{-ax}$

Sol. $$F_s[x e^{-ax}] = -\frac{d}{ds} F_c[e^{-ax}]$$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\{0\} - \left\{ \frac{1}{a^2+s^2}(-a+0) \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$$

$$F_s[x e^{-ax}] = -\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2} \right]$$

$$= -\sqrt{\frac{2}{\pi}} \left[\frac{-a}{(s^2+a^2)^2}(2s) \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{2as}{(s^2+a^2)^2}$$

$$F_c[x e^{-ax}] = \frac{d}{ds} F_s[e^{-ax}]$$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2}(-a\sin sx - s\cos sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\{0\} - \left\{ \frac{1}{a^2+s^2}(0-s) \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}$$

$$F_c[x e^{-ax}] = \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{(s^2+a^2)(1) - s(2s)}{(s^2+a^2)^2}$$

$$= \sqrt{\frac{2}{\pi}} \frac{a^2-s^2}{(s^2+a^2)^2}$$

11. Solve the integral equation $\int_0^\infty f(x) \cos \lambda x\, dx = e^{-\lambda}$

Sol. Given $\int_0^\infty f(x) \cos \lambda x\, dx = e^{-\lambda}$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\lambda x\,dx = \sqrt{\frac{2}{\pi}}\; e^{-\lambda}$$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\; e^{-\lambda}$$

$$f(x) = F_c^{-1}\left[\sqrt{\frac{2}{\pi}}\; e^{-\lambda}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}e^{-\lambda}\cos\lambda x\,d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty e^{-\lambda}\cos\lambda x\,d\lambda$$

$$= \frac{2}{\pi}\left[\frac{e^{-\lambda}}{1+x^2}(-\cos\lambda x + \lambda\sin\lambda x)\right]_0^\infty$$

$$= \frac{2}{\pi}\left[\{0\} - \left\{\frac{1}{1+x^2}(-1+0)\right\}\right]$$

$$(i.e.)\, f(x) = \frac{2}{\pi}\,\frac{1}{1+x^2}$$

12. Find the Fourier transform of $e^{-\frac{x^2}{2}}$

Sol. $F[f(x)] = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$F\left[e^{-\frac{x^2}{2}}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, e^{isx}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[x^2-2isx\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[(x-is)^2 - i^2s^2\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[(x-is)^2\right]}\, e^{-\frac{s^2}{2}}\, dx$$

$$= \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-is)^2}\, dx$$

$$= \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(\frac{x-is}{\sqrt{2}}\right)^2}\, dx$$

Put $\dfrac{x-is}{\sqrt{2}} = t$

$\dfrac{dx}{\sqrt{2}} = dt$

$$= \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2}\,\sqrt{2}\, dt$$

$\displaystyle\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}$

$$= \frac{e^{-\frac{s^2}{2}}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$= \frac{e^{-\frac{s^2}{2}}}{\sqrt{\pi}}\,\sqrt{\pi} = e^{-\frac{s^2}{2}}$$

(i.e.) $F[e^{-\frac{x^2}{2}}] = e^{-\frac{s^2}{2}}$

Note: If the transform of f(x) is equal to f(s), then the function f(x) is called self-reciprocal. In the above problem, $e^{-\frac{x^2}{2}}$ is self-reciprocal under Fourier transform.

13. Find the Fourier cosine transform of $e^{-a^2x^2}$ **and hence find** $F_s[x e^{-a^2x^2}]$

Sol. $$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos sx\, dx$$

$$F_c\left[e^{-a^2x^2}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{1}{2}\int_{-\infty}^{\infty} e^{-a^2x^2}\cos sx\, dx$$

$$= \frac{1}{\sqrt{2\pi}} R.P. \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} R.P. \int_{-\infty}^{\infty} e^{-\left[a^2x^2 - isx\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} R.P. \int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{is}{2a}\right)^2 - \frac{i^2s^2}{4a^2}\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} R.P. \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} e^{-\frac{s^2}{4a^2}}\, dx$$

$$= \frac{e^{-\frac{s^2}{4a^2}}}{\sqrt{2\pi}} R.P. \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2}\, dx$$

$$= \frac{e^{-\frac{s^2}{4a^2}}}{\sqrt{2\pi}} R.P. \int_{-\infty}^{\infty} e^{-t^2}\, \frac{dt}{a}$$

Put $ax - \frac{is}{2a} = t$ $a\, dx = dt$

$$= \frac{e^{-\frac{s^2}{4a^2}}}{a\sqrt{2\pi}} R.P. \sqrt{\pi}$$

$$(i.e.)\ F_c[e^{-a^2x^2}] = \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}}$$

$$F_s[x e^{-a^2x^2}] = -\frac{d}{ds} F_c[e^{-a^2x^2}]$$

$$= -\frac{d}{ds}\left[\frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}}\right]$$

$$= -\frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}} \left(\frac{-2s}{4a^2}\right) = \frac{s}{2\sqrt{2}\,a^3} e^{-\frac{s^2}{4a^2}}$$

14. Find the Fourier cosine transform of e^{-4x}. Hence deduce that $\int_0^\infty \frac{\cos 2x}{x^2+16}\,dx = \frac{\pi}{8}e^{-8}$ and

$\int_0^\infty \frac{x\sin 2x}{x^2+16}\,dx = \frac{\pi}{2}e^{-8}$

Sol. $F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$

$$F_c[e^{-4x}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-4x}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-4x}}{16+s^2}(-4\cos sx + s\sin sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[\{0\} - \left\{\frac{1}{16+s^2}(-4+0)\right\}\right]$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{4}{s^2+16}$$

Using ***inverse Fourier cosine transform***, we have

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c[f(x)]\cos sx\,ds$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{4}{s^2+16}\right)\cos sx\,ds$$

$$f(x) = \frac{8}{\pi}\int_0^\infty \frac{\cos sx}{s^2+16}\,ds$$

$$\int_0^\infty \frac{\cos sx}{s^2+16}\,ds = \frac{\pi}{8}f(x)$$

$$\int_0^\infty \frac{\cos sx}{s^2+16}\,ds = \frac{\pi}{8}e^{-4x} \quad --------- (1)$$

Put $x = 2$, *we get*

$$\int_0^\infty \frac{\cos 2s}{s^2+16}\,ds = \frac{\pi}{8}e^{-8}$$

$$\int_0^\infty \frac{\cos 2x}{x^2+16}\,dx = \frac{\pi}{8}e^{-8}$$

Differentiate (1) w.r.t. x, we get

$$\frac{d}{dx}\int_0^\infty \frac{\cos sx}{s^2+16}\,ds = \frac{\pi}{8}\frac{d}{dx}(e^{-4x})$$

$$\int_0^\infty \frac{\partial}{\partial x}\left(\frac{\cos sx}{s^2+16}\right)ds = \frac{\pi}{8}\frac{d}{dx}(e^{-4x})$$

$$\int_0^\infty \left(\frac{-\sin sx.s}{s^2+16}\right)ds = \frac{\pi}{8}(e^{-4x})(-4)$$

$$\int_0^\infty \frac{s\sin sx}{s^2+16}\,ds = \frac{\pi}{2}e^{-4x}$$

$$Put \quad x = 2, \quad we \quad get$$

$$\int_0^\infty \frac{s \sin 2s}{s^2+16}\, ds = \frac{\pi}{2} e^{-8}$$

$$\int_0^\infty \frac{x \sin 2x}{x^2+16}\, dx = \frac{\pi}{2} e^{-8}$$

15. Find the Fourier sine and cosine transform of e^{-x} and hence find the Fourier sine transform of $\frac{x}{1+x^2}$ and Fourier cosine transform of $\frac{1}{1+x^2}$

Sol. $$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx$$

$$F_c[e^{-x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+s^2} (-\cos sx + s \sin sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\{0\} - \left\{ \frac{1}{1+s^2} (-1+0) \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{s^2+1}$$

$$F_s[e^{-x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+s^2} (-\sin sx - s \cos sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\{0\} - \left\{ \frac{1}{1+s^2} (0-s) \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{s}{s^2+1}$$

Now, $$F_c\left[\frac{1}{1+x^2}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{1+x^2} \cos sx\, dx \quad ------ (1)$$

Using ***inverse Fourier cosine transform***, we have

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c[f(x)] \cos sx\, ds$$

$$e^{-x} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{1}{s^2+1} \right) \cos sx\, ds$$

$$e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{\cos sx}{s^2+1}\, ds$$

$$\int_0^\infty \frac{\cos sx}{s^2+1}\, ds = \frac{\pi}{2} e^{-x}$$

$$\int_0^\infty \frac{\cos sx}{x^2+1}\, dx = \frac{\pi}{2} e^{-s}$$

Put $x = s$
and $s = x$

Equation (1) becomes

$$F_c\left[\frac{1}{1+x^2}\right]=\sqrt{\frac{2}{\pi}}\ \frac{\pi}{2}\ e^{-s}$$

$$=\sqrt{\frac{\pi}{2}}\ e^{-s}$$

$$F_s\left[\frac{x}{1+x^2}\right]=-\frac{d}{ds}F_c\left[\frac{1}{1+x^2}\right]$$

$$=-\frac{d}{ds}\left[\sqrt{\frac{\pi}{2}}\ e^{-s}\right]$$

$$=-\sqrt{\frac{\pi}{2}}\ e^{-s}(-1)$$

$$=\sqrt{\frac{\pi}{2}}\ e^{-s}$$

16. Find the Fourier transform of $f(x)=e^{-a|x|}$, $a>0$. Hence deduce that

$(i)\ \int_0^\infty \frac{\cos xt}{a^2+t^2}dt=\frac{\pi}{2a}e^{-a|x|}$ $(ii)\ \int_0^\infty \frac{dx}{x^2+1}=\frac{\pi}{2}$ $(iii)\ \int_0^\infty \frac{dx}{(x^2+1)^2}=\frac{\pi}{4}$ **and also prove**

that $(iv)\ F\left[x\,e^{-a|x|}\right]=i\sqrt{\frac{2}{\pi}}\ \frac{2as}{(s^2+a^2)^2}$

Sol.

$$F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}(\cos sx+i\sin sx)\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\cos sx\,dx+i\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\sin sx\,dx$$

$$=\frac{2}{\sqrt{2\pi}}\int_0^{\infty} e^{-ax}\cos sx\,dx+0$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx+s\sin sx)\right]_0^\infty$$

$$=\sqrt{\frac{2}{\pi}}\left[\{0\}-\left\{\frac{1}{a^2+s^2}(-a+0)\right\}\right]$$

$$(i.e.)\ F[f(x)]=\sqrt{\frac{2}{\pi}}\ \frac{a}{s^2+a^2}$$

Using ***inverse Fourier transform***, we have

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]\,e^{-isx}\,ds$$

$$e^{-a|x|}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)(\cos sx-i\sin sx)\,ds$$

$$=\frac{a}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{s^2+a^2}\right)\cos sx\,ds-i\frac{a}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{s^2+a^2}\right)\sin sx\,ds$$

$$=\frac{2a}{\pi}\int_0^{\infty}\frac{\cos sx}{s^2+a^2}\,ds-0$$

$$\int_0^{\infty} \frac{\cos sx}{s^2+a^2}\,ds = \frac{\pi}{2a} e^{-a|x|}$$

$$(i.e.) \int_0^{\infty} \frac{\cos xt}{t^2+a^2}\,dt = \frac{\pi}{2a} e^{-a|x|} \quad This \ \ proves \ \ (i)$$

$Put \ \ x = 0 \ \ and \ \ a = 1, \ we \ \ get$

$$\int_0^{\infty} \frac{1}{t^2+a^2}\,dt = \frac{\pi}{2}$$

$$(i.e.) \int_0^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{2} \quad This \ \ proves \ \ (ii)$$

Using **Parseval's identity,** we have

$$\int_{-\infty}^{\infty} |F(s)|^2\,ds = \int_{-\infty}^{\infty} |f(x)|^2\,dx$$

$$\int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2} \right)^2 ds = \int_{-\infty}^{-\infty} \left[e^{-a|x|} \right]^2 dx$$

$$\frac{2a^2}{\pi} \int_{-\infty}^{\infty} \frac{1}{(s^2+a^2)^2}\,ds = \int_{-\infty}^{-\infty} \left[e^{-a|x|} \right]^2 dx$$

$$\frac{4a^2}{\pi} \int_0^{\infty} \frac{ds}{(s^2+a^2)^2} = 2\int_0^{\infty} e^{-2ax}\,dx$$

$$\frac{2a^2}{\pi} \int_0^{\infty} \frac{ds}{(s^2+a^2)^2} = \left[\frac{e^{-2ax}}{-2a} \right]_0^{\infty}$$

$$= \frac{1}{-2a}[0-1]$$

$$\frac{2a^2}{\pi} \int_0^{\infty} \frac{ds}{(s^2+a^2)^2} = \frac{1}{2a}$$

$$\int_0^{\infty} \frac{ds}{(s^2+a^2)^2} = \frac{\pi}{4a^3}$$

$put \ \ a = 1, \ \ we \ \ get$

$$\int_0^{\infty} \frac{ds}{(s^2+1)^2} = \frac{\pi}{4}$$

$$(i.e.) \int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{4} \qquad This \ \ proves \ \ (iii)$$

By the property, $F[x\,f(x)] = (-i)\dfrac{d}{ds} F[f(x)]$

$$F[x e^{-a|x|}] = (-i)\frac{d}{ds} F[e^{-a|x|}]$$

$$= (-i)\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2} \right]$$

$$= (-i)\sqrt{\frac{2}{\pi}} \left[\frac{-a}{(s^2+a^2)^2} (2s) \right]$$

$$= i \sqrt{\frac{2}{\pi}} \frac{2as}{(s^2+a^2)^2} \qquad This \ \ proves \ \ (iv)$$

17. Find the Fourier sine and cosine transform of $x^{n-1}, 0< n<1, x>0$ and hence prove that $\frac{1}{\sqrt{x}}$ is self reciprocal under both Fourier sine and cosine transforms. Also find $F\left[\frac{1}{\sqrt{|x|}}\right]$.

Sol. Consider $F_c[f(x)] - i\,F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx - i\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$

$$F_c[f(x)] - i\,F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)(\cos sx - i\sin sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\, e^{-isx}\,dx$$

$$F_c[x^{n-1}] - i\,F_s[x^{n-1}] = \sqrt{\frac{2}{\pi}}\int_0^\infty x^{n-1}\, e^{-isx}\,dx$$

$$\boxed{\int_0^\infty x^{n-1} e^{-ax}\,dx = \frac{\Gamma(n)}{a^n}}$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n)}{(is)^n}$$

$$= (-i)^n\sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n)}{s^n}$$

$$= \left(\cos\frac{n\pi}{2} - i\,\sin\frac{n\pi}{2}\right)\sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n)}{s^n}$$

$$-i = \cos\frac{\pi}{2} - i\,\sin\frac{\pi}{2}$$

$$(-i)^n = \left(\cos\frac{\pi}{2} - i\,\sin\frac{\pi}{2}\right)^n = \cos\frac{n\pi}{2} - i\,\sin\frac{n\pi}{2}$$

Equating R.P and I.P, we get

$$F_c[x^{n-1}] = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n)}{s^n}\cos\frac{n\pi}{2} \quad \text{--------} \quad (1)$$

$$F_s[x^{n-1}] = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n)}{s^n}\sin\frac{n\pi}{2} \quad \text{--------} \quad (2)$$

Put $n = \frac{1}{2}$ *in equation* (1), *we have*

$$F_c[x^{\frac{1}{2}-1}] = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(1/2)}{s^{1/2}}\cos\frac{\pi}{4}$$

$$F_c\left[\frac{1}{\sqrt{x}}\right] = \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{\pi}}{\sqrt{s}}\,\frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{s}}$$

Put $n = \frac{1}{2}$ *in equation* (1), *we have*

$$F_s[x^{\frac{1}{2}-1}] = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(1/2)}{s^{1/2}}\sin\frac{\pi}{4}$$

$$F_s\left[\frac{1}{\sqrt{x}}\right] = \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{\pi}}{\sqrt{s}}\,\frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{s}}$$

Hence $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier sine and cosine transforms.

Now, $F[f(x)] = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$F\left[\frac{1}{\sqrt{|x|}}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{|x|}} (\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{|x|}} \cos sx\, dx + i\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{|x|}} \sin sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{x}} \cos sx\, dx + 0$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{x}} \cos sx\, dx$$

$$= F_c\left[\frac{1}{\sqrt{x}}\right]$$

$$= \frac{1}{\sqrt{s}}$$

18. Verify Parseval's theorem of Fourier transform for the function $f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x > 0 \end{cases}$

Sol. $F(s) = F[f(x)] = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} 0.e^{isx}\, dx + \int_{0}^{\infty} e^{-x}.e^{isx}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(1-is)x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(1-is)x}}{-(1-is)} \right]_0^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 - \frac{1}{-(1-is)} \right]$$

(*i.e.*) $F(s) = \dfrac{1}{\sqrt{2\pi}} \dfrac{1}{1-is}$

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} F(s)\overline{F(s)}\, ds = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{1-is} \frac{1}{\sqrt{2\pi}} \frac{1}{1+is} ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+s^2} ds$$

$$= \frac{2}{2\pi} \int_{0}^{\infty} \frac{ds}{1+s^2}$$

$$= \frac{1}{\pi} \left[\frac{1}{1} \tan^{-1}\left(\frac{s}{1}\right) \right]_0^{\infty}$$

$$= \frac{1}{\pi} \left[\frac{\pi}{2} - 0 \right]$$

$$= \frac{1}{2}$$

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{0} 0.\, dx + \int_{0}^{\infty} (e^{-x})^2 \, dx$$

$$= \int_{0}^{\infty} e^{-2x} \, dx$$

$$= \left[\frac{e^{-2x}}{-2} \right]_0^{\infty}$$

$$= \left[0 - \frac{1}{-2} \right]$$

$$= \frac{1}{2}$$

$$\therefore \int_{-\infty}^{\infty} |F(s)|^2 \, ds = \int_{-\infty}^{\infty} |f(x)|^2 \, dx$$

Hence Parseval's theorem is verified.

19. Using Parseval's identity, calculate $i) \int_0^{\infty} \frac{dx}{(x^2+a^2)^2}$ $ii) \int_0^{\infty} \frac{x^2 dx}{(x^2+4)^2}$

Sol. (i) Let $f(x) = e^{-ax}$ then $F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$

Using Parseval's identity for Fourier cosine transform, we have

$$\int_0^{\infty} [F_c(s)]^2 \, ds = \int_0^{\infty} [f(x)]^2 \, dx$$

$$\int_0^{\infty} \left(\sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2} \right)^2 ds = \int_0^{\infty} (e^{-ax})^2 \, dx$$

$$\frac{2a^2}{\pi} \int_0^{\infty} \frac{ds}{(s^2+a^2)^2} = \int_0^{\infty} e^{-2ax} \, dx$$

$$= \left[\frac{e^{-2ax}}{-2a} \right]_0^{\infty}$$

$$= \left[0 - \frac{1}{-2a} \right]$$

$$\frac{2a^2}{\pi} \int_0^{\infty} \frac{ds}{(s^2+a^2)^2} = \frac{1}{2a}$$

$$(i.e.) \int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$$

(ii) Let $f(x) = e^{-2x}$ then $F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}} \frac{s}{s^2+4}$

Using Parseval's identity for Fourier sine transform, we have

$$\int_0^{\infty} [F_s(s)]^2 \, ds = \int_0^{\infty} [f(x)]^2 \, dx$$

$$\int_0^{\infty} \left(\sqrt{\frac{2}{\pi}} \frac{s}{s^2+4} \right)^2 ds = \int_0^{\infty} (e^{-2x})^2 \, dx$$

$$\frac{2}{\pi} \int_0^{\infty} \frac{s^2 ds}{(s^2+4)^2} = \int_0^{\infty} e^{-4x} \, dx$$

$$= \left[\frac{e^{-4x}}{-4} \right]_0^\infty$$

$$= \left[0 - \frac{1}{-4} \right]$$

$$\frac{2}{\pi} \int_0^\infty \frac{s^2 ds}{(s^2+4)^2} = \frac{1}{4}$$

$$(i.e.) \int_0^\infty \frac{x^2 dx}{(x^2+4)^2} = \frac{\pi}{8}$$

20. Use transform methods to evaluate $i) \int_0^\infty \frac{dx}{(x^2+1)(x^2+4)}$ $ii) \int_0^\infty \frac{x^2 dx}{(x^2+9)(x^2+25)}$

Sol. (i) Let $f(x) = e^{-x}$ and $g(x) = e^{-2x}$

Then $F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \frac{1}{s^2+1}$ and $G_c(s) = G_c[g(x)] = \sqrt{\frac{2}{\pi}} \frac{2}{s^2+4}$

We have $\int_0^\infty F_c(s) G_c(s)\, ds = \int_0^\infty f(x)\, g(x)\, dx$

$$\int_0^\infty \sqrt{\frac{2}{\pi}} \frac{1}{s^2+1} \sqrt{\frac{2}{\pi}} \frac{2}{s^2+4} ds = \int_0^\infty e^{-x} e^{-2x}\, dx$$

$$\frac{4}{\pi} \int_0^\infty \frac{ds}{(s^2+1)(s^2+4)} = \int_0^\infty e^{-3x}\, dx$$

$$= \left[\frac{e^{-3x}}{-3} \right]_0^\infty$$

$$= \left[0 - \frac{1}{-3} \right]$$

$$\frac{4}{\pi} \int_0^\infty \frac{ds}{(s^2+1)(s^2+4)} = \frac{1}{3}$$

$$(i.e.) \int_0^\infty \frac{dx}{(x^2+1)(x^2+4)} = \frac{\pi}{12}$$

(ii) Let $f(x) = e^{-3x}$ and $g(x) = e^{-5x}$

Then $F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}} \frac{s}{s^2+9}$ and $G_s(s) = G_s[g(x)] = \sqrt{\frac{2}{\pi}} \frac{s}{s^2+25}$

We have $\int_0^\infty F_s(s) G_s(s)\, ds = \int_0^\infty f(x)\, g(x)\, dx$

$$\int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{s^2+9} \sqrt{\frac{2}{\pi}} \frac{s}{s^2+25} ds = \int_0^\infty e^{-3x} e^{-5x}\, dx$$

$$\frac{2}{\pi} \int_0^\infty \frac{s^2 ds}{(s^2+9)(s^2+25)} = \int_0^\infty e^{-8x}\, dx$$

$$= \left[\frac{e^{-8x}}{-8} \right]_0^\infty = \left[0 - \frac{1}{-8} \right]$$

$$\frac{2}{\pi} \int_0^\infty \frac{s^2 ds}{(s^2+9)(s^2+25)} = \frac{1}{8} \quad \Rightarrow \int_0^\infty \frac{x^2 dx}{(x^2+9)(x^2+25)} = \frac{\pi}{16}$$

21. Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ **using transforms.**

Sol. Let $f(x)=e^{-ax}$ and $g(x)=e^{-bx}$

Then $F_c(s)=F_c[f(x)]=\sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}$ and $G_c(s)=G_c[g(x)]=\sqrt{\frac{2}{\pi}}\frac{b}{s^2+b^2}$

We have $\int_0^\infty F_c(s)G_c(s)\,ds=\int_0^\infty f(x)\,g(x)\,dx$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}\sqrt{\frac{2}{\pi}}\frac{b}{s^2+b^2}ds=\int_0^\infty e^{-ax}e^{-bx}\,dx$$

$$\frac{2ab}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)}=\int_0^\infty e^{-(a+b)x}\,dx$$

$$=\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty=\left[0-\frac{1}{-(a+b)}\right]$$

$$\frac{2ab}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)}=\frac{1}{a+b}$$

$$(i.e.)\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}=\frac{\pi}{2ab(a+b)}$$

22. Find the Fourier sine transform of $f(x)=\begin{cases}x, & 0<x<1\\ 2-x, & 1<x<2\\ 0, & x>2\end{cases}$

Sol. $F_s[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$

$$=\sqrt{\frac{2}{\pi}}\left[\int_0^1 x\sin sx\,dx+\int_1^2(2-x)\sin sx\,dx+\int_2^\infty 0.\sin sx\,dx\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[x\left(\frac{-\cos sx}{s}\right)-(1)\left(\frac{-\sin sx}{s^2}\right)\right]_0^1+\sqrt{\frac{2}{\pi}}\left[(2-x)\left(\frac{-\cos sx}{s}\right)-(-1)\left(\frac{-\sin sx}{s^2}\right)\right]_1^2$$

$$=\sqrt{\frac{2}{\pi}}\left[\left\{\frac{-\cos s}{s}+\frac{\sin s}{s^2}\right\}-\{0+0\}\right]+\sqrt{\frac{2}{\pi}}\left[\left\{0-\frac{\sin 2s}{s^2}\right\}-\left\{\frac{-\cos s}{s}-\frac{\sin s}{s^2}\right\}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{2\sin s}{s^2}-\frac{\sin 2s}{s^2}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{2\sin s-2\sin s\cos s}{s^2}\right]$$

$$=2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s\,(1-\cos s)}{s^2}\right]$$

Properties of Fourier Transform

1. Prove that F[af(x) + bg(x)] = aF(s) + bG(s) [Linearity property on Fourier transform]

Proof. We have $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = F(s)$

$$F[a\,f(x) + b\,g(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [a\,f(x) + b\,g(x)]\, e^{isx}\, dx$$

$$= a\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \;+\; b\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\, e^{isx}\, dx$$

$$= a\,F(s) + b\,G(s)$$

2. Prove (i)F_c[af(x) + bg(x)] = aF_c(s) + bG_c(s)[Linear property on Fourier cosine transform]

(ii)F_s[af(x) + bg(x)] = aF_s(s) + bG_s(s)[Linear property on Fourier sine transform]

Proof. (*i*) $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx\, dx = F_c(s)$

$$F_c[a\,f(x) + b\,g(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} [a\,f(x) + b\,g(x)] \cos sx\, dx$$

$$= a\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx\, dx + b\sqrt{\frac{2}{\pi}} \int_0^{\infty} g(x) \cos sx\, dx$$

$$= a\,F_c(s) + b\,G_c(s)$$

(*ii*) $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx\, dx = F_s(s)$

$$F_s[a\,f(x) + b\,g(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} [a\,f(x) + b\,g(x)] \sin sx\, dx$$

$$= a\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx\, dx + b\sqrt{\frac{2}{\pi}} \int_0^{\infty} g(x) \sin sx\, dx$$

$$= a\,F_s(s) + b\,G_s(s)$$

3. Prove that $F[f(x-a)] = e^{ias} F(s)$ [Time shifting property]

Proof. We have $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = F(s)$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(t+a)}\, dt$$

Put $x - a = t$
$dx = dt$

$$= e^{ias} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$$

$$= e^{ias} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= e^{ias} F(s)$$

4. Prove that $F[e^{iax} f(x)] = F(s+a)$ **[Frequency shifting property]**

Proof. We have $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = F(s)$

$$F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x}\, dx$$

$$= F(s+a)$$

5. Prove that $(i)\ F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right),\ a > 0$ **[Change of scale property]**

$(ii)\ F_s[f(ax)] = \frac{1}{a} F_s\left(\frac{s}{a}\right)$

$(iii)\ F_c[f(ax)] = \frac{1}{a} F_c\left(\frac{s}{a}\right)$

Proof. (i) We have $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = F(s)$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is\frac{t}{a}} \frac{dt}{a}$$

Put $a\,x = t$
$a\,dx = dt$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\frac{s}{a}t}\, dt$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right)$$

(ii) We have $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx\, dx = F_s(s)$

$$F_s[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(ax) \sin sx\, dx$$

Put $a\,x = t$
$a\,dx = dt$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin\left(\frac{st}{a}\right) \frac{dt}{a}$$

$$= \frac{1}{a} \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin\left(\frac{s}{a}\right)t\, dt$$

$$= \frac{1}{a} F_s\left(\frac{s}{a}\right)$$

(iii) We have $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx\, dx = F_c(s)$

$$F_c[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(ax) \cos sx\, dx$$

Put $a\,x = t$
$a\,dx = dt$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos\left(\frac{st}{a}\right) \frac{dt}{a}$$

$$= \frac{1}{a} \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos\left(\frac{s}{a}\right)t\, dt \quad = \frac{1}{a} F_c\left(\frac{s}{a}\right)$$

6. If $\overline{f}(\lambda)$ is the Fourier transform of $f(x)$, find the Fourier transform of $f(x-a)$ and $f(ax)$.

Proof. $F[f(x-a)] = e^{ia\lambda}\,\overline{f}(\lambda)$ [see property (3) and (5) (*i*)]

and $F[f(ax)] = \dfrac{1}{a}\,\overline{f}\left(\dfrac{\lambda}{a}\right)$

7. Prove that [Modulation property]

$(i)\ F[f(x)\cos ax] = \frac{1}{2}[F(s+a)+F(s-a)]$ $\quad (ii)\ F_s[f(x)\cos ax] = \frac{1}{2}[F_s(s+a)+F_s(s-a)]$

$(iii)\ F_s[f(x)\sin ax] = \frac{1}{2}[F_c(s-a)-F_c(s+a)]$ $\quad (iv)\ F_c[f(x)\cos ax] = \frac{1}{2}[F_c(s+a)+F_c(s-a)]$

$(v)\ F_c[f(x)\sin ax] = \frac{1}{2}[F_s(a+s)+F_s(a-s)]$

Proof. (*i*) We have $F[f(x)] = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx = F(s)$

$$F[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos ax\, e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left(\frac{e^{iax}+e^{-iax}}{2}\right)e^{isx}\,dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{i(s+a)x}\,dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{i(s-a)x}\,dx\right]$$

$$= \frac{1}{2}[F(s+a)+F(s-a)]$$

(*ii*) We have $F_s[f(x)] = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} f(x)\sin sx\,dx = F_s(s)$

2sinAcosB = sin(A + B) + sin(A – B)

$$F_s[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos ax\,\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\,\frac{1}{2}[\sin(s+a)x+\sin(s-a)x]\,dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(s+a)x\,dx + \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(s-a)x\,dx\right]$$

$$= \frac{1}{2}[F_s(s+a)+F_s(s-a)]$$

(*iii*) We have $F_s[f(x)] = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} f(x)\sin sx\,dx = F_s(s)$

2sinAsinB = cos(A – B) – cos(A + B)

$$F_s[f(x)\sin ax] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin ax\,\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\,\frac{1}{2}[\cos(s-a)x-\cos(s+a)x]\,dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(s-a)x\,dx - \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(s+a)x\,dx\right]$$

$$= \frac{1}{2}[F_c(s-a)-F_c(s+a)]$$

(iv) We have $F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx = F_c(s)$

$$F_c[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos ax\cos sx\,dx$$

2cosAcosB = cos(A + B) + cos(A – B)

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,\frac{1}{2}[\cos(s+a)x + \cos(s-a)x]\,dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s+a)x\,dx + \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(s-a)x\,dx\right]$$

$$= \frac{1}{2}[F_c(s+a) + F_c(s-a)]$$

(v) We have $F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx = F_c(s)$

$$F_c[f(x)\sin ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin ax\cos sx\,dx$$

2sinAcosB = sin(A + B) + sin(A – B)

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,\frac{1}{2}[\sin(a+s)x + \sin(a-s)x]\,dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(a+s)x\,dx + \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(a-s)x\,dx\right]$$

$$= \frac{1}{2}[F_s(a+s) + F_s(a-s)]$$

8. Prove that (i) $F[f(-x)] = F(-s)$ (ii) $F[\overline{f(x)}] = \overline{F(-s)}$ (iii) $F[\overline{f(-x)}] = \overline{F(s)}$

Proof. (i) We have $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx = F(s)$

$$F[f(-x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(-x)\,e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f(t)\,e^{-ist}\,(-dt)$$

Put $-x = t$
$-dx = dt$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{-ist}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{i(-s)t}\,dt$$

$$= F(-s)$$

(ii) We have $F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$

$$F(-s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-isx}\,dx$$

$$\overline{F(-s)} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{f(x)}\,e^{isx}\,dx$$

$$= F[\overline{f(x)}]$$

(*iii*) We have $F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$\overline{F(s)}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{f(x)}\, e^{-isx}\, dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} \overline{f(-t)}\, e^{ist}\, (-dt)$$

Put $-x = t$
$-dx = dt$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{f(-t)}\, e^{ist}\, dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overline{f(-x)}\, e^{isx}\, dx$$

$$=F[\overline{f(-x)}]$$

Convolution of two functions for Fourier transform.

The convolution of two functions f(x) and g(x) is defined by

$$(f*g)(x)=f(x)*g(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

Convolution theorem

Statement. If F[f(x)] = F(s) and F[g(x)] = G(s) then $F[f(x)*g(x)]=F(s).G(s)$

Proof. $F[f(x)*g(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} [f(x)*g(x)]\, e^{isx}\, dx$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt\right] e^{isx}\, dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-t)\, e^{isx}\, dx\right] dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-t)\, e^{isx}\, e^{ist}\, e^{-ist}\, dx\right] dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-t)\, e^{is(x-t)}\, d(x-t)\right] e^{ist}\, dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, G(s)\, e^{ist}\, dt$$

$$=G(s)\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$$

$$=G(s)\,F(s)$$

(i.e.) $F[f(x)*g(x)]=F(s).G(s)$

Parseval's identity for Fourier transform.

Statement: If F(s) is the Fourier transform of f(x) then

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds=\int_{-\infty}^{\infty} |f(x)|^2\, dx$$

Proof. By convolution theorem for Fourier transform, we have

$$F[f(x)*g(x)]=F(s).G(s)$$

$$\therefore F^{-1}[F(s)G(s)]=f(x)*g(x)$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) G(s)\, e^{-isx}\, ds = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

$$\Rightarrow \int_{-\infty}^{\infty} F(s) G(s)\, e^{-isx}\, ds = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

Putting $x = 0$, we get

$$\int_{-\infty}^{\infty} F(s) G(s)\, ds = \int_{-\infty}^{\infty} f(t)\, g(-t)\, dt \quad ------- (1)$$

$$Let \;\; g(-t) = \overline{f(t)} \quad ------- (2)$$

$$(i.e.)\;\; g(t) = \overline{f(-t)}$$

$$\begin{aligned} G(s) = F[g(x)] &= F[g(t)] \\ &= F[\overline{f(-t)}] \\ &= F[\overline{f(-x)}] \\ &= \overline{F(s)} \quad (by \;\; property) \end{aligned}$$

$$(i.e.)\;\; G(s) = \overline{F(s)} \quad ---------- (3)$$

Substituting (2) and (3) in equation (1) we have

$$\int_{-\infty}^{\infty} F(s) \overline{F(s)}\, ds = \int_{-\infty}^{\infty} f(t) \overline{f(t)}\, dt$$

(i.e.) $$\int_{-\infty}^{\infty} | F(s) |^2\, ds = \int_{-\infty}^{\infty} | f(x) |^2\, dx$$

University Questions:

1. State the Fourier transform of the derivatives of a function.

Sol.

$$F[f'(x)] = (-is) F(s)$$

$$F[f''(x)] = (-is)^2 F(s)$$

$$F[f'''(x)] = (-is)^3 F(s)$$

$$------------$$

$$Ingeneral, \quad F[f^{(n)}(x)] = (-is)^n F(s)$$

2. Give an example for self-reciprocal under Fourier transform.

Sol. $e^{-\frac{x^2}{2}}$ is self-reciprocal under Fourier transform.

3. Give an example for self-reciprocal under Fourier cosine transform.

Sol. $e^{-\frac{x^2}{2}}$ is self-reciprocal under Fourier cosine transform.

4. Give an example for self-reciprocal under both Fourier sine and cosine transform.

Sol. $\frac{1}{\sqrt{x}}$ is self-reciprocal under both Fourier sine and cosine transform.

5. Find the Fourier transform of $f(x)=\begin{cases} a-|x|, & |x|<a \\ 0, & |x|\geq a \end{cases}$

Hence deduce that $(i)\ \int_0^\infty \frac{\sin t}{t}\,dt=\frac{\pi}{2}$ $\quad (ii)\ \int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt=\frac{\pi}{3}$

Sol.
$$F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$$
$$=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} 0.e^{isx}\,dx+\int_{-a}^{a}(a-|x|)\,e^{isx}\,dx+\int_{a}^{\infty}0.e^{isx}\,dx\right]$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a-|x|)(\cos sx+i\sin sx)\,dx$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a-|x|)\cos sx\,dx+i\,\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a-|x|)\sin sx\,dx$$
$$=\frac{2}{\sqrt{2\pi}}\int_{0}^{a}(a-|x|)\cos sx\,dx+0$$
$$=\sqrt{\frac{2}{\pi}}\int_{0}^{a}(a-x)\cos sx\,dx$$
$$=\sqrt{\frac{2}{\pi}}\left[(a-x)\left(\frac{\sin sx}{s}\right)-(-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^a$$
$$=\sqrt{\frac{2}{\pi}}\left[\left\{0-\frac{\cos sa}{s^2}\right\}-\left\{0-\frac{1}{s^2}\right\}\right]$$
$$(i.e.)\ F[f(x)]=\sqrt{\frac{2}{\pi}}\,\frac{1-\cos as}{s^2}$$

Using ***inverse Fourier transform***, we have
$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F[f(x)]\,e^{-isx}\,ds$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos as}{s^2}\right)(\cos sx-i\sin sx)\,ds$$
$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1-\cos as}{s^2}\right)\cos sx\,ds-i\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1-\cos as}{s^2}\right)\sin sx\,ds$$
$$=\frac{2}{\pi}\int_{0}^{\infty}\left(\frac{1-\cos as}{s^2}\right)\cos sx\,ds-0$$
$$\int_{0}^{\infty}\left(\frac{1-\cos as}{s^2}\right)\cos sx\,ds=\frac{\pi}{2}f(x)$$

Put $x=0$ *we get*
$$\int_{0}^{\infty}\left(\frac{1-\cos as}{s^2}\right)ds=\frac{\pi}{2}f(0)$$

$$\int_0^\infty \left(\frac{1-\cos 2t}{\frac{4t^2}{a^2}} \right) \frac{2dt}{a} = \frac{\pi}{2}(a)$$

Put as = 2t
ads = 2dt

$f(x) = a - |x|$
$f(0) = a - 0 = a$

$$2a\int_0^\infty \left(\frac{2\sin^2 t}{4t^2} \right) dt = \frac{\pi a}{2}$$

$$\int_0^\infty \left(\frac{\sin t}{t} \right)^2 dt = \frac{\pi}{2}$$

This proves (i)

Using **Parseval's identity,** we have

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

$$\int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} \left[\frac{1-\cos as}{s^2} \right] \right)^2 ds = \int_{-\infty}^{-a} 0.dx + \int_{-a}^{a} (a-|x|)^2\, dx + \int_a^\infty 0.dx$$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{1-\cos as}{s^2} \right)^2 ds = \int_{-a}^{a} (a-|x|)^2\, dx$$

$$\frac{4}{\pi} \int_0^\infty \left(\frac{1-\cos as}{s^2} \right)^2 ds = 2\int_0^a (a-x)^2\, dx$$

$$\frac{4}{\pi} \int_0^\infty \left(\frac{1-\cos 2t}{4t^2/a^2} \right)^2 \frac{2dt}{a} = 2\left[\frac{(a-x)^3}{-3} \right]_0^a$$

Put as = 2t
ads = 2dt

$$\frac{8a^3}{16\pi} \int_0^\infty \left(\frac{1-\cos 2t}{t^2} \right)^2 dt = 2\left[\{0\} - \left\{ -\frac{a^3}{3} \right\} \right]$$

$$\frac{a^3}{2\pi} \int_0^\infty \left(\frac{2\sin^2 t}{t^2} \right)^2 dt = \frac{2a^3}{3}$$

$$\frac{4}{2\pi} \int_0^\infty \left(\frac{\sin^2 t}{t^2} \right)^2 dt = \frac{2}{3}$$

$$(i.e.)\ \int_0^\infty \left(\frac{\sin t}{t} \right)^4 dt = \frac{\pi}{3}$$

6. Find the Fourier sine and cosine transform of $f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x > a \end{cases}$

Sol. $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^a \sin x \sin sx\, dx + \int_a^\infty 0.\sin sx\, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \frac{1}{2} [\cos(s-1)x - \cos(s+1)x]\, dx$$

2sinAsinB = cos(A – B) – cos(A + B)

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s-1)x}{s-1} - \frac{\sin(s+1)x}{s+1} \right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left\{ \frac{\sin(s-1)a}{s-1} - \frac{\sin(s+1)a}{s+1} \right\} - \{0-0\} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s-1)a}{s-1} - \frac{\sin(s+1)a}{s+1} \right]$$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^a \sin x \cos sx \, dx + \int_a^\infty 0. \cos sx \, dx \right]$$

2cosAsinB = sin(A + B) – sin(A – B)

$$= \sqrt{\frac{2}{\pi}} \int_0^a \frac{1}{2} [\sin(s+1)x - \sin(s-1)x] \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{-\cos(s+1)x}{s+1} + \frac{\cos(s-1)x}{s-1} \right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left\{ \frac{-\cos(s+1)a}{s+1} + \frac{\cos(s-1)a}{s-1} \right\} - \left\{ \frac{-1}{s+1} + \frac{1}{s-1} \right\} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left\{ \frac{(s-1)[-\cos sa \cos a + \sin sa \sin a] + (s+1)[\cos sa \cos a + \sin sa \sin a]}{(s+1)(s-1)} \right\} - \left\{ \frac{-(s-1)+(s+1)}{(s+1)(s-1)} \right\} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2s \sin sa \sin a + 2\cos sa \cos a}{s^2-1} - \frac{2}{s^2-1} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s \sin sa \sin a + \cos sa \cos a - 1}{s^2-1} \right]$$

7. Find the Fourier transform of $f(x) = \begin{cases} 1-x^2, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$

Hence deduce that $(i) \int_0^\infty \frac{\sin s - s\cos s}{s^3} ds = \frac{\pi}{4}$ $\quad (ii) \int_0^\infty \frac{\sin s - s\cos s}{s^3} \cos\frac{s}{2} ds = \frac{3\pi}{16}$

$(iii) \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3} \right)^2 ds = \frac{\pi}{15}$

Sol.

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \, e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-1} 0.e^{isx} \, dx + \int_{-1}^{1} (1-x^2) \, e^{isx} \, dx + \int_1^\infty 0.e^{isx} \, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)(\cos sx + i \sin sx) \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2) \cos sx \, dx + i \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2) \sin sx \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1-x^2)\cos sx \, dx + 0$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x^2)\left(\frac{\sin sx}{s}\right) - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (-2)\left(\frac{-\sin sx}{s^3}\right) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\left\{ 0 - \frac{2\cos s}{s^2} + \frac{2\sin s}{s^3} \right\} - \{0 - 0 + 0\} \right]$$

$$(i.e.) \quad F[f(x)] = 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s\cos s}{s^3} \right]$$

Using ***inverse Fourier transform***, we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f(x)] \, e^{-isx} \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}} \left(\frac{\sin s - s\cos s}{s^3} \right) (\cos sx - i \sin sx) \, ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3} \right) \cos sx \, ds - i \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3} \right) \sin sx \, ds$$

$$= \frac{4}{\pi} \int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3} \right) \cos sx \, ds - 0$$

$$\int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3} \right) \cos sx \, ds = \frac{\pi}{4} f(x) \text{ -------- } (1)$$

Put $x = 0$ *in equation* (1) *we get*

$$\int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3} \right) ds = \frac{\pi}{4} f(0)$$

$$= \frac{\pi}{4}(1) = \frac{\pi}{4} \quad \textit{This proves } (i)$$

$f(x) = 1 - x^2$
$f(0) = 1 - 0 = 1$

Put $x = \frac{1}{2}$ *in equation* (1) *we get*

$$\int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3} \right) \cos\frac{s}{2} \, ds = \frac{\pi}{4} f\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{4}\left(1 - \frac{1}{4}\right) = \frac{\pi}{4}\left(\frac{3}{4}\right) = \frac{3\pi}{16} \quad \textit{This proves } (ii)$$

$f(x) = 1 - x^2$
$f(\frac{1}{2}) = 1 - \frac{1}{4} = \frac{3}{4}$

Using **Parseval's identity,** we have

$$\int_{-\infty}^{\infty} |F(s)|^2 \, ds = \int_{-\infty}^{\infty} |f(x)|^2 \, dx$$

$$\int_{-\infty}^{\infty} \left(2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s\cos s}{s^3} \right] \right)^2 ds = \int_{-\infty}^{-1} 0.dx + \int_{-1}^{1} (1-x^2)^2 \, dx + \int_1^{\infty} 0.dx$$

$$\frac{8}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3} \right)^2 ds = \int_{-1}^{1} (1-x^2)^2 \, dx$$

$$\frac{16}{\pi} \int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3} \right)^2 ds = 2\int_0^1 (1-x^2)^2 \, dx$$

$$= 2\int_0^1 (1 + x^4 - 2x^2) \, dx$$

$$= 2\left[x + \frac{x^5}{5} - \frac{2x^3}{3} \right]_0^1$$

$$= 2\left[\left\{ 1 + \frac{1}{5} - \frac{2}{3} \right\} - \{0 + 0 - 0\} \right]$$

$$= 2\left[\frac{8}{15} \right]$$

$$\frac{16}{\pi} \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3} \right)^2 ds = \frac{16}{15}$$

$$\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3} \right)^2 ds = \frac{\pi}{15} \qquad \textit{This proves (iii)}$$

UNIT V
Z-TRANSFORMS AND DIFFERENCE EQUATIONS

5.1. DEFINITION: (ONE-SIDED OR UNILATERAL)

Let $\{f(n)\}$ be a sequence defined for all positive integers $n = 0,1,2,\ldots\infty$, then Z-transform of $\{f(n)\}$ is defined as

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n} ,$$

where z is an arbitrary complex variable.

5.2. DEFINITION: (*Z*-TRANSFORM FOR DISCRETE VALUES OF *t*)

If $f(t)$ is a function defined for discrete values of t, where $t = nT$, $n = 0,1,2,\ldots$, T being the sampling period, then Z-transform of $f(t)$ is defined as

$$Z\big(f(t)\big) = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

5.3. NOTE:

(i) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$ if $|x| < 1$

(ii) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

(iii) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

(iv) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

(v) $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \cdots$

(vi) $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \cdots$

(vii) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ if $|x| < 1$

(viii) $-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$

(ix) $1 + a + a^2 + \cdots + a^r = \frac{a^{r+1} - 1}{a - 1}$.

5.4. EXAMPLES:

☑ Example: 1

$Z(1) = \frac{z}{z-1}$

PROOF: $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$\therefore \quad Z(1) = \sum_{n=0}^{\infty} (1)\, z^{-n} = \sum_{n=0}^{\infty} \frac{1}{z^n} = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots$$

$$= \left(1 - \frac{1}{z}\right)^{-1}, \quad \left|\frac{1}{z}\right| \le 1 \qquad \left(\because\ (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots\right)$$

$$= \left(\frac{z-1}{z}\right)^{-1}, \quad |z| > 1$$

$$\boxed{Z(1) = \left(\frac{z}{z-1}\right)}.$$

☑ Example: 2

$Z(a^n) = \frac{z}{z-a}$ if $|z| > |a|$

PROOF:

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

$$Z(a^n) = \sum_{n=0}^{\infty} a^n\, z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \cdots$$

$$= \left(1-\frac{a}{z}\right)^{-1}, \qquad \text{if } \left|\frac{a}{z}\right| < 1$$

$$= \left(\frac{z-a}{z}\right)^{-1}$$

$$\boxed{Z\left(a^n\right) = \left(\frac{z}{z-a}\right)}, \qquad \text{if } |a| < |z|$$

☑ Example: 3

$$Z(n) = \frac{z}{(z-1)^2}$$

PROOF:

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

$$\therefore \qquad Z(n) = \sum_{n=0}^{\infty} n\, z^{-n} = 0 + \frac{1}{z} + \frac{2}{z^2} + \cdots = \frac{1}{z}\left(1 + \frac{2}{z} + \frac{3}{z^2} + \cdots\right)$$

$$= \frac{1}{z}\left(1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \cdots\right)$$

$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-2} = \frac{1}{z}\left(\frac{z-1}{z}\right)^{-2} = \frac{1}{z}\left(\frac{z}{z-1}\right)^2$$

$$\boxed{Z(n) = \frac{z}{(z-1)^2}}.$$

☑ Example: 4

$$Z\left(\frac{1}{n}\right) = \log\left(\frac{z}{z-1}\right), \quad \text{if } |z| > 1, \; n > 0$$

PROOF: $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$\therefore \qquad Z\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n}\; z^{-n} = \sum_{n=1}^{\infty} \frac{1}{nz^n} = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \cdots$$

$$= \left(\frac{1}{z}\right) + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots$$

$$= -\log\left(1-\frac{1}{z}\right) \qquad \left(\because\ -\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right)$$

$$= -\log\left(\frac{z-1}{z}\right)$$

$$\boxed{Z\left(\frac{1}{n}\right) = \log\left(\frac{z}{z-1}\right)}$$

☑ Example: 5

$$Z\left(\frac{1}{n+1}\right) = z\log\left(\frac{z}{z-1}\right)$$

PROOF: $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$\therefore \qquad Z\left(\frac{1}{n+1}\right) = \sum_{n=0}^{\infty} \frac{1}{n+1}\ z^{-n} = \sum_{n=0}^{\infty} \frac{1}{(n+1)z^n} = 1 + \frac{1}{2z} + \frac{1}{3z^2} + \cdots$$

$$= z\left(\frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \cdots\right) \qquad (\times \text{ and } \div \text{ by } z)$$

$$= z\left(\left(\frac{1}{z}\right) + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots\right)$$

$$= z\left(-\log\left(1-\frac{1}{z}\right)\right) = -z\log\left(\frac{z-1}{z}\right)$$

$$\boxed{Z\left(\frac{1}{n+1}\right) = z\log\left(\frac{z}{z-1}\right)}.$$

☑ Example: 6

$$Z\left(\frac{1}{n-1}\right) = \frac{1}{z}\log\left(\frac{z}{z-1}\right), \qquad n > 1$$

PROOF: $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$\therefore \qquad Z\left(\frac{1}{n-1}\right) = \sum_{n=2}^{\infty} \frac{1}{n-1}\ z^{-n} = \sum_{n=2}^{\infty} \frac{1}{(n-1)z^n} = \frac{1}{z^2} + \frac{1}{2z^3} + \frac{1}{3z^4} + \cdots$$

$$= \frac{1}{z}\left(\left(\frac{1}{z}\right) + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots\right)$$

$$= \frac{1}{z}\left(-\log\left(1-\frac{1}{z}\right)\right) = \frac{1}{z}\left(-\log\left(\frac{z-1}{z}\right)\right)$$

$$\boxed{Z\left(\frac{1}{n-1}\right) = \frac{1}{z}\log\left(\frac{z}{z-1}\right)}$$

☑ Example: 7

$$Z\left(\frac{1}{n!}\right) = e^{\frac{1}{z}}$$

PROOF: $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$\therefore \quad Z\left(\frac{1}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\; z^{-n} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z^n}\right) = 1+\frac{1}{1!}\left(\frac{1}{z}\right)+\frac{1}{2!}\left(\frac{1}{z}\right)^2+\cdots$$

$$\boxed{Z\left(\frac{1}{n!}\right) = e^{\frac{1}{z}}} \qquad \left(\because \quad e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)$$

☑ Example: 8

$$Z\left(\frac{1}{(n+1)!}\right) = ze^{\frac{1}{z}} - z$$

PROOF: $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$\therefore \quad Z\left(\frac{1}{(n+1)!}\right) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\; z^{-n} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\left(\frac{1}{z^n}\right)$$

$$= \frac{1}{1!}+\frac{1}{2!}\left(\frac{1}{z}\right)+\frac{1}{3}\left(\frac{1}{z}\right)^2+\cdots$$

$$= z\left(\frac{1}{1!}\left(\frac{1}{z}\right)+\frac{1}{2!}\left(\frac{1}{z}\right)^2+\frac{1}{3}\left(\frac{1}{z}\right)^3+\cdots\right)$$

$$= z\left(e^{\frac{1}{z}} - 1\right) \qquad \left(\because \quad e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)$$

$$\boxed{Z\left(\frac{1}{(n+1)!}\right) = \left(ze^{\frac{1}{z}} - z\right)}.$$

☑ Example: 9

$Z(n\,a^n) = \dfrac{az}{(z-a)^2}$

PROOF:

$$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

$$\therefore \qquad Z\left(n\,a^n\right) = \sum_{n=0}^{\infty} n\,a^n\, z^{-n} = \sum_{n=0}^{\infty} n\left(\frac{a}{z}\right)^n$$

$$= 0 + \left(\frac{a}{z}\right) + 2\left(\frac{a}{z}\right)^2 + \cdots$$

$$= \frac{a}{z}\left(1 + 2\left(\frac{a}{z}\right) + 3\left(\frac{a}{z}\right)^2 + \cdots\right)$$

$$= \frac{a}{z}\left(1 - \frac{a}{z}\right)^{-2} = \frac{a}{z}\left(\frac{z-a}{z}\right)^{-2} = \frac{a}{z}\frac{z^2}{(z-a)^2}$$

$$\boxed{Z(n\,a^n) = \frac{az}{(z-a)^2}}$$

5.5. LINEAR PROPERTY:

$$Z\left(af(n) + bg(n)\right) = aZ\left(f(n)\right) + bZ\left(g(n)\right)$$
$$= a\,F(z) + b\,G(z)$$

5.6. NOTE:

(i) $Z(1) = \dfrac{z}{z-1}$ (ii) $Z(a^n) = \dfrac{z}{z-a}$, if $|z| > |a|$

☑ Example: 10

Find $Z(k)$

SOLUTION: $Z(k) = k\,Z(1) = k\left(\dfrac{z}{z-1}\right)$

☑ Example: 11

Find $Z\left((-1)^n\right)$

SOLUTION:

Since $Z(a^n) = \frac{z}{z-a}$

$\therefore$ $Z((-1)^n) = \frac{z}{z-(-1)} = \frac{z}{z+1}$

☑ Example: 12

Prove that $Z\left(e^{-an}\right) = \frac{z}{z-e^{-a}}$

PROOF:

Since $Z\left(a^n\right) = \frac{z}{z-a}$

$\therefore$ $Z\left(\left(e^{-a}\right)^n\right) = \frac{z}{z-e^{-a}}$.

☑ Example: 13

Find $Z(\cos n\theta)$ and $Z(\sin n\theta)$.

SOLUTION:

Let $a = e^{i\theta}$

$\therefore$ $a^n = e^{in\theta} = \cos n\theta + i\sin n\theta$

We know that $Z(a^n) = \frac{z}{z-a}$

$\therefore$ $Z(a^n) = Z\left((e^{i\theta})^n\right) = \frac{z}{z-e^{i\theta}}$

$$Z(\cos n\theta + i\sin n\theta) = \frac{z}{z-(\cos\theta + i\sin\theta)}$$

$$= \frac{z}{(z-\cos\theta) - i\sin\theta}$$

$$= \frac{z}{(z-\cos\theta) - i\sin\theta}\,\frac{(z-\cos\theta) + i\sin\theta}{(z-\cos\theta) + i\sin\theta}$$

$$= \frac{z\,(z-\cos\theta) + iz\sin\theta}{(z-\cos\theta)^2 + \sin^2\theta}$$

$$= \frac{z\,(z-\cos\theta) + iz\sin\theta}{z^2 + \cos^2\theta - 2z\cos\theta + \sin^2\theta}$$

$$= \frac{z\left(z-\cos\theta\right)+iz\sin\theta}{z^2-2z\cos\theta+1}$$

$$Z(\cos n\theta)+iZ(\sin n\theta) = \frac{z\left(z-\cos\theta\right)}{z^2-2z\cos\theta+1}+\frac{iz\sin\theta}{z^2-2z\cos\theta+1}$$

Equating real and imaginary parts, we get

$$Z(\cos n\theta) = \frac{z\left(z-\cos\theta\right)}{z^2-2z\cos\theta+1}$$

$$Z(\sin n\theta) = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

NOTE:

We know that $Z\left(f(t)\right) = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$

$\therefore$ $Z\left(\sin at\right) = \sum_{n=0}^{\infty} \sin anT \;\; z^{-n}$

$$= \sum_{n=0}^{\infty} \sin n\theta \,.\, z^{-n}, \quad \text{where } \theta = aT$$

$$= Z(\sin n\theta)$$

$$= \frac{z\sin\theta}{z^2-2z\cos\theta+1} = \frac{z\sin aT}{z^2-2z\cos aT+1}$$

$$lll^{ly} \quad Z(\cos at) = \frac{z(z-\cos aT)}{z^2-2z\cos aT+1}.$$

☑ Example: 14

Find $Z(r^n \cos n\theta)$ and $Z(r^n \sin n\theta)$

SOLUTION:

Hints: Let $a = re^{i\theta}$

$$a^n = r^n\, e^{in\theta} = r^n(\cos n\theta + i\sin n\theta)$$

Answer:

$$Z(r^n \cos n\theta) = \frac{z\left(z-r\cos\theta\right)}{z^2-2zr\cos\theta+r^2}$$

$$Z(r^n \sin n\theta) = \frac{zr\sin\theta}{z^2-2zr\cos\theta+r^2}.$$

☑ Example: 15

Find $Z(t)$

SOLUTION:

$$Z\left(f(t)\right)=\sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

$$\therefore \qquad Z\left(t\right)=\sum_{n=0}^{\infty} nT\, z^{-n}=T\sum_{n=0}^{\infty} n\, z^{-n}=T\, Z(n)$$

$$=T\,\frac{z}{\left(z-1\right)^2} \qquad \left(\because \quad Z(n)=\frac{z}{\left(z-1\right)^2}\right).$$

☑ Example: 16

Find $Z(e^{-at})$

SOLUTION:

$$Z\left(f(t)\right)=\sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

$$\therefore \qquad Z\left(e^{-at}\right)=\sum_{n=0}^{\infty} e^{-anT}\, z^{-n}=\sum_{n=0}^{\infty}\left(e^{-aT}\right)^n z^{-n}$$

$$=Z\left(e^{-aT}\right)^n=\frac{z}{z-e^{-aT}} \qquad \left(\because \quad Z(a^n)=\frac{z}{z-a}\right)$$

☑ Example: 17

Find $Z\left(\frac{1}{n(n+1)}\right)$

SOLUTION:

Now $\qquad \frac{1}{n(n+1)}=\frac{A}{n}+\frac{B}{n+1}$

$$=\frac{A(n+1)+B(n)}{n(n+1)}$$

$\Rightarrow \qquad A(n+1)+B(n)=1$

Put $n=0$ $\qquad A(1)+B(0)=1 \Rightarrow A=1$

Put $n=-1$: $\quad A(0)+B(-1)=1 \Rightarrow B=-1$

$$\therefore \qquad \frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$$

$$\therefore \qquad Z\left(\frac{1}{n(n+1)}\right) = Z\left(\frac{1}{n} - \frac{1}{n+1}\right) = Z\left(\frac{1}{n}\right) - Z\left(\frac{1}{n+1}\right)$$

$$= \log\left(\frac{z}{z-1}\right) - z\log\left(\frac{z}{z-1}\right) = (1-z)\log\left(\frac{z}{z-1}\right)$$

☑ Example: 18

Find the Z-transform of $\dfrac{1}{(n+1)(n+2)}$

SOLUTION:

$$\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2} = \frac{A(n+2)+B(n+1)}{(n+1)(n+2)}$$

$$\Rightarrow \qquad A(n+2)+B(n+1)=1$$

Put $z=-1$: $A(1)=1 \qquad \Rightarrow \boxed{A=1}$

Put $z=-2$: $B(-1)=1 \qquad \Rightarrow \boxed{B=-1}$

$$\therefore \qquad \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$$

$$\therefore \quad Z\left(\frac{1}{(n+1)(n+2)}\right) = Z\left(\frac{1}{n+1} - \frac{1}{n+2}\right) = Z\left(\frac{1}{n+1}\right) - Z\left(\frac{1}{n+2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{1}{z}\right)^n - \sum_{n=0}^{\infty} \frac{1}{n+2}\left(\frac{1}{z}\right)^n$$

$$= \left[1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \cdots\right] - \left[\frac{1}{2} + \frac{1}{3}\left(\frac{1}{z}\right) + \frac{1}{4}\left(\frac{1}{z}\right)^2 + \cdots\right]$$

$$= z\left[\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots\right] - z^2\left[\frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$= z\left[-\log\left(1-\frac{1}{z}\right)\right] - z^2\left[-\log\left(1-\frac{1}{z}\right) - \frac{1}{z}\right]$$

$$= -z\log\left(1-\frac{1}{z}\right) + z^2\log\left(1-\frac{1}{z}\right) + z$$

$$= \left(z^2 - z\right)\log\left(1-\frac{1}{z}\right) + z$$

☑ Example: 19

Find $Z\left(\cos^2 t\right)$

SOLUTION:

$$Z\left(\cos^2 t\right) = Z\left(\frac{1+\cos 2t}{2}\right) = \frac{1}{2}\left[Z(1) + Z(\cos 2t)\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-1} + \frac{z(z-\cos 2T)}{z^2 - 2z\cos 2T + 1}\right]$$

5.7. EXERCISE:

1. Find $Z(\sin^2 t)$
2. $Z(\cos^3 t)$
3. $Z(\sin^3 t)$

5.8. HINTS:

(i) $\sin 3A = 3\sin A - 4\sin^3 A$

(ii) $\cos 3A = 4\cos^3 A - 3\cos A$

5.9. PROPERTIES OF Z-TRANSFORM:

☑ Property: 1

FIRST SHIFTING THEOREM:

If $Z\left(f(t)\right) = F(z)$ then

(i) $Z\left(e^{-at} f(t)\right) = F(z\, e^{aT})$

(ii) $Z\left(e^{at} f(t)\right) = F(z\, e^{-aT})$

(iii) $Z\left(a^n f(t)\right) = F\left(\frac{z}{a}\right)$

(iv) $Z\left(a^n f(n)\right) = F\left(\frac{z}{a}\right)$

PROOF:

(i) We know that

$$Z\left(f(t)\right) = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

$$Z\left(e^{-at} f(t)\right) = \sum_{n=0}^{\infty} e^{-anT} f(nT)\, z^{-n} = \sum_{n=0}^{\infty} f(nT)\left(ze^{aT}\right)^{-n}$$

$$= Z\left(f(t)\right)_{z \to z\, e^{aT}} = \left[F(z)\right]_{z \to z^{aT}}$$

$$Z\left(e^{-at} f(t)\right) = F\left(ze^{aT}\right).$$

(iii) We know that

$$Z\left(f(t)\right) = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

$$Z\left(a^n f(t)\right) = \sum_{n=0}^{\infty} a^n f(nT)\, z^{-n} = \sum_{n=0}^{\infty} f(nT)\left(\frac{z}{a}\right)^{-n}$$

$$= Z\left(f(t)\right)_{z \to \frac{z}{a}} = \left[F(z)\right]_{z \to \frac{z}{a}}$$

$$Z\left(a^n f(t)\right) = F\left(\frac{z}{a}\right).$$

☑ Property: 2

DIFFERENTIATION IN *Z*-DOMAIN:

$Z\left(n\, f(n)\right) = -z\dfrac{d}{dz}\left(F(z)\right)$, where $F(z) = Z\left(f(n)\right)$.

PROOF:

$$F(z) = Z\left(f(n)\right)$$

$$F(z) = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

$$\frac{d}{dz}\left(F(z)\right) = \sum_{n=0}^{\infty} f(n)\,(-n)\, z^{-n-1} = -\frac{1}{z}\sum_{n=0}^{\infty} f(n)\, n\, z^{-n}$$

$$-z\,\frac{d}{dz}\left(F(z)\right) = \sum_{n=0}^{\infty} n\, f(n)\, z^{-n} = Z\left(n\, f(n)\right)$$

$$\therefore \qquad Z\left(n\, f(n)\right) = -z\frac{d}{dz}\left(F(z)\right)$$

☑ Property: 3

SECOND SHIFTING THEOREM:

(i) $Z\left(f(n+1)\right) = z\, F(z) - zf(0)$

PROOF:

$$Z(f(n+1)) = \sum_{n=0}^{\infty} f(n+1)\, z^{-n} = \sum_{n=0}^{\infty} f(n+1)\, z^{-n}\, z\, z^{-1}$$

$$Z(f(n+1)) = z \sum_{n=0}^{\infty} f(n+1)\, z^{-(n+1)}$$

put $n+1=m$

$$= z \sum_{m=1}^{\infty} f(m)\, z^{-m} = z\left[\sum_{m=0}^{\infty} f(m)\, z^{-m} - f(0)\right]$$

$$= z(F(z) - f(0))$$

$$Z(f(n+1)) = z\,F(z) - zf(0).$$

(ii) $Z(f(t+T)) = z\,F(z) - zf(0)$

PROOF:

$$Z(f(t+T)) = \sum_{n=0}^{\infty} f(nT+T)\, z^{-n} = \sum_{n=0}^{\infty} f(nT+T)\, z^{-n}\, z\, z^{-1}$$

$$Z(f(nT+T)) = z \sum_{n=0}^{\infty} f((n+1)T)\; z^{-(n+1)}$$

put $n+1=m$

$$= z \sum_{m=1}^{\infty} f(mT)\, z^{-(m)} = z\left[\sum_{m=0}^{\infty} f(mT)\, z^{-(m)} - f(0)\right]$$

$$= z(F(z) - f(0))$$

$$Z(f(t+T)) = z\,F(z) - zf(0).$$

☑ Property: 4

INITIAL VALUE THEOREM:

If $Z(f(t)) = F(z)$ then $f(0) = \lim_{z\to\infty} F(z)$

PROOF:

$$F(z) = Z(f(t)) = \sum_{n=0}^{\infty} f(nT)\, z^{-n} = f(0.\,T) + \frac{f(1.\,T)}{z} + \frac{f(2.\,T)}{z^2} + \cdots$$

$$= f(0) + \frac{f(T)}{z} + \frac{f(2T)}{z^2} + \cdots$$

$$\lim_{z \to \infty} F(z) = \lim_{z \to \infty}\left[f(0) + \frac{f(T)}{z} + \frac{f(2T)}{z^2} + \cdots \right] = f(0)$$

(i.e.) $f(0) = \lim_{z \to \infty} F(z)$.

NOTE:

If $Z\left(f(n)\right) = F(z)$ then $f(0) = \lim_{z \to \infty} F(z)$

☑ Property: 5

FINAL VALUE THEOREM:

If $Z\left(f(t)\right) = F(z)$ then $\lim_{t \to \infty} f(t) = \lim_{z \to 1}\left(z-1\right)F(z)$

PROOF:

$$Z\left[f(t+T) - f(t)\right] = \sum_{n=0}^{\infty}\left[f(nT+T) - f(nT)\right] z^{-n}$$

$$Z\left(f(t+T)\right) - Z\left(f(t)\right) = \sum_{n=0}^{\infty}\left[f(nT+T) - f(nT)\right] z^{-n}$$

$$zF(z) - zf(0) - F(z) = \sum_{n=0}^{\infty}\left[f(nT+T) - f(nT)\right] z^{-n}$$

$$\left(z-1\right)F(z) - z\,f(0) = \sum_{n=0}^{\infty}\left[f(nT+T) - f(nT)\right] z^{-n}$$

Taking limit as $z \to 1$ we get

$$\lim_{z \to 1}\left[\left(z-1\right)F(z) - z\,f(0)\right] = \lim_{z \to 1}\sum_{n=0}^{\infty}\left[f(nT+T) - f(nT)\right] z^{-n}$$

$$\lim_{z \to 1}\left(z-1\right)F(z) - f(0) = \sum_{n=0}^{\infty}\left[f(nT+T) - f(nT)\right]$$

$$= \lim_{z \to \infty}\left[\begin{array}{l} \cancel{f(T)} - f(0) + \cancel{f(2T)} - \cancel{f(T)} \\ \quad + \cancel{f(3T)} - \cancel{f(2T)} + \cdots + f(n+1)T - \cancel{f(nT)} \end{array}\right]$$

$$= \lim_{z \to \infty}\left[f(n+1)T - f(0)\right]$$

$$= \lim_{z \to \infty} f(n+1)T - f(0)$$

$$\lim_{z \to 1}\left(z-1\right)F(z) - \cancel{f(0)} = f(\infty) - \cancel{f(0)} = \lim_{t \to \infty} f(t).$$

(i.e.) $\lim_{t \to \infty} f(t) = \lim_{z \to 1} (z-1) F(z)$.

NOTE: If $Z(f(n)) = F(z)$ then $\lim_{n \to \infty} f(n) = \lim_{z \to 1} (z-1) F(z)$

5.10. EXAMPLES BASED ON PROPERTIES:

☑ Example: 1

Find $Z\left(e^{-at}\ t\right)$.

SOLUTION:

We know that $Z\left(e^{-at}\ f(t)\right) = Z[f(t)]_{z \to z\,e^{aT}} = F(z)_{z \to z\,e^{aT}}$

Since $$Z(f(t)) = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

$$Z(t) = \sum_{n=0}^{\infty} nT\ \ z^{-n} = T \sum_{n=0}^{\infty} n\, z^{-n}$$

$$= T\left(0 + \frac{1}{z} + \frac{2}{z^2} + \cdots\right) = T\left(\frac{1}{z} + 2\left(\frac{1}{z}\right)^2 + \cdots\right)$$

$$= T\left(\frac{1}{z}\right)\left(1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \cdots\right)$$

$$= \frac{T}{z}\left(1 - \frac{1}{z}\right)^{-2} = \frac{T}{z}\left(\frac{z-1}{z}\right)^{-2} = \frac{T}{z}\left(\frac{z}{z-1}\right)^2 = \frac{T\,z}{(z-1)^2}.$$

$$\therefore \quad Z\left(e^{-at}\ t\right) = \left[\frac{T\,z}{(z-1)^2}\right]_{z \to z\,e^{aT}} = \frac{T\,z\,e^{aT}}{\left(z\,e^{aT} - 1\right)^2}.$$

☑ Example: 2

Find $Z\left(a^n\ n\right)$.

SOLUTION:

We know that

$$Z\left(a^n\ f(n)\right) = F\left(\frac{z}{a}\right)$$

Now, $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$\therefore \qquad Z(n)=\sum_{n=0}^{\infty} n\, z^{-n} = 0+\frac{1}{z}+\frac{2}{z^2}+\cdots$$

$$=\frac{1}{z}\left(1+\frac{2}{z}+\frac{3}{z^2}+\cdots\right)=\frac{1}{z}\left(1+2\left(\frac{1}{z}\right)+3\left(\frac{1}{z}\right)^2+\cdots\right)$$

$$=\frac{1}{z}\left(1-\frac{1}{z}\right)^{-2}=\frac{1}{z}\left(\frac{z-1}{z}\right)^{-2}=\frac{1}{z}\left(\frac{z}{z-1}\right)^{2}$$

$$Z(n)=\frac{z}{(z-1)^2}.$$

$$\therefore \qquad Z(a^n\, n)=\left[\frac{z}{(z-1)^2}\right]_{z\to\frac{z}{a}}=\left[\frac{\frac{z}{a}}{\left(\frac{z}{a}-1\right)^2}\right]=\left[\frac{\frac{z}{a}}{\left(\frac{z-a}{a}\right)^2}\right]=\frac{z}{a}\left[\frac{a^2}{(z-a)^2}\right]=\frac{az}{(z-a)^2}$$

.

☑ Example: 3

Find $Z(n^2)$.

SOLUTION:

We know that

$$Z(n\, f(n))=-z\frac{d}{dz}(F(z))$$

$$Z(n^2)=Z(n\,.n)=-z\frac{d}{dz}(Z(n))$$

Now,

$$Z(n)=\sum_{n=0}^{\infty} n\, z^{-n} = 0+\frac{1}{z}+\frac{2}{z^2}+\cdots=\frac{1}{z}\left(1+\frac{2}{z}+\frac{3}{z^2}+\cdots\right)$$

$$=\frac{1}{z}\left(1+2\left(\frac{1}{z}\right)+3\left(\frac{1}{z}\right)^2+\cdots\right)=\frac{1}{z}\left(1-\frac{1}{z}\right)^{-2}=\frac{1}{z}\left(\frac{z-1}{z}\right)^{-2}=\frac{1}{z}\left(\frac{z}{z-1}\right)^{2}$$

$$Z(n)=\frac{z}{(z-1)^2}.$$

$$\therefore \qquad Z(n^2)=-z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]=-z\left[\frac{(z-1)^2(1)-z\,2(z-1)}{(z-1)^4}\right]=-z\left[\frac{(z-1)(z-1-2z)}{(z-1)^4}\right]$$

$$=z\left[\frac{(z+1)}{(z-1)^3}\right]=\frac{z^2+z}{(z-1)^3}$$

☑ Example: 4

Find the Z-transform of $\cos n\theta$ and $\sin n\theta$. Hence deduce that Z-transform of $\cos(n+1)\theta$ and $\sin(n+1)\theta$

SOLUTION:

We know that $Z(\cos n\theta) = \dfrac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$ and $Z(\sin n\theta) = \dfrac{z\ \sin\theta}{z^2 - 2z\cos\theta + 1}$

Since $Z(f(n+1)) = zF(z) - zf(0)$

$$\therefore \quad Z(\cos(n+1)\theta) = z\left[\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} - 1\right] = z\left[\frac{\cancel{z^2} - z\cos\theta - \cancel{z^2} + 2z\cos\theta - 1}{z^2 - 2z\cos\theta + 1}\right]$$

$$= \frac{z(z\cos\theta - 1)}{z^2 - 2z\cos\theta + 1}$$

And $$Z(\sin(n+1)\theta) = z\left[\frac{z\ \sin\theta}{z^2-2z\cos\theta+1} - 0\right] = \frac{z^2\sin\theta}{z^2 - 2z\cos\theta + 1}$$

☑ Example: 5

Find $Z(a^n\cos n\theta)$ and $Z(a^n\sin n\theta)$

SOLUTION:

We know that $Z(\cos n\theta) = \dfrac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$ and $Z(\sin n\theta) = \dfrac{z\ \sin\theta}{z^2 - 2z\cos\theta + 1}$

By scaling property, $Z(a^n\cos n\theta) = F\left(\dfrac{z}{a}\right)$

$$\therefore \quad Z(a^n\cos n\theta) = \frac{\frac{z}{a}\left(\frac{z}{a} - \cos\theta\right)}{\frac{z^2}{a^2} - 2\frac{z}{a}\cos\theta + 1} = \frac{\frac{z}{\cancel{a}}\left(\frac{z - a\cos\theta}{\cancel{a}}\right)}{\frac{z^2 - 2az\cos\theta + a^2}{\cancel{a^2}}} = \frac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}.$$

And $Z(a^n\sin n\theta) = F\left(\dfrac{z}{a}\right)$

$$\therefore \quad Z(a^n\sin n\theta) = \frac{\frac{z}{a}\sin\theta}{\frac{z^2}{a^2} - 2\frac{z}{a}\cos\theta + 1} = \frac{\frac{z}{\cancel{a}}\sin\theta}{\frac{z^2 - 2za\cos\theta + a^2}{a^{\cancel{2}}}} = \frac{az\ \sin\theta}{z^2 - 2az\cos\theta + a^2}$$

☑ Example: 6

Find $Z\left(\sin\left(\frac{n\pi}{2}\right)\right)$ and $Z\left(\cos\left(\frac{n\pi}{2}\right)\right)$ and also find $Z\left(a^n \sin\left(\frac{n\pi}{2}\right)\right)$ and $Z\left(a^n \cos\left(\frac{n\pi}{2}\right)\right)$

SOLUTION:

We know that $Z(\sin n\theta) = \frac{z \sin\theta}{z^2 - 2z\cos\theta + 1}$ and $Z(\cos n\theta) = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$

$$\therefore \quad Z\left(\sin\frac{n\pi}{2}\right) = \frac{z \sin\left(\frac{\pi}{2}\right)}{z^2 - 2z\cos\left(\frac{\pi}{2}\right) + 1} = \frac{z\,(1)}{z^2 - 2z(0) + 1} = \frac{z}{z^2+1}$$

And $$Z\left(a^n \sin\left(\frac{n\pi}{2}\right)\right) = \frac{\frac{z}{a}}{\frac{z^2}{a^2}+1} = \frac{\frac{z}{a}}{\frac{z^2+a^2}{a^2}} = \frac{az}{z^2+a^2}$$

Now $$Z\left(\cos\left(\frac{n\pi}{2}\right)\right) = \frac{z\left(z-\cos\left(\frac{\pi}{2}\right)\right)}{z^2 - 2z\cos\left(\frac{\pi}{2}\right)+1} = \frac{z(z-0)}{z^2-2z(0)+1} = \frac{z^2}{z^2+1}$$

$$\therefore \quad Z\left(a^n \cos\left(\frac{n\pi}{2}\right)\right) = \frac{\frac{z^2}{a^2}}{\frac{z^2}{a^2}+1} = \frac{\frac{z^2}{\cancel{a^2}}}{\frac{z^2+a^2}{\cancel{a^2}}} = \frac{z^2}{z^2+a^2}$$

☑ Example: 7

Find $Z\left(\sin^2\left(\frac{n\pi}{4}\right)\right)$ and $Z\left(\cos^2\left(\frac{n\pi}{4}\right)\right)$

SOLUTION:

We know that $Z\left(\cos\left(\frac{n\pi}{2}\right)\right) = \frac{z^2}{z^2+1}$

$$\therefore \quad Z\left(\sin^2\left(\frac{n\pi}{4}\right)\right) = Z\left[\frac{1-\cos 2\left(\frac{n\pi}{4}\right)}{2}\right] = \frac{1}{2}Z\left[1-\cos\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{2}\left[Z(1) - Z\left(\cos\left(\frac{n\pi}{2}\right)\right)\right] = \frac{1}{2}\left[\frac{z}{z-1} - \frac{z^2}{z^2+1}\right]$$

And
$$Z\left(\cos^2\left(\frac{n\pi}{4}\right)\right) = Z\left[\frac{1+\cos 2\left(\frac{n\pi}{4}\right)}{2}\right] = \frac{1}{2}Z\left[1+\cos\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{2}\left[Z(1) + Z\left(\cos\left(\frac{n\pi}{2}\right)\right)\right] = \frac{1}{2}\left[\frac{z}{z-1} + \frac{z^2}{z^2+1}\right]$$

☑ Example: 8

Find the Z-transform of $n a^n \sin n\theta$

SOLUTION:

Since
$$Z(\sin n\theta) = \frac{z \sin\theta}{z^2 - 2z\cos\theta + 1}$$

By scaling property,
$$Z\left(a^n \sin n\theta\right) = F\left(\frac{z}{a}\right)$$

$$Z\left(a^n \sin n\theta\right) = \frac{\frac{z}{a}\sin\theta}{\frac{z^2}{a^2} - 2\frac{z}{a}\cos\theta + 1} = \frac{\frac{z}{a}\sin\theta}{\frac{z^2 - 2za\cos\theta + a^2}{a^2}}$$

$$= \frac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$$

$$\therefore Z\left(n a^n \sin n\theta\right) = -z\frac{d}{dz}\left[\frac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}\right]$$

$$= -z\left[\frac{\left(z^2 - 2az\cos\theta + a^2\right)a\sin\theta - az\sin\theta(2z - 2a\cos\theta)}{\left(z^2 - 2az\cos\theta + a^2\right)^2}\right]$$

$$= -z\left[\frac{az^2\sin\theta - \cancel{2a^2 z\sin\theta\cos\theta} + a^3\sin\theta - 2az^2\sin\theta + \cancel{2a^2 z\sin\theta\cos\theta}}{\left(z^2 - 2az\cos\theta + a^2\right)^2}\right]$$

$$= -z\left[\frac{a^3\sin\theta - az^2\sin\theta}{\left(z^2 - 2az\cos\theta + a^2\right)^2}\right]$$

$$= \frac{za\sin\theta\left(z^2 - a^2\right)}{\left(z^2 - 2az\cos\theta + a^2\right)^2}$$

5.11. INVERSE *Z*-TRANSFORM:

5.11.1. DEFINITION:

If $Z(f(n)) = F(z)$ then inverse *Z*-transform is defined as

$$f(n) = Z^{-1}\left(F(z)\right)$$

5.11.2. NOTE:

(i) $Z^{-1}\left(\frac{z}{z-a}\right) = a^n$

(ii) $Z^{-1}\left(\frac{z}{(z-a)^2}\right) = na^{n-1}$

(iii) $Z^{-1}\left(\frac{1}{z-a}\right) = a^{n-1}$

(iv) $Z^{-1}\left(\frac{z^2}{(z-a)^2}\right) = (n+1)a^n$

(v) $Z^{-1}\left(\frac{z^2}{(z+a)^2}\right) = (n+1)(-a)^n$

(vi) $Z^{-1}\left(\frac{z^2}{z^2+a^2}\right) = a^n\cos\left(\frac{n\pi}{2}\right)$

(vii) $Z^{-1}\left(\frac{az}{z^2+a^2}\right) = a^n\sin\left(\frac{n\pi}{2}\right)$

(viii) $Z^{-1}\left(\frac{8z}{(z-a)^3}\right) = n(n-1)a^n$

5.11.3. TYPE: I (METHOD OF PARTIAL FRACTION)

☑ Example: 1

Find $Z^{-1}\left[\dfrac{10z}{(z-1)(z-2)}\right]$

SOLUTION:

Let $F(z)=\dfrac{10z}{(z-1)(z-2)}$

$\therefore \quad \dfrac{F(z)}{z}=\dfrac{10}{(z-1)(z-2)}$

$\therefore \quad \dfrac{F(z)}{z}=\dfrac{10}{(z-1)(z-2)}=\dfrac{A}{z-1}+\dfrac{B}{z-2}=\dfrac{A(z-2)+B(z-1)}{(z-1)(z-2)}$

$\Rightarrow \quad A(z-2)+B(z-1)=10$

put $z=1$: $A(-1)=10 \Rightarrow A=-10$

put $z=2$: $B(1)=10 \Rightarrow B=10$

$\therefore \quad \dfrac{F(z)}{z}=\dfrac{-10}{z-1}+\dfrac{10}{z-2}$

$\Rightarrow \quad F(z)=\dfrac{-10z}{z-1}+\dfrac{10z}{z-2}$

Taking Z^{-1} on both sides,

$$Z^{-1}(F(z))=-10Z^{-1}\left(\frac{z}{z-1}\right)+10\,Z^{-1}\left(\frac{z}{z-2}\right)$$

$$=-10\,(1^n)+10\,(2^n) \qquad \left(\therefore\ Z^{-1}\left(\frac{z}{z-a}\right)=a^n\right)$$

$$f(n)=10\,(2^n-1).$$

☑ Example: 2

Find $Z^{-1}\left[\dfrac{z^3}{(z-1)^2(z-2)}\right]$ using partial fraction

SOLUTION:

Let $F(z)=\dfrac{z^3}{(z-1)^2\,(z-2)}$

$\therefore \qquad \dfrac{F(z)}{z} = \dfrac{z^2}{(z-1)^2(z-2)}$

$\therefore \qquad \dfrac{F(z)}{z} = \dfrac{z^2}{(z-1)^2(z-2)} = \dfrac{A}{z-1} + \dfrac{B}{(z-1)^2} + \dfrac{C}{z-2}$

$$= \frac{A(z-1)(z-2) + B(z-2) + C(z-1)^2}{(z-1)^2(z-2)}$$

$\Rightarrow \qquad A(z-1)(z-2) + B(z-2) + C(z-1)^2 = z^2$

Put $z = 1$: $\qquad B(-1) = 1 \;\Rightarrow\; B = -1$

Put $z = 2$: $\qquad C(1) = 4 \;\Rightarrow\; C = 4$

Equating the coefficient of z^2:

$$A + C = 1 \;\Rightarrow\; A = 1 - C = 1 - 4 = -3$$

$\therefore \qquad \dfrac{F(z)}{z} = \dfrac{-3}{z-1} + \dfrac{-1}{(z-1)^2} + \dfrac{4}{z-2} \qquad \Rightarrow \qquad F(z) = \dfrac{-3z}{z-1} - \dfrac{z}{(z-1)^2} + \dfrac{4z}{z-2}$

Taking Z^{-1} on both sides,

$$Z^{-1}(F(z)) = -3Z^{-1}\left(\frac{z}{z-1}\right) - Z^{-1}\left(\frac{z}{(z-1)^2}\right) + 4\,Z^{-1}\left(\frac{z}{z-2}\right)$$

$$= -3(1^n) - n(1^{n-1}) + 4\,(2^n)$$

$$f(n) = -3 - n + 4\,(2^n).$$

☑ Example: 3

Find $Z^{-1}\left(\dfrac{z(z^2 - z + 2)}{(z+1)(z-1)^2}\right)$ using partial fraction method

SOLUTION:

Let $\qquad F(z) = \dfrac{z(z^2 - z + 2)}{(z+1)(z-1)^2}$

$\therefore \qquad \dfrac{F(z)}{z} = \dfrac{z^2 - z + 2}{(z+1)(z-1)^2} = \dfrac{A}{z+1} + \dfrac{B}{z-1} + \dfrac{C}{(z-1)^2}$

$$= \frac{A(z-1)^2 + B(z+1)(z-1) + C(z+1)}{(z+1)(z-1)^2}$$

$$\Rightarrow \quad A(z-1)^2 + B(z+1)(z-1) + C(z+1) = z^2 - z + 2$$

Put $z=-1$: $A(4)+0+0=1+1+2 \Rightarrow 4A=4 \Rightarrow \boxed{A=1}$

Put $z=1$: $0+0+2C=1-1+2 \Rightarrow 2C=2 \Rightarrow \boxed{C=1}$

Put $z=0$: $A-B+C=2 \Rightarrow B=A+C-2=1+1-2 \Rightarrow \boxed{B=0}$

$$\therefore \quad \frac{F(z)}{z} = \frac{1}{z+1} + 0 + \frac{1}{(z-1)^2}$$

$$\therefore \quad F(z) = \frac{z}{z+1} + \frac{z}{(z-1)^2}$$

$$\therefore \quad Z^{-1}(F(z)) = Z^{-1}\left(\frac{z}{z+1} + \frac{z}{(z-1)^2}\right) = Z^{-1}\left(\frac{z}{z+1}\right) + Z^{-1}\left(\frac{z}{(z-1)^2}\right)$$

$$f(n) = (-1)^n + n.$$

☑ Example: 4

Find $Z^{-1}\left(\frac{z^2}{(z+2)(z^2+4)}\right)$ using partial fraction

SOLUTION:

Let $$F(z) = \frac{z^2}{(z+2)(z^2+4)}$$

$$\therefore \quad \frac{F(z)}{z} = \frac{z}{(z+2)(z^2+4)} = \frac{A}{z+2} + \frac{(Bz+c)}{z^2+4} = \frac{A(z^2+4)+(Bz+c)(z+2)}{(z+2)(z^2+4)}$$

$$\Rightarrow \quad A(z^2+4)+(Bz+c)(z+2) = z$$

Put $z=-2$ $\quad 8A+0=-2 \Rightarrow A=\frac{-2}{8} \Rightarrow \boxed{A=-\frac{1}{4}}$

Equating the coeff. of z^2: $A+B=0 \Rightarrow B=-A \Rightarrow \boxed{B=\frac{1}{4}}$

Equating the constant term: $4A+2C=0 \Rightarrow 2C=-4A \Rightarrow \boxed{C=\frac{1}{2}}$

$$\therefore \qquad \frac{F(z)}{z} = \frac{-\frac{1}{4}}{z+2} + \frac{\left(\frac{1}{4}z + \frac{1}{2}\right)}{z^2+4}$$

$$F(z) = -\frac{1}{4}\frac{z}{z+2} + \frac{1}{4}\frac{z^2}{z^2+4} + \frac{1}{2}\frac{z}{z^2+4}$$

$$\therefore \qquad Z^{-1}\left(F(z)\right) = Z^{-1}\left(-\frac{1}{4}\frac{z}{z+2} + \frac{1}{4}\frac{z^2}{z^2+4} + \frac{1}{2}\frac{z}{z^2+4}\right)$$

$$= -\frac{1}{4}Z^{-1}\left(\frac{z}{z+2}\right) + \frac{1}{4}Z^{-1}\left(\frac{z^2}{z^2+4}\right) + \frac{1}{4}Z^{-1}\left(\frac{2z}{z^2+4}\right)$$

$$f(n) = -\frac{1}{4}(-2)^n + \frac{1}{4}2^n \cos\left(\frac{n\pi}{2}\right) + \frac{1}{4}2^n \sin\left(\frac{n\pi}{2}\right)$$

☑ Example: 5

Find $Z^{-1}\left(\frac{z^3+3z}{(z-1)^2(z^2+1)}\right)$ using partial fraction method

SOLUTION:

Let $$F(z) = \frac{z^3+3z}{(z-1)^2(z^2+1)}$$

$$\therefore \qquad \frac{F(z)}{z} = \frac{z^2+3}{(z-1)^2(z^2+1)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{(Cz+D)}{z^2+1}$$

$$= \frac{A(z-1)(z^2+1) + B(z^2+1) + (Cz+D)(z-1)^2}{(z-1)^2(z^2+1)}$$

$$\Rightarrow \qquad A(z-1)(z^2+1) + B(z^2+1) + (Cz+D)(z-1)^2 = z^2+3$$

Put $z=1$: $\quad 0+2B+0=4 \;\Rightarrow\; \boxed{B=2}$

Equating the coeff. of z^3 :

$$A+C=0 \qquad (1)$$

Equating the coeff. of z^2 :

$$-A+B-2C+D=1$$

$$-A-2C+D=1-B=1-2=-1 \qquad (2)$$

Put $z=0$:

$$A(-1)(1)+B(1)+D(1)=3$$

$$-A+D=3-B=3-2=1 \quad (3)$$

(2) ⇒ $-A-2(-A)+D=-1$

$$A+D=-1$$

(3) ⇒ $-A+D=1$

$$2D=0$$

$$\boxed{D=0}$$

(3) ⇒ $-A+D=1 \Rightarrow -A=1 \Rightarrow \boxed{A=-1}$

(1) ⇒ $A+C=0 \Rightarrow C=-A=-(-1)=1 \Rightarrow \boxed{C=1}$

∴ $$\frac{F(z)}{z}=\frac{-1}{z-1}+\frac{2}{(z-1)^2}+\frac{(z+0)}{z^2+1}$$

$$F(z)=-\frac{z}{z-1}+\frac{2z}{(z-1)^2}+\frac{z^2}{z^2+1}$$

∴ $$Z^{-1}(F(z))=-Z^{-1}\left(\frac{z}{z-1}\right)+2Z^{-1}\left(\frac{z}{(z-1)^2}\right)+Z^{-1}\left(\frac{z^2}{z^2+1}\right)$$

$$=-(1)^n+2n+1^n\cos\left(\frac{n\pi}{2}\right)$$

$$f(n)=-1+2n+\cos\left(\frac{n\pi}{2}\right).$$

☑ Example: 6

Find $Z^{-1}\left(\frac{z^3-20z}{(z-2)^3(z-4)}\right)$

SOLUTION:

Let $F(z)=\frac{z^3-20z}{(z-2)^3(z-4)}$

$$\frac{F(z)}{z}=\frac{z^2-20}{(z-2)^3(z-4)}=\frac{A}{z-2}+\frac{B}{(z-2)^2}+\frac{C}{(z-2)^3}+\frac{D}{z-4}$$

$$=\frac{A(z-2)^2(z-4)+B(z-2)(z-4)+C(z-4)+D(z-2)^3}{(z-2)^3(z-4)}$$

$$\Rightarrow \quad A(z-2)^2(z-4)+B(z-2)(z-4)+C(z-4)+D(z-2)^3=z^2-20$$

Put $z=4$: $\quad D(8)=16-20 \quad \Rightarrow \quad \boxed{D=-\frac{1}{2}}$

Put $z=2$: $\quad C(-2)=4-20 \quad \Rightarrow \quad \boxed{C=8}$

Equating the coeff. of z^3 : $\quad A+D=0 \quad \Rightarrow \quad A=-D=-\left(-\frac{1}{2}\right) \quad \Rightarrow \quad \boxed{A=\frac{1}{2}}$

Put $z=0$: $\quad A(4)(-4)+B(-2)(-4)+C(-4)+D(-8)=-20$

$$-16A+8B-4C-8D=-20$$

$$-4A+2B-C-2D=-5$$

$$2B=-5+4A+C+2D$$

$$=-5+4\left(\frac{1}{2}\right)+8+2\left(-\frac{1}{2}\right)$$

$$=-5+2+8-1$$

$$2B=4$$

$$\boxed{B=2}$$

$$\therefore \quad \frac{F(z)}{z}=\frac{1/2}{z-2}+\frac{2}{(z-2)^2}+\frac{8}{(z-2)^3}+\frac{-1/2}{z-4}$$

$$F(z)=\frac{1}{2}\frac{z}{z-2}+\frac{2z}{(z-2)^2}+\frac{8z}{(z-2)^3}-\frac{1}{2}\frac{z}{z-4}$$

$$\therefore \quad Z^{-1}(F(z))=Z^{-1}\left(\frac{1}{2}\frac{z}{z-2}+\frac{2z}{(z-2)^2}+\frac{8z}{(z-2)^3}-\frac{1}{2}\frac{z}{z-4}\right)$$

$$=\frac{1}{2}Z^{-1}\left(\frac{z}{z-2}\right)+2Z^{-1}\left(\frac{z}{(z-2)^2}\right)+Z^{-1}\left(\frac{8z}{(z-2)^3}\right)-\frac{1}{2}Z^{-1}\left(\frac{z}{z-4}\right)$$

$$=\frac{1}{2}\left(2^n\right)+2\left(n2^{n-1}\right)+n(n-1)2^n-\frac{1}{2}\left(4^n\right)$$

$$f(n)=\frac{1}{2}\left(2^n\right)+n2^n+n(n-1)2^n-\frac{1}{2}\left(4^n\right)$$

5.11.4. EXERCISE:

(i) $\quad Z^{-1}\left[\frac{z-4}{(z+2)(z+3)}\right]$

ANSWER: $Z^{-1}\left[\frac{z-4}{(z+2)(z+3)}\right] = Z^{-1}\left[\frac{-6}{z+2}+\frac{7}{z+3}\right] = -6(-2)^{n-1}+7(-3)^{n-1}$

(ii) $Z^{-1}\left[\frac{z-4}{(z-1)(z-2)^2}\right]$

ANSWER: $Z^{-1}\left[\frac{z-4}{(z-1)(z-2)^2}\right] = Z^{-1}\left[\frac{-3}{z-1}-\frac{2}{(z-2)^2}+\frac{3}{z-2}\right]$

$$= -3(1^{n-1}) - (n-1)\left(2^{n-1}\right) + 3\left(2^{n-1}\right)$$

5.11.5. TYPE: II (METHOD OF RESIDUES) (CAUCHY'S RESIDUE THEOREM)

5.11.6. DEFINITION:

If $Z(f(n)) = F(z)$ then

$$f(n) = Z^{-1}\left(F(z)\right)$$

$$= \frac{1}{2\pi i}\int_c z^{n-1}F(z)\,dz$$, where c is the closed contour

which encloses all the poles of the integrand.

where $\int_c z^{n-1}F(z)\,dz = 2\pi i \begin{pmatrix}\text{sum of the residues of } z^{n-1}F(z) \\ \text{at each of its poles}\end{pmatrix}$

5.11.7. NOTE:

(ii) If $z = a$ is a simple pole of $f(z)$ then residue at $z = a$ is

$$\lim_{z\to a}(z-a)f(z)$$

(ii) If $z = a$ is a pole of order m of $f(z)$ then residue at $z = a$ is

$$\frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}(z-a)^m\, f(z).$$

5.11.8. EXAMPLES:

☑ Example: 1

Find inverse Z-transform of $\frac{z}{(z-1)(z-2)}$ using residue theorem.

SOLUTION:

Let $Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right]=f(n)$

Then

$$f(n)=\frac{1}{2\pi i}\int_c z^{n-1}F(z)\,dz=\frac{1}{2\pi i}\int_c z^{n-1}\,\frac{z}{(z-1)(z-2)}\,dz$$

$$=\frac{1}{2\pi i}\int_c \frac{z^n}{(z-1)(z-2)}\,dz \qquad (1)$$

To find $\int_c \frac{z^n}{(z-1)(z-2)}\,dz$: $\int_c \frac{z^n}{(z-1)(z-2)}\,dz=2\pi i\begin{pmatrix}\text{sum of the residues of } \phi(z)\\ \text{at each of its poles}\end{pmatrix}$

where $\phi(z)=\frac{z^n}{(z-1)(z-2)}$.

The poles are $z=1, z=2$.

Residue at $z=1$:

$$\text{Res. }\left[\phi(z)\right]_{z=1}=\lim_{z\to1}(z-1)\,\phi(z)=\lim_{z\to1}(z-1)\frac{z^n}{(z-1)(z-2)}=\lim_{z\to1}\frac{z^n}{(z-2)}$$

$$=\frac{1^n}{-1}=-1.$$

Residue at $z=2$:

$$\text{Res. }\left[\phi(z)\right]_{z=2}=\lim_{z\to2}(z-2)\,\phi(z)=\lim_{z\to2}(z-2)\frac{z^n}{(z-1)(z-2)}=\lim_{z\to2}\frac{z^n}{(z-1)}=\frac{2^n}{1}=2^n.$$

$$\therefore \qquad \int_c \frac{z^n}{(z-1)(z-2)}\,dz=2\pi i\left[-1+2^n\right] \qquad (2)$$

Sub. (2) in (1) we get,

$$f(n)=\frac{1}{2\pi i}2\pi i\left[-1+2^n\right]$$

$$f(n)=2^n-1.$$

☑ Example: 2

Find $Z^{-1}\left[\dfrac{z(z+1)}{(z-1)^3}\right]$

SOLUTION:

Let $Z^{-1}\left[\dfrac{z(z+1)}{(z-1)^3}\right] = f(n)$

Then $f(n) = \dfrac{1}{2\pi i}\int_c z^{n-1}F(z)\,dz = \dfrac{1}{2\pi i}\int_c z^{n-1}\,\dfrac{z(z+1)}{(z-1)^3}\,dz$

$$= \frac{1}{2\pi i}\int_c \frac{z^n(z+1)}{(z-1)^3}\,dz \qquad (1)$$

To find $\int_c \dfrac{z^n(z+1)}{(z-1)^3}\,dz$:

$$\int_c \frac{z^n(z+1)}{(z-1)^3}\,dz = 2\pi i\begin{pmatrix}\text{sum of the residues of } \phi(z)\\ \text{at each of its poles}\end{pmatrix} \quad \text{where } \phi(z) = \frac{z^n(z+1)}{(z-1)^3}.$$

Here $z=1$ is a pole of order 3.

$$\text{Residue } [\phi(z)]_{z=1} = \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}(z-1)^3\,\frac{z^n(z+1)}{(z-1)^3} = \frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}\left(z^n(z+1)\right)$$

$$= \frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}\left(z^{n+1}+z^n\right) = \frac{1}{2}\lim_{z\to 1}\frac{d}{dz}\left((n+1)z^n + nz^{n-1}\right)$$

$$= \frac{1}{2}\lim_{z\to 1}\left((n+1)\,n\,z^{n-1} + n\,(n-1)\,z^{n-2}\right)$$

$$= \frac{1}{2}\left((n+1)\,n + n\,(n-1)\right) = \frac{1}{2}\left(n^2 + \cancel{n} + n^2 - \cancel{n}\right)$$

$$= \frac{1}{2}\left(2n^2\right) = n^2$$

$\therefore$
$$\int_c \frac{z^n(z+1)}{(z-1)^3}\,dz = 2\pi i\left(n^2\right) \qquad (2)$$

$\therefore$ (1) $\Rightarrow$ $f(n) = \dfrac{1}{2\pi i}\,2\pi i\,(n^2) = n^2.$

☑ Example: 3

Find $Z^{-1}\left(\frac{z^2}{(z-a)(z-b)}\right)$

SOLUTION:

Let $Z^{-1}\left(\frac{z^2}{(z-a)(z-b)}\right) = f(n)$

Then
$$f(n) = \frac{1}{2\pi i}\int_c z^{n-1} F(z)\, dz = \frac{1}{2\pi i}\int_c z^{n-1} \frac{z^2}{(z-a)(z-b)}\, dz$$
$$= \frac{1}{2\pi i}\int_c \frac{z^{n+1}}{(z-a)(z-b)}\, dz$$

To find $\int_c \frac{z^{n+1}}{(z-a)(z-b)}\, dz$:

$$\int_c \frac{z^{n+1}}{(z-a)(z-b)}\, dz = 2\pi i[\text{sum of the residues of } \phi(z) \text{ at each of its poles}]$$

where $\phi(z) = \frac{z^{n+1}}{(z-a)(z-b)}$

Here $z = a,\ z = b$ are simple poles.

Residue at $z = a$:

$$\text{Res.}[\phi(z)]_{z=a} = \lim_{z \to a}(z-a)\phi(z) = \lim_{z \to a}(z-a)\frac{z^{n+1}}{(z-a)(z-b)} = \frac{a^{n+1}}{a-b}$$

Residue at $z = b$:

$$\text{Res.}[\phi(z)]_{z=b} = \lim_{z \to b}(z-b)\phi(z) = \lim_{z \to b}(z-b)\frac{z^{n+1}}{(z-a)(z-b)} = \frac{b^{n+1}}{b-a}$$

$$\therefore \quad \int_c \frac{z^{n+1}}{(z-a)(z-b)}\, dz = 2\pi i\left[\frac{a^{n+1}}{a-b} + \frac{b^{n+1}}{b-a}\right]$$

$$\therefore \quad f(n) = \frac{1}{2\pi i}\, 2\pi i\left(\frac{a^{n+1}}{a-b} + \frac{b^{n+1}}{b-a}\right)$$

$$f(n) = \frac{1}{a-b}\left(a^{n+1} - b^{n+1}\right)$$

☑ Example: 4

Find $Z^{-1}\left(\frac{z\left(z^2-z+2\right)}{(z+1)(z-1)^2}\right)$ by using residue theorem

SOLUTION:

Let $$f(n)=Z^{-1}\left(\frac{z\left(z^2-z+2\right)}{(z+1)(z-1)^2}\right)$$

Then $$f(n)=\frac{1}{2\pi i}\int_c z^{n-1}\frac{z\left(z^2-z+2\right)}{(z+1)(z-1)^2}dz=\frac{1}{2\pi i}\int_c \frac{z^n\left(z^2-z+2\right)}{(z+1)(z-1)^2}\,dz$$

To find $\int_c \frac{z^n\left(z^2-z+2\right)}{(z+1)(z-1)^2}\,dz$:

$$\int_c \frac{z^n\left(z^2-z+2\right)}{(z+1)(z-1)^2}\,dz=2\pi i\left[\text{sum of the residues of } \phi(z) \text{ at each of its poles}\right]$$

where $\phi(z)=\frac{z^n\left(z^2-z+2\right)}{(z+1)(z-1)^2}$

Here $z=-1$ is a simple pole and $z=1$ is a pole of order 2.

Residue at $z=-1$:

$$\text{Res.}\left[\phi(z)\right]_{z=-1}=\lim_{z\to-1}(z+1)\phi(z)=\lim_{z\to-1}\cancel{(z+1)}\frac{z^n\left(z^2-z+2\right)}{\cancel{(z+1)}(z-1)^2}$$

$$=\frac{(-1)^n(1+1+2)}{(-1-1)^2}=(-1)^n.$$

Residue at $z=1$:

$$\text{Res.}\left[\phi(z)\right]_{z=1}=\frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}(z-a)^m\,\phi(z)$$

$$=\frac{1}{1!}\lim_{z\to1}\frac{d}{dz}\cancel{(z-1)^2}\frac{z^n\left(z^2-z+2\right)}{(z+1)\cancel{(z-1)^2}}$$

$$=\lim_{z\to1}\frac{d}{dz}\left[\frac{z^{n+2}-z^{n+1}+2z^n}{z+1}\right]$$

$$= \lim_{z \to 1} \left[\frac{(z+1)\left[(n+2)z^{n+1} - (n+1)z^n + 2nz^{n-1}\right] - \left(z^{n+2} - z^{n+1} + 2z^n\right)(1)}{(z+1)^2} \right]$$

$$= \left[\frac{2\left[(n+2)-(n+1)+2n\right]-(1-1+2)}{4} \right]$$

$$= \left[\frac{2\left[\cancel{n}+2-\cancel{n}-1+2n\right]-2}{4} \right] = \left(\frac{2[1+2n]-2}{4} \right)$$

$$= 2\left(\frac{\cancel{1}+2n-\cancel{1}}{4} \right) = n$$

$$\therefore \quad \int_c \frac{z^n\left(z^2-z+2\right)}{(z+1)(z-1)^2}\, dz = 2\pi i\left[(-1)^n + n\right]$$

$$\therefore \quad f(n) = \frac{1}{\cancel{2\pi i}}\, \cancel{2\pi i}\left[(-1)^n + n\right]$$

$$= (-1)^n + n\,.$$

☑ Example: 5

Find $Z^{-1}\left(\dfrac{2z^2+4z}{(z-2)^3}\right)$ by using residue theorem

SOLUTION:

Let $\quad f(n) = Z^{-1}\left(\dfrac{2z^2+4z}{(z-2)^3}\right)$

Then $\quad f(n) = \dfrac{1}{2\pi i}\displaystyle\int_c z^{n-1}\, \dfrac{2z^2+4z}{(z-2)^3}\, dz = \dfrac{1}{2\pi i}\displaystyle\int_c \dfrac{2z^n(z+2)}{(z-2)^3}\, dz$

To find $\displaystyle\int_c \dfrac{2z^n(z+2)}{(z-2)^3}\, dz$:

$$\int_c \frac{2z^n(z+2)}{(z-2)^3}\, dz = 2\pi i\left[\text{sum of the residue of } \phi(z) \text{ at each of its poles}\right]$$

where $\phi(z) = \dfrac{2z^n(z+2)}{(z-2)^3}$

Here $z = 2$ is a pole of order 3.

Residue at $z = 2$:

$$\text{Res.}\left[\phi(z)\right]_{z=2} = \frac{1}{(m-1)!}\lim_{z\to 2}\frac{d^{m-1}}{dz^{m-1}}(z-2)^m\,\phi(z)$$

$$= \frac{1}{2!}\lim_{z\to 2}\frac{d^2}{dz^2}(z-2)^3\,\frac{2z^n(z+2)}{(z-2)^3}$$

$$= \lim_{z\to 2}\frac{d^2}{dz^2}\left[z^{n+1}+2z^n\right] = \lim_{z\to 2}\frac{d}{dz}\left[(n+1)z^n+2nz^{n-1}\right]$$

$$= \lim_{z\to 2}\left[n(n+1)z^{n-1}+2n(n-1)z^{n-2}\right]$$

$$= n(n+1)2^{n-1}+2n(n-1)2^{n-2}$$

$$= n(n+1)2^{n-1}+n(n-1)2^{n-1} = 2^{n-1}\left(n^2+n+n^2-n\right)$$

$$= 2^{n-1}\left(2n^2\right) = n^2 2^n$$

$$\therefore \quad \int_c \frac{2z^n(z+2)}{(z-2)^3}\,dz = 2\pi i\left[2^n\ n^2\right]$$

$$\therefore \quad f(n) = \frac{1}{2\pi i}\,2\pi i\left[2^n\ n^2\right] = 2^n\ n^2.$$

5.11.9. DEFINITION: (CONVOLUTION)

The convolution of two sequence $\{f(n)\}$ and $\{g(n)\}$ is defined as

$$\{f(n) * g(n)\} = \sum_{r=0}^{n} f(r)\,g(n-r).$$

The convolution of two functions $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = \sum_{r=0}^{n} f(rT)\,g(n-r)T, \quad \text{where } T \text{ is the sampling period.}$$

5.11.10. CONVOLUTION THEOREM:

(i) $Z\left(f(n) * g(n)\right) = F(z).\, G(z)$

where $Z\left(f(n)\right) = F(z)$ and $Z\left(g(n)\right) = G(z)$

(ii) $Z\left(f(t) * g(t)\right) = F(z).\, G(z)$

where $Z\left(f(t)\right) = F(z)$ and $Z\left(g(t)\right) = G(z)$.

PROOF:

(i) $F(z) = z\left(f(n)\right) = \sum_{n=0}^{\infty} f(n)\, z^{-n} \qquad G(z) = z\left(g(n)\right) = \sum_{n=0}^{\infty} g(n)\, z^{-n}$

$$\therefore\ F(z).\, G(z) = \sum_{n=0}^{\infty} f(n)\, z^{-n} \sum_{n=0}^{\infty} g(n)\, z^{-n}$$

$$= \left(f(0) + f(1)\, z^{-1} + f(2)\, z^{-2} + \cdots f(n)\, z^{-n} + \cdots\right)$$

$$\left(g(0) + g(1)\, z^{-1} + g(2)\, z^{-2} + \cdots g(n)\, z^{-n} + \cdots\right)$$

$$= \left(f(0)\, g(0) + \left[f(0)\, g(1) + f(1) g(0)\right] z^{-1} + \cdots + \left[\sum_{r=0}^{n} f(r) g(n-r)\right] z^{-n} + \cdots \right)$$

$$= \sum_{n=0}^{\infty} \left[\sum_{r=0}^{n} f(r) g(n-r)\right] z^{-n} = \sum_{n=0}^{\infty} \left[f(n) * g(n)\right] z^{-n} = Z\left(f(n) * g(n)\right)$$

(i.e.) $Z\left(f(n) * g(n)\right) = F(z)\, G(z)$.

(ii) $F(z) = z\left(f(t)\right) = \sum_{n=0}^{\infty} f(nT)\, z^{-n} \qquad G(z) = z\left(g(t)\right) = \sum_{n=0}^{\infty} g(nT)\, z^{-n}$

$$\therefore \qquad F(z).\, G(z) = \sum_{n=0}^{\infty} f(nT)\, z^{-n} \sum_{n=0}^{\infty} g(nT)\, z^{-n}$$

$$= \left(f(0T) + f(1T)\, z^{-1} + f(2T)\, z^{-2} + \cdots f(nT)\, z^{-n} + \cdots\right)$$

$$\left(g(0T) + g(1T)\, z^{-1} + g(2T)\, z^{-2} + \cdots g(nT)\, z^{-n} + \cdots\right)$$

$$= \Big(f(0T)\, g(0T) + \left[f(0T)\, g(1T) + f(1T) g(0T)\right] z^{-1} +$$

$$\cdots + \left[\sum_{r=0}^{n} f(rT) g(n-r)T\right] z^{-n} + \cdots \Big)$$

$$= \sum_{n=0}^{\infty} \left[\sum_{r=0}^{n} f(rT) g(n-r)T\right] z^{-n} = \sum_{n=0}^{\infty} \left[f(t) * g(t)\right] z^{-n}$$

$$= Z\left(f(t) * g(t)\right)$$

(i.e.) $Z\left(f(t) * g(t)\right) = F(z)\, G(z)$.

5.11.11. NOTE:

(i) $Z\left(f(n) * g(n)\right) = F(z)\, G(z) \quad \Rightarrow \quad Z^{-1}\left(F(z)\, G(z)\right) = f(n) * g(n)$

(ii) $Z\left(f(t) * g(t)\right) = F(z)\, G(z) \quad \Rightarrow \quad Z^{-1}\left(F(z)\, G(z)\right) = f(t) * g(t)$.

5.11.12. TYPE III: CONVOLUTION METHOD:

☑ Example: 1

Using convolution theorem evaluate $Z^{-1}\left[\dfrac{z^2}{(z-1)(z-3)}\right]$.

SOLUTION:

$$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = Z^{-1}\left[\frac{z}{z-1}\cdot\frac{z}{z-3}\right]$$

$$= Z^{-1}\left[\frac{z}{z-1}\right] * Z^{-1}\left[\frac{z}{z-3}\right] = 1^n * 3^n$$

$$= \sum_{r=0}^{n} 1^r\, 3^{n-r} = 3^n + 3^{n-1} + 3^{n-2} + \cdots + 3^1 + 1$$

$$= 1 + 3 + 3^2 + \cdots + 3^n$$

$$= \frac{3^{n+1}-1}{3-1} \qquad \left(\because\ 1 + a + a^2 + \cdots + a^n = \frac{a^{n+1}-1}{a-1}\right)$$

$$= \frac{3^{n+1}-1}{2}.$$

☑ Example: 2

Using convolution theorem evaluate $Z^{-1}\left(\dfrac{z^2}{(z-4)(z-3)}\right)$

SOLUTION:

$$Z^{-1}\left(\frac{z^2}{(z-4)(z-3)}\right) = Z^{-1}\left(\frac{z}{(z-4)}\cdot\frac{z}{(z-3)}\right) = Z^{-1}\left(\frac{z}{z-4}\right) * Z^{-1}\left(\frac{z}{z-3}\right)$$

$$= 4^n * 3^n = \sum_{r=0}^{n} 4^r \; 3^{n-r} = 3^n \sum_{r=0}^{n} 4^r \; 3^{-r} = 3^n \sum_{r=0}^{n} \left(\frac{4}{3}\right)^r$$

$$= 3^n \left(1 + \left(\frac{4}{3}\right) + \left(\frac{4}{3}\right)^2 + \cdots + \left(\frac{4}{3}\right)^n\right)$$

$$= 3^n \left(\frac{\left(\frac{4}{3}\right)^{n+1} - 1}{\frac{4}{3} - 1}\right) = 3^n \left(\frac{\frac{4^{n+1} - 3^{n+1}}{3^{n+1}}}{\frac{1}{3}}\right)$$

$$= \cancel{3^{n+1}} \left(\frac{4^{n+1} - 3^{n+1}}{\cancel{3^{n+1}}}\right) = 4^{n+1} - 3^{n+1}.$$

☑ Example: 3

Using convolution theorem evaluate $Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]$

SOLUTION:

$$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = Z^{-1}\left[\frac{z}{z-a} \cdot \frac{z}{z-a}\right] = Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-a}\right]$$

$$= a^n * a^n = \sum_{r=0}^{n} a^r \; a^{n-r} = a^n + a\, a^{n-1} + a^2 \; a^{n-2} + \cdots$$

$$= (n+1)\, a^n.$$

☑ Example: 4

Using convolution theorem evaluate $Z^{-1}\left(\frac{z^2}{(z+a)(z+b)}\right)$

SOLUTION:

$$Z^{-1}\left(\frac{z^2}{(z+a)(z+b)}\right) = Z^{-1}\left(\frac{z}{z+a} \cdot \frac{z}{z+b}\right) = Z^{-1}\left(\frac{z}{z+a}\right) * Z^{-1}\left(\frac{z}{z+b}\right)$$

$$= (-a)^n * (-b)^n = \sum_{r=0}^{n} (-a)^r \, (-b)^{n-r}$$

$$= (-b)^n \sum_{r=0}^{n} (-a)^r (-b)^{-r} = (-b)^n \sum_{r=0}^{n} \left(\frac{a}{b}\right)^r$$

$$= (-b)^n \left(1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \cdots + \left(\frac{a}{b}\right)^n\right)$$

$$= (-b)^n \left(\frac{\left(\frac{a}{b}\right)^{n+1} - 1}{\left(\frac{a}{b}\right) - 1}\right) = (-b)^n \left(\frac{a^{n+1} - b^{n+1}}{b^{n+1}\left(\frac{a-b}{b}\right)}\right)$$

$$= (-1)^n \cancel{(b)^{n+1}} \left(\frac{a^{n+1} - b^{n+1}}{\cancel{b^{n+1}}(a-b)}\right) = (-1)^n \left(\frac{a^{n+1} - b^{n+1}}{a-b}\right)$$

☑ Example: 5

Using convolution theorem evaluate $Z^{-1}\left(\frac{z^2}{(z+a)^2}\right)$

SOLUTION:

$$Z^{-1}\left(\frac{z^2}{(z+a)^2}\right) = Z^{-1}\left(\frac{z}{z+a} \cdot \frac{z}{z+a}\right) = Z^{-1}\left(\frac{z}{z+a}\right) * Z^{-1}\left(\frac{z}{z+a}\right)$$

$$= (-a)^n * (-a)^n = \sum_{r=0}^{n} (-a)^r (-a)^{n-r}$$

$$= (-a)^n \sum_{r=0}^{n} (-a)^r (-a)^{-r} = (-a)^n \sum_{r=0}^{n} 1^r$$

$$= (-a)^n (n+1)$$

☑ Example: 6

Using convolution theorem evaluate $Z^{-1}\left(\frac{8z^2}{(2z-1)(4z-1)}\right)$

SOLUTION:

$$Z^{-1}\left(\frac{8z^2}{(2z-1)(4z-1)}\right) = Z^{-1}\left(\frac{\cancel{8}z^2}{\cancel{2}\left(z-\frac{1}{2}\right)\cancel{4}\left(z-\frac{1}{4}\right)}\right) = Z^{-1}\left(\frac{z}{z-\frac{1}{4}} \cdot \frac{z}{z-\frac{1}{2}}\right)$$

$$= Z^{-1}\left(\frac{z}{z-\frac{1}{4}}\right) * Z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right)$$

$$= \left(\frac{1}{4}\right)^n * \left(\frac{1}{2}\right)^n = \sum_{r=0}^{n}\left(\frac{1}{4}\right)^r\left(\frac{1}{2}\right)^{n-r}$$

$$= \left(\frac{1}{2}\right)^n \sum_{r=0}^{n}\left(\frac{1}{2}\right)^{2r}\left(\frac{1}{2}\right)^{-r} = \left(\frac{1}{2}\right)^n \sum_{r=0}^{n}\left(\frac{1}{2}\right)^r$$

$$= \left(\frac{1}{2}\right)^n\left(1+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2+\cdots+\left(\frac{1}{2}\right)^n\right)$$

$$= \left(\frac{1}{2}\right)^n\left(\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}}\right) = \left(\frac{1}{2}\right)^n\left(\frac{1-\left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}}\right)$$

$$= \left(\frac{1}{2}\right)^{n-1}\left(1-\left(\frac{1}{2}\right)^{n+1}\right)$$

☑ Example: 7

Using convolution theorem evaluate $Z^{-1}\left(\frac{12z^2}{(3z-1)(4z+1)}\right)$

SOLUTION:

$$Z^{-1}\left(\frac{12z^2}{(3z-1)(4z+1)}\right) = Z^{-1}\left(\frac{12z^2}{3\left(z-\frac{1}{3}\right)4\left(z+\frac{1}{4}\right)}\right) = Z^{-1}\left(\frac{z}{z-\frac{1}{3}}\cdot\frac{z}{z+\frac{1}{4}}\right)$$

$$= \left(\frac{1}{3}\right)^n * \left(-\frac{1}{4}\right)^n = \sum_{r=0}^{n}\left(\frac{1}{3}\right)^r\left(-\frac{1}{4}\right)^{n-r}$$

$$= \left(-\frac{1}{4}\right)^n \sum_{r=0}^{n}\left(\frac{1}{3}\right)^r(-4)^r = \left(-\frac{1}{4}\right)^n \sum_{r=0}^{n}\left(-\frac{4}{3}\right)^r$$

$$= \left(-\frac{1}{4}\right)^n\left(1+\left(-\frac{4}{3}\right)+\left(-\frac{4}{3}\right)^2+\cdots+\left(-\frac{4}{3}\right)^n\right)$$

$$= \left(-\frac{1}{4}\right)^n \left(\frac{1-\left(-\frac{4}{3}\right)^{n+1}}{1-\left(-\frac{4}{3}\right)}\right) = \left(-\frac{1}{4}\right)^n \left(\frac{1-\left(-\frac{4}{3}\right)^{n+1}}{\frac{7}{3}}\right)$$

$$= \left(\frac{3}{7}\right)\left(-\frac{1}{4}\right)^n \left(1-\left(-\frac{4}{3}\right)^{n+1}\right).$$

☑ Example: 8

Using convolution theorem evaluate $Z^{-1}\left(\frac{z}{z-4}\right)^3$

SOLUTION:

$$Z^{-1}\left(\frac{z}{z-4}\right)^3 = Z^{-1}\left(\frac{z}{z-4}\cdot\frac{z^2}{(z-4)^2}\right) = 4^n * (n+1)\ 4^n$$

$$= (n+1)\left(4^n * 4^n\right) = \sum_{r=0}^{n}(r+1)\ 4^r\ (4)^{n-r} = 4^n\sum_{r=0}^{n}(r+1)$$

$$= 4^n\left[1+2+3+\cdots+(n+1)\right] = 4^n\,\frac{(n+1)(n+2)}{2}.$$

5.12. FORMATION OF DIFFERENCE EQUATION:

☑ Example: 1

Form the difference equation from $y_n = a + b3^n$

SOLUTION:

Given $\quad y_n = a + b3^n \qquad (1)$

$$y_{n+1} = a + b3^{n+1} = a + 3b\ 3^n \qquad (2)$$

$$y_{n+2} = a + b3^{n+2} = a + 9b\ 3^n \qquad (3)$$

Eliminating a and b from (1), (2) and (3), we get

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 1 & 3 \\ y_{n+2} & 1 & 9 \end{vmatrix} = 0$$

$$y_n(9-3) - 1(9y_{n+1} - 3y_{n+2}) + (y_{n+1} - y_{n+2}) = 0$$

$$6y_n - 9y_{n+1} + 3y_{n+2} + y_{n+1} - y_{n+2} = 0$$

$$2y_{n+2} - 8y_{n+1} + 6y_n = 0.$$

☑ Example: 2

Derive the difference equation from $y_n = (A + Bn)2^n$.

SOLUTION:

Given $\quad y_n = A2^n + Bn2^n \qquad (1)$

$$y_{n+1} = A2^{n+1} + B(n+1)2^{n+1} = 2A\,2^n + 2B(n+1)\,2^n \qquad (2)$$

$$y_{n+2} = A2^{n+2} + B(n+2)2^{n+2} = 4A\,2^n + 4B(n+2)\,2^n \qquad (3)$$

Eliminating A and B we get,

$$\begin{vmatrix} y_n & 1 & n \\ y_{n+1} & 2 & 2(n+1) \\ y_{n+2} & 4 & 4(n+2) \end{vmatrix} = 0$$

$$y_n\left[8(n+2) - 8(n+1)\right] - 1\left[4(n+2)y_{n+1} - 2(n+1)y_{n+2}\right]$$
$$+ n\left[4y_{n+1} - 2y_{n+2}\right] = 0$$

$$y_n\left[\cancel{8n} + 16 - \cancel{8n} - 8\right] - \left[(4n+8)y_{n+1} - (2n+2)y_{n+2}\right] + 4ny_{n+1} - 2ny_{n+2} = 0$$

$$8y_n - (4n+8)y_{n+1} + (2n+2)y_{n+2} + 4ny_{n+1} - 2ny_{n+2} = 0$$

$$y_{n+2}\left[\cancel{2n} + 2 - \cancel{2n}\right] - y_{n+1}\left[\cancel{4n} + 8 - \cancel{4n}\right] + 8y_n = 0$$

$$2y_{n+2} - 8y_{n+1} + 8y_n = 0$$

$$y_{n+2} - 4y_{n+1} + 4y_n = 0.$$

☑ Example: 3

From $y_n = a\,2^n + b(-2)^n$, derive a difference equation by eliminating the constants.

SOLUTION:

Given $\quad y_n = a\,2^n + b(-2)^n \qquad (1)$

$$y_{n+1} = a\,2^{n+1} + b(-2)^{n+1} = 2a\,2^n - 2b(-2)^n \qquad (2)$$

$$y_{n+2} = a\,2^{n+2} + b(-2)^{n+2} = 4a\,2^n + 4b(-2)^n \qquad (3)$$

Eliminating a and b, we get

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 2 & -2 \\ y_{n+2} & 4 & 4 \end{vmatrix} = 0$$

$$y_n[8+8] - 1[4y_{n+1} + 2y_{n+2}] + 1[4y_{n+1} - 2y_{n+2}] = 0$$

$$16y_n - \cancel{4y_{n+1}} - 2y_{n+2} + \cancel{4y_{n+1}} - 2y_{n+2} = 0$$

$$-4y_{n+2} + 16y_n = 0$$

$$y_{n+2} - 4y_n = 0.$$

☑ Example: 4

Form a difference equation by eliminating the arbitrary constant A from $y_n = A3^n$

SOLUTION:

Given $\quad y_n = A3^n \qquad (1)$

$$y_{n+1} = A3^{n+1} = A3.3^n \qquad (2)$$

Eliminating A from (1) and (2), we get

$$\begin{vmatrix} y_n & 1 \\ y_{n+1} & 3 \end{vmatrix} = 0$$

$$3y_n - y_{n+1} = 0$$

$$y_{n+1} - 3y_n = 0.$$

☑ Example: 5

Form a difference equation by eliminating arbitrary constant from $U_n = a2^{n+1}$.

SOLUTION:

Given $\quad U_n = a2^{n+1} \qquad (1)$

$$U_{n+1} = a2^{n+2} = 2a2^{n+1} \qquad (2)$$

Eliminating a from (1) and (2), we get,

$$\begin{vmatrix} U_n & 1 \\ U_{n+1} & 2 \end{vmatrix} = 0$$

$$2U_n - U_{n+1} = 0$$

$$U_{n+1} - 2U_n = 0.$$

5.13. SOLUTION OF DIFFERENCE EQUATIONS USING *Z*-TRANSFORMS:

5.13.1. FORMULA:

$$Z\left(y(k)\right) = F(z)$$

$$Z\left(y(k+1)\right) = zF(z) - zy(0)$$

$$Z\left(y(k+2)\right) = z^2F(z) - z^2y(0) - zy(1)$$

$$Z\left(y(k+3)\right) = z^3F(z) - z^3y(0) - z^2y(1) - zy(2)$$

5.13.2. EXAMPLES:

☑ Example: 1

Solve the difference equation, $y(k+2) - 4\,y(k+1) + 4\,y(k) = 0$ where $y(0) = 1$, $y(1) = 0$

SOLUTION:

Given $y(k+2) - 4\,y(k+1) + 4\,y(k) = 0$

Taking Z-transform on both sides,

$$Z\left[y(k+2) - 4\,y(k+1) + 4\,y(k)\right] = Z(0)$$

$$Z\left[y(k+2)\right] - 4\,Z\left[y(k+1)\right] + 4\,Z\left[\,y(k)\right] = Z(0)$$

$$\left[z^2F(z) - z^2y(0) - zy(1)\right] - 4\left[zF(z) - zy(0)\right] + 4\,F(z) = 0$$

$$\left[z^2F(z) - z^2\,(1) - z\,(0)\right] - 4\left[zF(z) - z\,(1)\right] + 4\,F(z) = 0$$

$$F(z)\left[z^2 - 4z + 4\right] = z^2 - 4z$$

$$F(z) = \frac{z^2 - 4z}{z^2 - 4z + 4}$$

$$Z(y(k)) = \frac{z^2 - 4z}{z^2 - 4z + 4}$$

$$\Rightarrow \qquad y(k) = Z^{-1}\left(\frac{z^2 - 4z}{z^2 - 4z + 4}\right)$$

$$= Z^{-1}\left(\frac{z\,(z-4)}{(z-2)^2}\right)$$

To find $Z^{-1}\left(\frac{z\,(z-4)}{(z-2)^2}\right)$

Let $\quad F(z) = \frac{z\,(z-4)}{(z-2)^2}$

$$\therefore \quad \frac{F(z)}{z} = \frac{z-4}{(z-2)^2} = \frac{A}{z-2} + \frac{B}{(z-2)^2} = \frac{A(z-2)+B}{(z-2)^2}$$

$\Rightarrow \quad A(z-2)+B = z-4$

put $z=2$: $\quad A(0)+B = 2-4 \Rightarrow B=-2$

Equating the constant term: $\quad -2A+B=-4 \;\Rightarrow\; -2A=-4-B \;\Rightarrow\; A=1$

$$\therefore \quad \frac{F(z)}{z} = \frac{1}{z-2} - \frac{2}{(z-2)^2}$$

$$\Rightarrow \quad F(z) = \frac{z}{z-2} - \frac{2z}{(z-2)^2}$$

Taking Z^{-1} on both sides,

$$Z^{-1}\left(F(z)\right) = Z^{-1}\left[\frac{z}{z-2}\right] - Z^{-1}\left[\frac{2z}{(z-2)^2}\right] = 2^k - k2^k \qquad \left(\because\; Z^{-1}\left[\frac{az}{(z-a)^2}\right] = k\,a^k\right)$$

$$y(k) = 2^k\,(1-k).$$

☑ Example: 2

Solve $y(n+2) - 3\,y(n+1) + 2\,y(n) = 2^n$, given that $y(0)=0,\, y(1)=0$.

SOLUTION:

Given $\quad y(n+2) - 3\,y(n+1) + 2\,y(n) = 2^n$

Taking Z-transform on both sides,

$$Z\left[y(n+2) - 3\,y(n+1) + 2\,y(n)\right] = Z(2^n)$$

$$Z\left[y(n+2)\right] - 3\,Z\left[y(n+1)\right] + 2\,Z\left[\,y(n)\right] = Z(2^n)$$

$$\left[z^2F(z) - z^2\;y(0) - z\;y(1)\right] - 3\left[z\;F(z) - z\;y(0)\right] + 2\;F(z) = \frac{z}{z-2}$$

$$\left[z^2 F(z) - z^2\,(0) - z\,(0)\right] - 3\left[z\,F(z) - z\,(0)\right] + 2\,F(z) = \frac{z}{z-2}$$

$$F(z)\left[z^2 - 3z + 2\right] = \frac{z}{z-2}$$

$$F(z) = \frac{z}{(z-2)\left(z^2 - 3z + 2\right)}$$

$$Z(y(n)) = \frac{z}{(z-2)(z-2)(z-1)}$$

$$Z(y(n)) = \frac{z}{(z-2)^2 (z-1)}$$

$\Rightarrow$ $$y(n) = Z^{-1}\left(\frac{z}{(z-2)^2 (z-1)}\right)$$

To find $Z^{-1}\left(\frac{z}{(z-2)^2 (z-1)}\right)$

Let $\quad F(z) = \dfrac{z}{(z-2)^2 (z-1)}$

$\therefore \quad \dfrac{F(z)}{z} = \dfrac{1}{(z-1)(z-2)^2} = \dfrac{A}{z-1} + \dfrac{B}{z-2} + \dfrac{C}{(z-2)^2}$

$$= \frac{A(z-2)^2 + B(z-1)(z-2) + C(z-1)}{(z-1)(z-2)^2}$$

$\Rightarrow \quad A(z-2)^2 + B(z-1)(z-2) + C(z-1) = 1$

put $z = 2$: $\quad \boxed{C = 1}$

put $z = 1$: $\quad \boxed{A = 1}$

Equating the coeff. of z^2: $\quad A + B = 0 \Rightarrow \boxed{B = -1}$

$\therefore \quad \dfrac{F(z)}{z} = \dfrac{1}{z-1} + \dfrac{-1}{z-2} + \dfrac{1}{(z-2)^2} \quad \Rightarrow \quad F(z) = \dfrac{z}{z-1} - \dfrac{z}{z-2} + \dfrac{z}{(z-2)^2}$

Taking Z^{-1} on both sides,

$$Z^{-1}\left(F(z)\right) = Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z}{z-2}\right] + Z^{-1}\left[\frac{z}{(z-2)^2}\right]$$

$$y(n) = 1^n - 2^n + n2^{n-1} \qquad \left(\because \ Z^{-1}\left[\frac{z}{z-a}\right] = a^n, Z^{-1}\left[\frac{z}{(z-a)^2}\right] = n\,a^{n-1} \right)$$

☑ Example: 3

Solve the difference equation $y(n)+3\,y(n-1)-4\,y(n-2)=0,\ n\geq 2$, given that $y(0)=3,\ y(1)=-2$.

SOLUTION:

Changing n into $n+2$, then given equation becomes

$$y(n+2)+3\,y(n+1)-4\,y(n)=0,\ n\geq 0$$

Taking Z-transform on both sides,

$$Z\left[y(n+2)+3\,y(n+1)-4\,y(n)\right]=Z(0)$$

$$Z\left[y(n+2)\right]+3\,Z\left[y(n+1)\right]-4\,Z\left[\,y(n)\right]=Z(0)$$

$$\left[z^2F(z)-z^2\ y(0)-z\ y(1)\right]+3\left[z\ F(z)-z\ y(0)\right]-4\ F(z)=0$$

$$\left[z^2F(z)-z^2\ (3)-z\ (-2)\right]+3\left[z\ F(z)-z\ (3)\right]-4\ F(z)=0$$

$$F(z)\left[z^2+3z-4\right]=3z^2-2z+9z$$

$$F(z)=\frac{3z^2+7z}{z^2+3z-4}$$

$$F(z)=\frac{z(3z+7)}{(z+4)(z-1)}$$

$$Z(y(n))=\frac{z(3z+7)}{(z+4)(z-1)}$$

$$\Rightarrow \qquad y(n)=Z^{-1}\left(\frac{z(3z+7)}{(z+4)(z-1)}\right)$$

To find $Z^{-1}\left(\dfrac{z(3z+7)}{(z+4)(z-1)}\right)$

Let $\quad F(z)=\dfrac{z(3z+7)}{(z+4)(z-1)}$

$$\therefore \qquad \frac{F(z)}{z}=\frac{3z+7}{(z+4)(z-1)}=\frac{A}{z+4}+\frac{B}{z-1}=\frac{A(z-1)+B(z+4)}{(z+4)(z-1)}$$

$$\Rightarrow \qquad A(z-1)+B(z+4)=3z+7$$

put $z=-4$: $\quad -5A=-5 \Rightarrow A=1$

put $z=1$: $\quad 5B=10 \Rightarrow B=2$

$$\therefore \quad \frac{F(z)}{z}=\frac{1}{z+4}+\frac{2}{z-1} \qquad \Rightarrow \qquad F(z)=\frac{z}{z+4}+\frac{2z}{z-1}$$

Taking Z^{-1} on both sides,

$$\Rightarrow \qquad Z^{-1}\left(F(z)\right)=Z^{-1}\left[\frac{z}{z+4}\right]+2Z^{-1}\left[\frac{z}{z-1}\right]$$

$$y(n)=(-4)^n+2(-1)^n \qquad \left(\because \; Z^{-1}\left[\frac{z}{z-a}\right]=a^n\right).$$

☑ Example: 4

Solve $u_{n+2}-5u_{n+1}+6u_n=(-1)^n$, where $u_0=u_1=0$.

SOLUTION:

Given $\qquad u_{n+2}-5u_{n+1}+6u_n=(-1)^n$

Taking Z-transform on both sides,

$$Z\left[u_{n+2}-5u_{n+1}+6u_n\right]=Z\left[(-1)^n\right]$$

$$Z\left[u_{n+2}\right]-5Z\left[u_{n+1}\right]+6Z\left[u_n\right]=Z\left[(-1)^n\right]$$

$$\left[z^2F(z)-z^2\,u(0)-z\,u(1)\right]-5\left[z\,F(z)-z\,u(0)\right]+6\,F(z)=\frac{z}{z+1}$$

$$\left[z^2F(z)-z^2\,(0)-z\,(0)\right]-5\left[z\,F(z)-z\,(0)\right]+6\,F(z)=\frac{z}{z+1}$$

$$F(z)\left[z^2-5z+6\right]=\frac{z}{z+1}$$

$$F(z)=\frac{z}{(z+1)\left(z^2-5z+6\right)}$$

$$Z(u_n))=\frac{z}{(z+1)(z-3)(z-2)}$$

$$\Rightarrow \qquad u_n=Z^{-1}\left(\frac{z}{(z+1)(z-3)(z-2)}\right)$$

To find $Z^{-1}\left(\frac{z}{(z+1)(z-3)(z-2)}\right)$

Let $$F(z)=\frac{z}{(z+1)(z-3)(z-2)}$$

$$\therefore \quad \frac{F(z)}{z}=\frac{1}{(z+1)(z-3)(z-2)}=\frac{A}{z+1}+\frac{B}{z-3}+\frac{C}{z-2}$$

$$=\frac{A(z-3)(z-2)+B(z+1)(z-2)+C(z+1)(z-3)}{(z+1)(z-3)(z-2)}$$

$$\Rightarrow \quad A(z-3)(z-2)+B(z+1)(z-2)+C(z+1)(z-3)=1$$

put $z=-1$: $\quad A(-4)(-3)=1 \quad \Rightarrow \quad \boxed{A=\frac{1}{12}}$

put $z=3$: $\quad B(4)(1)=1 \quad \Rightarrow \quad \boxed{B=\frac{1}{4}}$

put $z=2$: $\quad C(3)(-1)=1 \quad \Rightarrow \quad \boxed{C=-\frac{1}{3}}$

$$\therefore \quad \frac{F(z)}{z}=\left(\frac{1}{12}\right)\frac{1}{z+1}+\left(\frac{1}{4}\right)\frac{1}{z-3}+\left(\frac{-1}{3}\right)\frac{1}{z-2}$$

$$\Rightarrow \quad F(z)=\left(\frac{1}{12}\right)\frac{z}{z+1}+\left(\frac{1}{4}\right)\frac{z}{z-3}-\left(\frac{1}{3}\right)\frac{z}{z-2}$$

Taking Z^{-1} on both sides,

$$\Rightarrow \quad Z^{-1}(F(z))=\left(\frac{1}{12}\right)Z^{-1}\left[\frac{z}{z+1}\right]+\left(\frac{1}{4}\right)Z^{-1}\left[\frac{z}{z-3}\right]-\left(\frac{1}{3}\right)Z^{-1}\left[\frac{z}{z-2}\right]$$

$$u_n=\frac{1}{12}(-1)^n+\frac{1}{4}3^n-\frac{1}{3}2^n \qquad \left(\because Z^{-1}\left[\frac{z}{z-a}\right]=a^n\right)$$

☑ Example: 5

Solve the difference equation $y(n+3)-3y(n+1)+2y(n)=0$, given that $y(0)=4$, $y(1)=0$ and $y(2)=8$.

SOLUTION:

Given $y(n+3)-3y(n+1)+2y(n)=0,$

Taking Z-transform on both sides we get,

$$Z\left(y(n+3)-3y(n+1)+2y(n)\right)=Z(0)$$

$$Z\left(y(n+3)\right)-3Z\left(y(n+1)\right)+2Z\left(y(n)\right)=0$$

$$\left[z^3F(z)-z^3y(0)-z^2y(1)-zy(2)\right]-3\left[zF(z)-zy(0)\right]+2F(z)=0$$

$$\left[z^3F(z)-z^3(4)-0-8z\right]-3\left[zF(z)-4z\right]+2F(z)=0$$

$$z^3F(z)-4z^3-8z-3zF(z)+12z+2F(z)=0$$

$$F(z)\left[z^3-3z+2\right]=4z^3-4z$$

$$F(z)=\frac{4z\left(z^2-1\right)}{z^3-3z+2}$$

$$Z\left(y(n)\right)=\frac{4z\left(z^2-1\right)}{(z-1)^2(z+2)}$$

$\therefore$ $$y(n)=Z^{-1}\left(\frac{4z\left(z^2-1\right)}{(z-1)^2(z+2)}\right)$$

Now $F(z)=\dfrac{4z\left(z^2-1\right)}{(z-1)^2(z+2)}$

$$\frac{F(z)}{z}=\frac{4\left(z^2-1\right)}{(z-1)^2(z+2)}=\frac{A}{z-1}+\frac{B}{(z-1)^2}+\frac{C}{z+2}$$

$$=\frac{A(z-1)(z+2)+B(z+2)+C(z-1)^2}{(z-1)^2(z+2)}$$

$\Rightarrow$ $$A(z-1)(z+2)+B(z+2)+C(z-1)^2=4\left(z^2-1\right)$$

Put $z=-2$: $0+0+C(9)=4(4-1) \Rightarrow 9C=12 \Rightarrow \boxed{C=\frac{4}{3}}$

Put $z=1$: $B(3)=0\Rightarrow \boxed{B=0}$

Put $z=0$: $$A(-1)(2)+B(2)+C(-1)^2=4(-1)$$

$$-2A+2B+C=-4$$

$$\Rightarrow \quad -2A = -4 - 2B - C = -4 - 0 - \frac{4}{3} \Rightarrow \boxed{A = \frac{8}{3}}$$

$$\therefore \quad \frac{F(z)}{z} = \frac{8/3}{z-1} + 0 + \frac{4/3}{z+2}$$

$$\therefore \quad F(z) = \frac{8}{3}\frac{z}{z-1} + \frac{4}{3}\frac{z}{z+2}$$

$$\therefore \quad y(n) = Z^{-1}\left(\frac{8}{3}\frac{z}{z-1} + \frac{4}{3}\frac{z}{z+2}\right) = \frac{8}{3}Z^{-1}\left(\frac{z}{z-1}\right) + \frac{4}{3}Z^{-1}\left(\frac{z}{z+2}\right)$$

$$= \frac{8}{3}\left(1^n\right) + \frac{4}{3}(-2)^n$$

5.13.3. EXERCISE:

(i) Solve $y_{n+2} + 4y_{n+1} + 3y_n = 2^n$ with $y_0 = 0,\ y_1 = 1$ using Z-transform

(ii) Solve $u_{n+2} + 6u_{n+1} + 9u_n = 2^n$ given that $u_0 = u_1 = 0$

(iii) Solve $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$ with $u_0 = 0, u_1 = 1$

(iv) Solve $y_{k+2} + 2y_{k+1} + y_k = k$, given that $y_0 = y_1 = 0$

(v) Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$, given that $y_0 = y_1 = 0$

(vi) Solve $y_{n+2} + y_n = 2$, with $y_0 = y_1 = 0$

TUTORIALS

1. Find the Z transform of

(i) $\frac{a^n}{n!}$ (ii) $\frac{1}{n!}$

(iii) $2^n \sinh 3n$ (iv) $\frac{1}{n(n-1)}$

(v) $\frac{2n+3}{(n+1)(n+2)}$ (vi) $\frac{(n+1)(n+2)}{2}$

(vii) $\sin^2\left(\frac{n\pi}{4}\right)$ (viii) $\sin^3\left(\frac{n\pi}{6}\right)$

(ix) $\cos^2 t$ (x) $\cos^3 t$

2. Find the Z transform of

(i) nC_2 (ii) n(n-1)(n-2)

(iii) nC_k (iv)(n+1)(n+2)

(v) $e^{3t} \cos t$ (vi) $e^t \sin 2t$

(vii) $e^{-2t} t^3$ (viii) $e^{-t} t^2$

3. Find the inverse Z transform by partial fraction method:

(i) $\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)}$ (ii) $\frac{z^2+2z}{(z^2+2z+4)}$

(iii) $\frac{2z^2}{(z+2)(z^2+4)}$ (iv) $\frac{5z}{(z-1)(z-2)}$

(v) $\frac{z}{(z+2)(z^2+4)}$ (vi) $\frac{5z}{(z^2+4z+3)}$

(vii) $\frac{7z}{(z^2+7z+10)}$

4. Find the inverse Z transform by residue method

(i) $\frac{5z}{(z-1)(z-2)}$

(ii) $\frac{3z^2}{(z+2)(z^2+4)}$

(iii) $\frac{5z}{(z^2+2z+2)}$

(iv) $\frac{z^2+z}{(z-1)^3}$

(v) $\frac{6z^2}{(z-a)(z-b)}$

(vi) $\frac{3z}{(z^2+2z+2)}$

(vii) $\frac{z(z+2)}{(z-2)^3}$

(viii) $\frac{2z(3z+1)}{z^3-3z^2+4}$

(ix) $\frac{z^3}{(3z-1)^2(z-2)}$

5. Using Z transform solve the difference equations

(i) $u_{n+2}-7u_{n+1}+12u_n=2^n,\ u_0=0=u_1$

(ii) u_{n+3}-3u_n+2u_n=0 , , u_0=4 , u_1=0,u_2=8

(iii)f(n)+3f(n-1)-4f(n-2)=0 , $n \geq 2, f(0)=3, f(1)=-2$

(iv)y_{n+2}-5y_{n+1}+6y_n=36 , y_0=y_1=0

(v)u_{n+2}+4u_{n+1}+4u_n= n , u_0=0 , u_1=1

(vi)u_{k+2}+u_k=2^k.k

(vii) u_{n+2}-4u_{n+1}+4u_n=0 , u_0=1,u_1=0

(viii)u_{n+2}+4u_{n+1}+3u_n= 3 , u_0=1 , u_1=1

UNIT – III

APPLICATION OF
PARTIAL DIFFERENTIAL EQUATIONS

Classification of a Partial Differential Equation

The second order partial differential equation in the function u of the two independent variable is

$$A(x,y)\frac{\partial^2 u}{\partial x^2}+B(x,y)\frac{\partial^2 u}{\partial x\partial y}+c(x,y)\frac{\partial^2 u}{\partial y^2}+f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0$$

It is classified as (i) Elliptic if $B^2-4AC<0$

(ii) Parabolic $B^2-4AC=0$

(iii) hyperbolic $B^2-4AC>0$

1. Classify the following partial differential equations

(a) $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}$

(b) $\frac{\partial^2 u}{\partial x\partial y}=\left(\frac{\partial u}{\partial x}\right)+\left(\frac{\partial u}{\partial y}\right)+xy$

Solution:

(a)A=1, B=0, C=-1

$B^2-4AC=0+4=4>0$

Equation is hyperbolic

(b) A=0, B=1, C=0

$B^2 - 4AC = 1 - 0 = 1 > 0$

Equation is hyperbolic

2. Classify the following partial differential equations

(a) $$4\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} - 4\frac{\partial u}{\partial x} - 8\frac{\partial u}{\partial y} - 16u = 0$$

(b) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

Solution:

(a) A=4, B=4, C=1

$B^2 - 4AC = 16 - 16 = 0$

Equation is parabolic

(b) A=1, B=0, C=1

$B^2 - 4AC = -4 < 0$

Equation is Elliptic.

3. Classify the following partial differential equations

$x^2 f_{xx} + (1 - y^2) f_{xx} = 0 \; for -1 < y < 1, -\infty < x < \infty$

Solution:

$A = x^2, B = 0, C = 1 - y^2$

$B^2 - 4AC = -4x^2(1 - y^2)$

$= 4x^2(y^2 - 1)$

$x^2 \; is \; always \; +ve \; in \; -\infty < x < \infty, \;\; x \neq 0$

$In \; -1 < y < 1, \;\; y^2 - 1 \, is - ve$

Equation is Elliptic

If x = 0, B^2-4AC=0, the equation is **Parabolic.**

4. Classify the following partial differential equations:

$(a)\ y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} + 2u_x - 3u = 0$

$(b)\ y^2u_{xx} + u_{yy} + u_x^{\ 2} + u_y^{\ 2} + 7 = 0$

Solution:

(a)Here A = y^2 ,B = $-2xy$, C = x^2

$B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$

Equation is parabolic

(b) Here A = y^2 ,B =0, C = 1

$B^2 - 4AC = 0 - 4y^2 = -4y^2 < 0$

Here y^2 is always positive

Equation is elliptic.

5. Classify $(1+x)^2u_{xx} - 4xu_{xy} + u_{yy} = x$

Solution:

Here A = $(1+x)^2$,B = $-4x$, C = 1

$B^2 - 4AC = 16x^2 - 4(1+x)^2$

$= 16x^2 - 4 - 8x - 4x^2$

$=4(3x^2 - 2x - 1)$

If $x=1$,then $B^2 - 4AC = 0$

Given PDE is Parabolic.

If $x<0$ (or) $x>0$ Then $B^2 - 4AC > 0$

Given PDE is Hyperbolic.

One Dimensional Wave Equation

One dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

Assumptions made in the derivation of one dimensional wave equation or equation of vibration of strings

(i) The mass of the string per unit length is constant.

(ii) The string is perfectly elastic and does not offer any resistance to bending.

(iii) The tension caused by stretching the string before fixing it at end points is so large that the action of the gravitational force on the string can be neglected.

(iv) The string performs a small transverse motion in a vertical plane that is every particle of the string moves strictly vertically and so that the deflection and the slope at every point of the string remain small in absolute value.

The possible solution of one dimensional wave equation

$$y(x,t) = \left(c_1 e^{px} + c_2 e^{-px}\right)\left(c_3 e^{apt} + c_4 e^{-apt}\right)$$

$$y(x,t) = \left(c_5 \cos px + c_6 \sin px\right)\left(c_7 \cos apt + c_8 \sin apt\right)$$

$$y(x,t) = \left(c_9 x + c_{10}\right)\left(c_{11} t + c_{12}\right)$$

1. What is the constant a^2 in wave equation $u_{tt} = a^2 u_{xx}$?

Solution:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

$$where \;\; a^2 = \frac{tension}{mass\ per\ unit\ length\ of\ the\ string}$$

2. **What is the correct solution of one dimensional equation?**
   Solution:
   $y(x,t) = (c_5 \cos px + c_6 \sin px)(c_7 \cos apt + c_8 \sin apt)$

Boundary and Initial conditions

One dimensional wave equation is given by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

$(i)\, y(0,t) = 0$

$(ii)\, y(l,t) = 0$

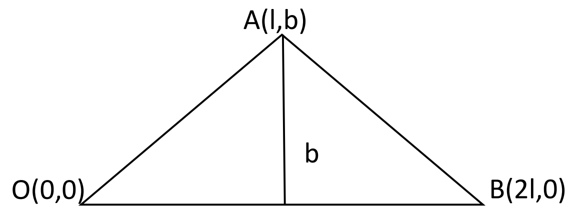
$(iii)\, \frac{\partial y(x,0)}{\partial t} = f(x)$

$(iv)\, y(x,0) = g(x)$

3. **A tightly stretched string of length 2l is fastened at both ends. The midpoint of the string is displaced to a distance b and released from rest in this position. Write the initial conditions.**
   Solution:
   The one dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$
   The equation of OA is

$$\frac{y-0}{b}=\frac{x-0}{l-0}$$

$$\Rightarrow y=\frac{bx}{l},0<x<l$$



$The\, equation\, of\ AB\, is\, B(2l,0)$

$$\frac{y-b}{-b}=\frac{x-l}{l}$$

$$\Rightarrow y-b=\frac{lb-bx}{l}$$

$$\Rightarrow y=\frac{lb-bx+lb}{l}=\frac{b}{l}(2l-x),\ l<x<2l$$

The initial boundary conditions are

$(i) y(0,t)=0$

$(ii) y(l,t)=0$

$(iii) \dfrac{\partial y(x,0)}{\partial t}=0$

$$(iv) y(x,0)=\begin{cases}\dfrac{bx}{l},\ 0<x<l\\ \dfrac{2}{l}(2l-x), l<x<2l\end{cases}$$

4. A string is stretched and fastened to two points x = 0 & x = l apart. Motion is started by displacing the string into the form $y=k(lx-x^2)$ from which it is released at time t = 0. Write the most general solution to this problem.

Solution:

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l}$$

5. A string is stretched and fastened to two points x = 0 & x = l apart. Is initially at rest in its equilibrium position. If it is set Vibrating by giving each point velocity $y = k(lx - x^2)$ **from which it is released at time t = 0. Write the most general solution to this problem.**

Solution:

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \sin\frac{n\pi a t}{l}$$

6. Write the boundary conditions for the following boundary value problem "If a string of length l **initially at rest in its equilibrium position and each of its point is given the velocity** $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3\frac{\pi x}{l}, 0 < x < l$ **Determine the displacement function y(x,t)"**

Solution:

The boundary conditions are

(i) $y(0,t) = 0, t > 0,$

(ii) $y(l,t) = 0, t > 0,$

(iii) $y(x,0) = 0, 0 < x < l$,

(iv) $\frac{\partial y}{\partial t}(x,0) = v_0 \sin^3\frac{\pi x}{l}, 0 < x < l$

7. Write the boundary conditions for solving the string equation if the string is subjected to initial displacement f(x) and initial velocity g(x).

Solution:

(i) $y(0,t) = 0$

(ii) $y(l,t) = 0$

(iii) $\frac{\partial y}{\partial t}(x,0) = g(x), 0 < x < l$

(iv) $y(x,0) = f(x), 0 < x < l$

Problems on vibrating on strings with initial velocity zero

1. A string is stretched and fastened to two points x = 0 and x = l apart motion is started by displacing the string into the form $y = k(lx - x^2)$ from which it is released at time t = 0. Find the displacement of any point on the string at a distance of x from one end at time t.

Solution:

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

From the given problem, we get the following boundary conditions.

$(i) y(0,t) = 0 \forall t > 0$

$(ii) y(l,t) = 0 \forall t > 0$

$(iii) \dfrac{\partial y(x,0)}{\partial t} = 0$

$(iv) y(x,0) = k(lx - x^2) = f(x)$

The correct solution which satisfies our boundary conditions is given by

$y(x,t) = (A\cos px + B\sin px)(C\cos apt + D\sin apt) \to (1)$

Apply (i) in (1) we get,

$y(0,t) = A(C\cos apt + D\sin apt)$

$\Rightarrow 0 = A(C\cos apt + D\sin apt)$

$\Rightarrow A = 0,\ C\cos apt + D\sin apt) \neq 0$

Putting A= 0 in (1) we get

$y(x,t) = (B\sin px)(C\cos apt + D\sin apt) \to (2)$

Applying (ii) in (2) we get

$y(l,t) = (B\sin pl)(C\cos apt + D\sin apt)$

$0 = (B\sin pl)(C\cos apt + D\sin apt)$

$If\ B = 0\ then\ we\ get\ trivial\ solution$

$B \neq 0,\ (C\cos apt + D\sin apt) \neq 0$

$\therefore \sin pl = 0$

$\sin pl = \sin n\pi$

$\Rightarrow p = \dfrac{n\pi}{l}$

$(2) \Rightarrow y(x,t) = B\sin\dfrac{n\pi x}{l}\left(C\cos\dfrac{n\pi at}{l} + D\sin\dfrac{n\pi at}{l}\right) \to (3)$

Before applying condition (iii) differentiate (3) partially with respect to t

$$\frac{\partial y(x,t)}{\partial t} = B\sin\frac{n\pi x}{l}\left(-C\sin\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \to (I)$$

Apply (iii) in (I) we get

$$\frac{\partial y(x,0)}{\partial t} = B\sin\frac{n\pi x}{l}\left(D\frac{n\pi a}{l}\right)$$

$$\Rightarrow 0 = B\sin\frac{n\pi x}{l}\left(D\frac{n\pi a}{l}\right)$$

$$Here\ B \neq 0,\ \sin\frac{n\pi x}{l} \neq 0,\ \frac{n\pi a}{l} \neq 0$$

$$\therefore D = 0$$

$$(3) \Rightarrow y(x,t) = BC\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$y(x,t) = B_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \to (II),\ where\ B_n = BC$$

Since the partial differential equation is linear, any linear combination of solutions of the form (4) with n = 1, 2, 3, … is also a solution of the equation.

The solution (II) can be written as

$$y(x,t) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \to (4)$$

Apply (iv) in (4)

$$y(x,0) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}$$

$$f(x)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \rightarrow (5)$$

(5) represents half range fourier sine series in the interval $(0,l)$

$$\therefore B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{2}{l}\int_0^l k(lx-x^2)\sin\frac{n\pi x}{l}$$

$$=\frac{2k}{l}\left[\left(lx-x^2\right)\frac{l}{n\pi}\left(-\cos\frac{n\pi x}{l}\right)+\left(l-2x\right)\sin\frac{n\pi x}{l}.\frac{l^2}{n^2\pi^2}-2\cos\frac{n\pi x}{l}\frac{l^3}{n^3\pi^3}\right]_0^l$$

$$=\frac{2k}{l}\left[-2(-1)^n\frac{l^3}{n^3\pi^3}+\frac{2l^3}{n^3\pi^3}\right]=\frac{4kl^2}{n^3\pi^3}\left[1-(-1)^n\right]=\begin{cases}\dfrac{8kl^2}{n^3\pi^3}, & n \text{ is odd}\\ 0, & n \text{ is even}\end{cases}$$

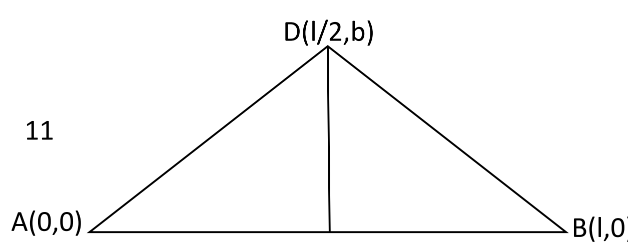
$$(4)\Rightarrow y(x,t)=\sum_{n=1,3,5,\dots}^{\infty}\frac{8kl^2}{n^3\pi^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

2. **A string is stretched and its ends are fastened at two points *x=0* and *x=l* the midpoint of the string is displaced transversely through a small distance b and string is released from the rest in that position. Find an expression for the transverse displacement of the string at anytime during the subsequent motion.**

   Solution:

   To find the equation of the string in its initial position.

   The equation of the string AD is



$$\frac{x-0}{l/2-0} = \frac{y-0}{b-0}$$

$$\frac{2x}{l} = \frac{y}{b}$$

$$y = \frac{2bx}{l}\ ,\ 0 < x < l/2$$

The equation of the string DB is

$$\frac{x-l/2}{l-l/2} = \frac{y-b}{0-b}$$

$$\frac{2x-l}{l} = \frac{y-b}{-b}$$

$$y = \frac{2b}{l}(l-x)\ ,\ l/2 < x < l$$

Hence initially the displacement of the string is in the form

$$y(x,0) = \begin{cases} \dfrac{2bx}{l}\ ,\ 0 < x < l/2 \\ \dfrac{2b}{l}(l-x)\ ,\ l/2 < x < l \end{cases} = f(x)$$

The wave equation is $\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$

From the given problem, we get the following boundary conditions.

$(i)\, y(0,t) = 0 \;\forall t > 0$

$(ii)\, y(l,t) = 0 \forall t > 0$

$(iii)\, \dfrac{\partial y(x,0)}{\partial t} = 0$

$$(iv)\, y(x,0) = \begin{cases} \dfrac{2bx}{l}, \; 0 < x < l/2 \\ \dfrac{2b}{l}(l-x), \; l/2 < x < l \end{cases} = f(x)$$

The correct solution which satisfies our boundary conditions is given by

$$y(x,t) = (A\cos px + B\sin px)(C\cos apt + D\sin apt) \to (1)$$

Apply (i) in (1) we get,

$y(0,t) = A(C\cos apt + D\sin apt)$

$\Rightarrow 0 = A(C\cos apt + D\sin apt)$

$\Rightarrow A = 0,\; C\cos apt + D\sin apt) \neq 0$

Putting A= 0 in (1) we get

$y(x,t) = (B\sin px)(C\cos apt + D\sin apt) \to (2)$

Applying (ii) in (2) we get

$y(l,t) = (B\sin pl)(C\cos apt + D\sin apt)$

$0 = (B\sin pl)(C\cos apt + D\sin apt)$

$If\; B = 0\, then\; we\, get\, trivial\, solution$

$B \neq 0,\; (C\cos apt + D\sin apt) \neq 0$

$\therefore \sin pl = 0$

$\sin pl = \sin n\pi$

$\Rightarrow p = \dfrac{n\pi}{l}$

$$(2) \Rightarrow y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \to (3)$$

Before applying condition (iii) differentiate (3) partially with respect to t

$$\frac{\partial y(x,t)}{\partial t} = B\sin\frac{n\pi x}{l}\left(-C\sin\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \rightarrow (I)$$

Apply (iii) in (I) we get

$$\frac{\partial y(x,0)}{\partial t} = B\sin\frac{n\pi x}{l}\left(D\frac{n\pi a}{l}\right)$$

$$\Rightarrow 0 = B\sin\frac{n\pi x}{l}\left(D\frac{n\pi a}{l}\right)$$

$$Here\ B \neq 0,\ \sin\frac{n\pi x}{l} \neq 0,\ \frac{n\pi a}{l} \neq 0$$

$$\therefore D = 0$$

$$(3) \Rightarrow y(x,t) = BC\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$y(x,t) = B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \rightarrow (II),\ where\ B_n = BC$$

Since the partial differential equation is linear, any linear combinationof solutions of the form (4) with n = 1, 2, 3, … is also a solution of the equation.

The solution (II) can be written as

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \rightarrow (4)$$

Apply (iv) in (4)

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$f(x)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \rightarrow (5)$$

(5) represents half range fourier sine series in the interval $(0,l)$

$$\therefore B_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{2}{l}\frac{2b}{l}\left\{\int_0^{l/2} x\sin\left(\frac{n\pi x}{l}\right)dx+\int_{l/2}^{l}(l-x)\sin\left(\frac{n\pi x}{l}\right)dx\right\}$$

$$=\frac{4b}{l^2}\left\{\begin{array}{l}\left[(x)\left(-\cos\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi}\right)-1\left(-\sin\left(\frac{n\pi x}{l}\right)\right)\left(\frac{l}{n\pi}\right)^2\right]_0^{l/2}\\+\left[(l-x)\left(-\cos\left(\frac{n\pi x}{l}\right)\frac{l}{n\pi}\right)-(-1)\left(-\sin\left(\frac{n\pi x}{l}\right)\right)\left(\frac{l}{n\pi}\right)^2\right]_{l/2}^{l}\end{array}\right\}$$

$$=\frac{4b}{l^2}\left\{\begin{array}{l}\left[-\frac{l}{2}\frac{l}{n\pi}\cos\left(\frac{n\pi}{2}\right)+\sin\left(\frac{n\pi}{2}\right)\frac{l^2}{n^2\pi^2}-0+0\right]\\+\left[0-0+\frac{l}{2}\frac{l}{n\pi}\cos\left(\frac{n\pi}{2}\right)+\sin\left(\frac{n\pi}{2}\right)\frac{l^2}{n^2\pi^2}\right]\end{array}\right\}$$

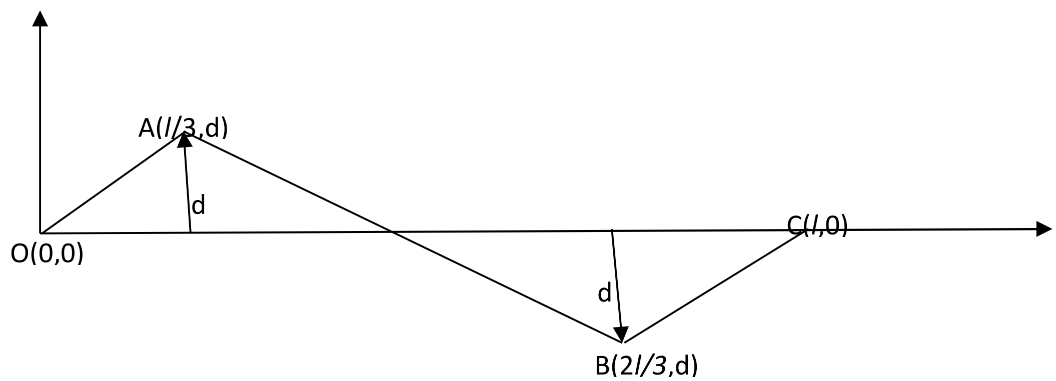
$$=\frac{4b}{l^2}\left[2\sin\left(\frac{n\pi}{2}\right)\frac{l^2}{n^2\pi^2}\right]$$

$$B_n=\frac{8b}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$(4)\Rightarrow y(x,t)=\sum_{n=1}^{\infty}\frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

3. The points of trisection of a tight stretched string of length l with fixed ends are parallel aside through a distance d on opposite sides of a position of equilibrium, and the string is released from rest. Obtain an expression for the displacement of the string at any subsequent time and show that the midpoint of the string always remains at rest.

Solution:



To find the equation of the string into initial position OABC

The equation of the string OA is

$$\frac{x-0}{l/3} = \frac{y-0}{d}$$

$$\frac{3x}{l} = \frac{y}{d}$$

$$y = \frac{3xd}{l},\ 0 < x < \frac{l}{3}$$

The equation of the string AB is

$$\frac{x-l/3}{l/3}=\frac{y-d}{-2d}$$

$$\frac{3x-l}{l}=\frac{y-d}{-2d}$$

$$y-d=\frac{-2d}{l}(3x-l)=\frac{2d}{l}(l-3x)$$

$$y=\frac{2d}{l}(l-3x)+d=\frac{3d}{l}(l-2x),\ l/3<x<2l/3$$

The equation of the string BC is

$$\frac{x-2l/3}{l-2l/3}=\frac{y+d}{d}$$

$$\frac{3x-2l}{l}=\frac{y+d}{d}$$

$$y+d=\frac{d}{l}(3x-2l)$$

$$y=\frac{3xd-2dl+dl}{l}=\frac{3xd-dl}{l}=\frac{3d}{l}(x-l),\ 2l/3<x<l$$

The wave equation is $\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$

From the given problem, we get the following boundary conditions.

$(i)\, y(0,t)=0\ \forall t>0$

$(ii)\, y(l,t)=0\ \forall t>0$

$(iii)\, \frac{\partial y(x,0)}{\partial t}=0$

$$(iv)\, y(x,0)=f(x)=\begin{cases}\frac{3xd}{l},\ 0<x<\frac{l}{3}\\ \frac{3d}{l}(l-2x),\ \frac{l}{3}<x<\frac{2l}{3}\\ \frac{3d}{l}(x-l),\ \frac{2l}{3}<x<l\end{cases}$$

The correct solution which satisfies our boundary conditions is given by

$$y(x,t) = (A\cos px + B\sin px)(C\cos apt + D\sin apt) \rightarrow (1)$$

Apply (i) in (1) we get,

$$y(0,t) = A(C\cos apt + D\sin apt)$$

$$\Rightarrow 0 = A(C\cos apt + D\sin apt)$$

$$\Rightarrow A = 0,\ C\cos apt + D\sin apt) \neq 0$$

Putting A= 0 in (1) we get

$$y(x,t) = (B\sin px)(C\cos apt + D\sin apt) \rightarrow (2)$$

Applying (ii) in (2) we get

$$y(l,t) = (B\sin pl)(C\cos apt + D\sin apt)$$

$$0 = (B\sin pl)(C\cos apt + D\sin apt)$$

$$If\ B = 0\ then\ we\ get\ trivial\ solution$$

$$B \neq 0,\ (C\cos apt + D\sin apt) \neq 0$$

$$\therefore \sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$\Rightarrow p = \frac{n\pi}{l}$$

$$(2) \Rightarrow y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \rightarrow (3)$$

Before applying condition (iii) differentiate (3) partially with respect to t

$$\frac{\partial y(x,t)}{\partial t} = B\sin\frac{n\pi x}{l}\left(-C\sin\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \rightarrow (I)$$

Apply (iii) in (I) we get

$$\frac{\partial y(x,0)}{\partial t} = B\sin\frac{n\pi x}{l}\left(D\frac{n\pi a}{l}\right)$$

$$\Rightarrow 0 = B\sin\frac{n\pi x}{l}\left(D\frac{n\pi a}{l}\right)$$

$$Here\ B \neq 0,\ \sin\frac{n\pi x}{l} \neq 0,\ \frac{n\pi a}{l} \neq 0$$

$$\therefore D = 0$$

$$(3) \Rightarrow y(x,t) = BC\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$y(x,t) = B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \rightarrow (II),\ where\ B_n = BC$$

Since the partial differential equation is linear, any linear combinationof solutions of the form (4) with n = 1, 2, 3, … is also a solution of the equation.

The solution (II) can be written as

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \rightarrow (4)$$

Apply (iv) in (4)

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \rightarrow (5)$$

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$B_n = \frac{6d}{l^2}\left[\int_0^{l/3} x\sin\frac{n\pi x}{l}dx + \int_{l/3}^{2l/3}(l-2x)\sin\frac{n\pi x}{l}dx + \int_{2l/3}^{l}(x-l)\sin\frac{n\pi x}{l}dx\right]$$

$$B_n = \frac{6d}{l^2}\left[\begin{array}{l}\left(-x\cos\frac{n\pi x}{l}.\frac{l}{n}+\sin\frac{n\pi x}{l}.\frac{l^2}{n^2\pi^2}\right)_0^{l/3}+\left(-(l-2x)\cos\frac{n\pi x}{l}.\frac{l}{n\pi}-2\sin\frac{n\pi x}{l}.\frac{l^2}{n^2\pi^2}\right)_{l/3}^{2l/3}+\\ \left(-(x-l)\cos\frac{n\pi x}{l}.\frac{l}{n\pi}+\sin\frac{n\pi x}{l}.\frac{l^2}{n^2\pi^2}\right)_{2l/3}^{l}\end{array}\right]$$

$$=\frac{6d}{l^2}\left[\begin{array}{l}\frac{-l^2}{3n\pi}\cos\frac{n\pi}{3}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3}+\frac{l^2}{3n\pi}\cos\frac{2n\pi}{3}+\frac{l^2}{3n\pi}\cos\frac{n\pi}{3}+\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{3}\\ -\frac{l^2}{3n\pi}\cos\frac{2n\pi}{3}-\frac{2l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}-\frac{2l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\end{array}\right]$$

$$=\frac{6d}{l^2}\left[\frac{3l^2}{n^2\pi^2}\sin\frac{n\pi}{3}-\frac{3l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right]$$

$$=\frac{18d}{n^2\pi^2}\left[\sin\frac{n\pi}{3}-\sin\frac{2n\pi}{3}\right]$$

$$=\frac{18d}{n^2\pi^2}\left[\sin\frac{n\pi}{3}-\sin\left(n\pi-\frac{n\pi}{3}\right)\right]$$

$$=\frac{18d}{n^2\pi^2}\left[\sin\frac{n\pi}{3}-\left(\sin n\pi\cos\frac{n\pi}{3}-\cos n\pi\sin\frac{n\pi}{3}\right)\right]$$

$$=\frac{18d}{n^2\pi^2}\sin\frac{n\pi}{3}\left[(-1)^n+1\right]$$

$$B_n=\begin{cases}\frac{36d}{n^2\pi^2}\sin\frac{n\pi}{3} & : n\,is\,even\\ 0 & : n\,is\,odd\end{cases}$$

$$\therefore y(x,t)=\frac{36d}{\pi^2}\sum_{n=2,4,6,..}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}\rightarrow(7)$$

The displacement at the midpoint is got by putting $x = \frac{l}{2}$ in (7)

i.e., $y\left(\frac{l}{2}, t\right) = 0$

There is no displacement at $x = \frac{l}{2}$

The midpoint of the string is rest.

4. A string is stretched with fixed end points x = 0 and x = *l* is initially in a position given by $y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$. It is released from rest from this position. Find the displacement y at any distance x from one end at any time t.

Solution:

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

From the given problem, we get the following boundary conditions.

$(i) y(0,t) = 0 \, \forall t > 0$

$(ii) y(l,t) = 0 \forall t > 0$

$(iii) \frac{\partial y(x,0)}{\partial t} = 0$

$(iv) y(x,0) = y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$

The correct solution which satisfies our boundary conditions is given by

$y(x,t) = (A \cos px + B \sin px)(C \cos apt + D \sin apt) \rightarrow (1)$

Apply (i) in (1) we get,

$$y(0,t) = A(C\cos apt + D\sin apt)$$

$$\Rightarrow 0 = A(C\cos apt + D\sin apt)$$

$$\Rightarrow A = 0,\ C\cos apt + D\sin apt) \neq 0$$

Putting A= 0 in (1) we get

$$y(x,t) = (B\sin px)(C\cos apt + D\sin apt) \rightarrow (2)$$

Applying (ii) in (2) we get

$$y(l,t) = (B\sin pl)(C\cos apt + D\sin apt)$$

$$0 = (B\sin pl)(C\cos apt + D\sin apt)$$

$$If\ B = 0\ then\ we\ get\ trivial\ solution$$

$$B \neq 0,\ (C\cos apt + D\sin apt) \neq 0$$

$$\therefore \sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$\Rightarrow p = \frac{n\pi}{l}$$

$$(2) \Rightarrow y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \rightarrow (3)$$

Before applying condition (iii) differentiate (3) partially with respect to t

$$\frac{\partial y(x,t)}{\partial t} = B\sin\frac{n\pi x}{l}\left(-C\sin\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \rightarrow (I)$$

Apply (iii) in (I) we get

$$\frac{\partial y(x,0)}{\partial t} = B\sin\frac{n\pi x}{l}\left(D\frac{n\pi a}{l}\right)$$

$$\Rightarrow 0 = B\sin\frac{n\pi x}{l}\left(D\frac{n\pi a}{l}\right)$$

$$Here\ B \neq 0,\ \sin\frac{n\pi x}{l} \neq 0,\ \frac{n\pi a}{l} \neq 0$$

$$\therefore D = 0$$

$$(3) \Rightarrow y(x,t) = BC\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$y(x,t) = B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \to (II),\ where\ B_n = BC$$

Since the partial differential equation is linear, any linear combination of solutions of the form (4) with n = 1, 2, 3, … is also a solution of the equation.

The solution (II) can be written as

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \to (4)$$

Apply (iv) in (4)

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$y_0 \sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$\frac{y_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right] = B_1 \sin\frac{\pi x}{l} + B_2 \sin\frac{2\pi x}{l} + B_3 \sin\frac{3\pi x}{l} + ...$$

By Equating like coefficients

$$B_1 = \frac{3y_0}{4}, B_2 = 0, B_3 = -\frac{y_0}{4}, B_4 = B_5 = ... = 0$$

Substitute these values in (4) we get

$$y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi at}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l}$$

5. A string is stretched and fastened to two points at a distance *l* apart. Motion is started by displacing the string in the form $y(x,0)=\sin\frac{\pi x}{l}$ **from which it is released at time *t=0*. Show that the displacement of any point at a distance *x* from one end at time *t* is given by** $y(x,t)=a\sin\frac{\pi x}{l}\cos\frac{\pi x}{l}$**.**

Solution:

The wave equation is $\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$

From the given problem, we get the following boundary conditions.

$(i)\, y(0,t)=0\ \forall t>0$

$(ii)\, y(l,t)=0\forall t>0$

$(iii)\, \frac{\partial y(x,0)}{\partial t}=0$

$(iv)\, y(x,0)=a\sin\frac{\pi x}{l}=f(x)$

The correct solution which satisfies our boundary conditions is given by

$y(x,t)=\left(A\cos px+B\sin px\right)(C\cos apt+D\sin apt)\rightarrow(1)$

Apply (i) in (1) we get,

$y(0,t)=A(C\cos apt+D\sin apt)$

$\Rightarrow 0=A(C\cos apt+D\sin apt)$

$\Rightarrow A=0,\ C\cos apt+D\sin apt)\neq 0$

Putting A= 0 in (1) we get

$y(x,t)=\left(B\sin px\right)(C\cos apt+D\sin apt)\rightarrow(2)$

Applying (ii) in (2) we get

$$y(l,t) = (B\sin pl)(C\cos apt + D\sin apt)$$

$$0 = (B\sin pl)(C\cos apt + D\sin apt)$$

$$If\ B = 0\,then\ we\,get\,trivial\,solution$$

$$B \neq 0,\ (C\cos apt + D\sin apt) \neq 0$$

$$\therefore \sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$\Rightarrow p = \frac{n\pi}{l}$$

$$(2) \Rightarrow y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \rightarrow (3)$$

Before applying condition (iii) differentiate (3) partially with respect to t

$$\frac{\partial y(x,t)}{\partial t} = B\sin\frac{n\pi x}{l}\left(-C\sin\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \rightarrow (I)$$

Apply (iii) in (I) we get

$$\frac{\partial y(x,0)}{\partial t} = B\sin\frac{n\pi x}{l}\left(D\frac{n\pi a}{l}\right)$$

$$\Rightarrow 0 = B\sin\frac{n\pi x}{l}\left(D\frac{n\pi a}{l}\right)$$

$$Here\ B \neq 0,\ \sin\frac{n\pi x}{l} \neq 0,\ \frac{n\pi a}{l} \neq 0$$

$$\therefore D = 0$$

$$(3) \Rightarrow y(x,t) = BC\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$y(x,t) = B_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \rightarrow (II),\ where\ B_n = BC$$

Since the partial differential equation is linear, any linear combination of solutions of the form (4) with n = 1, 2, 3, … is also a solution of the equation.

The solution (II) can be written as

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l} \rightarrow (4)$$

Apply (iv) in (4)

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$a\sin\frac{\pi x}{l} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$a\sin\frac{\pi x}{l} = B_1\sin\frac{\pi x}{l} + B_2\sin\frac{2\pi x}{l} + B_3\sin\frac{3\pi x}{l} + \ldots$$

By Equating like coefficients

$$B_1 = a, B_2 = 0, B_3 = 0, B_4 = B_5 = \ldots = 0$$

Substitute these values in (4) we get

$$y(x,t) = \sin\frac{\pi x}{l}\cos\frac{\pi at}{l}$$

Problems on vibrating on strings with non – zero initial velocity

1. **A tightly stretched string with fixed end points x = 0 and x = *l* is initially at rest in its equilibrium position. If it is set vibrating by giving each point a velocity kx(*l*-x). Find the displacement of the string at any time.**

   Solution:

   The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

   From the given problem, we get the following boundary conditions.

   $$(i)\, y(0,t) = 0 \;\forall t > 0$$
   $$(ii)\, y(l,t) = 0 \forall t > 0$$
   $$(iii)\, y(x,0) = 0$$
   $$(iv)\, \frac{\partial y(x,0)}{\partial t} = kx(l-x) = f(x)$$

   The correct solution which satisfies our boundary conditions is given by

   $$y(x,t) = (A\cos px + B\sin px)(C\cos apt + D\sin apt) \rightarrow (1)$$

   Apply (i) in (1) we get,

   $$y(0,t) = A(C\cos apt + D\sin apt)$$
   $$\Rightarrow 0 = A(C\cos apt + D\sin apt)$$
   $$\Rightarrow A = 0,\; C\cos apt + D\sin apt) \neq 0$$

   Putting A= 0 in (1) we get

   $$y(x,t) = (B\sin px)(C\cos apt + D\sin apt) \rightarrow (2)$$

   Applying (ii) in (2) we get

$$y(l,t) = (B\sin pl)(C\cos apt + D\sin apt)$$

$$0 = (B\sin pl)(C\cos apt + D\sin apt)$$

$$If\ B = 0\ then\ we\ get\ trivial\ solution$$

$$B \neq 0,\ (C\cos apt + D\sin apt) \neq 0$$

$$\therefore \sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$\Rightarrow p = \frac{n\pi}{l}$$

$$(2) \Rightarrow y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \rightarrow (3)$$

Apply (iii) in (3)

$$y(x,0) = B\sin\frac{n\pi x}{l}C$$

$$0 = BC\sin\frac{n\pi x}{l}$$

$$\Rightarrow C = 0$$

$$(3) \Rightarrow y(x,0) = BD\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$y(x,0) = B_n\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

Since the partial differential equation is lin ear, any linear combition of solutions of the form (4) with n = 1, 2, 3, … is also a solution of the equation.The general solution can be written as

$$y(x,t) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \rightarrow (4)$$

Differentiate Partially (4) with respect to t we get,

$$\frac{\partial y(x,t)}{\partial t} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}.\frac{n\pi a}{l} \rightarrow (5)$$

Apply (iv) in (5) we get

$$\frac{\partial y(x,0)}{\partial t} = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{l}\sin\frac{n\pi x}{l}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \rightarrow (6),\ where\ \ b_n = B_n \frac{n\pi a}{l}$$

Equation (6) Represents Half rane sine series.

$$\therefore b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\int_0^l k(lx - x^2)\sin\frac{n\pi x}{l}$$

$$= \frac{2k}{l}\left[\left(lx - x^2\right)\frac{l}{n\pi}\left(-\cos\frac{n\pi x}{l}\right) + \left(l - 2x\right)\sin\frac{n\pi x}{l}.\frac{l^2}{n^2\pi^2} - 2\cos\frac{n\pi x}{l}\frac{l^3}{n^3\pi^3}\right]_0^l$$

$$= \frac{2k}{l}\left[-2(-1)^n\frac{l^3}{n^3\pi^3} + \frac{2l^3}{n^3\pi^3}\right] = \frac{4kl^2}{n^3\pi^3}\left[1 - (-1)^n\right] = \begin{cases} \frac{8kl^2}{n^3\pi^3},\ n\ is\ odd \\ 0, \quad\ n\ is\ even \end{cases}$$

$$B_n = \frac{l}{n\pi a}b_n$$

$$\Rightarrow B_n = \frac{l}{n\pi a}\begin{cases} \frac{8kl^2}{n^3\pi^3},\ n\ is\ odd \\ 0, \quad\ n\ is\ even \end{cases}$$

$$\Rightarrow B_n = \begin{cases} \frac{8kl^3}{an^4\pi^4},\ n\ is\ odd \\ 0, \quad\ n\ is\ even \end{cases}$$

$$(4) \Rightarrow y(x,t) = \sum_{n=1,3,5,..}^{\infty} \frac{8kl^4}{an^4\pi^4} \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}$$

2. If a string of length l is initially at rest in equilibrium position and each of its point is given velocity $V_0 \sin^3\left(\frac{\pi x}{l}\right),\ \ 0 < x < l$. Determine the displacement function $y(x,t)$

Solution:

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

From the given problem, we get the following boundary conditions.

$(i)\, y(0,t) = 0\ \forall t > 0$

$(ii)\, y(l,t) = 0 \forall t > 0$

$(iii)\, y(x,0) = 0$

$(iv)\, \frac{\partial y(x,0)}{\partial t} = V_0 \sin^3\left(\frac{\pi x}{l}\right) = f(x)$

The correct solution which satisfies our boundary conditions is given by

$$y(x,t) = \left(A\cos px + B\sin px\right)(C\cos apt + D\sin apt) \rightarrow (1)$$

Apply (i) in (1) we get,

$$y(0,t) = A(C\cos apt + D\sin apt)$$

$$\Rightarrow 0 = A(C\cos apt + D\sin apt)$$

$$\Rightarrow A = 0,\ C\cos apt + D\sin apt) \neq 0$$

Putting A= 0 in (1) we get

$$y(x,t) = (B\sin px)(C\cos apt + D\sin apt) \to (2)$$

Applying (ii) in (2) we get

$$y(l,t) = (B\sin pl)(C\cos apt + D\sin apt)$$

$$0 = (B\sin pl)(C\cos apt + D\sin apt)$$

$$If\ B = 0\, then\ we\, get\, trivial\, solution$$

$$B \neq 0,\ (C\cos apt + D\sin apt) \neq 0$$

$$\therefore \sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$\Rightarrow p = \frac{n\pi}{l}$$

$$(2) \Rightarrow y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \to (3)$$

Apply (iii) in (3)

$$y(x,0) = B\sin\frac{n\pi x}{l}C$$

$$0 = BC\sin\frac{n\pi x}{l}$$

$$\Rightarrow C = 0$$

$$(3) \Rightarrow y(x,0) = BD\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

$$y(x,0) = B_n\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l}$$

Since the partial differential equation is lin ear, any linear combination of solutions of the form (4) with n = 1, 2, 3, … is also a solution of the equation. The general solution can be written as

$$y(x,t) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}\sin\frac{n\pi at}{l} \to (4)$$

Differentiate Partially (4) with respect to t we get,

$$\frac{\partial y(x,t)}{\partial t}=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}.\frac{n\pi a}{l}\to(5)$$

$$Apply\,(iv)\,in\,(5)\,we\,get$$

$$\frac{\partial y(x,0)}{\partial t}=\sum_{n=1}^{\infty}B_n\frac{n\pi a}{l}\sin\frac{n\pi x}{l}$$

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}\to(6),\ where\ \ b_n=B_n\frac{n\pi a}{l}$$

$$v_0\sin^3\frac{\pi x}{l}=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$$

$$\frac{v_0}{4}\left[3\sin\frac{\pi x}{l}-\sin\frac{3\pi x}{l}\right]=b_1\sin\frac{\pi x}{l}+b_2\sin\frac{2\pi x}{l}+b_3\sin\frac{3\pi x}{l}+...$$

By Equating like coefficients

$$b_1=\frac{3V_0}{4},b_2=0,b_3=-\frac{V_0}{4},b_4=b_5=...=0$$

$$But\quad B_n=b_n\frac{l}{n\pi a}$$

$$\Rightarrow B_1=\frac{3V_0 l}{4\pi a},\quad B_2=0,\quad B_3=\frac{-Vl_0}{12\pi a},\quad B_5=0,B_6=0.....$$

Substitute these values in (4) we get

$$y(x,t)=\frac{3V_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi at}{l}-\frac{V_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l}$$

Problems on vibrating on strings when initial velocity and initial displacement are given

1. **Find the displacement of a tightly stretched string of a length 7cms vibrating between fixed end points if initial displacement is $10\sin\left(\frac{3\pi x}{7}\right)$ and initial velocity is $15\sin\left(\frac{9\pi x}{l}\right)$**

Solution :

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

From the given problem, we get the following boundary conditions.

$(i) y(0,t) = 0 \, \forall t > 0$

$(ii) y(l,t) = 0 \forall t > 0$

$(iii) y(x,0) = 10\sin\frac{3\pi x}{7}$

$(iv) \frac{\partial y(x,0)}{\partial t} = 15\sin\frac{9\pi x}{7}$

The correct solution which satisfies our boundary conditions is given by

$$y(x,t) = (A\cos px + B\sin px)(C\cos apt + D\sin apt) \to (1)$$

Apply (i) in (1) we get,

$y(0,t) = A(C\cos apt + D\sin apt)$

$\Rightarrow 0 = A(C\cos apt + D\sin apt)$

$\Rightarrow A = 0,\ C\cos apt + D\sin apt) \neq 0$

Putting A= 0 in (1) we get

$$y(x,t) = (B\sin px)(C\cos apt + D\sin apt) \rightarrow (2)$$

Applying (ii) in (2) we get

$$y(l,t) = (B\sin pl)(C\cos apt + D\sin apt)$$

$$0 = (B\sin pl)(C\cos apt + D\sin apt)$$

$$\text{If } B = 0 \text{ then we get trivial solution}$$

$$B \neq 0,\ (C\cos apt + D\sin apt) \neq 0$$

$$\therefore \sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$\Rightarrow p = \frac{n\pi}{l}$$

$$(2) \Rightarrow y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \rightarrow (3)$$

$$y(x,t) = \sin\frac{n\pi x}{l}\left(BC\cos\frac{n\pi at}{l} + BD\sin\frac{n\pi at}{l}\right)$$

$$y(x,t) = \sin\frac{n\pi x}{l}\left(B_n\cos\frac{n\pi at}{l} + C_n\sin\frac{n\pi at}{l}\right)$$

The general solution is

$$y(x,t) = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{l}\left(B_n\cos\frac{n\pi at}{l} + C_n\sin\frac{n\pi at}{l}\right) \rightarrow (4)$$

Apply (iii) in (4) we get

$$y(x,0) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l} = 10\sin\frac{3\pi x}{7}$$

$$B_1\sin\frac{\pi x}{l} + B_2\sin\frac{2\pi x}{l} + B_3\sin\frac{3\pi x}{7} + \ldots = 10\sin\left(\frac{3\pi x}{7}\right)$$ Equating like coefficients,

we get $B_3 = 10 \rightarrow (I)$

From (4) we get

$$\frac{\partial y(x,t)}{\partial t} = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(-B_n \cos\frac{n\pi at}{l}\frac{n\pi a}{l} + C_n \sin\frac{n\pi at}{l}.\frac{n\pi a}{l}\right) \to (5)$$

Apply (*iv*) *in* (5) *we get*

$$\frac{\partial y(x,0)}{\partial t} = \sum_{n=1}^{\infty} C_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l} = 15\sin\frac{9\pi x}{7}$$

$$\Rightarrow C_9 \frac{9\pi a}{l} = 15$$

$$C_9 = \frac{15l}{9\pi a} \to (II)$$

The remaining C_n*'s are zero*

Substitute (*I*) & (*II*) *in* (4)

$$\Rightarrow y(x,t) = 10\cos\frac{3\pi at}{l}\sin\frac{3\pi x}{l} + \frac{15l}{9\pi a}\sin\frac{9\pi at}{l}\sin\frac{9\pi x}{l}$$

$$l = 7 \Rightarrow y(x,t) = 10\cos\frac{3\pi at}{7}\sin\frac{3\pi x}{7} + \frac{105}{9\pi a}\sin\frac{9\pi at}{7}\sin\frac{9\pi x}{7}$$

One Dimensional Heat Equation

One dimensional heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$where\ \alpha^2 = \frac{k}{s\rho} is\ diffusivity\ of\ the\ material\ of\ the\ bar$$

1. **State any two laws which are assumed to derive one dimensional heat equation?**

   Solution:

   (i) Heat flows from higher to lower temperature

   (ii) The amount of Heat required to produce a given temperature change in a body is proportionality is known as the specific heat(s) of the conducting material.

2. **State the Fourier law of heat conduction.**

   Solution:

   The rate at which heat flows across any area is proportional to the area and to the temperature gradient normal to the curve. This constant of proportionality is known as thermal conductivity(k) of material. It is known as fourier law of heat conduction.

3. **Define temperature gradient**

   Solution:

   The rate of change of temperature with respect to the distance is called as temperature gradient.

4. **In one dimensional heat equation** $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ **what does** α^2 **refer to?**

Solution:

$\alpha^2 = \frac{k}{s\rho}$ *is diffusivity of the material*

where $k -$ *thermal conductivity*

$S -$ *specific heat capacity*

$\rho -$ *Density.*

5. **What are the possible solution of one dimensional heat equation?**

Solution:

The possible solutions are

$$(i)\, u(x,t) = Ae^{-\alpha^2 p^2 t}\left(B\cos px + C\sin px\right)$$

$$(ii)\, u(x,t) = Ae^{-\alpha^2 p^2 t}\left(Be^{px} + Ce^{-px}\right)$$

$$(iii)\, u(x,t) = Ae^{-\alpha^2 p^2 t}\left(Bx + c\right)$$

6. **What is the correct solution of one dimensional heat equation?**

Solution:

The correct solution is

$$u(x,t) = Ae^{-\alpha^2 p^2 t}\left(B\cos px + C\sin px\right)$$

Boundary and Initial conditions

One dimensional heat equation is given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$(i)\ u(0,t) = 0$

$(ii)\ u(l,t) = 0$

$(iii)\ u(x,0) = f(x)$

7. Derive the steady state solution of one dimensional heat equation

Solution:

The one dimensional heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

$$u(x,t) = u(x)$$

$$\Rightarrow \frac{\partial u}{\partial t} = 0$$

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} = 0$$

$$\Rightarrow \frac{d^2 u}{dx^2} = 0$$

$$u(x) = ax + b$$

8. What is the steady state temperature of a rod of length l whose ends are kept at 30^0 and 40^0

Solution:

The heat flow equation is $\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}$

When the steady state condition exist, $\frac{\partial u}{\partial t}=0$ ($\because$ u is independent of t)

Then the heat flow equation becomes $\frac{\partial^2 u}{\partial x^2}=0$

$$u=ax+b \qquad \rightarrow (1)$$

$$x=0, u=30 \Rightarrow b=30$$

$$x=l, u=40 \Rightarrow 40=al+30$$

$$a=\frac{10}{l}$$

$$(1) \Rightarrow u=\frac{10}{l}x+30$$

9. The bar of length 50 cm has its ends kept at 20 C and 100 C until steady state conditions prevails. Find the steady state temperature of the rod.

Solution:

The steady state equation of the one dimensional heat equation is

$$\frac{d^2 y}{dx^2}=0$$

$$\Rightarrow u(x)=ax+b \rightarrow (1)$$

$\textit{The boundary conditions are } (a)\, u(0)=20 \,\&\, (b)\, u(l)=100$

$$Applying\ (a)\ in\ (1)$$
$$u(0) = 20 \Rightarrow b = 20$$
$$substitute\ this\ value\ in\ (1)\ we\ get$$
$$u(x) = ax + 20 \rightarrow (2)$$
$$Applying\ (b)\ in(2)$$
$$u(l) = 100 \Rightarrow al + 20 = 100$$
$$\Rightarrow al = 80$$
$$\Rightarrow a = \frac{80}{l}$$

Substitute this value in (2) we get

$$u(x) = \frac{80x}{l} + 20$$
$$l = 50 \Rightarrow u(x) = \frac{80x}{50} + 20$$
$$u(x) = \frac{8x}{5} + 20$$

10. Write down the appropriate solution of one dimensional heat flow equation. How is it chosen?

Solution:

$$u(x,t) = (A\cos px + B\sin px)e^{-\alpha^2 p^2 t} \rightarrow (i)$$

Here as we are dealing with the heat equation u(x, t) representing the temperature at time t, u(x, t) must decreases as t increases.

i.e., u(x,t) cannot be define as $t \rightarrow \infty$

solution (i) is correct solution.

11. Define steady state.

Solution:

Steady state is the state in which the temperature does not vary with respect to the time.

Type I Zero Boundary Conditions

1. **A rod 1 cm long with insulated lateral surface is initially at temperature u_0 at an inner point distance *x cm* from one end. If both ends are kept at zero temperature find the temperature at any point of the rod at any time t.**

   Solution:

   The temperature function *u(x,t)* satisfies the partial differential equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

   The heat equation satisfying the boundary conditions are

   $(i)\ u(0,t) = 0$

   $(ii) u(l,t) = 0$

   $(iii) u(x,0) = u_0$

   The correct solution is

   $$u(x,t) = Ae^{-\alpha^2 p^2 t}\left(B\cos px + C\sin px\right) \to (1)$$

   $Apply\ (i)\, in\, (1)\, we\, get$

   $$u(0,t) = ABe^{-\alpha^2 p^2 t}$$

   $$0 = ABe^{-\alpha^2 p^2 t}$$

   $$B = 0\ \&\ e^{-\alpha^2 p^2 t} \neq 0, A \neq 0$$

   $$(1) \Rightarrow u(x,t) = AC\sin pxe^{-\alpha^2 p^2 t} \to (2)$$

   Applying (ii) in (2) we get

$$u(l,t) = AC\sin px.e^{-\alpha^2 p^2 t}$$

$$0 = AC\sin px.e^{-\alpha^2 p^2 t}$$

$$\Rightarrow \sin pl = 0$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

$$(2) \Rightarrow u(x,t) = AC\sin \frac{n\pi x}{l} e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)}$$

$$u(x,t) = B_n \sin \frac{n\pi x}{l} e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)}$$

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)} \rightarrow (3)$$

Apply (iii) in (3) we get,

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \rightarrow (4)$$

(4) Represents half range sine series

$$\text{where} \quad B_n = \frac{2}{l}\int_0^l u_0 \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2u_0}{l}\left[-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{-2u_0}{l}\left[(-1)^n - 1\right]$$

$$= 0 \quad \text{if} \quad n \quad \text{is} \quad \text{even}$$

$$= \frac{4u_0}{n\pi} \quad \text{if} \quad n \quad \text{is} \quad \text{odd}$$

$$(3) \Rightarrow u(x,t) = \frac{4u_0}{\pi}\sum_{n=1,3,5...}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l}e^{-\alpha^2\frac{n^2\pi^2 t}{l^2}}$$

$$u(x,t) = \frac{4u_0}{\pi}\sum_{n=1.}^{\infty}\frac{1}{(2n-1)}\sin\frac{(2n-1)\pi x}{l}e^{-\alpha^2\frac{n^2(2n-1)^2 t}{l^2}}$$

2. A uniform bar of length *l* through which heat flow is insulated at its sides. The ends are kept at zero temperature. If the initial temperature at the interior points of the bar is given by $k\left(lx - x^2\right),\ 0 < x < l$. Find the temperature distribution in the bar after time t.

Solution:

The heat equation is $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$

The heat equation satisfying the boundary conditions are

$$(i) u(0,t)=0$$
$$(ii) u(l,t)=0$$
$$(iii) u(x,0)=f(x)=k(lx-x^2)$$

The correct solution is

$$u(x,t)=Ae^{-\alpha^2 p^2 t}\left(B\cos px + C\sin px\right)\rightarrow(1)$$

Apply (i) *in* (1) *we get*

$$u(0,t)=ABe^{-\alpha^2 p^2 t}$$
$$0=ABe^{-\alpha^2 p^2 t}$$
$$B=0 \,\&\, e^{-\alpha^2 p^2 t}\neq 0, A\neq 0$$
$$(1)\Rightarrow u(x,t)=AC\sin px e^{-\alpha^2 p^2 t}\rightarrow(2)$$

Applying (ii) in (2) we get

$$u(l,t)=AC\sin px.e^{-\alpha^2 p^2 t}$$
$$0=AC\sin px.e^{-\alpha^2 p^2 t}$$
$$\Rightarrow \sin pl=0$$
$$pl=n\pi$$
$$p=\frac{n\pi}{l}$$
$$(2)\Rightarrow u(x,t)=AC\sin\frac{n\pi x}{l}e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)}$$
$$u(x,t)\ =B_n\sin\frac{n\pi x}{l}e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)}$$

The general solution is

$$u(x,t)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{l}e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)}\rightarrow(3)$$

Apply (iii) in (3) we get,

$$u(x,0)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$f(x)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \rightarrow (4)$$

(4) Represents half range sine series

$$B_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\int_0^l k(lx-x^2)\sin\frac{n\pi x}{l}$$

$$=\frac{2k}{l}\left[\left(lx-x^2\right)\frac{l}{n\pi}\left(-\cos\frac{n\pi x}{l}\right)+\left(l-2x\right)\sin\frac{n\pi x}{l}.\frac{l^2}{n^2\pi^2}-2\cos\frac{n\pi x}{l}\frac{l^3}{n^3\pi^3}\right]_0^l$$

$$=\frac{2k}{l}\left[-2(-1)^n\frac{l^3}{n^3\pi^3}+\frac{2l^3}{n^3\pi^3}\right]=\frac{4kl^2}{n^3\pi^3}\left[1-(-1)^n\right]=\begin{cases}\frac{8kl^2}{n^3\pi^3}, & n \text{ is odd}\\ 0, & n \text{ is even}\end{cases}$$

$$(3)\Rightarrow u(x,t)=\sum_{n=1,3,5,..}^{\infty}\frac{8kl^2}{n^3\pi^3}\sin\frac{n\pi x}{l}e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)}$$

3. A uniform bar of length *l* through which heat flow is insulated at its sides. The ends are kept at zero temperature. If the initial temperature at the interior points of the bar is given by $k\sin^3\frac{\pi x}{l},\ 0<x<l$. Find the temperature distribution in the bar after time t.

Solution:

The heat equation is $\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}$

The heat equation satisfying the boundary conditions are

$(i) u(0,t)=0$

$(ii) u(l,t)=0$

$(iii) u(x,0)=f(x)=k\sin^3\frac{\pi x}{l}$

The correct solution is

$$u(x,t)=Ae^{-\alpha^2p^2t}\left(B\cos px+C\sin px\right)\rightarrow(1)$$

$Apply\ (i)\ in\ (1)\ we\ get$

$$u(0,t)=ABe^{-\alpha^2p^2t}$$

$$0=ABe^{-\alpha^2p^2t}$$

$$B=0\ \&\ e^{-\alpha^2p^2t}\neq 0, A\neq 0$$

$$(1)\Rightarrow u(x,t)=AC\sin pxe^{-\alpha^2p^2t}\rightarrow(2)$$

Applying (ii) in (2) we get

$$u(l,t)=AC\sin px.e^{-\alpha^2p^2t}$$

$$0=AC\sin px.e^{-\alpha^2p^2t}$$

$$\Rightarrow\sin pl=0$$

$$pl=n\pi$$

$$p=\frac{n\pi}{l}$$

$$(2)\Rightarrow u(x,t)=AC\sin\frac{n\pi x}{l}e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)}$$

$$u(x,t)\ =B_n\sin\frac{n\pi x}{l}e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)}$$

The general solution is

$$u(x,t)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)} \rightarrow (3)$$

Apply (iii) in (3) we get,

$$u(x,0)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$f(x)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \rightarrow (4)$$

$$k\sin^3\frac{\pi x}{l}=B_1\sin\frac{\pi x}{l}+B_2\sin\frac{2\pi x}{l}+B_3\sin\frac{3\pi x}{l}+\ldots$$

$$\frac{k}{4}\left[3\sin\frac{\pi x}{l}-\sin\frac{3\pi x}{l}\right]=B_1\sin\frac{\pi x}{l}+B_2\sin\frac{2\pi x}{l}+B_3\sin\frac{3\pi x}{l}+\ldots$$

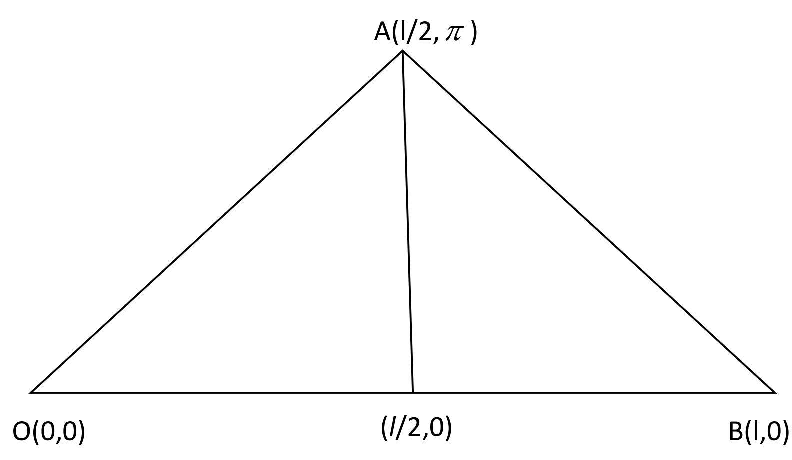
Equating the like coefficients we get

$$B_1=\frac{3k}{4}, B_2=0, B_3=\frac{k}{4}, \text{The remaining } B_n's \text{ are zero}$$

$$(3)\Rightarrow u(x,t)=\frac{3k}{4}\sin\frac{\pi x}{l}e^{-\left(\frac{\pi^2\alpha^2}{l^2}\right)t}-\frac{k}{4}\sin\frac{3\pi x}{l}e^{-\left(\frac{9\pi^2\alpha^2}{l^2}\right)t}$$

4. Find the temperature distribution in a homogeneous bar of length π which is insulated laterally, if the ends are kept at zero temperature and if initially, the temperature is k at the centre of the bar and falls uniformly to zero at its ends.

Solution:



The Equation of OA is

$$\frac{x}{\pi} = \frac{y}{k} \Rightarrow y = \frac{2kx}{\pi}$$

The Equation of AB is

$$\frac{x - \frac{\pi}{2}}{\frac{\pi}{2}} = \frac{y - k}{-k}$$

$$\Rightarrow y = \frac{2k}{\pi}(\pi - x)$$

$$u(x,0) = \begin{cases} \frac{2kx}{\pi}, 0 < x < \frac{\pi}{2} \\ \frac{2k}{\pi}(\pi - x), \pi < x < \pi \end{cases}$$

The heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

The heat equation satisfying the boundary conditions are

$$(i) u(0,t) = 0$$

$$(ii) u(l,t) = 0$$

$$(iii) u(x,0) = f(x) = \begin{cases} \frac{2kx}{\pi}, 0 < x < \frac{\pi}{2} \\ \frac{2k}{\pi}(\pi - x), \pi < x < \pi \end{cases}$$

The correct solution is

$$u(x,t) = Ae^{-\alpha^2 p^2 t}\left(B\cos px + C\sin px\right) \to (1)$$

$$Apply\ (i)\, in\, (1)\, we\, get$$

$$u(0,t) = ABe^{-\alpha^2 p^2 t}$$

$$0 = ABe^{-\alpha^2 p^2 t}$$

$$B = 0\ \&\ e^{-\alpha^2 p^2 t} \neq 0, A \neq 0$$

$$(1) \Rightarrow u(x,t) = AC\sin pxe^{-\alpha^2 p^2 t} \to (2)$$

Applying (ii) in (2) we get

$$u(l,t) = AC\sin px.e^{-\alpha^2 p^2 t}$$

$$0 = AC\sin px.e^{-\alpha^2 p^2 t}$$

$$\Rightarrow \sin pl = 0$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

$$(2) \Rightarrow u(x,t) = AC\sin\frac{n\pi x}{l} e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)}$$

$$u(x,t) \ = B_n \sin\frac{n\pi x}{l} e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)}$$

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)} \to (3)$$

$$l = \pi \Rightarrow u(x,t) = \sum_{n=1}^{\infty} B_n \sin nxe^{-\left(n^2\alpha^2 t\right)} \to (4)$$

$$Apply\, (iii)\, in\, (4)\, we\, get$$

$$f(x)=\sum_{n=1}^{\infty}B_n\sin nx$$

$$B_n=\int_0^{\pi}f(x)\sin nx\,dx$$

$$=\frac{4k}{\pi^2}\left[\int_0^{\frac{\pi}{2}}x\sin nx\,dx+\int_{\frac{\pi}{2}}^{\pi}(\pi-x)\sin nx\,dx\right]$$

$$=\frac{4k}{\pi^2}\left[\left(-x\cos nx.\frac{1}{n}+\sin nx\frac{1}{n^2}\right)_0^{\frac{\pi}{2}}+\left[-(\pi-x)\cos nx.\frac{1}{n}-\sin nx.\frac{1}{n^2}\right]_{\frac{\pi}{2}}^{\pi}\right]$$

$$=\frac{4k}{\pi^2}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2}+\frac{1}{n^2}\sin\frac{n\pi}{2}+\frac{\pi}{2n}\cos\frac{n\pi}{2}+\frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$=\frac{8k}{\pi^2}\left[\frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$(4)\Rightarrow u(x,t)=\frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin nx e^{-\alpha^2n^2t}$$

Type II Steady State Conditions and Zero Boundary Conditions

1. **An insulated rod of length 30 cm has its ends A and B kept at 20^0C and 80^0C respectively until steady state conditions prevail. The temperature at each end is then suddenly reduced to 0^0C and kept so. Find the resulting temperature distribution *u(x,t)* taking origin at A.**

   Solution:

The temperature function *u(x,t)* satisfies the partial differential equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ --------(1)

In steady state

First let us find the temperature distribution at any distance *x*, before the ends A and B are reduced to zero. Prior to the temperature change at the ends A and B, when *t =0*, the heat flow was independent of time (steady state conditions). When the temperature *u* depends only on *x* and not on *t*, (1) reduces to

$$\frac{d^2u}{dx^2} = 0 \qquad ------(2)$$

$$Integrating \;\; w.r.to \;\; x, \;\; \frac{du}{dx} = A$$

$$Again \;\; \text{int}\,egrating$$

$$u = Ax + B \qquad ----(3)$$

Where A and B are arbitrary constants.

Given that *u=20* when *x=0* ie, *u(0)=20*

From (3), we get *B=20*

u=80 when *x=30* ie, *u(30)=80*

From (3), *30A+B=80* we get *A=2*

$$\therefore (3) \;\; becomes \;\; u = 2x + 20$$

When the temperature at A and B are reduced to 0^0C, the state is no more steady state. For this **transient state** the boundary conditions are

$(i)\; u(0,t) = 0$

$(ii) u(l,t) = 0 \quad , l = 30$

$(iii) u(x,0) = 2x + 20$

The correct solution is

$$u(x,t) = Ae^{-\alpha^2 p^2 t}\left(B\cos px + C\sin px\right) \to (1)$$

$Apply\ (i)\, in\, (1)\, we\, get$

$$u(0,t) = ABe^{-\alpha^2 p^2 t}$$

$$0 = ABe^{-\alpha^2 p^2 t}$$

$$B = 0\ \&\ e^{-\alpha^2 p^2 t} \neq 0, A \neq 0$$

$$(1) \Rightarrow u(x,t) = AC\sin pxe^{-\alpha^2 p^2 t} \to (2)$$

Applying (ii) in (2) we get

$$u(l,t) = AC\sin px.e^{-\alpha^2 p^2 t}$$

$$0 = AC\sin px.e^{-\alpha^2 p^2 t}$$

$$\Rightarrow \sin pl = 0$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

$$(2) \Rightarrow u(x,t) = AC\sin\frac{n\pi x}{l}e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)}$$

$$u(x,t) \ = B_n \sin\frac{n\pi x}{l}e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)}$$

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}e^{-\left(\frac{n^2\pi^2\alpha^2}{l^2}t\right)} \to (3)$$

Apply (iii) in (3) we get,

$$u(x,0)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$f(x)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \rightarrow (4)$$

(4) Represents half range sine series

$$where \quad B_n = \frac{2}{l}\int_0^l (2x+20)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{2u_0}{l}\left[(2x+20)\frac{l}{n\pi}\left(-\cos\frac{n\pi x}{l}\right)-2\frac{l^2}{n^2\pi^2}\left(-\sin\frac{n\pi x}{l}\right)\right]_0^l$$

$$=\frac{-2u_0}{l}\left[\left\{-(2l+20)\frac{l}{n\pi}(-1)^n+0\right\}-\left\{-20\frac{l}{n\pi}+0\right\}\right]$$

$$=2\left[-(2l+20)\frac{(-1)^n}{n\pi}+\frac{20}{n\pi}\right]$$

$$=2\left[-80\frac{(-1)^n}{n\pi}+\frac{20}{n\pi}\right]$$

$$=\frac{40}{n\pi}\left[1-4(-1)^n\right]$$

Substituting in (3)

$$(3)\Rightarrow \qquad u(x,t)=\frac{4u_0}{\pi}\sum_{n=1.}^{\infty}\frac{1}{n}\left[1-4(-1)^n\right]\sin\frac{n\pi x}{30}e^{-\alpha^2\frac{n^2(2n-1)^2 t}{900}}$$

Type III Steady State conditions and Non-zero Boundary

1. **A rod of 30 cm has its ends A and B are kept at $20^0 c$ and $40^0 c$ respectively until steady state conditions prevails. The temperature at A is then suddenly raised to $90^0 c$ and the same**

time that B is lowered to 30^0c. Find the temperature distribution in the rod at time t. also show that the temperature at the midpoint of the rod remains unaltered for all time, regardless of the material of the rod.

Solution:

The heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \to (1)$$

when the steady state conditions prevails $\frac{\partial u}{\partial t} = 0$

$$(1) \Rightarrow \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$\Rightarrow \frac{d^2 u}{dx^2} = 0$$

$$\Rightarrow u(x) = ax + b \to (2)$$

Now the boundary conditions are

$(i)\, u(0) = 20$

$(ii)\, u(10) = 40$

Applying (i) in (2) we get

b = 20

$(2) \Rightarrow u(x) = ax + 20 \to (3)$

Applying (ii) in (3) we get

$a = 2$

$(3) \Rightarrow u(x) = 2x + 20$

Hence the steady state, the temperature function is given by

$$u(x) = 2x + 20$$

Now the temperature at A is raised to $50^0 c$ and the temperature at B is lowered to $10^o c$. That is , the steady state is changed to unsteady state. For this unsteady state the temperature distribution is given by

$$u(x) = 2x + 20$$

Now the new boundary conditions are

$$(i) u(0,t) = 50 \; \forall t > 0$$

$$(ii) u(10,t) = 10 \; \forall t > 0$$

$$(iii) u(x,0) = 2x + 20$$

The correct solution is

$$u(x,t) = (A\cos px + B\sin px)e^{-\alpha^2 p^2 t} \rightarrow (4)$$

Apply (i) and (ii) in (4) we get

$$u(x,t) = Ae^{-\alpha^2 p^2 t} = 50$$

$$u(l,t) = (A\cos pl + B\sin pl)e^{-\alpha^2 p^2 t} = 10$$

It is not possible to find the constants A & B. since we have infinite number of values for A & B. Therefore in this case we split the solution u(x,t) into two parts

$$u(x,t) = u_s(x) + u_t(x,t) \rightarrow (5)$$

Where $u_s(x)$ is a solution of the equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ and is a function of x alone and satisfying the conditions $u_s(0) = 50 \; \& \; u_s(10) = 10 \; \& \; u_t(x,t)$ is a transient solution satisfying (5) which decreases at t increases.

To find $u_s(x)$:

$u_s(x) = a_1 x + b_1 \rightarrow (6)$

Applying the condition $u_s(0) = b_1 = 50$

$(6) \Rightarrow a_1 x + 50 \rightarrow (7)$

Applying the condition $u_s(10) = 10a_1 + 50 = 10$

$\Rightarrow a_1 = -4$

$(7) \Rightarrow u_s(x) = -4x + 50$

To find $u_t(x,t)$:

$u(x,t) = u_s(x) + u_t(x,t)$

$\Rightarrow u_t(x,t) = u(x,t) - u_s(x) \rightarrow (9)$

Now we have to find the boundary conditions for $u_t(x,t)$

Putting x = 0 in (9) we get

$u_t(0,t) = u(0,t) - u_s(0)$

$= 50 - 50$

$u_t(0,t) = 0$

Putting $x = 10$ *in* (9) *we get*

$u_t(10,t) = u(10,t) - u_s(0)$

$= 10 - 10$

$u_t(10,t) = 0$

Putting t = 0 in (9) we get

$u_t(x,0) = u(x,0) - u_s(x)$

$= 2x + 20 + 4x - 50$

$u_t(x,0) = 6x - 30$

Now the function $u_t(x,t)$ we have the following boundary conditions

$(i)\, u_t(0,t) = 0$

$(ii)\, u_t(10,t) = 0$

$(iii)\, u_t(x,0) = 6x - 30$

Applying the first two conditions we get the general solution as

$$u_t(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} e^{-\left(\frac{n^2\pi^2\alpha^2}{100}t\right)} \rightarrow (10)$$

Applying the (iii) in equation (10) we get

$$u_t(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} = 6x - 30$$

$$b_n = \frac{2}{10}\int_0^{10} (6x-30)\sin\frac{n\pi x}{10}\, dx$$

$$= \frac{6}{5}\int_0^{10} (x-5)\sin\frac{n\pi x}{10}\, dx$$

$$= \frac{6}{5}\left[-(x-5)\cos\frac{n\pi x}{10}.\frac{10}{n\pi} + \sin\frac{n\pi x}{10}.\frac{100}{n^2\pi^2}\right]_0^{10}$$

$$= \frac{6}{5}\left[\frac{-50}{n\pi}(-1)^n - \frac{50}{n\pi}\right]$$

$$= -\frac{60}{n\pi}\left[1 + (-1)^n\right]$$

$$= \begin{cases} 0 & :\text{n is odd} \\ -\dfrac{120}{n\pi} & :\text{ n is even} \end{cases}$$

$$(10) \Rightarrow u_t(x,t) = \sum_{n=2,4,6,..}^{\infty} \frac{-120}{n\pi}\sin\frac{n\pi x}{10} e^{-\left(\frac{n^2\pi^2\alpha^2}{100}t\right)} \rightarrow (11)$$

Substitute (8) and (11) in (5) we get

$$\text{u}(x,t) = -4x + 50 + \sum_{n=2,4,6,..}^{\infty} \frac{-120}{n\pi}\sin\frac{n\pi x}{10} e^{-\left(\frac{n^2\pi^2\alpha^2}{100}t\right)}$$

2. Find the solution of the one dimensional diffusion equation satisfying the boundary conditions

(i) u is bounded as $t \to \infty$

$(ii)\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0 \; \forall t$

$(iii)\left(\frac{\partial u}{\partial x}\right)_{x=a} = 0 \forall t$

$(iv) u(x,0) = x(a-x)$

Solution:

The correct solution is

$$u(x,t) = (A\cos px + B\sin px)e^{-\alpha^2 p^2 t} \to (A)\alpha^2$$

Diff partially w.r.t x we get

$$\frac{\partial u(x,t)}{\partial x} = (-Ap\sin px + Bp\cos px)e^{-\alpha^2 p^2 t} \to (1)$$

Apply (ii) *in* (1) *we get*

$$\frac{\partial u(0,t)}{\partial x} = Bpe^{-\alpha^2 p^2 t} = 0$$

$$\Rightarrow B = 0, p \neq 0 \;\&\; e^{-\alpha^2 p^2 t} = 0$$

$$\Rightarrow \sin ap = 0$$

$$p = \frac{n\pi}{a}$$

substitute this value of $p = \frac{n\pi}{a}$ & $B = 0$ *in* (A) *we get*

$$u(x,t) = A\cos\frac{n\pi x}{a}e^{-\frac{\alpha^2 n^2 \pi^2}{a^2}t}$$

The general solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{a}e^{-\frac{\alpha^2 n^2 \pi^2}{a^2}t}$$

$$= A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{a}e^{-\frac{\alpha^2 n^2 \pi^2}{a^2}t}$$

Applying (iv) in this equation we get

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{a} = x(a-x)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{a} = x(a-x)$$

$$a_0 = \frac{2}{a}\int_0^a x(a-x)\,dx$$

$$= \frac{2}{a}\int_0^a \left(ax - x^2\right)dx$$

$$= \frac{2}{a}\left[\frac{ax^2}{2} - \frac{x^3}{3}\right]_0^a$$

$$= \frac{2}{a}\left[\frac{a^3}{2} - \frac{a^3}{3}\right] = \frac{2a^3}{6a} = \frac{a^2}{3}$$

$$a_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx$$

$$= \frac{2}{a}\int_0^a \left(ax - x^2\right)\cos\frac{n\pi x}{a}\,dx$$

$$= \frac{2}{a}\left[\left(ax - x^2\right)\sin\frac{n\pi x}{a}.\frac{a}{n\pi} + (a-2x)\cos\frac{n\pi x}{a}.\frac{a^2}{n^2\pi^2} + 2\sin\frac{n\pi x}{a}.\frac{a^3}{n^3\pi^3}\right]_0^a$$

$$= \frac{2}{a}\left[\frac{-a^3}{n^2\pi^2}(-1)^n - \frac{a^3}{n^2\pi^2}\right]$$

$$= -\frac{2a^2}{n^2\pi^2}\left[(-1)^n + 1\right]$$

$$= \begin{cases} -\dfrac{4a^2}{n^2\pi^2} & : n \text{ is even} \\ 0 & : n \text{ is odd} \end{cases}$$

$$u(x,t) = \frac{a^2}{6} + \sum_{2,4,6,\ldots}^{\infty} -\frac{4a^2}{n^2\pi^2}\cos\frac{n\pi x}{a}\, e^{-\left(\frac{\alpha^2 n^2 \pi^2}{a^2}t\right)}$$

3. Solve the equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ subject to the following conditions:

$(i) u \, is \; finite \, when \; t \to \infty$

$(ii) \left(\frac{\partial u}{\partial x} \right) = 0, when \, x = 0 \; \forall t > 0$

$(iii) u = 0 \, when \, x = l \; \forall t > 0$

$(iv) u = u_0 \; when \, t = 0 \; for \, all \, values \, of \; x \, between \, 0 \, \& \, l$

Solution:

The correct solution is

$$u(x,t) = (A\cos px + B\sin px)e^{-\alpha^2 p^2 t} \to (1)$$

Diff partially w.r.t. to x we get

$$\frac{\partial u(x,t)}{\partial x} = (-Ap\sin px + Bp\cos px)e^{-\alpha^2 p^2 t} \to (2)$$

$Apply \, (ii) \, in \, (2) \, we \, get$

$$\frac{\partial u(0,t)}{\partial x} = Bpe^{-\alpha^2 p^2 t} = 0$$

$$\Rightarrow B = 0, p \neq 0 \,\&\, e^{-\alpha^2 p^2 t} \neq 0$$

$$(2) \Rightarrow u(x,t) = A\cos pxe^{-\alpha^2 p^2 t} \to (3)$$

$Apply \, (iii) \, in \, (3) \, we \, get$

$$u(x,t) = A\cos ple^{-\alpha^2 p^2 t} = 0$$

$$\Rightarrow \cos pl = 0$$

$$pl = an \, odd \, multiple \, of \, \frac{\pi}{2}$$

$$pl = (2n-1)\frac{\pi}{2}$$

$$\Rightarrow p = \frac{(2n-1)\pi}{2l}$$

Substitute this value in (3) we get

$$u(x,t)=\sum_{n=1}^{\infty} A\cos\frac{(2n-1)\pi x}{2l}e^{-\alpha^2\frac{(2n-1)^2\pi^2}{4l^2}t}\to(4)$$

$Apply\,(iv)\,in\,(4)\,we\,get,$

$$u(x,0)=\sum_{n=1}^{\infty} A_n\cos\frac{(2n-1)\pi x}{2l}=u_0$$

$$u_0=A_1\cos\frac{\pi x}{2l}+A_2\cos\frac{3\pi x}{2l}+\dots+A_n\cos\frac{(2n-1)\pi x}{2l}+\dots\to(5)$$

Multiplying both sides of (5) by $\cos\frac{(2n-1)\pi x}{2l}$ and then integrating from 0 to l we get

$$\int_0^l u_0\cos\frac{(2n-1)\pi x}{2l}dx=\int_0^l A_1\cos\frac{\pi x}{2l}\cos\frac{(2n-1)\pi x}{2l}dx+\int_0^l A_2\cos\frac{3\pi x}{2l}\cos\frac{(2n-1)\pi x}{2l}dx$$

$$+\dots+\int_0^l A_n\cos^2\frac{(2n-1)\pi x}{2l}dx\to(6)$$

W.K.T $\int_0^l \cos mx\cos nx=0\ if\ \ m\neq n.$ Applying this in equation (6) we get all terms in R.H.S. of (6) becomes zero except the term $\int_0^l A_n\cos^2\frac{(2n-1)\pi x}{2l}dx$

Hence the equation (6) reduces to

$$\int_0^l u_0\cos\frac{(2n-1)\pi x}{2l}dx=A_n\int_0^l\cos^2\frac{(2n-1)\pi x}{2l}dx$$

$$L.H.S. = u_0 \int_0^l \cos\frac{(2n-1)\pi x}{2l}\,dx = \left[\frac{\sin\frac{(2n-1)\pi x}{2l}}{\frac{(2n-1)\pi x}{2l}}\right]_0^l$$

$$= \frac{2lu_0}{(2n-1)\pi}\sin\frac{(2n-1)\pi}{2}$$

$$= \frac{2lu_0}{(2n-1)\pi}\sin\left(n\pi - \frac{\pi}{2}\right)$$

$$= \frac{2lu_0}{(2n-1)\pi}\left[\sin n\pi\cos\frac{\pi}{2} - \cos n\pi\sin\frac{\pi}{2}\right]$$

$$= \frac{2lu_0}{(2n-1)\pi}(-1)^{n+1}$$

$$R.H.S = A_n \int_0^l \cos^2\frac{(2n-1)\pi x}{2l}dx$$

$$= A_n \int_0^l \frac{1}{2}\left[1 + \frac{\cos(2n-1)\pi x}{l}\,dx\right]$$

$$= \frac{A_n}{2}\left[x + \frac{\sin\frac{(2n-1)\pi x}{l}}{\frac{(2n-1)\pi}{l}}\right]_0^l$$

$$= \frac{A_n l}{2}$$

$$\Rightarrow \frac{2lu_0}{(2n-1)\pi}(-1)^{n+1} = \frac{A_n l}{2}$$

$$A_n = \frac{4u_0(-1)^{n+1}}{(2n-1)\pi}$$

$$(4) \Rightarrow u(x,t) = \sum_{n=1}^{\infty}\frac{4u_0(-1)^{n+1}}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2l}\,e^{-\alpha^2\frac{(2n-1)^2\pi^2}{4l^2}t}$$

Two dimensional heat flow equation

The two dimensional heat equation is $\frac{\partial u}{\partial t} = \alpha^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

But in steady state

$$u(x,t) = u(x)$$

$$\Rightarrow \frac{\partial u}{\partial t} = 0$$

$$\alpha^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \alpha^2 \neq 0$$

Which is the Laplace equation in two dimensions.

1. **What are the possible solutions of two dimensional heat equation or laplace equation?**

Solution:

$$(i)\, u(x,y) = \left(c_1 e^{px} + c_2 e^{-px}\right)\left(c_3 \cos py + c_4 \sin py\right)$$

$$(ii)\, u(x,y) = \left(c_5 \cos px + c_6 \sin px\right)\left(c_7 e^{py} + c_8 e^{-py}\right)$$

$$(iii)\, u(x,y) = \left(c_9 x + c_{10}\right)\left(c_{11} y + c_{12}\right)$$

2. **Write any two solutions of the laplace equation** $u_{xx} + u_{yy} = 0$ **involving exponential term x or y**

   Solution :

$$(i)\, u(x,y) = \left(c_1 e^{px} + c_2 e^{-px}\right)\left(c_3 \cos py + c_4 \sin py\right)$$

$$(ii)\, u(x,y) = \left(c_5 \cos px + c_6 \sin px\right)\left(c_7 e^{py} + c_8 e^{-py}\right)$$

Type-I: Temperature Distribution in Infinite Plates

Type – A:Temperature distribution along *x*-axis or parallel to *x*- axis.

1. **An infinitely long plane is bounded by two parallel edges *x=0* and *x=l* and an end at right angles to them. The breadth of this edge *y=0* is *l* and is maintained at temperature *f(x)*. All the other three edges are at temperature zero. Find the steady state temperature at any interior point of the plate.**

Solution:

Let $u(x, y)$ be the temperature at any point (x , y) in the steady state. Then u satisfies the differential equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

The boundary conditions are

(i) $u(0, y)=0 , \forall y$

(ii) $u(l, y) = 0 \ \ \forall y$

(iii) $u(x,\infty)=0 , 0< x< 10$

(iv) $u(x,0) = f(x) \quad 0 < x < l$

The correct solution should be

$$u(x, y)=(A\cos px + B\sin px)(Ce^{py} + De^{-py}) \rightarrow (1)$$

Applying (i) in (1), we get

$$u(0, y)=A(Ce^{py} + De^{-py}) =0$$

$$\Rightarrow A=0$$

Substitute A = 0 in eqn(1)

$$u\left(x, y\right)= B\sin px\left(Ce^{py} + De^{-py}\right)\rightarrow\left(2\right)$$

Applying (i) in (2), we get

$$u(l, y) = B\sin pl(Ce^{py} + De^{-py})$$

$$0 = B\sin lp(Ce^{py} + De^{-py}), \;\; here \;\; (Ce^{py} + De^{-py}) \neq 0, \;\; B \neq 0$$

$$\Rightarrow \sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$p\pi = n\pi$$

$$p = \frac{n\pi}{l} \qquad n = 1,2,3......$$

$$u(x, y) = B\sin nx\left(Ce^{ny} + De^{-ny}\right) \qquad ----(3)$$

Applying (iii) in eqn (3)

$$u(x,\infty) = B\sin\frac{n\pi x}{l}\left[Ce^{\infty} + De^{-\infty}\right] = 0$$

$$= B\sin\frac{n\pi x}{l}\left[Ce^{\infty}\right] = 0 \qquad \because e^{-\infty} = 0$$

Here, $B \neq 0$, $\sin\frac{n\pi x}{l} \neq 0$, $e^{-\infty} \neq 0$

$$\therefore C = 0$$

Substitute C=0 in eqn (3)

$$u(x, y) = B\sin\frac{n\pi x}{l}De^{\frac{-n\pi y}{l}}$$

$$= BD\sin\frac{n\pi x}{l}e^{\frac{-n\pi y}{l}} \qquad -----(4)$$

The most general solution is

$$u(x,y)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{l}e^{\frac{-n\pi y}{l}} \qquad ------(5)$$

Applying condition (iv) in eqn (5)

$$u(x,0)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{l} = f(x) \quad ------(6)$$

To find B_n then we expand $f(x)$ as a Fourier half range sine series in (0, 10)

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l} \qquad ------(7)$$

Where $b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$

Substituting this value in (5) , we get the required solution as

$$u(x,y)=\sum_{n=1}^{\infty}\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\sin\frac{n\pi x}{l}e^{\frac{-n\pi y}{l}}dx$$

2. **A rectangular plate with insulated surfaces is 10cm width and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along are short edge** $y=0$ **is** $u(x,y)=4(10x-x^2)\ \ for\, 0<x<10$ **while the two long edges as well as the short edge are kept at** 0^0 **C , find the steady state temperature function** $u(x,y)$ **.**

   Solution:

Let $u(x, y)$ be the temperature at any point (x , y) in the steady state. Then u satisfies the differential equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

The boundary conditions are

(i) $u(0, y)=0, \forall y$

(ii) $u(10, y)=0, \forall y$

(iii) $u(x, \infty)=0, 0<x<10$

(iv) $u(x,0)=4\left(10x-x^2\right), 0<x<10$

The correct solution should be

$$u(x, y)=(A\cos px+B\sin px)(Ce^{py}+De^{-py}) \rightarrow (1)$$

Applying (i) in (1), we get

$$u(0, y)=A(Ce^{py}+De^{-py})=0$$

$$\Rightarrow A=0$$

Substitute A = 0 in eqn(1)

$$u(x, y)=B\sin px\left(Ce^{py}+De^{-py}\right) \rightarrow (2)$$

Applying (ii) in (2) , we get

$$u(10, y)=B\sin 10p\left(Ce^{py}+De^{-py}\right)$$

$$0=B\sin 10p\left(Ce^{py}+De^{-py}\right), \quad Here\ \left(Ce^{py}+De^{-py}\right)\neq 0, B\neq 0$$

$$\Rightarrow \sin 10p=0$$

$$\sin 10p=\sin n\pi$$

$$10p=n\pi$$

$$p=\frac{n\pi}{10}$$

$$\therefore u(x,y) = B\sin\frac{n\pi x}{10}\left(Ce^{\frac{n\pi y}{10}} + De^{-\frac{n\pi y}{10}}\right) \rightarrow (3)$$

Applying (iii) in eqn (3)

$$u(x,\infty) = B\sin\frac{n\pi x}{10}\left[Ce^{\infty} + De^{-\infty}\right] = 0$$

$$= B\sin\frac{n\pi x}{10}\left[Ce^{\infty}\right] = 0 \qquad \because e^{-\infty} = 0$$

Here, $B \neq 0$, $\sin\frac{n\pi x}{10} \neq 0$, $e^{-\infty} \neq 0$

$$\therefore C = 0$$

Substitute C=0 in eqn (3)

$$u(x,y) = B\sin\frac{n\pi x}{10} De^{\frac{-n\pi y}{10}}$$

$$= BD\sin\frac{n\pi x}{10} e^{\frac{-n\pi y}{10}} \qquad -----(4)$$

The most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} e^{\frac{-n\pi y}{10}} \qquad -----(5)$$

Applying condition (iv) in eqn (5)

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} = 4(10x - x^2) \quad -----(6)$$

To find B_n then we expand $f(x)$ as a Fourier half range sine series in (0, 10)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad -----(7)$$

Where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$

From (6)&(7) , we get $B_n = b_n$

$$B_n = \frac{2}{10}\int_0^{10} 4(10x - x^2)\sin\frac{n\pi x}{10}dx$$

$$= \frac{4}{5}\left\{(10x - x^2)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - (10-2x)\left(\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{10}}{\frac{n^3\pi^3}{1000}}\right)\right\}_0^{10}$$

$$= \frac{4}{5}\left\{-\frac{2000}{n^3\pi^3}(-1)^n + \frac{2000}{n^3\pi^3}\right\}$$

$$= \frac{1600}{n^3\pi^3}\left[1 - (-1)^n\right]$$

$$\therefore B_n = \begin{cases} \frac{3200}{n^3\pi^3} \ , & \text{for } n \text{ is odd} \\ 0 \ , & \text{for } n \text{ is even} \end{cases}$$

Substituting the value of B_n in equation (5)

$$\therefore u(x,y) = \frac{3200}{\pi^3}\sum_{n=1,3,..}\frac{1}{n^3}\sin\frac{n\pi x}{10}e^{-\frac{n\pi y}{10}}$$

3. A rectangular plate with insulated surfaces is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along the short edge *y=0* is given by $u(x,0) = 20x,\ \ 0 \le x \le 5,\ u(x,0) = 20(10-x),\ \ 5 \le x \le 10,$ **while the two long edges *x=0* and *x=10* as well as the other short**

edge are kept at 0^0 C. Find the temperature function $u(x, y)$ in the steady state at any point of the plate.

Solution:

Let $u(x, y)$ be the temperature at any point (x , y) in the steady state. Then u satisfies the differential equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

The boundary conditions are

(i) $u(0, y)=0, \forall y$

(ii) $u(10, y) = 0 \ \forall y$

(iii) $u(x,\infty)=0, 0< x< 10$

(iv) $u(x,0) = \begin{cases} 20x & 0 \le x \le 5 \\ 20(10-x) & 5 \le x \le 10 \end{cases} \quad = f(x)$

The correct solution should be

$$u(x, y)=(A\cos px+B\sin px)(Ce^{py}+De^{-py}) \rightarrow (1)$$

Applying (i) in (1), we get

$$u(0, y)=A(Ce^{py}+De^{-py}) =0$$

$$\Rightarrow A=0$$

Substitute A = 0 in eqn(1)

$$u\left(x, y\right)= B\sin px\left(Ce^{py}+De^{-py}\right)\rightarrow\left(2\right)$$

Applying (ii) in (2) , we get

$$u(10, y) = B\sin 10p\left(Ce^{py} + De^{-py}\right)$$

$$0 = B\sin 10p\left(Ce^{py} + De^{-py}\right), \quad Here \left(Ce^{py} + De^{-py}\right) \neq 0, B \neq 0$$

$$\Rightarrow \sin 10p = 0$$

$$\sin 10p = \sin n\pi$$

$$10p = n\pi$$

$$p = \frac{n\pi}{10}$$

$$\therefore u(x, y) = B\sin\frac{n\pi x}{10}\left(Ce^{\frac{n\pi y}{10}} + De^{-\frac{n\pi y}{10}}\right) \to (3)$$

Applying (iii) in eqn (3)

$$u(x, \infty) = B\sin\frac{n\pi x}{10}\left[Ce^{\infty} + De^{-\infty}\right] = 0$$

$$= B\sin\frac{n\pi x}{10}\left[Ce^{\infty}\right] = 0 \qquad \because e^{-\infty} = 0$$

Here, $B \neq 0$, $\sin\frac{n\pi x}{10} \neq 0$, $e^{-\infty} \neq 0$

$$\therefore C = 0$$

Substitute C=0 in eqn (3)

$$u(x, y) = B\sin\frac{n\pi x}{10} De^{\frac{-n\pi y}{10}}$$

$$= BD\sin\frac{n\pi x}{10} e^{\frac{-n\pi y}{10}} \qquad ------(4)$$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} e^{\frac{-n\pi y}{10}} \qquad ------(5)$$

Applying condition (iv) in eqn (5)

$$u(x,0)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{10} \quad = f(x) \;\; ------(6)$$

To find B_n then we expand $f(x)$ as a Fourier half range sine series in (0, 10)

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad\qquad ------(7)$$

Where $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$

$$=\frac{1}{5}\int_0^{10} f(x)\sin\frac{n\pi x}{10}dx$$

$$=\frac{1}{5}\left[\int_0^{10} 20x\sin\frac{n\pi x}{10}dx+\int_5^{10} 20(10-x)\sin\frac{n\pi x}{10}dx\right]$$

$$=4\left[\begin{array}{l}\left\{x.\frac{10}{n\pi}\left(-\cos\frac{n\pi x}{10}\right)-(1)\frac{100}{n^2\pi^2}\left(-\sin\frac{n\pi x}{10}\right)\right\}_0^5+\\ \left\{(10-x).\frac{10}{n\pi}\left(-\cos\frac{n\pi x}{10}\right)-(-1)\frac{100}{n^2\pi^2}\left(-\sin\frac{n\pi x}{10}\right)\right\}_5^{10}\end{array}\right]$$

$$=4\left[\frac{-50}{n\pi}\cos\frac{n\pi}{2}+\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}-(0+0)+(0+0)-\left(\frac{-50}{n\pi}\cos\frac{n\pi}{2}+\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]$$

$$=\frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$A_n=0 \quad if \quad n=2,4,6.....$$

$$\therefore A_n=\frac{800}{(2n-1)^2\pi^2}\sin\frac{(2n-1)\pi}{2}$$

$$\therefore (5)\Rightarrow u(x,y)=\frac{800}{\pi^2}\sum_{n=1,3,5...}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}e^{-\frac{n\pi y}{10}}$$

$$u(x,y)=\frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\sin\frac{(2n-1)\pi}{2}\sin\frac{(2n-1)\pi x}{10}e^{-\frac{(2n-1)\pi y}{10}}$$

4. **An infinitely long uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π. This end is maintained at a temperature at a temperature u_0 at all points and other edges are kept at zero temperature.**

Determine the temperature at any point of the plate in the steady state.

Solution:

Let $u(x,y)$ be the temperature at any point (x , y) in the steady state. Then u satisfies the differential equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

The boundary conditions are

(i) $u(0,y)=0, \forall y$

(ii) $u(\pi,y)=0 \ \forall y$

(iii) $u(x,\infty)=0 \quad ,0<x<\pi$

(iv) $u(x,0)=u_0 \quad ,0<x<\pi$

The correct solution should be

$$u(x,y)=(A\cos px+B\sin px)(Ce^{py}+De^{-py}) \rightarrow (1)$$

Applying (i) in (1), we get

$$u(0,y)=A(Ce^{py}+De^{-py})=0$$

$$\Rightarrow A=0$$

Substitute A = 0 in eqn(1)

$$u(x,y)=B\sin px\left(Ce^{py}+De^{-py}\right)\rightarrow(2)$$

Applying (ii) in (2) , we get

$$u(\pi, y) = B\sin p\pi(Ce^{py} + De^{-py})$$

$$0 = B\sin \pi p(Ce^{py} + De^{-py}), \ \ here \ \ (Ce^{py} + De^{-py}) \neq 0, \ B \neq 0$$

$$\Rightarrow \sin p\pi = 0$$

$$\sin p\pi = \sin n\pi$$

$$p\pi = n\pi$$

$$p = n \qquad n = 1,2,3......$$

$$u(x, y) = B\sin nx\left(Ce^{ny} + De^{-ny}\right) \qquad ----(3)$$

Applying (iii) in eqn (3)

$$u(x, \infty) = B\sin nx\left[Ce^{\infty} + De^{-\infty}\right] = 0$$

$$= B\sin nx\left[Ce^{\infty}\right] = 0 \qquad \because e^{-\infty} = 0$$

Here, $B \neq 0$, $\sin\dfrac{n\pi x}{10} \neq 0$, $e^{-\infty} \neq 0$

$$\therefore C = 0$$

Substitute C=0 in eqn (3)

$$u(x, y) = B\sin nxDe^{-ny}$$

$$= BD\sin nxe^{-ny} \qquad -----(4)$$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin nxe^{-ny} \qquad -----(5)$$

Applying condition (iv) in eqn (5)

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin nx \ \ = u_0 \ \ -----(6)$$

To find B_n then we expand $f(x)$ as a Fourier half range sine series in (0, π)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad -----(7)$$

Where $b_n = \frac{2}{l}\int_0^l f(x)\sin nx dx$

$$= \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx dx$$

$$= \frac{2u_0}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}$$

$$= \left.\begin{array}{l} 0 \quad when \;\; n \;\; is \;\; even \\ \frac{4u_0}{n\pi} \quad when \;\; n \;\; is \;\; odd \end{array}\right\} \qquad ------(6)$$

Substituting (6) in (5), we get

$$u(x,y) = \frac{4u_0}{\pi} \sum_{n=1,3,5...}^{\infty} \frac{\sin nx}{n} e^{-ny}$$

when n is odd only

$$\therefore \quad u(x,y) = \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)} e^{-(2n-1)y}$$

5. A rectangular plate with insulated surfaces is 8 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along the short edge *y=0* is given by

$u(x,0)=100\sin\frac{\pi x}{8},\ \ 0\le x\le 8,$ **while the two long edges *x=0* and *x=l* as well as the other short edge are kept at 0^0 C. Find the temperature function $u(x,y)$ in the steady state at any point of the plate.**

Solution:

Let $u(x,y)$ be the temperature at any point (x , y) in the steady state. Then u satisfies the differential equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

The boundary conditions are

(i) $u(0,y)=0, \forall y$

(ii) $u(8,y)=0 \ \ \forall y$

(iii) $u(x,\infty)=0 \ \ , \ \ 0<x<8$

(iv) $u(x,0)=100\sin\frac{\pi x}{8} \quad 0<x<8$

The correct solution should be

$$u(x,y)=(A\cos px+B\sin px)(Ce^{py}+De^{-py})\rightarrow(1)$$

Applying (i) in (1), we get

$$u(0,y)=A(Ce^{py}+De^{-py})=0$$

$$\Rightarrow A=0$$

Substitute A = 0 in eqn(1)

$$u\left(x,y\right)=B\sin px\left(Ce^{py}+De^{-py}\right)\rightarrow\left(2\right)$$

Applying (ii) in (2) , we get

$$u(8,y)=B\sin 8p(Ce^{py}+De^{-py})$$

$$0=B\sin 8p(Ce^{py}+De^{-py}),\ \ here\ \ (Ce^{py}+De^{-py})\neq 0,\ \ B\neq 0$$

$$\Rightarrow \sin 8p=0$$

$$\sin 8p=\sin n\pi$$

$$8p=n\pi$$

$$p=\frac{n\pi}{8} \qquad n=1,2,3......$$

$$u(x,y)=B\sin\frac{n\pi x}{8}\left(Ce^{\frac{n\pi y}{8}}+De^{-\frac{n\pi y}{8}}\right) \qquad ----(3)$$

Applying (iii) in eqn (3)

$$u(x,\infty)=B\sin\frac{n\pi x}{8}\left[Ce^{\infty}+De^{-\infty}\right]=0$$

$$=B\sin\frac{n\pi x}{8}\left[Ce^{\infty}\right]=0 \qquad \because e^{-\infty}=0$$

Here, $B\neq 0$, $\sin\frac{n\pi x}{8}\neq 0$, $e^{-\infty}\neq 0$

$$\therefore C=0$$

Substitute C=0 in eqn (3)

$$u(x,y)=B\sin\frac{n\pi x}{8}De^{\frac{-n\pi y}{8}}$$

$$=BD\sin\frac{n\pi x}{8}e^{\frac{-n\pi y}{8}} \qquad -----(4)$$

The most general solution is

$$u(x,y)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{8}e^{\frac{-n\pi y}{8}} \qquad -----(5)$$

Applying condition (iv) in eqn (5)

$$u(x,0)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{8}$$

$$100\sin\frac{nx}{8}=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{8}$$

$$\therefore A_1 = 100, \quad A_2 = A_3 = ... = 0$$

$$u(x,y)=100\sin\frac{\pi x}{8} e^{-\frac{\pi y}{8}}$$

Type – B Temperature distribution along *y*- axis or parallel to *y* – axis

1. **A infinitely long rectangular plate with insulated surfaces is 10 cm wide. The two long edges and short edge are kept at zero temperature, while the other short edge *x=0* kept at temperature by $u(y,0)=20y, \ 0\le y\le 5,$ and $u(y,0)=20(10-y), \ 5\le y\le 10$ Find the function $u(x,y)$ in the steady state at any point of the plate.**

Solution:

Let $u(x,y)$ be the temperature at any point (x , y) in the steady state. Then u satisfies the differential equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

The boundary conditions are

(i) $u(x,0)=0 \quad \forall y$

(ii) $u(x,10)=0 \quad \forall y$

(iii) $u(\infty, y) = 0$

(iv) $u(0, y) = \begin{cases} 20y & 0 \le y \le 5 \\ 20(10-y) & 5 \le y \le 10 \end{cases} \quad = f(y) \ say$

The correct solution should be

$$u(x, y) = \left(Ae^{px} + Be^{-px}\right)\left(C\cos py + D\sin py\right) \quad \rightarrow (1)$$

Applying (i) in (1), we get

$$u(x,0) = \left(Ae^{px} + Be^{-px}\right)C = 0$$

$$\Rightarrow C = 0$$

Substitute C = 0 in eqn(1)

$$u(x, y) = \left(Ae^{px} + Be^{-px}\right)D\sin py \rightarrow (2)$$

Applying (ii) in (2) , we get

$$u(x,10) = \left(Ae^{px} + Be^{-px}\right)D\sin 10p$$

$$0 = \left(Ae^{px} + Be^{-px}\right)D\sin 10p \qquad Here \ \left(Ae^{px} + Be^{-px}\right) \ne 0,\ D \ne 0$$

$$\Rightarrow \sin 10P = 0$$

$$10p = n\pi$$

$$p = \frac{n\pi}{10}$$

$$\therefore u(x, y) = \left(Ae^{\frac{n\pi x}{10}} + Be^{-\frac{n\pi x}{10}}\right)D\sin\frac{n\pi x}{10} \rightarrow (3)$$

Applying (iii) in eqn (3)

$$u(\infty, y) = \left(Ae^{\infty} + Be^{-\infty}\right)D\sin\frac{n\pi x}{10} = 0$$

$$= \left[Ae^{\infty}\right] = 0 \qquad \because e^{-\infty} = 0$$

Here, $D \ne 0$, $\sin\frac{n\pi y}{10} \ne 0$, $e^{-\infty} \ne 0$

$$\therefore A = 0$$

Substitute A=0 in eqn (3)

$$u(x,y) = D\sin\frac{n\pi y}{10} Be^{\frac{-n\pi x}{10}}$$

$$= BD\sin\frac{n\pi y}{10} e^{\frac{-n\pi x}{10}} \qquad ------(4)$$

The most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi y}{10} e^{\frac{-n\pi x}{10}} \qquad ------(5)$$

Applying condition (iv) in eqn (5)

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi y}{10} = f(y) \quad ------(6)$$

To find B_n then we expand $f(y)$ as a Fourier half range sine series in (0, 10)

$$f(y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{l} \qquad ------(7)$$

Where $b_n = \frac{2}{l}\int_0^l f(y)\sin\frac{n\pi y}{l} dy$

$$= \frac{1}{5}\int_0^{10} f(y)\sin\frac{n\pi y}{10}\,dy$$

$$= \frac{1}{5}\left[\int_0^{10} 20y\sin\frac{n\pi y}{10}\,dy + \int_5^{10} 20(10-y)\sin\frac{n\pi y}{10}\,dy\right]$$

$$= 4\left[\begin{array}{l}\left\{x.\frac{10}{n\pi}\left(-\cos\frac{n\pi y}{10}\right)-(1)\frac{100}{n^2\pi^2}\left(-\sin\frac{n\pi y}{10}\right)\right\}_0^5 + \\ \left\{(10-x).\frac{10}{n\pi}\left(-\cos\frac{n\pi y}{10}\right)-(-1)\frac{100}{n^2\pi^2}\left(-\sin\frac{n\pi y}{10}\right)\right\}_5^{10}\end{array}\right]$$

$$= 4\left[\frac{-50}{n\pi}\cos\frac{n\pi}{2}+\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}-(0+0)+(0+0)-\left(\frac{-50}{n\pi}\cos\frac{n\pi}{2}+\frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$A_n = 0 \quad if \quad n = 2,4,6.....$$

$$\therefore A_n = \frac{800}{(2n-1)^2\pi^2}\sin\frac{(2n-1)\pi}{2}$$

$$\therefore (5) \Rightarrow u(x,y) = \frac{800}{\pi^2}\sum_{n=1,3,5...}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi y}{10}e^{-\frac{n\pi x}{10}}$$

$$u(x,y) = \frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\sin\frac{(2n-1)\pi}{2}\sin\frac{(2n-1)\pi y}{10}e^{-\frac{(2n-1)\pi x}{10}}$$

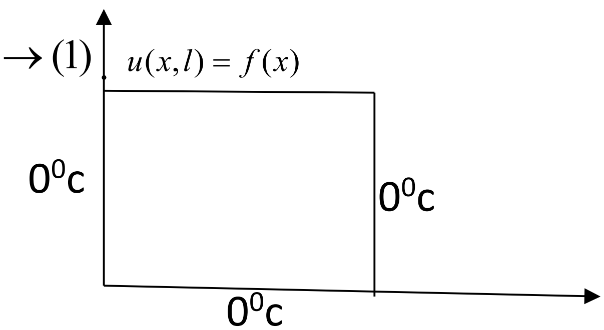
Type II Temperature Distribution in Finite Plates

Type-A Temperature distribution along *x*-axis or parallel to *x*-axis

1. **A square plate is bounded by the lines *x=0, y=0, x=l,y=l*. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x,l) = f(x)$ while the other three edges are kept at 0^0 C. Find the steady state temperature in the plate.**

Solution:

Let us take the sides of the plate be *l*. Let $u(x, y)$ satisfies the Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \rightarrow (1)$



The boundary conditions are

(i) $u(0, y) = 0, 0 < x < l$

(ii) $u(l, y) = 0, 0 < x < l$

(iii) $u(x, 0) = 0, 0 < x < l$

(iv) $u(x,l) = f(x),\ 0 < x < l$

The correct solution should be

$$u(x, y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \rightarrow (2)$$

Applying (i) in (2), we get

$$u(0, y) = A(Ce^{py} + De^{-py}) = 0$$

$$\Rightarrow A = 0$$

Applying (ii) in (2), we get

$$u(x,y) = B\sin px\left(Ce^{py} + De^{-py}\right) \to (2)$$

$Apply\,(ii)\,in\,(2)\,we\,get,$

$$u(l,y) = B\sin pl\left(Ce^{py} + De^{-py}\right)$$

$$0 = B\sin pl\left(Ce^{py} + De^{-py}\right), \quad Here\ \left(Ce^{py} + De^{-py}\right) \neq 0, B \neq 0$$

$$\Rightarrow \sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

$$\therefore u(x,y) = B\sin\frac{n\pi x}{l}\left(Ce^{\frac{n\pi y}{l}} + De^{-\frac{n\pi y}{l}}\right) \to (3)$$

Apply (iii) in (3) we get

$$u(x,0) = B\sin\frac{n\pi x}{l}(C+D)$$

$$0 = B\sin\frac{n\pi x}{l}(C+D)$$

$$C+D=0.\ \sin ce \sin\frac{n\pi x}{l} \neq 0, B \neq 0$$

$$\Rightarrow D = -C$$

$$(3) \Rightarrow u(x,y) = B\sin\frac{n\pi x}{l}C\left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right)$$

$$u(x,y) = 2BC\sin\frac{n\pi x}{l}.\frac{1}{2}\left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right)$$

$$Consider\ B_n = 2BC,\ and\ \frac{1}{2}\left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right) = \sinh\frac{n\pi y}{l}$$

$$\Rightarrow u(x,y) = B_n \sin\frac{n\pi x}{l}\sinh\frac{n\pi y}{l}$$

Most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\sinh\frac{n\pi y}{l} \to (4)$$

Aplly (iv) *in* (4) *we get*

$$u(x,l) = \sum_{n=1}^{\infty} B_n \sinh n\pi.\sin\frac{n\pi x}{l}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \to (5)\ \ where\ b_n = B_n \sinh n\pi$$

(5) represents Half range Fourier Sine series

$$\therefore b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$B_n \sinh n\pi = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$B_n = \frac{2}{l\sinh n\pi}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

Substituting this value of B_n in (4), we get the required temperature distribution.

2. A square plate is bounded by the lines $x=0, y=0$, $x=20 \,\&\, y=20$ .Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x,20)=x(20-x)$ when $0 < x < 20$ while the other three edges are kept at 0^0 C. Find the steady state temperature in the plate.

Solution:

Let us take the sides of the plate be $l=20$ .Let $u(x,y)$ satisfies the Laplace's equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \rightarrow (1)$.

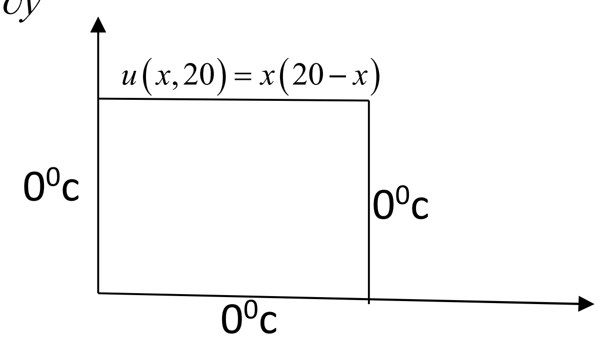
The boundary conditions are

(i) $u(0,y)=0, 0<x<l$

(ii) $u(l,y)=0, 0<x<l$

(iii) $u(x,0)=0, 0<x<l$

(iv) $u(x,l)=x(l-x), 0<x<l$



The correct solution should be

$$u(x,y)=(A\cos px+B\sin px)(Ce^{py}+De^{-py}) \rightarrow (2)$$

Applying (i) in (2), we get

$$u(0,y) = A(Ce^{py} + De^{-py}) = 0$$

$$\Rightarrow A = 0$$

Applying (ii) in (2), we get

$$u(x,y) = B\sin px\left(Ce^{py} + De^{-py}\right) \to (2)$$

$$Apply\,(ii)\,in\,(2)\,we\,get,$$

$$u(l,y) = B\sin pl\left(Ce^{py} + De^{-py}\right)$$

$$0 = B\sin pl\left(Ce^{py} + De^{-py}\right), \quad Here\ \left(Ce^{py} + De^{-py}\right) \neq 0, B \neq 0$$

$$\Rightarrow \sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

$$\therefore u(x,y) = B\sin\frac{n\pi x}{l}\left(Ce^{\frac{n\pi y}{l}} + De^{-\frac{n\pi y}{l}}\right) \to (3)$$

Apply (iii) in (3) we get

$$u(x,0) = B\sin\frac{n\pi x}{l}(C+D)$$

$$0 = B\sin\frac{n\pi x}{l}(C+D)$$

$$C+D=0.\ \sin ce \sin\frac{n\pi x}{l} \neq 0, B \neq 0$$

$$\Rightarrow D = -C$$

$$(3) \Rightarrow u(x,y) = B\sin\frac{n\pi x}{l}C\left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right)$$

$$u(x,y) = 2BC\sin\frac{n\pi x}{l}.\frac{1}{2}\left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right)$$

$$Consider\ B_n = 2BC,\ and\ \frac{1}{2}\left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right) = \sinh\frac{n\pi y}{l}$$

$$\Rightarrow u(x,y) = B_n \sin\frac{n\pi x}{l}\sinh\frac{n\pi y}{l}$$

Most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\sinh\frac{n\pi y}{l} \rightarrow (4)$$

Aplly (iv) *in* (4) *we get*

$$u(x,l) = \sum_{n=1}^{\infty} B_n \sinh n\pi.\sin\frac{n\pi x}{l}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \rightarrow (5)\ \ where\ b_n = B_n \sinh n\pi$$

(5) represents Half range Fourier Sine series

$$\therefore b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\int_0^l (lx-x^2)\sin\frac{n\pi x}{l}$$

$$=\frac{2}{l}\left[\left(lx-x^2\right)\frac{l}{n\pi}\left(-\cos\frac{n\pi x}{l}\right)+\left(l-2x\right)\sin\frac{n\pi x}{l}.\frac{l^2}{n^2\pi^2}-2\cos\frac{n\pi x}{l}\frac{l^3}{n^3\pi^3}\right]_0^l$$

$$=\frac{2}{l}\left[-2(-1)^n\frac{l^3}{n^3\pi^3}+\frac{2l^3}{n^3\pi^3}\right]=\frac{4l^2}{n^3\pi^3}\left[1-(-1)^n\right]=\begin{cases}\frac{8l^2}{n^3\pi^3}, & n \text{ is odd}\\ 0, & n \text{ is even}\end{cases}$$

$$\Rightarrow B_n=\begin{cases}\frac{8l^2}{n^3\pi^3}\cos ech n\pi & : n \text{ is odd}\\ 0 & : n \text{ is even}\end{cases}$$

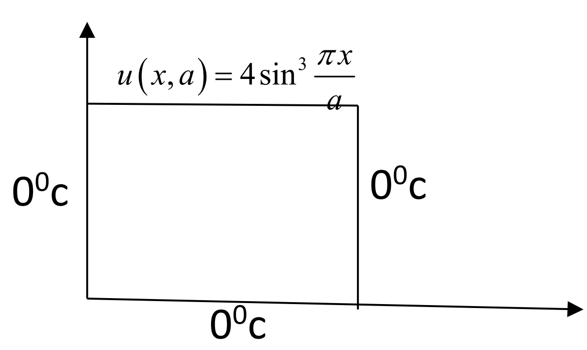
$$\therefore u\left(x,y\right)=\sum_{n=1,3,5,..}^{\infty}\frac{8l^2}{n^3\pi^3}\cos ech n\pi\sin\frac{n\pi x}{l}\sinh\frac{n\pi y}{l}$$

$$l=20\Rightarrow u\left(x,y\right)=\frac{3200}{\pi^3}\sum_{n=1,3,5,..}^{\infty}\frac{1}{n^3}\cos ech n\pi\sin\frac{n\pi x}{20}\sinh\frac{n\pi y}{20}$$

3. **A square plate is bounded by the lines $x=0, y=0$, $x=a \;\&\; y=a$ .Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x,a)=4\sin^3\left(\frac{\pi x}{a}\right)$ when 0 < x < a while the other three edges are kept at 0^0 C. Find the steady state temperature in the plate.**

Solution:

Let us take the sides of the plate be $l=a$ .Let $u(x,y)$ satisfies the Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.



The boundary conditions are

(i) $u(0,y)=0, 0<y<a$

(ii) $u(a,y)=0, 0<y<a$

(iii) $u(x,0)=0, 0<x<a$

(iv) $u(x,a)=4\sin^3\dfrac{\pi x}{a}, 0<x<a$

The correct solution should be

$$u(x,y)=(A\cos px+B\sin px)(Ce^{py}+De^{-py})\rightarrow(1)$$

Applying (i) in (1), we get

$$u(0,y)=A(Ce^{py}+De^{-py})=0$$

$$\Rightarrow A=0$$

Applying (ii) in (2), we get

$$u(x,y)=B\sin px\left(Ce^{py}+De^{-py}\right)\rightarrow(2)$$

Apply (ii) in (2) we get,

$$u(a,y)=B\sin pl\left(Ce^{py}+De^{-py}\right)$$

$$0=B\sin pl\left(Ce^{py}+De^{-py}\right),\quad Here\ \left(Ce^{py}+De^{-py}\right)\neq 0, B\neq 0$$

$$\Rightarrow \sin pa=0$$

$$\sin pa=\sin n\pi$$

$$pa=n\pi$$

$$p=\frac{n\pi}{a}$$

$$\therefore u(x,y)=B\sin\frac{n\pi x}{a}\left(Ce^{\frac{n\pi y}{a}}+De^{-\frac{n\pi y}{a}}\right)\rightarrow(3)$$

Apply (iii) in (3) we get

$$u(x,0) = B\sin\frac{n\pi x}{a}(C+D)$$

$$0 = B\sin\frac{n\pi x}{a}(C+D)$$

$$C+D=0.\ \sin ce \sin\frac{n\pi x}{a} \neq 0, B \neq 0$$

$$\Rightarrow D=-C$$

$$(3) \Rightarrow u(x,y) = B\sin\frac{n\pi x}{a} C\left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right)$$

$$u(x,y) = 2BC\sin\frac{n\pi x}{a}.\frac{1}{2}\left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right)$$

$$Consider\ B_n = 2BC,\ and\ \frac{1}{2}\left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right) = \sinh\frac{n\pi y}{a}$$

$$\Rightarrow u(x,y) = B_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

Most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a} \rightarrow (4)$$

Aplly (iv) in (4) we get

$$u(x,a) = \sum_{n=1}^{\infty} B_n \sinh n\pi.\sin\frac{n\pi x}{a}$$

$Apply(iv)\, in(4)\, we\, get$

$$u(x,a) = B_1 \sinh \pi \sin \frac{n\pi x}{a} + B_2 \sinh 2\pi \sin \frac{2\pi x}{a} + B_3 \sinh 3\pi \sin \frac{3\pi x}{a} + ...$$

$$4\sin^3 \frac{\pi x}{a} = B_1 \sinh \pi \sin \frac{n\pi x}{a} + B_2 \sinh 2\pi \sin \frac{2\pi x}{a} + B_3 \sinh 3\pi \sin \frac{3\pi x}{a} + ...$$

$$4.\frac{1}{4}\left[3\sin \frac{\pi x}{a} - \sin \frac{3\pi x}{a}\right] = B_1 \sinh \pi \sin \frac{n\pi x}{a} + B_2 \sinh 2\pi \sin \frac{2\pi x}{a} + B_3 \sinh 3\pi \sin \frac{3\pi x}{a} + ...$$

$$\Rightarrow B_1 = 3\cos ech\pi, B_2 = 0, B_3 = -\cos ech 3\pi, B_4 = B_5 = ... = 0$$

$$(4) \Rightarrow u(x,y) = 3\cos ech\pi \sin \frac{\pi x}{a} \sinh \frac{\pi y}{a} - \cos ech 3\pi \sin \frac{3\pi x}{a} \sinh \frac{3\pi y}{a}$$

4. **Find the steady state state temperature distribution in a rectangular plate of sides a and b insulated at the lateral surface and satisfying the boundary conditions** $u(0,y) = 0 = u(a,y)\; for\; 0 < y < b, and\; u(x,b) = 0, u(x,0) = x(a-x)\; for\; 0 < x < a$

Solution:

Let us take the sides of the plate be $l = a$ .Let $u(x, y)$ satisfies the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

The boundary conditions are

(i) $u(0, y) = 0, 0 < y < b$

(ii) $u(a, y) = 0, 0 < y < b$

(iii) $u(x,b) = 0, 0 < x < a$

(iv) $u(x,0) = x(a-x), 0 < x < a$

The correct solution should be

$$u(x, y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \rightarrow (1)$$

Applying (i) in (1), we get

$$u(0,y)=A(Ce^{py}+De^{-py})=0$$

$$\Rightarrow A=0$$

Applying (ii) in (2), we get

$$u(x,y)=B\sin px\left(Ce^{py}+De^{-py}\right)\rightarrow(2)$$

$Apply\,(ii)\,in\,(2)\,we\,get,$

$$u(a,y)=B\sin pl\left(Ce^{py}+De^{-py}\right)$$

$$0=B\sin pl\left(Ce^{py}+De^{-py}\right),\quad Here\left(Ce^{py}+De^{-py}\right)\neq 0, B\neq 0$$

$$\Rightarrow \sin pa=0$$

$$\sin pa=\sin n\pi$$

$$pa=n\pi$$

$$p=\frac{n\pi}{a}$$

$$\therefore u(x,y)=B\sin\frac{n\pi x}{a}\left(Ce^{\frac{n\pi y}{a}}+De^{-\frac{n\pi y}{a}}\right)\rightarrow(3)$$

Apply (iii) in (3) we get

$$u(x,b) = B\sin\frac{n\pi x}{a}\left(Ce^{\frac{n\pi b}{a}} + De^{-\frac{n\pi b}{a}}\right)$$

$$0 = B\sin\frac{n\pi x}{a}\left(Ce^{\frac{n\pi b}{a}} + De^{-\frac{n\pi b}{a}}\right)$$

$$Ce^{\frac{n\pi b}{a}} + De^{-\frac{n\pi b}{a}} = 0. \ \sin ce \sin\frac{n\pi x}{a} \neq 0, B \neq 0$$

$$\Rightarrow D = -Ce^{\frac{2n\pi b}{a}}$$

$$(3) \Rightarrow u(x,y) = B\sin\frac{n\pi x}{a}\left(Ce^{\frac{n\pi y}{a}} - Ce^{\frac{2n\pi b}{a}}e^{-\frac{n\pi y}{a}}\right)$$

$$u(x,y) = 2BC\sin\frac{n\pi x}{a}.\frac{1}{2}e^{\frac{n\pi b}{a}}\left(e^{\frac{n\pi(y-b)}{a}} - e^{-\frac{n\pi(y-b)}{a}}\right)$$

$$Consider\ B_n = 2BCe^{\frac{n\pi b}{a}},\ and\ \frac{1}{2}\left(e^{\frac{n\pi(y-b)}{a}} - e^{-\frac{n\pi(y-b)}{a}}\right) = \sinh\frac{n\pi(y-b)}{a}$$

$$\Rightarrow u(x,y) = B_n\sin\frac{n\pi x}{a}\sinh\frac{n\pi(y-b)}{a}$$

Most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{a}\sinh\frac{n\pi(y-b)}{a} \rightarrow (4)$$

$Apply\,(iv)\,in\,(4)\,we\,get$

$$u(x,0) = \sum_{n=1}^{\infty} -B_n\sin\frac{n\pi x}{a}\sinh\frac{n\pi b}{a}$$

$$f(x) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{a},\ where\ b_n = -B_n\sinh\frac{n\pi b}{a}$$

$$\therefore b_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx$$

$$=\frac{2}{a}\int_0^a (lx-x^2)\sin\frac{n\pi x}{a}dx$$

$$=\frac{2}{a}\left[\left(ax-x^2\right)\frac{a}{n\pi}\left(-\cos\frac{n\pi x}{a}\right)+\left(a-2x\right)\sin\frac{n\pi x}{l}.\frac{a^2}{n^2\pi^2}-2\cos\frac{n\pi x}{a}\frac{a^3}{n^3\pi^3}\right]_0^l$$

$$=\frac{2}{a}\left[-2(-1)^n\frac{a^3}{n^3\pi^3}+\frac{2l^3}{n^3\pi^3}\right]=\frac{4a^2}{n^3\pi^3}\left[1-(-1)^n\right]=\begin{cases}\frac{8a^2}{n^3\pi^3}, & n \text{ is odd}\\ 0, & n \text{ is even}\end{cases}$$

5. A rectangular plate is bounded by the lines $x=0, x=a, y=0 \,\&\, y=b$ and the edge temperatures are $u(0,y)=0$, $u(a,y)=0$, $u(x,b)=0$ & $u(x,0)=5\sin\frac{4\pi x}{a}+3\sin\frac{3\pi x}{a}$. Find the temperature distribution.

Solution:

W.K.T the Laplace equation satisfies the equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

The boundary conditions are

(i) $u(0,y)=0, 0<y<b$

(ii) $u(a,y)=0, 0<y<b$

(iii) $u(x,b)=0, 0<x<a$

(iv) $u(x,0)=5\sin\frac{4\pi x}{a}+3\sin\frac{3\pi x}{a}$ $,0<x<a$

The correct solution should be

$$u(x,y)=(A\cos px+B\sin px)(Ce^{py}+De^{-py})\rightarrow(1)$$

Applying (i) in (1), we get

$$u(0,y)=A(Ce^{py}+De^{-py})=0$$

$\Rightarrow A = 0$

Substitute A = 0 in eqn(1)

$$u(x,y) = B\sin px\left(Ce^{py} + De^{-py}\right) \rightarrow (2)$$

Apply (ii) in (2) we get,

$$u(10,y) = B\sin 10p\left(Ce^{py} + De^{-py}\right)$$

$$0 = B\sin 10p\left(Ce^{py} + De^{-py}\right), \quad Here\ \left(Ce^{py} + De^{-py}\right) \neq 0, B \neq 0$$

$$\Rightarrow \sin 10p = 0$$

$$\sin 10p = \sin n\pi$$

$$10p = n\pi$$

$$p = \frac{n\pi}{10}$$

$$\therefore u(x,y) = B\sin\frac{n\pi x}{a}\left(Ce^{\frac{n\pi y}{a}} + De^{-\frac{n\pi y}{a}}\right) \rightarrow (3)$$

Apply (iii) in (3) we get

$$u(x,b) = B\sin\frac{n\pi x}{a}\left(Ce^{\frac{n\pi b}{a}} + De^{-\frac{n\pi b}{a}}\right)$$

$$0 = B\sin\frac{n\pi x}{a}\left(Ce^{\frac{n\pi b}{a}} + De^{-\frac{n\pi b}{a}}\right)$$

$$Ce^{\frac{n\pi b}{a}} + De^{-\frac{n\pi b}{a}} = 0.\ \sin ce \sin\frac{n\pi x}{a} \neq 0, B \neq 0$$

$$\Rightarrow D = -Ce^{\frac{2n\pi b}{a}}$$

$$(3) \Rightarrow u(x,y) = B\sin\frac{n\pi x}{a}\left(Ce^{\frac{n\pi y}{a}} - Ce^{\frac{2n\pi b}{a}}e^{-\frac{n\pi y}{a}}\right)$$

$$u(x,y) = 2BC\sin\frac{n\pi x}{a}.\frac{1}{2}e^{\frac{n\pi b}{a}}\left(e^{\frac{n\pi(y-b)}{a}} - e^{-\frac{n\pi(y-b)}{a}}\right)$$

$$Consider\ B_n = 2BCe^{\frac{n\pi b}{a}},\ and\ \frac{1}{2}\left(e^{\frac{n\pi(y-b)}{a}} - e^{-\frac{n\pi(y-b)}{a}}\right) = \sinh\frac{n\pi(y-b)}{a}$$

$$\Rightarrow u(x,y) = B_n\sin\frac{n\pi x}{a}\sinh\frac{n\pi(y-b)}{a}$$

Most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{a}\sinh\frac{n\pi(y-b)}{a} \rightarrow (4)$$

Apply (iv) in (4) we get

$$u(x,0) = \sum_{n=1}^{\infty} -B_n\sin\frac{n\pi x}{a}\sinh\frac{n\pi b}{a} = 5\sin\frac{4\pi x}{a} + 3\sin\frac{3\pi x}{a}$$

$$-B_1\sin\frac{\pi x}{a}\sinh\frac{\pi b}{a} - B_2\sin\frac{2\pi x}{a}\sinh\frac{2\pi b}{a} - B_3\sin\frac{3\pi x}{a}\sinh\frac{3\pi b}{a} - B_4\sin\frac{4\pi x}{a}\sinh\frac{4\pi b}{a} + ... = 5\sin\frac{4\pi x}{a} + 3\sin\frac{3\pi x}{a}$$

Equating the like coefficients

$$B_1 = 0, B_2 = 0, B_3 = -3\cos ech\frac{3\pi b}{a}, B_4 = -5\cos ech\frac{4\pi b}{a}$$

& the remaining B_n 's are zero.

$$u(x,y) = -3\operatorname{cosech}\frac{3\pi b}{a}\sin\frac{3\pi x}{a}\sinh\frac{3\pi}{a}(y-b) - 5\operatorname{cosech}\frac{4\pi b}{a}\sin\frac{4\pi x}{a}\sinh\frac{4\pi}{a}(y-b)$$

6. A square plate is bounded by the lines *x=0, x=a, y=0* and *y=b*. Its faces are insulated and the temperature along *y=b* is kept at 100^0C, while the other three edges are kept at 0^0 C. Find the steady state temperature in the plate.

Solution:

Let $u(x,y)$ satisfies the Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \rightarrow (1)$

The boundary conditions are

(i) $u(0,y) = 0^0,\ 0 < y < b$

(ii) $u(a,y) = 0^0,\ 0 < y < b$

(iii) $u(x,0) = 0^0,\ 0 < x < a$

(iv) $u(x,b) = 100^0,\ 0 < x < a$

The correct solution should be

$$u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \rightarrow (2)$$

Applying (i) in (2), we get

$u(0,y) = A(Ce^{py} + De^{-py}) = 0$

$\Rightarrow A = 0$

Applying (ii) in (2), we get

$$u(a, y) = B\sin pa\left(Ce^{py} + De^{-py}\right)$$

$$0 = B\sin pa\left(Ce^{py} + De^{-py}\right) \qquad Here \ \left(Ce^{py} + De^{-py}\right) \neq 0, \ B \neq 0$$

$$\Rightarrow \sin pa = 0$$

$$pa = n\pi$$

$$p = \frac{n\pi}{a}$$

$$\therefore u(x, y) = B\sin\frac{n\pi x}{a}\left(Ce^{\frac{n\pi y}{a}} + Be^{-\frac{n\pi y}{a}}\right) \rightarrow (3)$$

Apply (iii) in (3) we get

$$u(x,0) = B\sin\frac{n\pi x}{l}(C + D)$$

$$0 = B\sin\frac{n\pi x}{l}(C + D)$$

$$C + D = 0, \quad \sin ce \ \sin\frac{n\pi x}{l} \neq 0, \ B \neq 0$$

$$\Rightarrow D = -C$$

$$(3) \Rightarrow u(x, y) = B\sin\frac{n\pi x}{a}C\left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right)$$

$$u(x, y) = B\sin\frac{n\pi x}{a}C.2\sin\frac{n\pi y}{a}$$

$$Consider \quad B_n = 2BC$$

$$\Rightarrow u(x, y) = B_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a} \qquad \rightarrow (4)$$

Applying condition (iv) in (4)

$$u(x,b)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi b}{a}$$

$$100=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi b}{a}$$

$$where \quad b_n = B_n \sinh\frac{n\pi b}{a}$$

$$\Rightarrow B_n = \frac{b_n}{\sinh\frac{n\pi b}{a}}$$

$$\Rightarrow 100=\sum_{n=1}^{\infty} b_n \sinh\frac{n\pi b}{a} \qquad \rightarrow (5)$$

(5) represents Half range Fourier Sine series

$$\therefore b_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx$$

$$B_n \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a 100\sin\frac{n\pi x}{a}dx$$

$$B_n = \frac{200}{a\sinh\frac{n\pi b}{a}}\left[\frac{-\cos\frac{n\pi x}{a}}{\frac{n\pi}{a}}\right]_0^a$$

$$=\frac{200}{a\sinh\frac{n\pi b}{a}}\times\frac{a}{n\pi}\left[-(-1)^n+1\right]$$

$$B_n = \frac{200}{n\pi\sinh\frac{n\pi b}{a}}\left[-(-1)^n+1\right]$$

$When\ \ n\ \ is\ \ odd,\ B_n = 0$

$$When\ \ n\ is\ even\ B_n = \frac{400}{n\pi \sinh \frac{n\pi b}{a}}$$

Substituting this value of B_n in (4), we get the required temperature distribution.

$$u(x,y) = \sum_{n=1,3,5..}^{\infty} \frac{400}{n\pi \sinh \frac{n\pi b}{a}} \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$$

Type-B Temperature distribution along *y*-axis or parallel to *y*-axis.

1. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ **which satisfies the conditions** $u(0,y) = 0,\ u(x,0) = 0,\ u(x,b) = 0\quad and\ \ u(a,y) = 100^0 C$.

Solution:

Let $u(x,y)$ be the temperature at any point (x , y) in the steady state. Then u satisfies the differential equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

The boundary conditions are

(i) $u(x,0) = 0\quad \forall y$

(ii) $u(x,b) = 0\quad \forall y$

(iii) $u(0,y) = 0$

(iv) $u(a,y) = 100,\quad 0 \le y \le b$

The correct solution should be

$$u(x,y) = \left(Ae^{px} + Be^{-px}\right)\left(C\cos py + D\sin py\right) \qquad \to (1)$$

Applying (i) in (1), we get

$$u(x,0) = \left(Ae^{px} + Be^{-px}\right)C = 0$$

$$\Rightarrow C = 0$$

Substitute C = 0 in eqn(1)

$$u(x,y) = \left(Ae^{px} + Be^{-px}\right)D\sin py \to (2)$$

Applying (ii) in (2) , we get

$$u(x,b) = \left(Ae^{px} + Be^{-px}\right)D\sin bp$$

$$0 = \left(Ae^{px} + Be^{-px}\right)D\sin 10p \qquad Here\ \left(Ae^{px} + Be^{-px}\right) \neq 0,\ D \neq 0$$

$$\Rightarrow \sin bP = 0$$

$$bp = n\pi$$

$$p = \frac{n\pi}{b}$$

$$\therefore u(x,y) = \left(Ae^{\frac{n\pi x}{b}} + Be^{-\frac{n\pi x}{b}}\right)D\sin\frac{n\pi x}{b} \to (3)$$

Applying (iii) in eqn (3)

$$u(0,y) = \left(A + B\right)D\sin\frac{n\pi y}{b} = 0 \Rightarrow A + B = 0$$

$$B = -A$$

(4) reduces to

$$u(x,y) = B_n \sinh\frac{n\pi x}{b}\sinh\frac{n\pi y}{b} \qquad n = 1,2,3.....$$

Hence the most general form of the solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b} \qquad -----(4)$$

Applying condition (iv) in eqn (4)

$$100=\sum_{n=1}^{\infty} B_n \sinh\frac{n\pi a}{b}\sin\frac{n\pi y}{b} \qquad ------(6)$$

$$\therefore B_n \sinh\frac{n\pi a}{b}=\frac{2}{b}\int_0^b 100\sinh\frac{n\pi y}{b}\,dy$$

$$=\frac{200}{b}\frac{b}{n\pi}\left[-\cos\frac{n\pi y}{b}\right]_0^b=\frac{200}{b}\frac{b}{n\pi}\left[1-(-1)^n\right]$$

$$=0 \qquad , n \;\; is \;\; odd$$

$$=\frac{400}{n\pi}, \quad n \;\; is \;\; even$$

$$B_n=0 \qquad if \;\; n \;\; is \;\; even$$

$$=\frac{400}{n\pi\sinh\frac{n\pi a}{b}}, \quad n \; is \; odd$$

$$\therefore (5)\Rightarrow u(x,y)=\frac{400}{\pi}\sum_{n=1,3,5...}^{\infty}\frac{1}{n.\sinh\frac{n\pi a}{b}}\sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b}$$

$$u(x,y)=\frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1).\sinh\frac{(2n-1)\pi a}{b}}\sinh\frac{(2n-1)\pi x}{b}\sin\frac{(2n-1)\pi y}{b}$$

Type-C Temperature distribution given on *x* and *y* aixs

1. **A rectangular plate is bounded by the lines *x=0, y=0, x=a,y=b.* Its surfaces are insulated and the temperature along two adjacent edges are kept at 100^0C, while the temperature along other two edges are at 0^0C. Find the steady state temperature at any point in the plate. Also find the steady state temperature at any point of side 'a' if two adjacent edges are kept at 100^0C and the others at 0^0C.**

Solution:

Let $u(x, y)$ be the temperature at any point (x , y) in the steady state. Then u satisfies the differential equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$. --------(1)

The boundary conditions are

(i) $u(0, y)=0 \quad 0 \leq y \leq b$

(ii) $u(x,0)=0 \quad 0 \leq x \leq a$

(iii) $u(a, y)=100 \quad 0 \leq y \leq b$

(iv) $u(x,b)=100, \quad 0 \leq x \leq a$

Where $u_1(x,y)$ and $u_2(x,y)$ are solutions of (1) and further $u_1(x,y)$ is the temperature at P with edge BC kept at 100^0 C and the other three sides at 0^0C while $u_2(x,y)$ is the temperature at P with the edge AB maintained at 100^0C and the other three edges at 0^0C.

Boundary conditions for the functions $u_1(x,y)$ *and* $u_2(x,y)$ are

$$u_1(0,y)=0 \qquad\qquad u_2(x,0)=0$$
$$u_1(a,y)=0 \qquad\qquad u_2(x,b)=0$$
$$u_1(x,0)=0 \qquad\qquad u_2(0,y)=0$$
$$u_1(x,b)=100^0 C \;------(2) \qquad\qquad u_2(a,y)=100^0 C------(3)$$

The most general solution for $u_1(x,y)$ satisfying the equation (1) and 1st three boundary conditions in (2) is given by

$$u(x,y)=\sum_{n=1}^{\infty} B_n \sinh\frac{n\pi x}{a}\sin\frac{n\pi y}{a} \qquad -----(4)$$

$$Now \quad u_1(x,y)=\sum_{n=1}^{\infty} B_n \sinh\frac{n\pi x}{a}\sin\frac{n\pi b}{a}$$

By applying the last boundary condition in (2)

$$100=\sum_{n=1}^{\infty} B_n \sinh\frac{n\pi b}{a}\sin\frac{n\pi x}{a}$$

$$\therefore B_n \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a 100\sinh\frac{n\pi yx}{a}\,dx$$

$$= \frac{200}{n\pi}\left[-\cos\frac{n\pi x}{a}\right]_0^a = \frac{200}{n\pi}\left[1-(-1)^n\right]$$

$$= 0 \quad , \; n \;\; is \;\; odd$$

$$= \frac{400}{n\pi}, \quad n \;\; is \;\; even$$

$$B_n = 0 \qquad if \;\; n \;\; is \;\; even$$

$$= \frac{400}{n\pi\sinh\frac{n\pi a}{b}}, \quad n \;\; is \;\; odd$$

$$\therefore (5) \Rightarrow u_1(x,y) = \frac{400}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n.\sinh\frac{n\pi b}{a}}\sinh\frac{n\pi y}{a}\sin\frac{n\pi x}{a}$$

$$u_1(x,y) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1).\sinh\frac{(2n-1)\pi b}{a}}\sinh\frac{(2n-1)\pi x}{a}\sin\frac{(2n-1)\pi y}{a}$$

Similarly we get

$$u_2(x,y) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1).\sinh\frac{(2n-1)\pi a}{b}}\sinh\frac{(2n-1)\pi x}{b}\sin\frac{(2n-1)\pi y}{b}$$

$$\therefore u(x,y) = u_1(x,y) + u_2(x,y)$$