

WEIGHTED RESIDUAL METHODS

RITZ TECHNIQUE $\Rightarrow$

Mohan S R, AP/Mech

UNIT-1

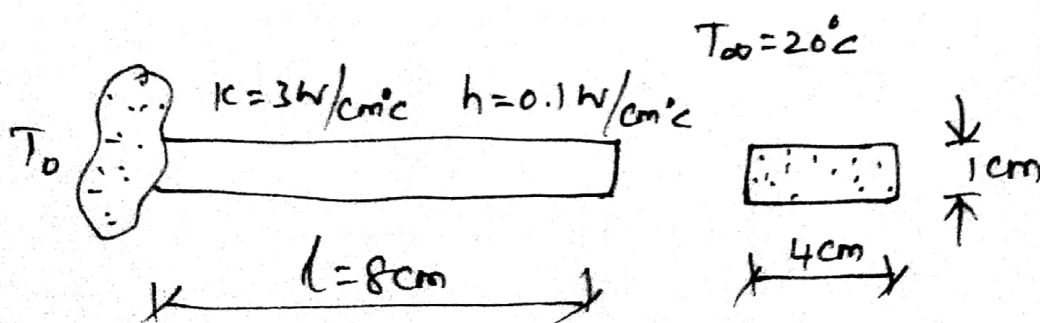
Functional Approximation Method

- * Weighted Residual Method
- * Variational Method (Rayleigh Ritz)

① The governing equation for one dimensional heat transfer through a fin of length l attached to a hot source as shown in fig. is given by

$$d/dx \left[-kA \frac{dT}{dx} \right] + h(T - T_{\infty}) = 0$$

If the free end of the fin is insulated, give the boundary conditions and determine using the collocation technique the temperature distribution in the fin. Report the temperature at the free end.



Apr/May 2018 & 2019.

Sol:

$$d/dn \left[-kA \frac{dT}{dn} \right] + hp(T - T_{\infty}) = 0$$

$$L = 8 \text{ cm}, t = 1 \text{ cm} \quad b = 4 \text{ cm} \quad k = 3 \text{ W/cm}^2\text{C}$$

$$h = 0.1 \text{ W/cm}^2\text{C} \quad T_{\infty} = 20^{\circ}\text{C} = 293 \text{ K}$$

$$T_1 = 200^{\circ}\text{C} = 473 \text{ K}.$$

$$\begin{aligned} \text{Perimeter of the fin} &= 2(b+t) \\ &= 2(4+1) = 10 \text{ cm}. \end{aligned}$$

$$\text{Area} = b \times t = 4 \text{ cm}^2.$$

Assume 2nd order polynomial eqn,

$$T = a_0 + a_1x + a_2x^2 \quad \text{--- (1)}$$

$$\text{at } x=0, \quad T = 473 \text{ K}$$

$$a_0 = 473$$

$$\therefore T = 473 + a_1x + a_2x^2 \quad \text{--- (2)}$$

$$\text{at } x=L, \quad H \cdot T = 0 \text{ or } \frac{dT}{dn} = 0.$$

$$\frac{dT}{dn} = a_1 + 2a_2x = 0$$

$$a_1 = -16a_2$$

$$T = 473 - 16a_2x + a_2x^2$$

$$T = 473 + (x^2 - 16x) a_2 \quad \text{--- (3)}$$

from above eqn,

$$\frac{dT}{dx} = (2x - 16) a_2$$

$$\frac{d^2T}{dx^2} = 2a_2$$

Substitute the value of $\frac{d^2T}{dx^2}$ in governing eqn to find residual R,

$$R = (x^2 - 16x - 24) a_2 + 180$$

Solving by point allocation,

$$R = 0$$

$$(x^2 - 16x - 24) a_2 + 180 = 0$$

$$x = \frac{16}{2}$$

$$(16 - 64 - 24) a_2 = -180$$

$$x = 4 \text{ cm.}$$

$$-72a_2 = -180$$

$$a_2 = 2.5$$

Sub in trial function,

$$T = 473 + (x^2 - 16x) 2.5$$

Temperature at free end $x = L = 8$,

$$T = 473 + (8^2 - 16(8)) 2.5 = 313K = 40^\circ C.$$

- ② A physical phenomenon is governed by the differential equation $\frac{d^2w}{dx^2} - 10x^2 = 5$ for $0 \leq x \leq 1$. The boundary conditions are given by $w(0) = w(1) = 0$. Assuming a trial solution $w(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, determine using Galerkin method the variation of 'w' with respect to x.

Nov/Dec 2016.

Sol:

$$\frac{d^2w}{dx^2} - 10x^2 = 5 \quad 0 \leq x \leq 1$$

boundary conditions are $w(0) = 0$, $w(1) = 0$

Assume trial function,

$$w(x) = a_1x(x-1)$$

$$\text{at } x=0, \quad w=0$$

$$x=1, \quad w=0$$

Hence it satisfies the boundary conditions.

$$w = a_1 x(x-1) = a_1 (x^2 - x)$$

$$\frac{dw}{dx} = a_1 (2x - 1)$$

$$\frac{d^2w}{dx^2} = 2a_1$$

$$\therefore R = 2a_1 - 10x^2 - 5$$

The trial function $x(x-1)$ is considered as weighting function,

$$F_1(x) = x(x-1)$$

Using Galerkin method,

$$\int_0^1 F_1(x) \cdot R \cdot dx = 0$$

$$\int_0^1 x(x-1) (2a_1 - 10x^2 - 5) dx = 0$$

$$\int_0^1 (x^2 - x) (2a_1 - 10x^2 - 5) dx = 0$$

$$\int_0^1 (2a_1 x^2 - 10x^4 - 5x^2 + 2a_1 x + 10x^3 + 5x) dx = 0$$

$$2a_1 \left(\frac{x^3}{3} \right)_0^1 - 10 \left(\frac{x^5}{5} \right)_0^1 - 5 \left(\frac{x^3}{3} \right)_0^1 + 2a_1 \left(\frac{x^2}{2} \right)_0^1 + 10 \left(\frac{x^4}{4} \right)_0^1 + 5 \left(\frac{x^2}{2} \right)_0^1 = 0$$

$$\frac{2a_1}{3} - 2 - \frac{5}{3} - a_1 + \frac{5}{2} + \frac{5}{2} = 0$$

$$\frac{2a_1}{3} - a_1 = -\frac{4}{3}$$

$$-\frac{a_1}{3} = -\frac{4}{3}$$

$$\boxed{a_1 = 4}$$

Hence,

$$\begin{aligned} W &= a_n (n-1) \\ &= 4n(n-1) \\ W &= 4n^2 - 4n. \end{aligned}$$

- ③ The following differential equation is available for a physical phenomenon.

$$\frac{d^2y}{dx^2} + 50 = 0 \quad ; \quad 0 \leq x \leq 10$$

The trial function is $y = a_1 x(10-x)$

The boundary Conditions are $y(0)=0$ $y(10)=0$.

Find the value of the parameter a_1 by the following methods.

i) Least Square

ii) Galerkin's

Sol:

$$\frac{d^2 y}{dn^2} + 50 = 0$$

$$y(0) = 0$$

$$y(10) = 0$$

Trial fn,

$$y = a_1 n(10-n)$$

$$\text{At } n=0, \quad y=0$$

$$n=10, \quad y=0.$$

It satisfies the boundary conditions.

To find Residual,

$$y = a_1 n(10-n)$$

$$y = 10na_1 - n^2a_1$$

$$\frac{dy}{dn} = 10a_1 - 2na_1$$

$$\frac{d^2 y}{dn^2} = -2a_1$$

$$\therefore R = -2a_1 + 50.$$

i) Least Square method

$$\frac{\partial S}{\partial a_1} = 0.$$

$$S = \int R^2 \cdot dn.$$

$$\frac{\partial S}{\partial a_1} = \int \frac{\partial R^2}{\partial a_1} dn = 0$$

$$\Rightarrow \int R \cdot \frac{\partial R}{\partial a_1} dn = 0$$

$$\int_0^{10} (-2a_1 + 50) (-2) dn = 0$$

$$\int_0^{10} (4a_1 - 100) dn = 0$$

$$(4a_1 - 100) (n)_0^{10} = 0$$

$$(4a_1 - 100) 10 = 0$$

$$4a_1 = 100$$

$$a_1 = 25$$

$$\therefore \text{trial fn, } y = -25n(10-n)$$

ii) Galerkin's

$$\int_0^{10} f_1(n) \cdot R dn = 0$$

$$\int_0^{10} a_1 x(10-n) \cdot (-25n + 50) dn = 0$$

$$\int_0^{10} (10a_1 n - a_1 n^2) (-25n + 50) dn = 0$$

$$\int_0^{10} (-20a_1 n^2 + 2a_1 n^3 + 500a_1 n - 50a_1 n^2) dn = 0$$

$$\left[-20 a_1^2 \frac{n^2}{2} + 2 a_1^2 \frac{n^3}{3} + 500 a_1 \frac{n^2}{2} - 50 a_1 \frac{n^3}{3} \right]_0^{10} = 0$$

$$-20 a_1^2 \left(\frac{100}{2} \right) + 2 a_1^2 \left(\frac{1000}{3} \right) + 500 a_1 \left(\frac{100}{2} \right) - 50 a_1 \left(\frac{1000}{3} \right) = 0$$

$$-1000 a_1^2 + 666.66 a_1^2 + 25000 a_1 - 16666.66 a_1 = 0$$

$$\therefore a_1, \quad -1000 a_1 + 666.66 a_1 + 25000 - 16666.66 = 0$$

$$-333.34 a_1 = -8333.34$$

$$a_1 = 24.99$$

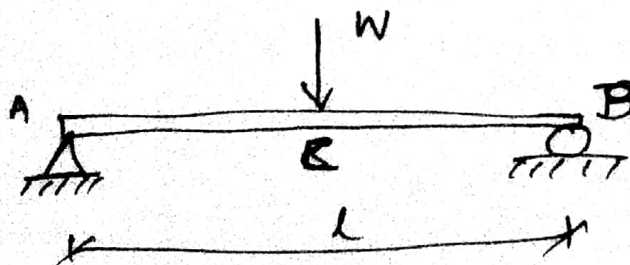
$\therefore$ trial function,

$$y = a_1 x(10-x)$$

$$y = 24.99 x(10-x)$$

$$y \approx 25x(10-x)$$

- (4) A beam AB of span 'L' simply supported at ends and carrying a concentrated load W at the centre 'C' as shown in fig. Determine the deflection at midspan by using Rayleigh-Ritz method and compare with exact solution.



Sol:

The total potential energy,

$$\Pi = U - W.$$

$U \rightarrow$ Strain Energy

$W \rightarrow$ Work done

Strain energy for beam,

$$U = \frac{EI}{2} \int_0^l \left(\frac{d^2 y}{dn^2} \right)^2 dn.$$

$y \rightarrow$ deflection

$E \rightarrow$ Young's modulus

$I \rightarrow$ area M.I.

polynomial eqn,

$$y = a_1 + a_2 n + a_3 n^2 \quad (3 \text{ term})$$

at $n=0$

$$y=0$$

$n=l$

$$y=0$$

$$\boxed{0 = a_1}$$

$$0 = a_1 + a_2 l + a_3 l^2$$

$$a_2 l + a_3 l^2 = 0$$

$$\boxed{a_2 = -a_3 l}$$

$$\therefore y = -a_3 l n + a_3 x^2$$

$$y = a_3 (x^2 - l n)$$

$$\frac{dy}{dn} = a_3 (2n - 1)$$

$$\frac{d^2y}{dn^2} = 2a_3$$

$$\therefore U = \frac{EI}{2} \int_0^l (2a_3)^2 dn$$

$$= \frac{EI}{2} 4a_3^2 l = 2EI a_3^2 l$$

Work done $W = W \cdot y_{\max}$

$$= W [a_3 (x^2 - l n)]_{n=l/2}$$

$$y_{\max} @ n = l/2$$

$$= W a_3 \left[(l/2)^2 - l(l/2) \right]$$

$$= W a_3 \left[\frac{l^2}{4} - \frac{l^2}{2} \right]$$

$$W = -W a_3 \frac{l^2}{4}$$

Total P.E,

$$\pi = U - W$$

$$\pi = 2EI a_3^2 l - (-W a_3 \frac{l^2}{4})$$

$$\pi = 2EI a_3^2 l + W a_3 \frac{l^2}{4}$$

for minimum P.E,

$$\frac{\partial \Pi}{\partial a_3} = 0$$

$$4EI a_3 l = - \frac{wl^2}{4}$$

$$a_3 = - \frac{wl^2}{4} \times \frac{1}{4EI l}$$

$$a_3 = - \frac{wl}{16EI}$$

$$\therefore y = a_3 (x^2 - lx)$$

$$y = - \frac{wl}{16EI} (x^2 - lx)$$

for max. deflection, $x = l/2$,

$$y_{\max} = - \frac{wl}{16EI} \left(\frac{l^2}{4} - \frac{l^2}{2} \right)$$

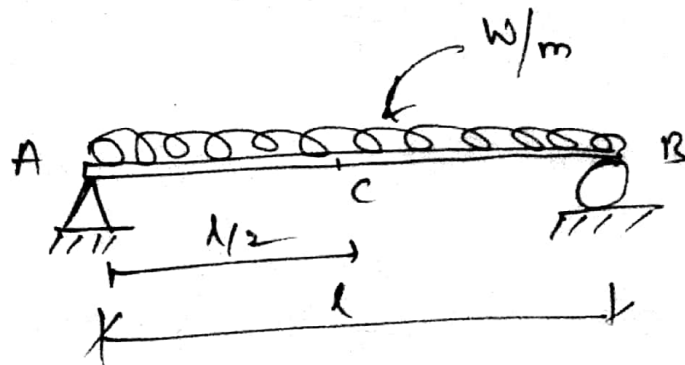
$$y_{\max} = \frac{wl^3}{64EI}$$

↑ approximate solution.

Exact Solution :

$$y_{\max} = \frac{Pl^3}{48EI} \approx \frac{wl^3}{48EI}$$

- ⑤ Find the deflection at the centre of the simply supported beam of span length 'l' subjected to uniformly distributed load throughout its length as shown in fig. using i) Point Collocation ii) Subdomain method.



Apr / May 2017.

Sol:

governing eqn, for udl,

$$EI \frac{d^4 y}{dx^4} - w = 0 \quad 0 \leq x \leq l$$

boundary Conditions are Satisfy the eqns.

$$\begin{aligned} @ \quad x=0 \quad y=0 \\ \quad \quad x=l \quad y=0 \end{aligned}$$

Let trial function,

$$y = a \sin \frac{\pi x}{l}$$

$$\frac{dy}{dn} = a \frac{\pi}{l} \cos \frac{\pi n}{l}$$

$$\frac{d^2y}{dn^2} = -a \frac{\pi^2}{l^2} \sin \frac{\pi n}{l}$$

$$\frac{d^3y}{dn^3} = -a \frac{\pi^3}{l^3} \cos \frac{\pi n}{l}$$

$$\frac{d^4y}{dn^4} = a \frac{\pi^4}{l^4} \sin \frac{\pi n}{l}$$

from governing eqn,

$$R = EI a \frac{\pi^4}{l^4} \sin \frac{\pi n}{l} - w$$

i) Point Collocation :

$$R = 0$$

$$EI a \frac{\pi^4}{l^4} \sin \frac{\pi n}{l} - w = 0$$

$$EI a \frac{\pi^4}{l^4} \sin \frac{\pi n}{l} = w$$

to get max. deflection, $n = l/2$.

$$EI a \frac{\pi^4}{l^4} \sin \frac{\pi \cdot (l/2)}{l} = w$$

$$EI a \frac{\pi^4}{l^4} = w$$

$$a = \frac{w l^4}{\pi^4 \cdot EI}$$

$$\therefore \text{ Trial fn, } y = \frac{w l^4}{\pi^4 EI} \sin \frac{\pi n}{l}$$

$$\text{At } n = l/2, \quad y_{\max} = \frac{w l^4}{\pi^4 EI} \sin \frac{\pi (l/2)}{l}$$

$$y_{\max} = \frac{w l^4}{\pi^4 \cdot EI} = \frac{w l^4}{97.4 EI}$$

ii) Sub domain :

$$\int_0^l R \, dn = 0.$$

$$\int_0^l \left(EI a \frac{\pi^4}{l^4} \sin \frac{\pi n}{l} - w \right) dn = 0$$

$$\left[a EI \frac{\pi^4}{l^4} \left(-\cos \frac{\pi n}{l} \right) \left(\frac{l}{\pi} \right) - wn \right]_0^l = 0$$

$$-a EI \frac{\pi^3}{l^3} (\cos \pi - \cos 0) - wl = 0$$

$$-a EI \frac{\pi^3}{l^3} (-1 - 1) = wl$$

$$a = \frac{w l^4}{2 \pi^3 EI} = \frac{w l^4}{62 EI}$$

$$\therefore y = \frac{w l^4}{62 EI} \sin \frac{\pi n}{l} \quad @ n = l/2$$

$$y_{\max} = \frac{w l^4}{62 EI} \sin \frac{\pi}{2} = \frac{w l^4}{62 EI} \quad 1.$$

ONE DIMENSIONAL ELEMENTS



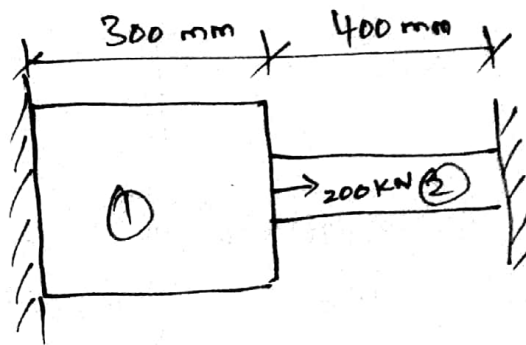
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UNIT-II

- ① Consider a bar as shown in fig. An axial load of 200 kN is applied at point P. Take $A_1 = 2400 \text{ mm}^2$, $E_1 = 70 \times 10^9 \text{ N/m}^2$, $A_2 = 600 \text{ mm}^2$, $E_2 = 200 \times 10^9 \text{ N/m}^2$.

Calculate the following,

- The nodal displacement at point P
- Stress in each material
- Reaction force.



Apr/May 2017.

Sol:

$$A_1 = 2400 \text{ mm}^2$$

$$A_2 = 600 \text{ mm}^2$$

$$L_1 = 300 \text{ mm}$$

$$L_2 = 400 \text{ mm}$$

$$E_1 = 70 \times 10^9 \text{ N/m}^2 = 70 \times 10^3 \text{ N/mm}^2$$

$$E_2 = 200 \times 10^9 \text{ N/m}^2 = 200 \times 10^3 \text{ N/mm}^2$$

$$P = 200 \text{ kN} = 200 \times 10^3 \text{ N}$$

$$\{F\} = [K] \{u\}$$

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Element ①

$$K_1 = \frac{A_1 E_1}{L_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{2400 \times 20 \times 10^3}{300} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_1 = 5.6 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{bmatrix} 5.6 & -5.6 \\ -5.6 & 5.6 \end{bmatrix}$$

Element ②

$$K_2 = \frac{A_2 E_2}{L_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{600 \times 200 \times 10^3}{400} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_2 = 3 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$K = K_1 + K_2$$

$$= 10^5 \begin{bmatrix} 5.6 & -5.6 & 0 \\ -5.6 & 8.6 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

Force Vector,

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \times 10^5 \\ 0 \end{bmatrix}$$

Displacement Vector,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ u_2 \\ 0 \end{bmatrix}$$

$$[K][u] = [F]$$

$$10^5 \begin{bmatrix} 5.6 & -5.6 & 0 \\ -5.6 & 8.6 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \times 10^5 \\ 0 \end{bmatrix}$$

$$\therefore (8.6 u_2) 10^5 = 2 \times 10^5$$

$$\boxed{u_2 = 0.2325 \text{ mm}}$$

Stresses in element ①,

$$\begin{aligned} \sigma_1 &= E_1 \times \frac{u_2 - u_1}{L_1} \\ &= 70 \times 10^3 \left(\frac{0.2325 - 0}{300} \right) \end{aligned}$$

$$\sigma_1 = 54.25 \text{ N/mm}^2$$

Stresses in element ②,

$$\begin{aligned}\sigma_2 &= E_2 \times \frac{u_3 - u_2}{L_2} \\ &= 200 \times 10^3 \left(\frac{0 - 0.2325}{400} \right) \\ \sigma_2 &= -116.25 \text{ N/mm}^2.\end{aligned}$$

Reaction forces,

$$[R] = [K][u] - [F]$$

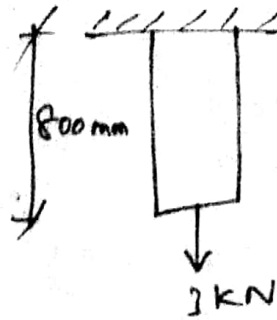
$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = 10^5 \begin{bmatrix} 5.6 & -5.6 & 0 \\ -5.6 & 8.6 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0.2325 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \times 10^5 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = 10^5 \begin{bmatrix} -1.302 \\ 1.99 \\ -0.697 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \times 10^5 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} -1.302 \times 10^5 \\ 0 \\ -0.697 \times 10^5 \end{bmatrix} \text{ N}$$

- ② A Steel bar of length 800 mm is subjected to an axial load of 3 kN. Find the nodal displacements of the bar and load vectors.

Take $E = 2 \times 10^5 \text{ N/mm}^2$. $A = 300 \text{ mm}^2$ $\rho = 7800 \text{ kg/m}^3$.



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Sol:

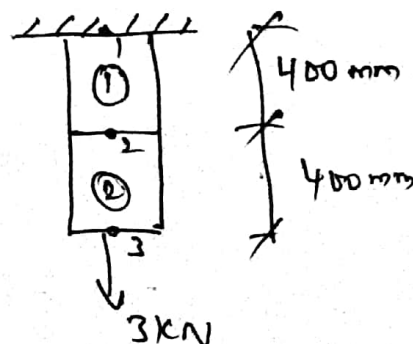
$$L = 800$$

$$F = 3 \times 10^3 \text{ N}$$

$$E = 2 \times 10^5 \text{ N/mm}^2$$

$$A = 300 \text{ mm}^2$$

$$\rho = 7800 \text{ kg/m}^3 = 76518 \text{ N/m}^3 = 7.65 \times 10^5 \text{ N/mm}^3$$



Force Vector,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = EA \begin{bmatrix} 1/L \\ 1/L \\ 2/L \end{bmatrix}$$

$$= 7.6518 \times 10^{-5} \times 300 \times 800$$

$$\begin{bmatrix} 1/6 \\ 1/6 \\ 2/3 \end{bmatrix}$$

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 3.06 \\ 3.06 \\ 12.24 \end{bmatrix}$$

Point load 3000 is acting on node 3.

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 3.06 \\ 3.06 \\ 12.24 + 3 \times 10^3 \end{bmatrix} = \begin{bmatrix} 3.06 \\ 3.06 \\ 3012.24 \end{bmatrix}$$

$$F = K \cdot u$$

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \frac{AE}{3L} \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & 8 & 16 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\begin{bmatrix} 3.06 \\ 3.06 \\ 3012.24 \end{bmatrix} = \frac{2 \times 10^8 \times 300}{3 \times 800} \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & 8 & 16 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$u_1 = 0 \text{ (fixed)}$$

$$\begin{bmatrix} 3.06 \\ 3.06 \\ 3012.24 \end{bmatrix} = 2500 \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & 8 & 16 \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\begin{bmatrix} 3.06 \\ 3012.24 \end{bmatrix} = 2500 \begin{bmatrix} -8 & 16 \\ -8 & 16 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}$$

$$2500(7u_2 - 8u_3) = 3.06 \quad \text{--- ①}$$

$$2500(-8u_2 + 46u_3) = 3012.24 \quad \text{--- ②}$$

Some,

$$2500(-56u_2 + 112u_3) = 2.168 \times 10^4$$

$$2500(56u_2 - 112u_3) = 24.48$$

$$2500(48u_3) = 2.168 \times 10^4$$

$$u_3 = 0.175 \text{ mm.}$$

Sub in ①

$$2500(7u_2 - 8(0.175)) = 3.06$$

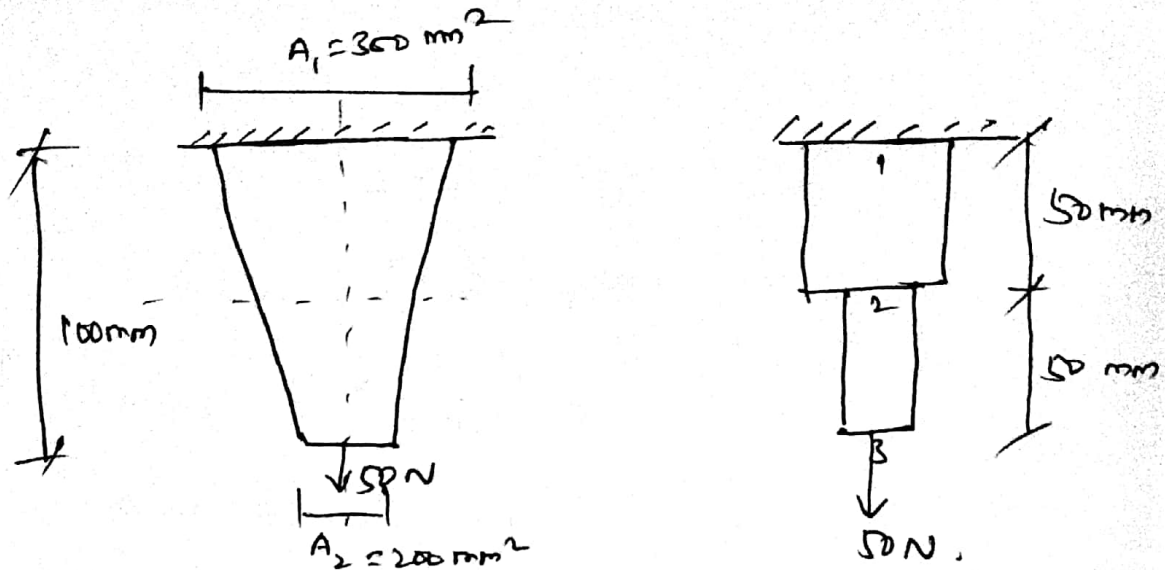
$$17500u_2 - 35000 = 3.06$$

$$u_2 = 2.00 \text{ mm.}$$

- ③ Using two finite elements, find the Stress distribution in a uniformly tapering bar a circular cross sectional area 3 cm^2 (300 mm^2) & 2 cm^2 (200 mm^2) at their ends, length 100 mm , Subjected to an axial tensile load of 50 N at smaller end and fixed at larger end. Take the Value of Young's modulus $2 \times 10^5 \text{ N/mm}^2$.

Nov/Dec 2017

Sol:



at node 1, $A_1 = 3 \text{ cm}^2 = 300 \text{ mm}^2$

at node 2, $A_2 = 2 \text{ cm}^2 = 200 \text{ mm}^2$

Area at node 3,

$$A_2 = \frac{A_1 + A_2}{2} = \frac{3 + 2}{2}$$

$$A_2 = 2.5 \text{ cm}^2$$

Avg. Area for ①, $A_1 = \frac{A_1 + A_2}{2} = \frac{3 + 2.5}{2}$

$$A_1 = 2.75 \text{ cm}^2 = 2.75 \times 10^2 \text{ mm}^2$$

Avg. Area for ②, $A_2 = \frac{A_2 + A_1}{2} = \frac{2.5 + 2}{2}$

$$A_2 = 2.25 \text{ cm}^2 = 2.25 \times 10^2 \text{ mm}^2$$

Tensile load at node 3, = 50 N,

$$E = 2 \times 10^5 \text{ N/mm}^2$$

for element ①,

$$K_1 = \frac{A_1 E_1}{L_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$
$$= \frac{275 \times 2 \times 10^5}{50} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_1 = 10^5 \begin{bmatrix} 11 & -11 \\ -11 & 11 \end{bmatrix}$$

for element ②,

$$K_2 = \frac{A_2 E_2}{L_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$
$$= \frac{225 \times 2 \times 10^5}{50} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_2 = 10^5 \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

Global Stiffness Matrix,

$$K = K_1 + K_2$$

$$[K] = 10^5 \begin{bmatrix} 11 & -11 & 0 \\ -11 & 20 & -9 \\ 0 & -9 & 9 \end{bmatrix}$$

Applying boundary Conditions,

$$u_1 = 0.$$

Neglecting rows & columns,

$$10^5 \begin{bmatrix} 20 & -9 \\ -9 & 7 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 50 \end{bmatrix}$$

$$10^5 [20u_2 - 9u_3] = 0 \quad \text{--- (1)}$$

$$10^5 [-9u_2 + 7u_3] = 50 \quad \text{--- (2)}$$

Solve,

$$10^5 [11u_2] = 50$$

$$u_2 = 4.5 \times 10^{-5} \text{ mm.}$$

$$10^5 [20(4.5 \times 10^{-5}) - 9u_3] = 0.$$

$$u_3 = 1.01 \times 10^{-4} \text{ mm.}$$

Stresses:

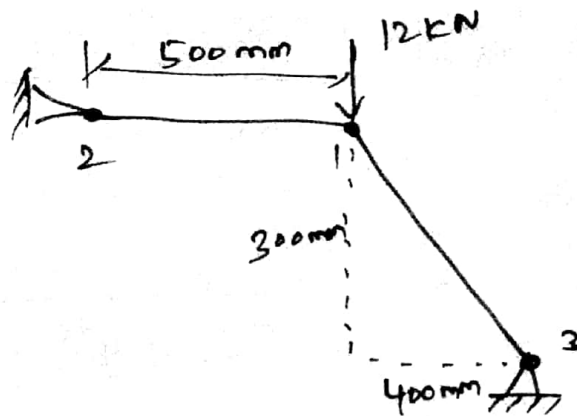
$$\sigma_1 = \frac{E(u_2 - u_1)}{l_1} = \frac{2 \times 10^5 (4.5 \times 10^{-5} - 0)}{50}$$

$$\sigma_1 = 0.1816 \text{ N/mm}^2$$

$$\sigma_2 = \frac{E(u_3 - u_2)}{l_2} = \frac{2 \times 10^5 (1.01 \times 10^{-4} - 4.5 \times 10^{-5})}{50}$$

$$\sigma_2 = 0.222 \text{ N/mm}^2.$$

- ④ For the two bar truss shown in fig. determine the displacements of node 1 and the stress in the element 1-3.

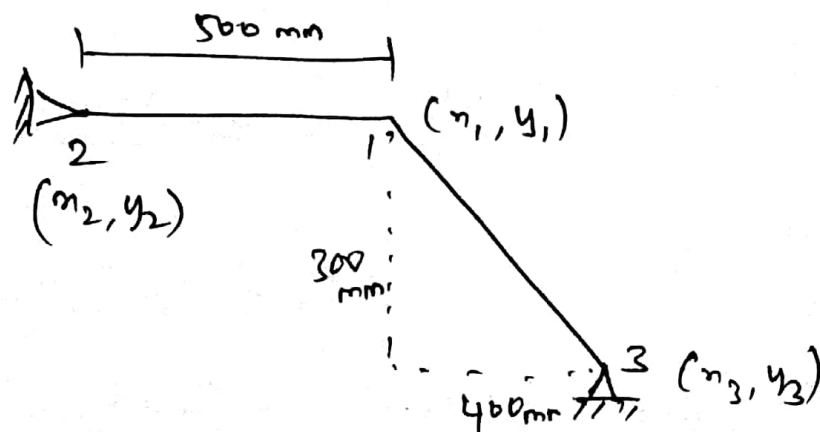


$$E = 70 \text{ GPa}$$

$$A = 200 \text{ mm}^2$$

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Sol:



$$E = 70 \text{ GPa} = 70 \times 10^9 \text{ N/m}^2$$

$$E = 70 \times 10^3 \text{ N/mm}^2$$

$$A = 200 \text{ mm}^2$$

For element ①

$$L_{e1} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(-500 - 0)^2 + (0 - 0)^2}$$

$$L_{e1} = 500 \text{ mm}$$

Direction of cosines $l_1 = \frac{x_2 - x_1}{L_{e1}}$

$$= \frac{-500 - 0}{500}$$

$$l_1 = -1$$

$$m_1 = \frac{y_2 - y_1}{L_{e1}} = \frac{0 - 0}{500} = 0.$$

For element ②,

$$L_{e2} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$$

$$= \sqrt{(400 - 0)^2 + (-300 - 0)^2}$$

$$L_{e2} = 500 \text{ mm.}$$

Direction of cosines, $l_2 = \frac{x_3 - x_1}{L_{e2}} = \frac{400}{500}$

$$l_2 = 0.8$$

$$m_2 = \frac{y_3 - y_1}{L_{e2}} = \frac{-300 - 0}{500} = -0.6.$$

$$[K]_1 = \frac{A_1 E_1}{L_{e1}} \begin{bmatrix} l_1^2 & l_1 m_1 & -l_1^2 & -l_1 m_1 \\ l_1 m_1 & -m_1^2 & -l_1 m_1 & -m_1^2 \\ -l_1^2 & -l_1 m_1 & l_1^2 & -l_1 m_1 \\ -l_1 m_1 & -m_1^2 & -l_1 m_1 & -m_1^2 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix}$$

$$[k] = \frac{2000 \times 70 \times 10^3}{500} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k_1] = 28 \times 10^3 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k_2] = \frac{200 \times 70 \times 10^3}{500} \begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix}$$

$$[k] = 28 \times 10^3 \begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix}$$

$$[k] = [k_1] + [k_2]$$

$$[k] = 28 \times 10^3 \begin{bmatrix} 1.64 & -0.48 & -1 & 0 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0 & 0 & 0.48 & -0.36 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.64 & 0.48 & 0 & 0 & 0.64 & -0.48 \\ -0.48 & -0.36 & 0 & 0 & -0.48 & 0.36 \end{bmatrix}$$

$$[K] [u] = [F]$$

$$28 \times 10^3 \begin{bmatrix} 1.64 & -0.48 & -1 & 0 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0 & 0 & 0.48 & -0.36 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.64 & 0.48 & 0 & 0 & 0.64 & -0.48 \\ -0.48 & -0.36 & 0 & 0 & -0.48 & 0.36 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix}$$

$$F_2 = -12 \times 10^3 \text{ N}$$

$$u_3 = u_4 = u_5 = u_6 = 0.$$

$$28 \times 10^3 \begin{bmatrix} 1.64 & -0.48 \\ -0.48 & 0.36 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -12 \times 10^3 \end{bmatrix}$$

$$28 \times 10^3 (1.64 u_1 - 0.48 u_2) = 0 \quad \text{--- (1)}$$

$$28 \times 10^3 (-0.48 u_1 + 0.36 u_2) = -12 \times 10^3 \quad \text{--- (2)}$$

by solve,

$$u_2 = -1.952 \text{ mm}$$

$$u_1 = -0.571 \text{ mm}$$

Stresses, 1-2,

$$\sigma = \frac{E}{l_{e1}} [-1 \quad -m \quad 1 \quad m] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$= \frac{70 \times 10^3}{500} [\quad \quad 1 \quad 0 \quad -1 \quad 0] \begin{bmatrix} -0.571 \\ -1.952 \\ 0 \\ 0 \end{bmatrix}$$

$$\sigma = 79.94 \text{ N/mm}^2 \text{ (Tensile load)}.$$

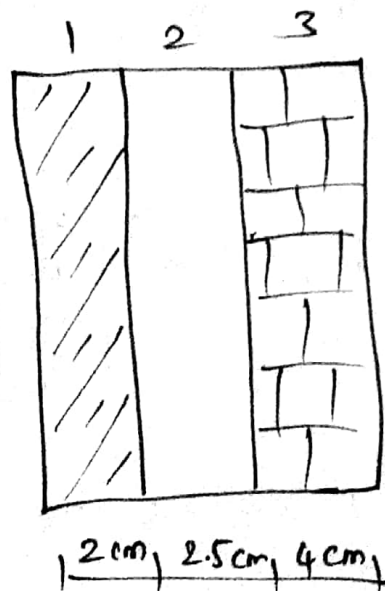
Stresses 1-3,

$$\sigma = \frac{E}{l_{e2}} [-1 \quad -m \quad 1 \quad m] \begin{bmatrix} u_1 \\ u_2 \\ u_5 \\ u_6 \end{bmatrix}$$

$$= \frac{70 \times 10^3}{500} [-0.8 \quad 0.6 \quad 0.8 \quad -0.6] \begin{bmatrix} -0.571 \\ -1.952 \\ 0 \\ 0 \end{bmatrix}$$

$$\sigma = -100 \text{ N/mm}^2 \text{ (compressive load)}$$

- ⑤ A Composite Wall consists of three materials as shown in fig. The inside wall temperature is 100°C and the outside air temperature is 50°C with a convection Co-efficient h of $10\text{ W/m}^2\text{C}$. Determine the temperature along the composite wall if the Conduction Co-efficients of the three materials are as follows in the fig.



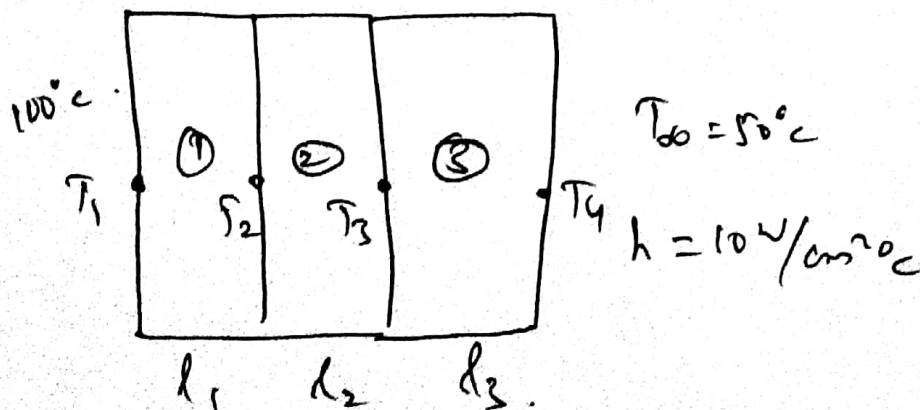
$$k_1 = 70\text{ W/m}^2\text{K}$$

$$k_2 = 40\text{ W/m}^2\text{K}$$

$$k_3 = 20\text{ W/m}^2\text{K}$$

Nov/Dec 2019.

Sol:



$$\lambda_1 = 2 \text{ cm} \quad \lambda_2 = 2.5 \text{ cm} \quad \lambda_3 = 4 \text{ cm}.$$

element ① :

$$[k] = \frac{k_1 A_1}{\lambda_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A = 1 \text{ cm}^2.$$

$$= \frac{70 \times 1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 35 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k_1] = \begin{bmatrix} 35 & -35 \\ -35 & 35 \end{bmatrix}$$

element ②

$$[k_2] = \frac{k_2 A_2}{\lambda_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{40 \times 1}{2.5} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k_2] = \begin{bmatrix} 16 & -16 \\ -16 & 16 \end{bmatrix}$$

element ③

$$[k_3] = \frac{20 \times 1}{4} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (\text{due to Convection})$$

$$[k_3] = \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ -5 & 15 \end{bmatrix}$$

FEA, eqn,

$$[K] [T] = [F]$$

$$F_1 = F_2 = F_3 = 0,$$

$$F_4 = hA T_\infty = 10 \times 1 \times 323 = 3230 \text{ W}$$

global matrix, $T_1 = 100^\circ\text{C} = 373 \text{ K}$.

$$\begin{bmatrix} 35 & -35 & 0 & 0 \\ -35 & (35+16) & -16 & 0 \\ 0 & -16 & 16+5 & -5 \\ 0 & 0 & -5 & 15 \end{bmatrix} \begin{bmatrix} 373 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3230 \end{bmatrix}$$

~~-8585~~ $\leq 4 \times 10^3$

$$-13055 + 51 T_2 - 16 T_3 = 0 \quad \text{--- (1)}$$

$$-16 T_2 + 21 T_3 - 5 T_4 = 0 \quad \text{--- (2)}$$

$$-5 T_3 + 15 T_4 = 3230 \quad \text{--- (3)}$$

$$5 T_3 = -3230 + 15 T_4$$

$$T_3 = 3 T_4 - 646 \quad \text{--- (4)}$$

Sub in (2),

$$-16 T_2 + 21 (3 T_4 - 646) - 5 T_4 = 0$$

$$-16 T_2 + 58 T_4 - 13566 = 0$$

$$T_2 = 3.62 T_4 - 847.8 \quad \text{--- (5)}$$

Sub (6) in (7)

$$-13055 + 51 T_2 - 16 (3 T_4 - 646) = 0$$

$$-13055 + 51 T_2 - 48 T_4 + 10336 = 0$$

$$51 T_2 = 2719 + 48 T_4$$

$$T_2 = 53.31 + 0.94 T_4 \quad \text{--- (6)}$$

Equate eqn (5) & (6)

$$3.62 T_4 - 847.8 = 53.31 + 0.94 T_4$$

$$2.68 T_4 = 901.11$$

$$\boxed{T_4 = 336.2 \text{ K}}$$

∴

$$T_2 = 53.31 + 0.94 (336.2)$$

$$\boxed{T_2 = 369.33 \text{ K}}$$

∴

$$T_3 = 3 (336.2) - 646$$

$$\boxed{T_3 = 362.6 \text{ K}}$$

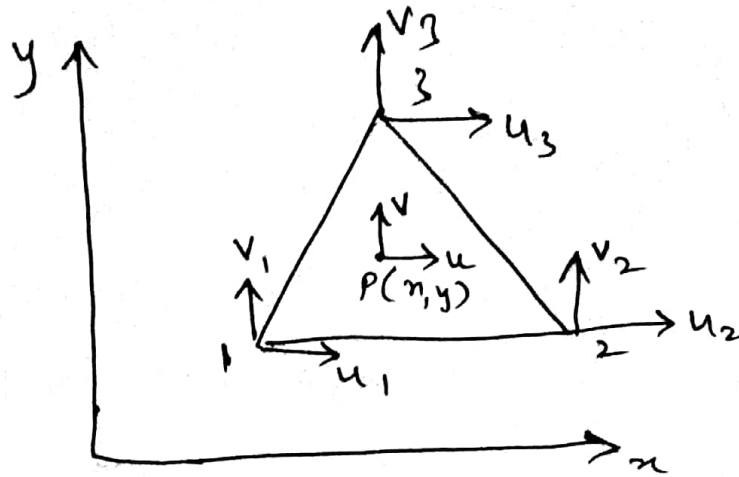
TWO DIMENSIONAL

SCALAR / VECTOR VARIABLE $\Rightarrow$

Mohan S R, AP/Mech

UNIT-3

Derivation of Shape functions for Two dimensional Linear Element :



Consider CST element,

$\delta \rightarrow$ displacement

$u \rightarrow$ displacement component in x axis

$v \rightarrow$ " " in y axis

$x, y \rightarrow$ Co-ordinates of nodes.

polynomial eqn,

$$u(x, y) = a_1 + a_2 x + a_3 y$$

$$v(x, y) = a_4 + a_5 x + a_6 y.$$

apply nodal conditions,

$$u = u_1, v = v_1, x = x_1, y = y_1,$$

$$u = u_2, v = v_2, x = x_2, y = y_2$$

$$u = u_3, v = v_3, x = x_3, y = y_3$$

So,

$$u_1 = a_1 + a_2 x_1 + a_3 y_1$$

$$u_2 = a_1 + a_2 x_2 + a_3 y_2$$

$$u_3 = a_1 + a_2 x_3 + a_3 y_3$$

&

$$v_1 = a_4 + a_5 x_1 + a_6 y_1$$

$$v_2 = a_4 + a_5 x_2 + a_6 y_2$$

$$v_3 = a_4 + a_5 x_3 + a_6 y_3$$

in matrix,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

(or)

$$[a] = [D]^{-1} [u]$$

$[D] \rightarrow$ Co-ordinate matrix

$$D^{-1} = \frac{C^T}{|D|}$$

$$[c] = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

↳ Co-factor matrix

$$c_{ij} = (-1)^{i+j} \cdot |D_m|$$

↳ minor matrix

$$c_{11} = (-1)^{1+1} \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} = x_2 y_3 - x_3 y_2$$

Similarly calculate for all co-factors.

$$[c] = \begin{bmatrix} x_2 y_3 - x_3 y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3 y_1 - x_1 y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1 y_2 - x_2 y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}$$

$$[c]^T = \begin{bmatrix} x_2 y_3 - x_3 y_2 & x_3 y_1 - x_1 y_3 & x_1 y_2 - x_2 y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}$$

$$|D| = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$|D| = 2A \quad [\text{Triangular area}]$$

$$\therefore [a] = [D^{-1}] [u]$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2 y_3 - x_3 y_2 & x_3 y_1 - x_1 y_3 & x_1 y_2 - x_2 y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} d_1 & d_2 & d_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

where,

$$d_1 = x_2 y_3 - x_3 y_2$$

$$d_2 = x_3 y_1 - x_1 y_3$$

$$d_3 = x_1 y_2 - x_2 y_1$$

$$\beta_1 = y_2 - y_3$$

$$\beta_2 = y_3 - y_1$$

$$\beta_3 = y_1 - y_2$$

$$\gamma_1 = x_3 - x_2$$

$$\gamma_2 = x_1 - x_3$$

$$\gamma_3 = x_2 - x_1$$

$$\therefore u(x, y) = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$u(x, y) = \begin{bmatrix} 1 & x & y \end{bmatrix} \frac{1}{2A} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} \alpha_1 + \beta_1 x + \gamma_1 y & \alpha_2 + \beta_2 x + \gamma_2 y & \alpha_3 + \beta_3 x + \gamma_3 y \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$u(x, y) = N_1 u_1 + N_2 u_2 + N_3 u_3$$

where,

$$N_1 = \frac{\alpha_1 + \beta_1 x + \gamma_1 y}{2A}$$

$$N_2 = \frac{\alpha_2 + \beta_2 x + \gamma_2 y}{2A}$$

$$N_3 = \frac{\alpha_3 + \beta_3 x + \gamma_3 y}{2A}$$

Similarly for, $v(x, y)$,

$$v(x, y) = N_1 v_1 + N_2 v_2 + N_3 v_3$$

finally,

displacement,

$$f(x, y) = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

$$f(x, y) = [N] \{f\}$$

Note:

$$N_1 + N_2 + N_3 = 1$$

Formulae :

* Stress - Strain relation,

$$[\sigma] = [D] [e]$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [D] \begin{bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{bmatrix}$$

$$D = \left[\frac{E}{1-\mu^2} \right] \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \quad (\text{plane stress})$$

$$D = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & 0 \\ \mu & 1-\mu & 0 \\ 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix} \quad (\text{plane strain})$$

$\mu \rightarrow$ poisson ratio

$E \rightarrow$ Young's modulus

$\sigma_x \rightarrow$ Stress in x axis

$\sigma_y \rightarrow$ " in y axis

$\tau_{xy} \rightarrow$ Shear stress

$e \rightarrow$ Strain

* Strain - displacement relation

$$\{e\} = [B]\{\delta\}$$

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$\{\delta\} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

$$e = \begin{bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{bmatrix}$$

* Stress - displacement relation

$$[e] = [D][e]$$

$$[e] = [D][B]\{\delta\}$$

Maximum normal stress,

$$\sigma_1 = \frac{1}{2} \left[(\sigma_x + \sigma_y) + \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \right]$$

Minimum normal stress,

$$\sigma_2 = \frac{1}{2} \left[(\sigma_x + \sigma_y) - \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \right]$$

Maximum Shear Stress,

$$\tau_{\max} = \frac{1}{2} (\sigma_1 - \sigma_2)$$

Principle angle θ_p ,

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

* Stiffness Matrix $[K]$

$$[K] = [B]^T [D] [B] A \cdot t$$

$$A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$t \rightarrow$ Thickness of element.

* Temperature effects :

$$\epsilon_0 = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{bmatrix} \text{ (plane stress)}$$

$$\epsilon_0 = (1+\mu) \begin{bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{bmatrix} \text{ (plane strain)}$$

$\alpha \rightarrow$ Thermal expansion Co-efficient

$\Delta T \rightarrow$ change in temperature.

$$[F_0] = [B]^T [D] [e_0] A \cdot t.$$

$$(6 \times 1) = (6 \times 3) (3 \times 3) (3 \times 1)$$

$$(6 \times 1) = (6 \times 1)$$

$$F_0 = \begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{bmatrix}$$

Note:

$$[F] = B^T D e_0 A t$$

$$= B^T D B \delta A t$$

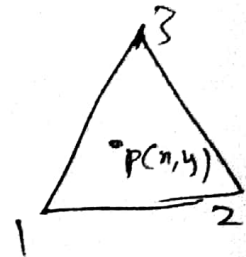
$$[F] = [K] \cdot [\delta]$$

Isoparametric Representation:

WKT,

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3.$$



Co-ordinates of point P,

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3.$$

$$(N_1 + N_2 + N_3 = 1), \quad N_3 = 1 - N_1 - N_2.$$

Substitute,

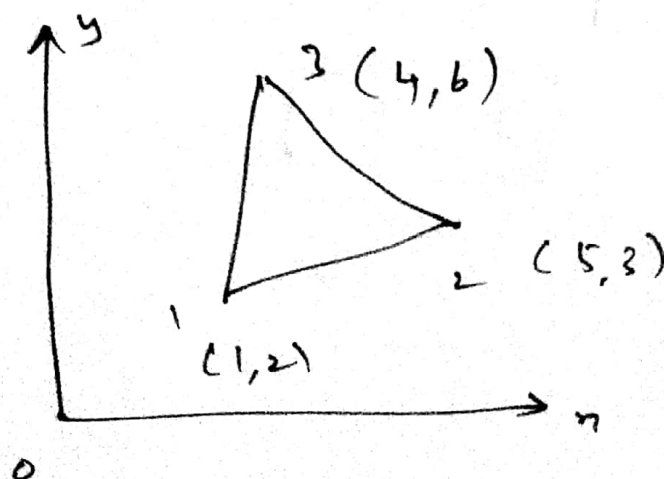
$$x = N_1 x_1 + N_2 x_2 + (1 - N_1 - N_2) x_3$$

$$x = (x_1 - x_3) N_1 + (x_2 - x_3) N_2 + x_3$$

Similarly,

$$y = (y_1 - y_3) N_1 + (y_2 - y_3) N_2 + y_3$$

- ① The nodal co-ordinates of a triangular element are shown in fig. At the point P inside the element, the x co-ordinate is 3.3 and the shape function $N_1 = 0.3$. Determine the shape functions N_2 , N_3 & y co-ordinate of the point P.



Sol:

$$x_1 = 1 \quad y_1 = 2$$

$$x_2 = 5 \quad y_2 = 3$$

$$x_3 = 4 \quad y_3 = 6$$

WKT,

$$x = (x_1 - x_3)N_1 + (x_2 - x_3)N_2 + N_3$$

$$3.2 = (1 - 4)0.3 + (5 - 4)N_2 + 4$$

$$\boxed{N_2 = 0.2}$$

$$\therefore N_1 + N_2 + N_3 = 1$$

$$0.3 + 0.2 + N_3 = 1$$

$$\boxed{N_3 = 0.5}$$

Then,

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

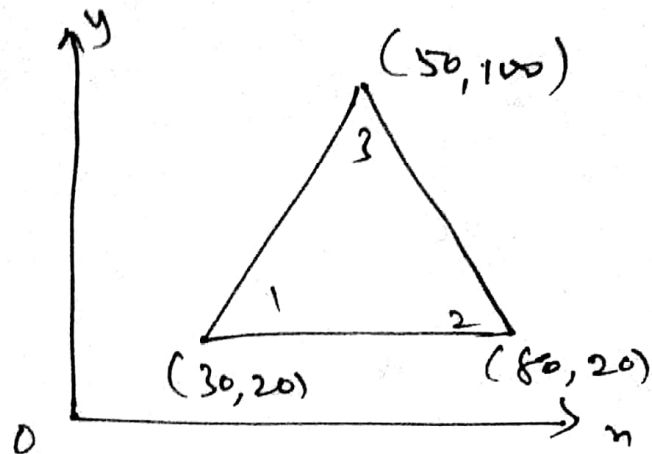
$$y = 0.3(2) + 0.2(3) + 0.5(6)$$

$$\boxed{y = 4.2}$$

② For the plane stress element shown in fig. evaluate the stiffness matrix. Assume

$$E = 210 \times 10^8 \text{ N/mm}^2, \quad \mu = 0.25, \quad t = 10 \text{ mm}.$$

The Co-ordinates are given in mm.



Sol:

$$x_1 = 30$$

$$y_1 = 20$$

$$x_2 = 80$$

$$y_2 = 20$$

$$x_3 = 50$$

$$y_3 = 100.$$

for CST,

$$[K] = [B]^T [D] [B] A t.$$

$$A = \frac{1}{2} \begin{vmatrix} x & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$= \frac{1}{2} \begin{vmatrix} 1 & 30 & 20 \\ 1 & 80 & 20 \\ 1 & 50 & 100 \end{vmatrix}$$

$$A = 2000 \text{ mm}^2$$

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$\beta_1 = y_2 - y_3 = 20 - 100 = -80$$

$$\beta_2 = y_3 - y_1 = 100 - 20 = 80$$

$$\beta_3 = y_1 - y_2 = 20 - 20 = 0$$

$$\gamma_1 = x_3 - x_2 = 50 - 80 = -30$$

$$\gamma_2 = x_1 - x_3 = 30 - 50 = -20$$

$$\gamma_3 = x_2 - x_1 = 80 - 30 = 50$$

$$\therefore [B] = \frac{1}{2 \times 2000} \begin{bmatrix} -80 & 0 & 80 & 0 & 0 & 0 \\ 0 & -30 & 0 & -20 & 0 & 50 \\ -30 & -80 & -20 & 80 & 50 & 0 \end{bmatrix}$$

$$= \frac{1}{4000} \begin{bmatrix} -8 & 0 & 8 & 0 & 0 & 0 \\ 0 & -3 & 0 & -2 & 0 & 5 \\ -3 & -8 & -2 & 8 & 5 & 0 \end{bmatrix}$$

$$[B]^T = \frac{1}{40} \begin{bmatrix} -8 & 0 & -3 \\ 0 & -3 & -8 \\ 8 & 0 & -2 \\ 0 & -2 & 8 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{bmatrix}$$

for plane stress condition,

$$D = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$

$$= \frac{210 \times 10^3}{1-0.25^2} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix}$$

$$[D] = 56 \times 10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

finally,

$$[K] = \frac{1}{40} \begin{bmatrix} -8 & 0 & -3 \\ 0 & -3 & -8 \\ 8 & 0 & -2 \\ 0 & -2 & 8 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{bmatrix} 56 \times 10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \times$$

$$\frac{1}{400} \begin{bmatrix} -8 & 0 & 8 & 0 & 0 & 0 \\ 0 & -3 & 0 & -2 & 0 & 5 \\ -3 & -8 & -2 & 8 & 5 & 0 \end{bmatrix} \quad 2000 \times 10.$$

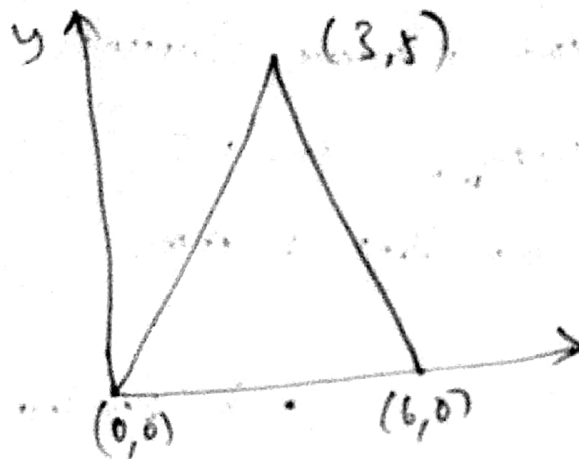
$$K = 7 \times 10^3 \begin{bmatrix} 269.5 & 60 & -247 & -20 & -22.5 & -40 \\ 60 & 132 & 0 & -72 & -60 & -60 \\ -247 & 0 & 262 & -40 & -15 & 40 \\ -20 & -72 & -40 & 112 & 60 & -40 \\ -22.5 & -60 & -15 & 60 & 37.5 & 0 \\ -40 & -60 & 40 & -40 & 0 & 100 \end{bmatrix}$$

Since the above matrix is symmetric.

$\therefore$ the sum of the values at any column is 0.

Therefore, Calculated stiffness matrix is correct.

③ Evaluate the element stiffness matrix for the triangular element as shown in fig. under plane strain condition. Assume, $E = 200 \text{ GPa}$, $\mu = 0.4$ $t = 1 \text{ mm}$.



Sol:

$$A = 15 \text{ mm}^2 \quad t = 1$$

$$[B] = \frac{1}{30} \begin{bmatrix} -5 & 0 & 5 & 0 & 0 & 0 \\ 0 & -3 & 0 & -3 & 0 & 6 \\ -3 & -5 & -3 & 5 & 6 & 0 \end{bmatrix}$$

$$[D] = 8 \times 10^4 \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$K = 1.33 \times 10^3$$

$$\begin{bmatrix} 64 & 30 & -16 & 0 & -18 & -30 \\ 30 & 52 & 0 & 2 & -30 & -54 \\ -16 & 0 & 84 & -30 & -18 & 30 \\ 0 & 2 & -30 & 52 & 30 & -54 \\ -18 & -30 & -18 & 30 & 36 & 0 \\ -30 & -54 & 30 & -54 & 0 & 108 \end{bmatrix}$$

Ans.

④ For the plane stress element, the nodal displacements are,

$$u_1 = 2 \text{ mm} \quad v_1 = 1 \text{ mm}$$

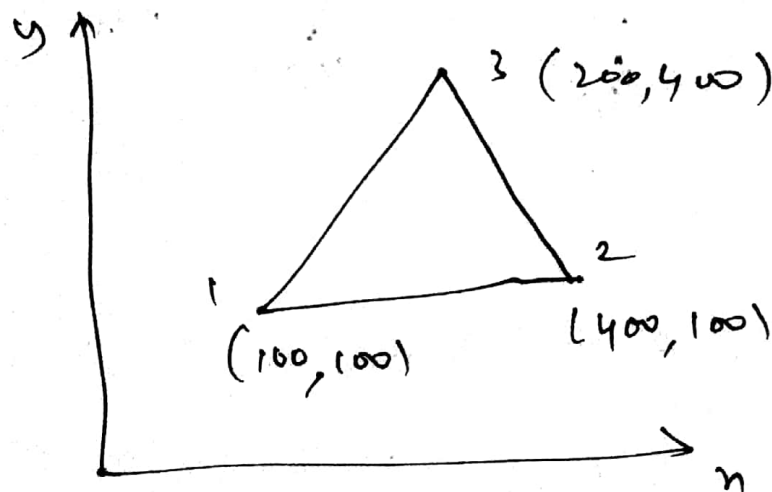
$$u_2 = 1 \text{ mm} \quad v_2 = 1.5 \text{ mm}$$

$$u_3 = 2.5 \text{ mm} \quad v_3 = 0.5 \text{ mm}$$

Determine the element stresses. Assume

$$E = 200 \text{ GPa/m}^2 \quad \mu = 0.3 \quad t = 10 \text{ mm}$$

all dimensions are in mm.



Sol:

$$[\sigma] = [D] \{e\}$$

$$[\sigma] = [D] [B] [\delta]$$

$$x_1 = 100$$

$$y_1 = 100$$

$$x_2 = 400$$

$$y_2 = 100$$

$$x_3 = 200$$

$$y_3 = 400$$

For plane stress,

$$[D] = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$

$$= \frac{2 \times 10^5}{1-0.3^2} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1-0.3}{2} \end{bmatrix}$$

$$[D] = \frac{2 \times 10^6}{91} \begin{bmatrix} 10 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & 3.5 \end{bmatrix}$$

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$\beta_1 = y_2 - y_3 = 100 - 400 = -300$$

$$\beta_2 = y_3 - y_1 = 400 - 100 = 300$$

$$\beta_3 = y_1 - y_2 = 100 - 100 = 0$$

$$\gamma_1 = x_3 - x_2 = 200 - 400 = -200$$

$$\gamma_2 = x_1 - x_3 = 100 - 200 = -100$$

$$\gamma_3 = x_2 - x_1 = 400 - 100 = 300$$

$$[A] = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 100 & 100 \\ 1 & 400 & 100 \\ 1 & 200 & 400 \end{vmatrix}$$

$$A = 45000 \text{ mm}^2$$

$$[B] = \frac{1}{2 \times 45000} \begin{bmatrix} -300 & 0 & 300 & 0 & 0 & 0 \\ 0 & -200 & 0 & -100 & 0 & 300 \\ -200 & -300 & -100 & 300 & 300 & 0 \end{bmatrix}$$

$$[B] = \frac{1}{900} \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 3 \\ -2 & -3 & -1 & 3 & 3 & 0 \end{bmatrix}$$

$$\delta = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1.5 \\ 2.5 \\ 0.5 \end{bmatrix}$$

$$\begin{bmatrix} \delta_x \\ \delta_y \\ \delta_{xy} \end{bmatrix} = \frac{2 \times 10^6}{91} \begin{bmatrix} 10 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} \frac{1}{900} \begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 3 \\ -2 & -3 & -1 & 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1.5 \\ 2.5 \\ 0.5 \end{bmatrix}$$

$$= \frac{2 \times 10^6}{91 \times 900} \begin{bmatrix} 10 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} \begin{bmatrix} -3 \\ -2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{2 \times 10^4}{819} \begin{bmatrix} -36 \\ -29 \\ 14 \end{bmatrix} = \begin{bmatrix} -879.12 \\ -708.18 \\ 341.88 \end{bmatrix} \text{ N/mm}^2$$

Maximum normal stress,

$$\begin{aligned} \sigma_1 &= \frac{1}{2} \left[(\sigma_x + \sigma_y) + \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \right] \\ &= \frac{1}{2} \left[(-879.12 - 708.18) + \sqrt{(-879.12 + 708.18)^2 + 4(341.88)^2} \right] \\ &= \frac{1}{2} \left[-1587.3 + 704.81 \right] \\ \sigma_1 &= -441.24 \text{ N/mm}^2 \end{aligned}$$

Minimum normal stress,

$$\begin{aligned} \sigma_2 &= \frac{1}{2} \left[(\sigma_x + \sigma_y) - \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \right] \\ &= \frac{1}{2} \left[-1587.3 - 704.81 \right] \end{aligned}$$

$$\sigma_2 = -1146 \text{ N/mm}^2$$

Principle angle $\theta = \frac{1}{2} \tan^{-1} \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \right)$

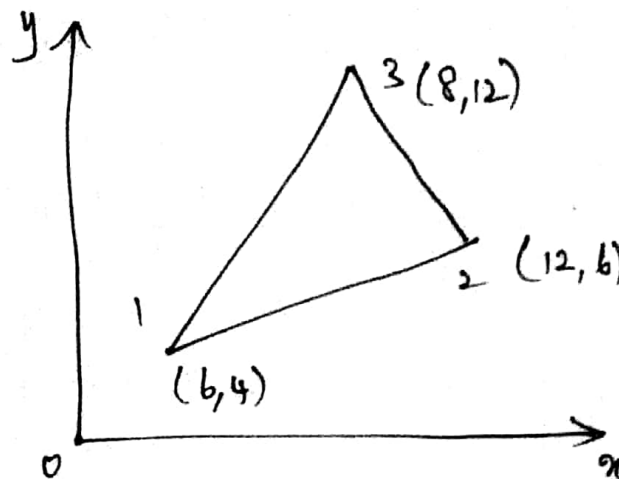
$$= \frac{1}{2} \tan^{-1} \left(\frac{2 \times 341.88}{-879.12 + 708.18} \right) = -37.98^\circ$$

5) The element shown in fig is subjected to a temperature change 10°C . Find the load due to temperature change.

Take $E = 200 \text{ GPa}$, $\mu = 0.3$, $t = 2 \text{ mm}$

$\alpha = 7 \times 10^{-6} / ^\circ\text{C}$. Assume plane stress conditions.

The Co-ordinates in mm.



Sol:

$$x_1 = 6 \quad y_1 = 4$$

$$x_2 = 12 \quad y_2 = 6$$

$$x_3 = 8 \quad y_3 = 12$$

$$\Delta T = 10^\circ\text{C}$$

$$\alpha = 7 \times 10^{-6} / ^\circ\text{C} \quad \mu = 0.3 \quad t = 2 \text{ mm}$$

$$E = 200 \text{ GPa} = 200 \times 10^3 \text{ N/mm}^2 = 2 \times 10^5 \text{ N/mm}^2$$

$$[F_0] = [B]^T [D] [e_0] A t$$

↳ Thermal Load Vector.

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 6 & 4 \\ 1 & 12 & 6 \\ 1 & 8 & 12 \end{vmatrix}$$

$$A = 22 \text{ mm}^2$$

$$\beta_1 = y_2 - y_3 = 6 - 12 = -6$$

$$\beta_2 = y_3 - y_1 = 12 - 4 = 8$$

$$\beta_3 = y_1 - y_2 = 4 - 6 = -2$$

$$\gamma_1 = x_3 - x_2 = 8 - 12 = -4$$

$$\gamma_2 = x_1 - x_3 = 6 - 8 = -2$$

$$\gamma_3 = x_2 - x_1 = 12 - 6 = 6$$

$$[B] = \frac{1}{44} \begin{bmatrix} -6 & 0 & 8 & 0 & -2 & 0 \\ 0 & -4 & 0 & -2 & 0 & 6 \\ -4 & -6 & -2 & 8 & 6 & -2 \end{bmatrix}$$

$$[B]^T = \frac{1}{44} \begin{bmatrix} -6 & 0 & -4 \\ 0 & -4 & -6 \\ 8 & 0 & -2 \\ 0 & -2 & 8 \\ -2 & 0 & 6 \\ 0 & 6 & -2 \end{bmatrix}$$

For plane stress,

$$D = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$

$$= \frac{2 \times 10^5}{1-0.3^2} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1-0.3}{2} \end{bmatrix}$$

$$[D] = \frac{2 \times 10^6}{91} \begin{bmatrix} 10 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & 3.5 \end{bmatrix}$$

$$[e] = \begin{bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \times 10^{-5} \\ 7 \times 10^{-5} \\ 0 \end{bmatrix}$$

$$\therefore [F_0] = [B]^T [D] [e] \text{ A t.}$$

$$\begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{bmatrix} = \frac{1}{44} \begin{bmatrix} -6 & 0 & -4 \\ 0 & -4 & -6 \\ 8 & 0 & -2 \\ 0 & -2 & 8 \\ -2 & 0 & 6 \\ 0 & 6 & -2 \end{bmatrix} \frac{2 \times 10^6}{91} \begin{bmatrix} 10 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & 3.5 \end{bmatrix} \begin{bmatrix} 7 \times 10^{-5} \\ 7 \times 10^{-5} \\ 0 \end{bmatrix} (22 \times 2)$$

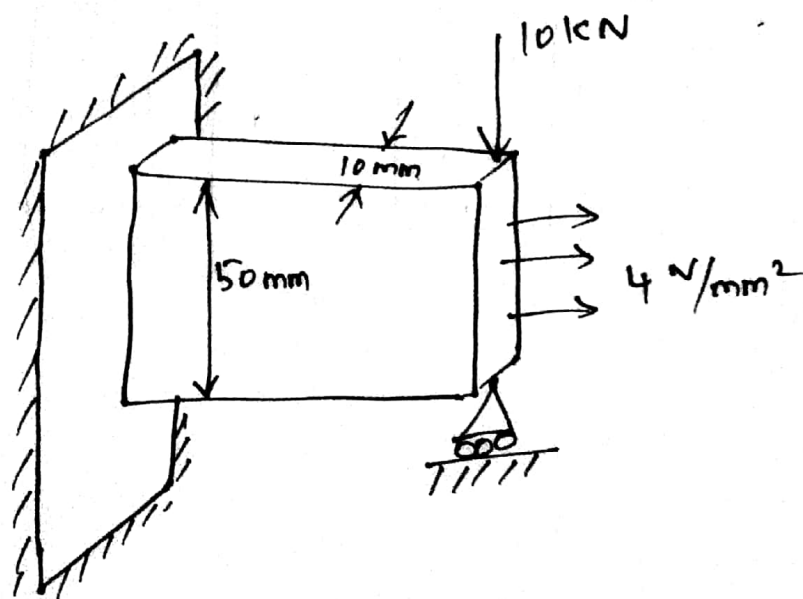
$$= \frac{2 \times 10^6 \times 22 \times 2}{94 \times 91} \begin{bmatrix} -60 & -18 & -14 \\ -12 & -40 & -21 \\ 80 & 24 & -7 \\ -6 & -20 & 28 \\ -20 & -6 & 21 \\ 18 & 60 & -7 \end{bmatrix} \begin{bmatrix} 7 \times 10^{-5} \\ 7 \times 10^{-5} \\ 0 \end{bmatrix}$$

$$= \frac{2 \times 10^6}{91} \times 7 \times 10^{-5} \begin{bmatrix} -60 & -18 & -14 \\ -12 & -40 & -21 \\ 80 & 24 & -7 \\ -6 & -20 & 28 \\ -20 & -6 & 21 \\ 18 & 60 & -7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

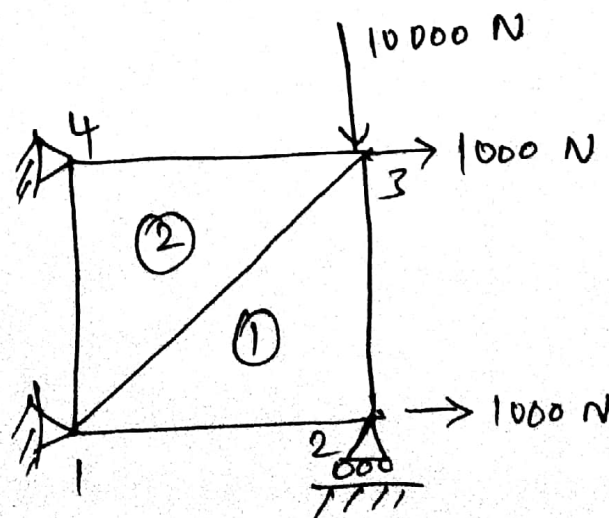
$$= \frac{140}{91} \begin{bmatrix} -78 \\ -52 \\ 104 \\ -26 \\ -26 \\ 78 \end{bmatrix} = \begin{bmatrix} -120 \\ -80 \\ 60 \\ -40 \\ -40 \\ 120 \end{bmatrix} N$$

University Problem :

For a thin plate subjected to the surface traction force along with an external load as shown in fig. determine the nodal displacements and element stresses. Take $t = 10 \text{ mm}$ $E = 210 \text{ GPa}$ $\mu = 0.25$. Analyse the plate using two element model.



Sol:



$$\text{Nodal Force} = \frac{1}{2} TA.$$

$T \rightarrow$ Traction force. (N/mm^2)

$$F = \frac{1}{2} \times 4 \times (50 \times 10)$$

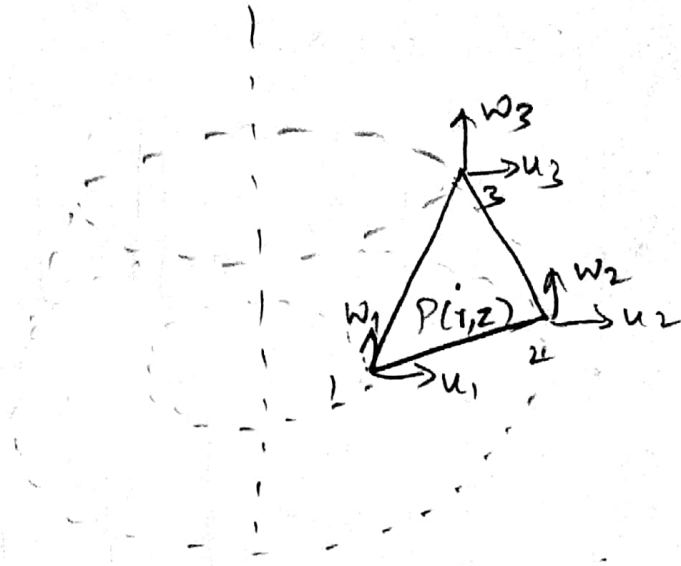
$$F = 1000 \text{ N. @ node 2 \& 3.}$$

TWO DIMENSIONAL

AXI-SYMMETRIC COMPONENTS $\Rightarrow$

Mohan S R, AP/Mech

Derivation of shape functions for Axisymmetric Triangular Element



Consider Axisymmetric triangular element.

Let $u_1, w_1, u_2, w_2, u_3, w_3$ be nodal displacements

$u_1, u_2, u_3 \rightarrow$ radial direction (r)

$w_1, w_2, w_3 \rightarrow$ Axial direction (z)

$$\{ \delta \} = \begin{bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{bmatrix}$$

from polynomial eqn,

$$u(r, z) = a_1 + a_2 r + a_3 z$$

$$v(r, z) = a_4 + a_5 r + a_6 z$$

in matrix, under $u \rightarrow$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$[a] = [D]^{-1} [u] \quad \text{--- (1)}$$

then

$D \rightarrow$ Coordinate matrix.

$$D^{-1} = \frac{[C]^T}{|D|}$$

$C \rightarrow$ Co-factor Matrix.

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$c_{ij} = (-1)^{i+j} |D_{ji}|$$

$\rightarrow$ Minor matrix of $|D|$.

$$c_{11} = (-1)^{1+1} \begin{vmatrix} r_2 & z_2 \\ r_3 & z_3 \end{vmatrix} = r_2 z_3 - r_3 z_2.$$

Similarly calculate for remaining factors,

$$\therefore [C] = \begin{bmatrix} r_2 z_3 - r_3 z_2 & z_2 - z_3 & r_3 - r_2 \\ r_3 z_1 - r_1 z_3 & z_3 - z_1 & r_1 - r_3 \\ r_1 z_2 - r_2 z_1 & z_1 - z_2 & r_2 - r_1 \end{bmatrix}$$

$$[C]^T = \begin{bmatrix} r_2 z_3 - r_3 z_2 & r_3 z_1 - r_1 z_3 & r_1 z_2 - r_2 z_1 \\ z_2 - z_3 & z_3 - z_1 & z_1 - z_2 \\ r_3 - r_2 & r_1 - r_3 & r_2 - r_1 \end{bmatrix}$$

$$|D| = \begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix}$$

$$|D| = 2A$$

from eqn ①,

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} r_2 z_3 - r_3 z_2 & r_3 z_1 - r_1 z_3 & r_1 z_2 - r_2 z_1 \\ z_2 - z_3 & z_3 - z_1 & z_1 - z_2 \\ r_3 - r_2 & r_1 - r_3 & r_2 - r_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\alpha_1 = r_2 z_3 - r_3 z_2 \quad \alpha_2 = r_3 z_1 - r_1 z_3 \quad \alpha_3 = r_1 z_2 - r_2 z_1$$

$$\beta_1 = z_2 - z_3 \quad \beta_2 = z_3 - z_1 \quad \beta_3 = z_1 - z_2$$

$$\gamma_1 = r_3 - r_2 \quad \gamma_2 = r_1 - r_3 \quad \gamma_3 = r_2 - r_1$$

WKT,

$$u(r, z) = a_1 + a_2 r + a_3 z.$$

$$u(r, z) = [1 \ r \ z] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$u(r, z) = [1 \ r \ z] \frac{1}{2A} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\alpha_1 + \beta_1 r + \gamma_1 z}{2A} & \frac{\alpha_2 + \beta_2 r + \gamma_2 z}{2A} & \frac{\alpha_3 + \beta_3 r + \gamma_3 z}{2A} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3.$$

Similarly for, $w(x, z)$.

$$w = N_1 w_1 + N_2 w_2 + N_3 w_3.$$

Finally displacement,

$$\delta_p = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

$$[\delta]_p = [N] \cdot [\delta]$$

Formula

* Stress-Strain relationship Matrix,

$$[\sigma] = [D] \cdot [e]$$

$$\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{bmatrix} = [D] \begin{bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{bmatrix}$$

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 \\ \mu & 1-\mu & \mu & 0 \\ \mu & \mu & 1-\mu & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix}$$

$\sigma_r \rightarrow$ Radial Stress

$\sigma_\theta \rightarrow$ Circumferential Stress

$\sigma_z \rightarrow$ Axial Stress

$\tau_{rz} \rightarrow$ Shear Stress.

Similarly for Strain $[e]$.

* Strain - Displacement Matrix

$$[e] = [B] [\delta]$$

$$[\delta] = \begin{bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{bmatrix}$$

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1 + \beta_1 r + \gamma_1 z}{r} & 0 & \frac{\alpha_2 + \beta_2 r + \gamma_2 z}{r} & 0 & \frac{\alpha_3 + \beta_3 r + \gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$r = \frac{r_1 + r_2 + r_3}{3}$$

$$z = \frac{z_1 + z_2 + z_3}{3}$$

* Stress - Displacement Matrix.

$$[\sigma] = [D] [e]$$

$$[\sigma] = [D] [B] [\delta]$$

* Stiffness Matrix:

$$[K] = [B]^T [D] [B] 2\pi r. A$$

- ① Compute the Strain-displacement Matrix for the axisymmetric triangular element shown in fig. Also determine the element strains. The nodal displacements are found out as

$$u_1 = 0.002$$

$$w_1 = 0.001$$

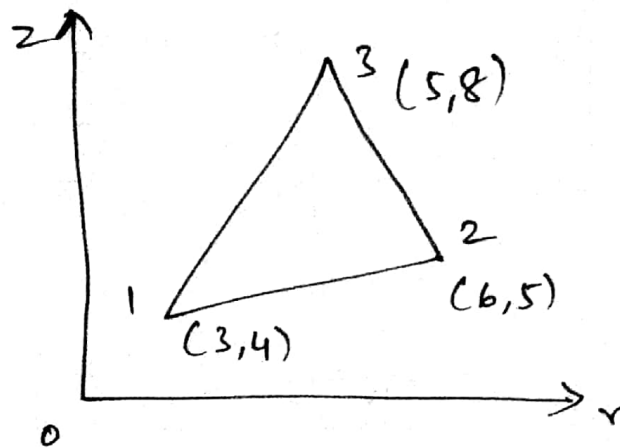
$$u_2 = 0.001$$

$$w_2 = -0.004$$

$$u_3 = -0.003$$

$$w_3 = 0.007$$

All dimensions are in cm.



Sol:

$$r_1 = 3$$

$$z_1 = 4$$

$$r_2 = 6$$

$$z_2 = 5$$

$$r_3 = 5$$

$$z_3 = 8$$

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1 + \beta_1 r + \gamma_1 z}{r} & 0 & \frac{\alpha_2 + \beta_2 r + \gamma_2 z}{r} & 0 & \frac{\alpha_3 + \beta_3 r + \gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$A = \frac{1}{2} \begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix}$$

$$= \frac{1}{2} \begin{vmatrix} 1 & 3 & 4 \\ 1 & 6 & 5 \\ 1 & 5 & 8 \end{vmatrix}$$

$$A \approx 5 \text{ cm}^2$$

$$\alpha_1 = r_2 z_3 - r_3 z_2 = (6 \times 8) - (5 \times 5) = 23$$

$$\alpha_2 = r_3 z_1 - r_1 z_3 = 5(4) - 3(8) = 20 - 24 = -4$$

$$\alpha_3 = r_1 z_2 - r_2 z_1 = 3(5) - 6(4) = 15 - 24 = -9$$

$$\beta_1 = z_2 - z_3 = 5 - 8 = -3$$

$$\beta_2 = z_3 - z_1 = 8 - 4 = 4$$

$$\beta_3 = z_1 - z_2 = 4 - 5 = -1$$

$$\gamma_1 = r_3 - r_2 = 5 - 6 = -1$$

$$\gamma_2 = r_1 - r_3 = 3 - 5 = -2$$

$$\gamma_3 = r_2 - r_1 = 6 - 3 = 3$$

$$\text{Coordinate } r = \frac{r_1 + r_2 + r_3}{3} = \frac{3 + 6 + 5}{3} = 4.7$$

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{4 + 5 + 8}{3} = 5.7$$

Now,

$$\frac{\alpha_1 + \beta_1 r + \gamma_1 z}{r} = 0.68$$

$$\frac{\alpha_2 + \beta_2 r + \gamma_2 z}{r} = 0.72$$

$$\frac{\alpha_3 + \beta_3 r + \gamma_3 z}{r} = 0.72$$

Hence,

$$B = \frac{1}{10} \begin{bmatrix} -3 & 0 & 4 & 0 & -1 & 0 \\ 0.68 & 0 & 0.72 & 0 & 0.72 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -3 & -2 & 4 & 3 & -1 \end{bmatrix}$$

$$\{e\} = [B] \{f\}$$

$$\begin{bmatrix} e_r \\ e_\theta \\ e_z \\ \phi_{rz} \end{bmatrix} = \frac{1}{10} \begin{bmatrix} -3 & 0 & 4 & 0 & -1 & 0 \\ 0.68 & 0 & 0.72 & 0 & 0.72 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -3 & -2 & 4 & 3 & -1 \end{bmatrix} \begin{bmatrix} 0.002 \\ 0.001 \\ 0.001 \\ -0.004 \\ -0.003 \\ 0.007 \end{bmatrix}$$

$$\begin{bmatrix} e_r \\ e_\theta \\ e_z \\ \phi_{rz} \end{bmatrix} = \begin{bmatrix} 0.0001 \\ -0.000008 \\ 0.0028 \\ -0.0039 \end{bmatrix}$$

② Calculate the element stresses for the axisymmetric element shown in fig. The nodal displacements are,

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$$u_1 = 0.02 \text{ mm}$$

$$w_1 = 0.03 \text{ mm}$$

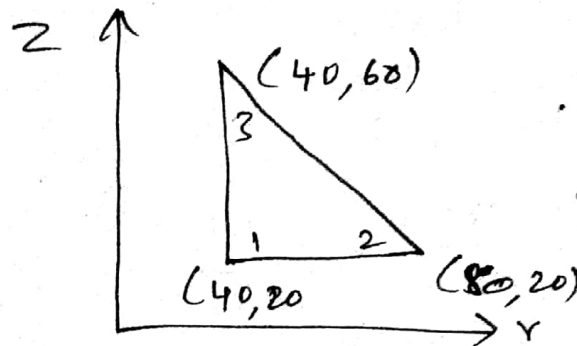
$$u_2 = 0.01 \text{ mm}$$

$$w_2 = 0.06 \text{ mm}$$

$$u_3 = 0.04 \text{ mm}$$

$$w_3 = 0.01 \text{ mm}$$

Take $E = 210 \text{ GPa}$ $\mu = 0.25$



Sol:

$$r_1 = 40$$

$$z_1 = 20$$

$$r_2 = 80$$

$$z_2 = 20$$

$$r_3 = 40$$

$$z_3 = 60$$

WILT,

$$\{\epsilon\} = [D][B]\{\delta\}$$

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 \\ \mu & 1-\mu & \mu & 0 \\ \mu & \mu & 1-\mu & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix}$$

$$= \frac{210 \times 10^3}{(1+0.25)(1-0.5)} \begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$[D] = 84 \times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1 + \beta_1 r + \beta_2 z}{r} & 0 & \frac{\alpha_2 + \beta_2 r + \beta_3 z}{r} & 0 & \frac{\alpha_3 + \beta_3 r + \beta_1 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

~~for α and β~~

$$\alpha_1 = r_2 z_3 - r_3 z_2 = 80(60) - 40(20) = 4000$$

$$\alpha_2 = r_3 z_1 - r_1 z_3 = 40(20) - 40(60) = -1600$$

$$\alpha_3 = r_1 z_2 - r_2 z_1 = 40(20) - 80(20) = -800$$

$$\beta_1 = z_2 - z_3 = 20 - 60 = -40$$

$$\beta_2 = z_3 - z_1 = 60 - 20 = 40$$

$$\beta_3 = z_1 - z_2 = 20 - 20 = 0$$

$$r_1 = r_3 - r_2 = 40 - 80 = -40$$

$$r_2 = r_1 - r_3 = 40 - 40 = 0$$

$$r_3 = r_2 - r_1 = 80 - 40 = 40$$

$$A = \frac{1}{2} \begin{vmatrix} 1 & 40 & 20 \\ 1 & 80 & 20 \\ 1 & 40 & 60 \end{vmatrix}$$

$$= \frac{1}{2} \left[1(4800 - 800) - 40(60 - 20) + 20(40 - 80) \right]$$

$$A = 800 \text{ mm}^2$$

$$B = \frac{1}{2 \times 800} \begin{bmatrix} -40 & 0 & 40 & 0 & 0 & 0 \\ \frac{4800 - 40(53.3) - 40(33.3)}{53.3} & 0 & \frac{-1600 + 40(53.3) + 0}{53.3} & 0 & \frac{-800 + 0 + 40(33.3)}{53.3} & 0 \\ 0 & -40 & 0 & 0 & 0 & 40 \\ -40 & -40 & 0 & 40 & 40 & 0 \end{bmatrix}$$

$$r = \frac{r_1 + r_2 + r_3}{3} = \frac{40 + 80 + 40}{3} = 53.33$$

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{20 + 20 + 60}{3} = 33.33$$

$$B = \frac{1}{1600} \begin{bmatrix} -40 & 0 & 40 & 0 & 0 & 0 \\ 10 & 0 & 10 & 0 & 10 & 0 \\ 0 & -40 & 0 & 0 & 0 & 40 \\ -40 & -40 & 0 & 40 & 40 & 0 \end{bmatrix}$$

$$B = \frac{1}{160} \begin{bmatrix} -4 & 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -4 & 0 & 0 & 0 & 4 \\ -4 & -4 & 0 & 4 & 4 & 0 \end{bmatrix}$$

$$\sigma = D B f$$

$$\begin{bmatrix} \sigma_y \\ \sigma_\theta \\ \sigma_z \\ \tau_{yz} \end{bmatrix} = \frac{84 \times 10^3}{160} \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -4 & 0 & 0 & 0 & 4 \\ -4 & -4 & 0 & 4 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0.02 \\ 0.03 \\ 0.01 \\ 0.06 \\ 0.04 \\ 0.01 \end{bmatrix}$$

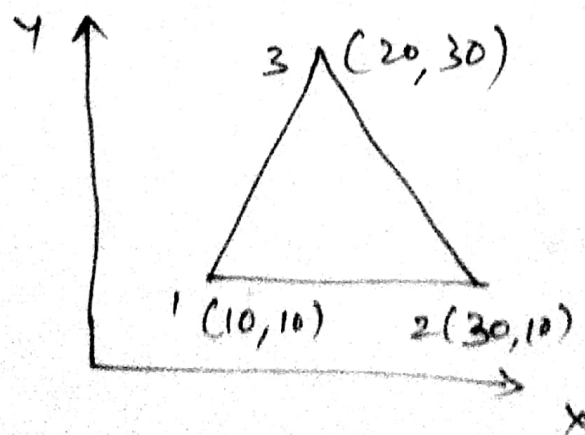
$$= \frac{84 \times 10^3}{160} \begin{bmatrix} -11 & -4 & 13 & 0 & 1 & 4 \\ -1 & -4 & 7 & 0 & 3 & 4 \\ -3 & -12 & 5 & 0 & 1 & 12 \\ -4 & -4 & 0 & 4 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0.02 \\ 0.03 \\ 0.01 \\ 0.06 \\ 0.04 \\ 0.01 \end{bmatrix}$$

$$= 525 \begin{bmatrix} -0.13 \\ 0.08 \\ -0.21 \\ 0.2 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{bmatrix} = \begin{bmatrix} -68.25 \\ 42 \\ -110.25 \\ 105 \end{bmatrix} \text{ N/mm}^2$$

- ③ With at least two examples explain what is meant by axisymmetric analysis. For the 3 noded triangular axisymmetric element shown in fig. derive the strain displacement matrix $[B]$ and also Constitutive Matrix $[D]$.

Apr/may 2018



Sol:

In some 3D solids like cylinders, flywheels, turbine, discs etc, the material content is symmetric with respect to their axes. (i.e, material content is equal in equal distances of opposite sides). Hence the stress developed, displacement produced etc. are considered as symmetric. Such elements are known as Axisymmetric elements.

Due to Axisymmetric Condition, these three dimensional solids can be treated as two dimensional solids for analysis.

$$r_1 = 10$$

$$z_1 = 10$$

$$r_2 = 30$$

$$z_2 = 10$$

$$r_3 = 20$$

$$z_3 = 30$$

$$r = \frac{r_1 + r_2 + r_3}{3} = \frac{10 + 30 + 20}{3} = 20$$

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{10 + 10 + 30}{3} = 16.67$$

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 \\ \mu & 1-\mu & \mu & 0 \\ \mu & \mu & 1-\mu & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix}$$

$$= \frac{210 \times 10^3}{(1+0.25)(1-2 \times 0.25)} \begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$[D] = 84 \times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1 + \beta_1 r_1 + \gamma_1 z_1}{r} & 0 & \frac{\alpha_2 + \beta_2 r_2 + \gamma_2 z_2}{r} & 0 & \frac{\alpha_3 + \beta_3 r_3 + \gamma_3 z_3}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$\alpha_1 = r_2 z_3 - r_3 z_2 = 30(30) - 20(10) = 700$$

$$\alpha_2 = r_3 z_1 - r_1 z_3 = 20(10) - 10(30) = -100$$

$$\alpha_3 = r_1 z_2 - r_2 z_1 = 10(10) - 30(10) = -200$$

$$\beta_1 = z_2 - z_3 = 10 - 30 = -20$$

$$\beta_2 = z_3 - z_1 = 30 - 10 = 20$$

$$\beta_3 = z_1 - z_2 = 10 - 10 = 0$$

$$\gamma_1 = r_3 - r_2 = 20 - 30 = -10$$

$$\gamma_2 = r_1 - r_3 = 10 - 20 = -10$$

$$\gamma_3 = r_2 - r_1 = 30 - 10 = 20$$

$$\frac{\alpha_1 + \beta_1 r + \gamma_1 z}{r} = \frac{700 + (-20)(20) + (-10)(16.67)}{20} = 6.66$$

$$\frac{\alpha_2 + \beta_2 r + \gamma_2 z}{r} = \frac{700 + 20(20) - 10(16.67)}{20} = 6.66$$

$$\frac{\alpha_3 + \beta_3 r + \gamma_3 z}{r} = \frac{-200 + 0 + 20(16.67)}{20} = 6.66$$

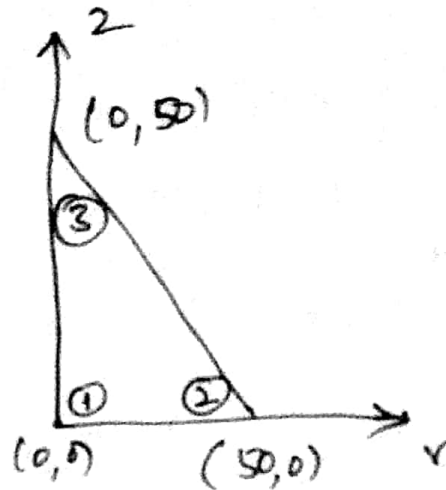
$$\therefore [A] = \frac{1}{2} \begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 10 & 10 \\ 1 & 30 & 10 \\ 1 & 20 & 30 \end{vmatrix} = \frac{1}{2} \begin{bmatrix} 1(700) - 10(20) \\ +10(-10) \end{bmatrix}$$

$$A = 200 \text{ mm}^2$$

$$[B] = \frac{1}{400} \begin{bmatrix} 700 & 0 & -100 & 0 & -200 & 0 \\ -20 & 0 & 20 & 0 & 0 & 0 \\ 6.66 & 0 & 6.66 & 0 & 6.66 & 0 \\ 0 & -10 & 0 & -10 & 0 & 20 \\ -10 & -20 & -10 & 20 & 20 & 0 \end{bmatrix}$$

④ For the axisymmetric element shown in fig. determine the stiffness matrix. Let $E = 2.1 \times 10^5 \text{ N/mm}^2$ & $\mu = 0.25$. The Co-ordinates are in mm.

Nov/Dec 2016.



Sol:

$$\begin{aligned} r_1 &= 0 & z_1 &= 0 \\ r_2 &= 50 & z_2 &= 0 \\ r_3 &= 0 & z_3 &= 50 \end{aligned}$$

$$r = \frac{r_1 + r_2 + r_3}{3} = \frac{50}{3} = 16.66$$

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{50}{3} = 16.66$$

$$A = \frac{1}{2} \begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 50 & 0 \\ 1 & 0 & 50 \end{vmatrix}$$

$$= \frac{1}{2} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 50 & 0 \\ 1 & 0 & 50 \end{vmatrix} = \frac{1}{2} (2500) = 1250 \text{ mm}^2$$

$$D = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 \\ \mu & 1-\mu & \mu & 0 \\ \mu & \mu & 1-\mu & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix}$$

$$= \frac{2.1 \times 10^5}{(1+0.25)(1-2(0.25))} \begin{bmatrix} 0.75 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0.25 & 0 \\ 0.25 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$[D] = 84 \times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1 + \beta_1 x + \gamma_1 z}{r} & 0 & \frac{\alpha_2 + \beta_2 x + \gamma_2 z}{r} & 0 & \frac{\alpha_3 + \beta_3 x + \gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$\alpha_1 = x_2 z_3 - x_3 z_2 = 0$$

$$\alpha_2 = x_3 z_1 - x_1 z_3 = 0$$

$$\alpha_3 = x_1 z_2 - x_2 z_1 = 0$$

$$\beta_1 = z_2 - z_3 = -50$$

$$\beta_2 = z_3 - z_1 = 50$$

$$\beta_3 = z_1 - z_2 = 0$$

$$\gamma_1 = r_3 - r_2 = -50$$

$$\gamma_2 = r_1 - r_3 = 0$$

$$\gamma_3 = r_2 - r_1 = 50$$

$$\frac{\alpha_1 + \beta_1 r + \gamma_1 z}{r} = \frac{0 + (-50) 16.66 + (-50) 16.66}{16.66} = -100$$

$$\frac{\alpha_2 + \beta_2 r + \gamma_2 z}{r} = \frac{0 + 50(16.66) + 0}{16.66} = 50$$

$$\frac{\alpha_3 + \beta_3 r + \gamma_3 z}{r} = \frac{0 + 0 + 50(16.66)}{16.66} = 50$$

$$[B] = \begin{bmatrix} -50 & 0 & 50 & 0 & 0 & 0 \\ -100 & 0 & 50 & 0 & 50 & 0 \\ 0 & -50 & 0 & 0 & 0 & 50 \\ -50 & -50 & 0 & 50 & 50 & 0 \end{bmatrix}$$

$$[B] = 50 \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$[B]^T = \frac{1}{10} \begin{bmatrix} -1 & -2 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Stiffness Matrix,

$$[K] = [B]^T [D] [B] \cdot 2\pi r \cdot A$$

$$[D][B] = 84 \times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{1}{10} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

4×4 4×6

$$= 42 \times 10^5 \begin{bmatrix} -5 & -1 & 4 & 0 & 1 & 1 \\ -7 & -1 & 4 & 0 & 3 & 1 \\ -3 & -3 & 2 & 0 & 1 & 3 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$[K] = 50 \begin{bmatrix} -1 & -2 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \times 10^5 \begin{bmatrix} -5 & -1 & 4 & 0 & 1 & 1 \\ -7 & -1 & 4 & 0 & 3 & 1 \\ -7 & -3 & 2 & 0 & 1 & 3 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$[K] = 2.7 \times 10^{13} \begin{bmatrix} 20 & 4 & -12 & -1 & -8 & -3 \\ 4 & 4 & -2 & -1 & -2 & -3 \\ -12 & -2 & 8 & 0 & 4 & 2 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ -8 & -2 & 4 & 1 & 4 & 1 \\ -3 & -3 & 2 & 0 & 1 & 3 \end{bmatrix}$$

(6x6) (4x6) x 2π x 13.16 x 1250

The above found Stiffness matrix is Correct.
Because of

- * Any Column Sum is zero
- * Sum of any row is also zero.
- * Symmetrical Matrix.

ISO PARAMETRIC ELEMENTS

AND

NUMERICAL INTEGRATION $\Rightarrow$

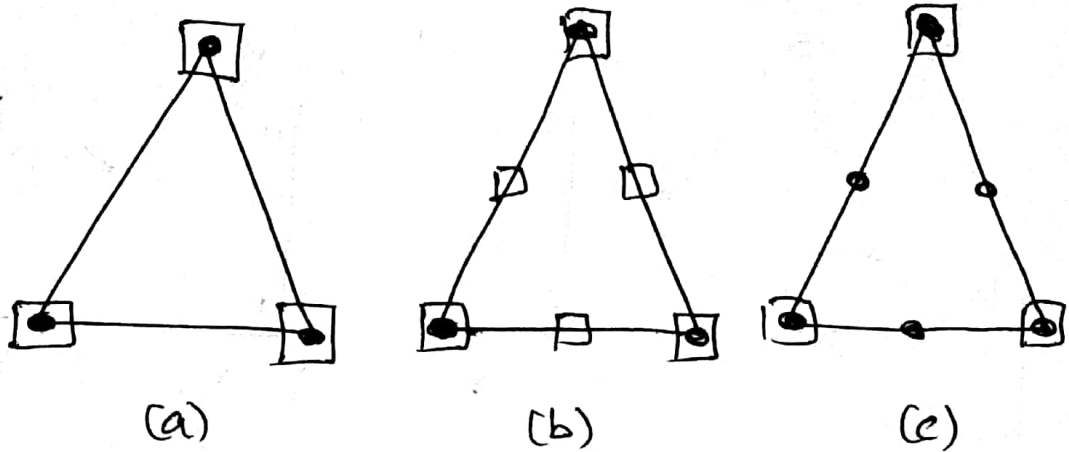
Mohan S R, AP/Mech

UNIT-5

→ Isoparametric (a)

→ Super parametric (b)

→ Sub parametric (c)



- → Nodes for defining displacements
- → Nodes for defining geometry.

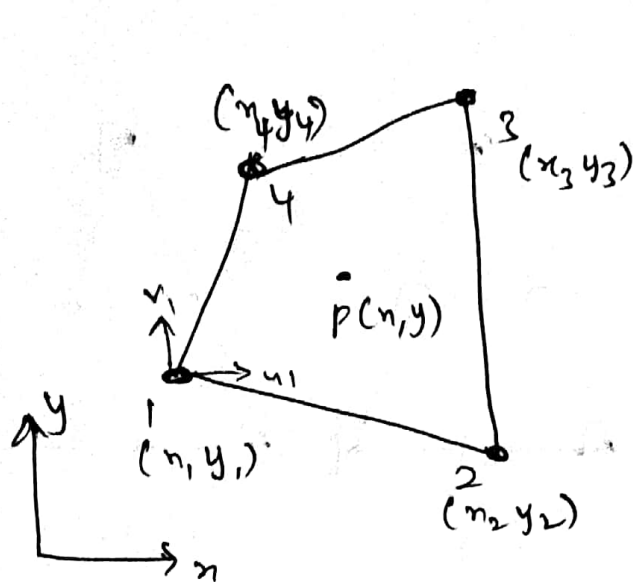
Higher order elements :

For any element, if the interpolation polynomial is of order two or more, the element is known as higher order elements.

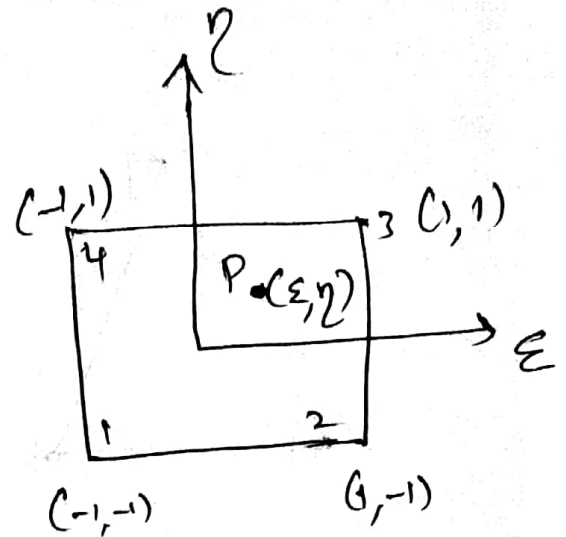
Exp:

$$\phi(x, y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + \dots + a_n y^n.$$

Shape functions 4-noded Quadrilateral Elements (Natural Co-ordinates)



a) Quadrilateral element



b) Isoparametric element
(Master element)

$$u(\xi, \eta) = a_1 + a_2 \xi + a_3 \eta + a_4 \xi \eta$$

$$v(\xi, \eta) = a_5 + a_6 \xi + a_7 \eta + a_8 \xi \eta$$

by solving,

$$N_1 = \frac{1}{4} (1 - \xi) (1 - \eta)$$

$$N_2 = \frac{1}{4} (1 + \xi) (1 - \eta)$$

$$N_3 = \frac{1}{4} (1 + \xi) (1 + \eta)$$

$$N_4 = \frac{1}{4} (1 - \xi) (1 + \eta)$$

&

$$N_1 + N_2 + N_3 + N_4 = 1$$

Jacobian Matrix

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{11} = \frac{1}{4} \left[-(1-\eta) x_1 + (1-\eta) x_2 + (1+\eta) x_3 - (1+\eta) x_4 \right]$$

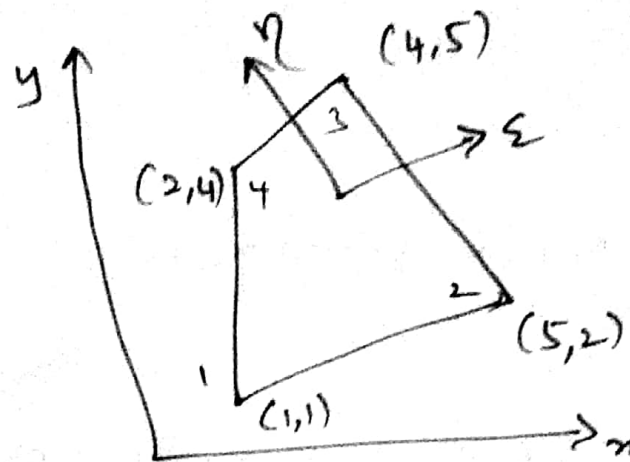
$$J_{12} = \frac{1}{4} \left[-(1-\eta) y_1 + (1-\eta) y_2 + (1+\eta) y_3 - (1+\eta) y_4 \right]$$

$$J_{21} = \frac{1}{4} \left[-(1-\varepsilon) x_1 - (1+\varepsilon) x_2 + (1+\varepsilon) x_3 + (1-\varepsilon) x_4 \right]$$

$$J_{22} = \frac{1}{4} \left[-(1-\varepsilon) y_1 - (1+\varepsilon) y_2 + (1+\varepsilon) y_3 + (1-\varepsilon) y_4 \right]$$

① For the element shown in fig. determine the Jacobian Matrix.

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Sol:

$$\begin{array}{ll} x_1 = 1 & y_1 = 1 \\ x_2 = 5 & y_2 = 2 \\ x_3 = 4 & y_3 = 5 \\ x_4 = 2 & y_4 = 4 \end{array}$$

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$\begin{aligned} J_{11} &= \frac{1}{4} \left[-(1-\eta) x_1 + (1-\eta) x_2 + (1+\eta) x_3 - (1+\eta) x_4 \right] \\ &= \frac{1}{4} \left[-(1-\eta) 1 + (1-\eta) 5 + (1+\eta) 4 - (1+\eta) 2 \right] \\ &= \frac{1}{4} \left[-1 + \eta + 5 - 5\eta + 4 + 4\eta - 2 - 2\eta \right] \\ &= \frac{1}{4} \left[-2\eta + 6 \right] = -0.5\eta + 1.5 \end{aligned}$$

$$\begin{aligned}
 J_{12} &= \frac{1}{4} \left[-(1-\eta) y_1 + (1-\eta) y_2 + (1+\eta) y_3 - (1+\eta) y_4 \right] \\
 &= \frac{1}{4} \left[-(1-\eta) 1 + (1-\eta) 2 + (1+\eta) 5 - (1+\eta) 4 \right] \\
 &= \frac{1}{4} \left[-1 + \eta + 2 - 2\eta + 5 + 5\eta - 4 - 4\eta \right] \\
 &= \frac{1}{4} [2] = 0.5
 \end{aligned}$$

$$\begin{aligned}
 J_{21} &= \frac{1}{4} \left[-(1-\varepsilon) x_1 - (1+\varepsilon) x_2 + (1+\varepsilon) x_3 + (1-\varepsilon) x_4 \right] \\
 &= \frac{1}{4} \left[-(1-\varepsilon) 1 - (1+\varepsilon) 5 + (1+\varepsilon) 4 + (1-\varepsilon) 2 \right] \\
 &= \frac{1}{4} \left[-1 + \varepsilon - 5 - 5\varepsilon + 4 + 4\varepsilon + 2 - 2\varepsilon \right] \\
 &= \frac{1}{4} [-2\varepsilon] = -0.5\varepsilon
 \end{aligned}$$

$$\begin{aligned}
 J_{22} &= \frac{1}{4} \left[-(1-\varepsilon) y_1 - (1+\varepsilon) y_2 + (1+\varepsilon) y_3 + (1-\varepsilon) y_4 \right] \\
 &= \frac{1}{4} \left[-(1-\varepsilon) 1 - (1+\varepsilon) 2 + (1+\varepsilon) 5 + (1-\varepsilon) 4 \right] \\
 &= \frac{1}{4} \left[-1 + \varepsilon - 2 - 2\varepsilon + 5 + 5\varepsilon + 4 - 4\varepsilon \right] \\
 &= \frac{1}{4} [6] = 1.5
 \end{aligned}$$

$$\therefore (J) = \begin{bmatrix} -0.5\eta + 1.5 & 0.5 \\ -0.5\varepsilon & 1.5 \end{bmatrix}$$

$$J_{22} = \frac{1}{4} \left[-(1-\eta) y_1 + (1-\eta) y_2 + (1+\eta) y_3 - (1+\eta) y_4 \right]$$

$$= \frac{1}{4} \left[-(1-\eta) 1 + (1-\eta) 0 + (1+\eta) 2.5 - (1+\eta) 3 \right]$$

$$= \frac{1}{4} \left[-1 + \eta + 2.5 + 2.5\eta - 3 - 3\eta \right]$$

$$= \frac{1}{4} \left[0.5\eta - 1.5 \right]$$

$$= \frac{1}{4} \left[0.5 \left(\frac{1}{3} \right) - 1.5 \right]$$

$$J_{12} = -0.333$$

$$J_{21} = \frac{1}{4} \left[-(1-\varepsilon) x_1 - (1+\varepsilon) x_2 + (1+\varepsilon) x_3 + (1-\varepsilon) x_4 \right]$$

$$= \frac{1}{4} \left[-(1-\varepsilon) 1 - (1+\varepsilon) 3 + (1+\varepsilon) 3.5 + (1-\varepsilon) 2 \right]$$

$$= \frac{1}{4} \left[-1 + \varepsilon - 3 - 3\varepsilon + 3.5 + 3.5\varepsilon + 2 - 2\varepsilon \right]$$

$$= \frac{1}{4} \left[-0.5\varepsilon + 1.5 \right]$$

$$= \frac{1}{4} \left[-0.5 \left(\frac{1}{3} \right) + 1.5 \right] = 0.333$$

$$J_{22} = \frac{1}{4} \left[-(1-\varepsilon) y_1 - (1+\varepsilon) y_2 + (1+\varepsilon) y_3 + (1-\varepsilon) y_4 \right]$$

$$= \frac{1}{4} \left[-(1-\varepsilon) 1 - (1+\varepsilon) 0 + (1+\varepsilon) 2.5 + (1-\varepsilon) 3 \right]$$

$$= \frac{1}{4} \left[-1 + \varepsilon + 2.5 + 2.5\varepsilon + 3 - 3\varepsilon \right] = \frac{1}{4} \left[0.5\varepsilon + 4.5 \right]$$

$$J_{22} = \frac{1}{4} \left[0.5 \left(\frac{1}{3} \right) + 4.5 \right] = 1.16$$

$$\therefore [J] = \begin{bmatrix} 0.83 & -0.33 \\ 0.33 & 1.16 \end{bmatrix}$$

- ③ Evaluate the integral by two point Gaussian quadrature $I = \int_{-1}^1 \int_{-1}^1 (2x^2 + 3xy + 4y^2) dx dy$.
Gauss points are $+0.57735$ and -0.57735
each of weight 1.000.

Nov/Dec 2018

Sol:

$$\int_{-1}^1 \int_{-1}^1 f(x, y) dx dy = \sum_{j=1}^n \sum_{i=1}^n w_j w_i f(x_i, y_i) \dots$$

$$f(x, y) = 2x^2 + 3xy + 4y^2$$

$$\int_{-1}^1 \int_{-1}^1 f(x, y) dx dy = w_1^2 (f(x_1, y_1)) + w_1 w_2 f(x_1, y_2) + w_2^2 (f(x_2, y_2)) + w_1 w_2 f(x_2, y_1)$$

for 2 points,

$$w_1 = w_2 = 1.0$$

$$x_1, y_1 = +0.57735, +0.57735$$

$$x_2, y_2 = -0.57735, -0.57735$$

$$w_1^2 f(x_1, y_1) = w_1^2 (2x_1^2 + 3x_1 y_1 + 4y_1^2)$$

$$= (1.0)^2 (2(0.57735)^2 + 3(0.57735)(0.57735) + 4(0.57735)^2)$$

$$= 3.0$$

$$w_1 w_2 f(x_1, y_2) = w_1 w_2 (2x_1^2 + 3x_1 y_2 + 4y_2^2)$$

$$= 1.0 (1.0) (2(0.57735)^2 + 3(0.57735)(-0.57735) + 4(-0.57735)^2)$$

$$= 1.0$$

$$w_2 w_1 f(x_2, y_1) = w_2 w_1 (2x_2^2 + 3x_2 y_1 + 4y_1^2)$$

$$= 1.0 (1.0) [2(-0.57735)^2 + 3(-0.57735)(0.57735) + 4(0.57735)^2]$$

$$= 1.0$$

$$w_2^2 f(x_2, y_2) = w_2^2 (2x_2^2 + 3x_2 y_2 + 4y_2^2)$$

$$= 1.0^2 [2(-0.57735)^2 + 3(-0.57735)(-0.57735) + 4(-0.57735)^2]$$

$$= 3.0$$

Hence,

$$\int_{-1}^1 \int_{-1}^1 (2x^2 + 3xy + 4y^2) dx dy = 3 + 1 + 1 + 1$$

$$= 8.0 //$$

④ Using Gauss Quadrature evaluate the following integral.

$$I = \int_{-1}^1 (4x^3 - 2x^2 + 3x + 6) dx.$$

Apr/May 2018

Sol:

To find no of gauss points, 3-degree of fn.

$$\text{Let } 2n-1 = 3.$$

$$n = \frac{3+1}{2} = 2 \text{ points.}$$

$$\therefore W_1 = W_2 = 1.0$$

$$x_1 = x_2 = \pm 0.57735.$$

$$\int_{-1}^1 f(x) dx = W_1 f(x_1) + W_2 f(x_2)$$

$$\begin{aligned} W_1 f(x_1) &= W_1 (4x_1^3 - 2x_1^2 + 3x_1 + 6) \\ &= 1 [4(0.57735)^3 - 2(0.57735)^2 + 3(0.57735) + 6] \\ &= 7.83 \end{aligned}$$

$$\begin{aligned} W_2 f(x_2) &= W_2 (4x_2^3 - 2x_2^2 + 3x_2 + 6) \\ &= 1 [4(-0.57735)^3 - 2(-0.57735)^2 + 3(-0.57735) + 6] \\ &= 4.16 \end{aligned}$$

$$I = 7.83 + 4.16 = 11.99 \approx 12.$$

⑤ Using Gauss Quadrature evaluate the following integral.

$$I = \int_{-1}^1 \int_{-1}^1 \frac{12 + 2\xi^2 + \eta}{1 + \eta^2} d\xi d\eta.$$

NW/Dec 2019

Sol:

$$f(\xi, \eta) = \frac{12 + 2\xi^2 + \eta}{1 + \eta^2} d\xi d\eta.$$

$$\int_{-1}^1 \int_{-1}^1 f(\xi, \eta) d\xi d\eta = w_1^2 f(\xi_1, \eta_1) + w_2^2 f(\xi_2, \eta_2) + w_1 w_2 f(\xi_1, \eta_2) + w_1 w_2 f(\xi_2, \eta_1)$$

For 2 point scheme,

$$w_1 = w_2 = 1.0$$

$$\xi_1, \eta_1 = 0.57735, 0.57735$$

$$\xi_2, \eta_2 = -0.57735, -0.57735$$

$$\begin{aligned} w_1^2 f(\xi_1, \eta_1) &= w_1^2 \left(\frac{12 + 2\xi_1^2 + \eta_1}{1 + \eta_1^2} \right) \\ &= 1^2 \left(\frac{12 + 2(0.57735)^2 + 0.57735}{1 + 0.57735^2} \right) = 9.93 \end{aligned}$$

$$w_2^2 f(\varepsilon_2, \eta_2) = w_2^2 \left[\frac{12 + 2\varepsilon_2^2 + \eta_2}{1 + \eta_2^2} \right]$$

$$= 1^2 \left[\frac{12 + 2(-0.57735)^2 + (-0.57735)}{1 + (-0.57735)^2} \right]$$

$$= 8.06 = 9.09$$

$$w_1 w_2 f(\varepsilon_1, \eta_2) = w_1 w_2 \left(\frac{12 + 2\varepsilon_1^2 + \eta_2}{1 + \eta_2^2} \right)$$

$$= 1 \times 1 \left[\frac{12 + 2(0.57735)^2 + (-0.57735)}{1 + (-0.57735)^2} \right]$$

$$= 8.08 = 9.09$$

$$w_1 w_2 f(\varepsilon_2, \eta_1) = w_1 w_2 \left(\frac{12 + 2\varepsilon_2^2 + \eta_1}{1 + \eta_1^2} \right)$$

$$= 1 \times 1 \left[\frac{12 + 2(-0.57735)^2 + 0.57735}{1 + (0.57735)^2} \right]$$

$$= 9.93$$

$$\therefore I = 9.93 + 9.09 + 9.09 + 9.93$$

$$I = 38.04 //$$