MOHAMED SATHAK A.J. COLLEGE OF ENGINEERING

(Approved by AICTE, New Delhi and Affiliated to Anna University, Chennai)

ISO 9001:2015 Certified by IRQS

NABCB QM 006

MA3151-MATRICES AND CALCULUS

NOTES OF LESSON

MA3151 MATRICES AND CALCULUS L T P C 3 1 0 4

UNIT I MATRICES 9+3

Eigen values and Eigenvectors of a real matrix – Characteristic equation – Properties of Eigen values and Eigenvectors – Cayley - Hamilton theorem – Diagonalization of matrices by orthogonal transformation – Reduction of a quadratic form to canonical form by orthogonal transformation – Nature of quadratic forms – Applications : Stretching of an elastic membrane.

UNIT II DIFFERENTIAL CALCULUS 9+3

Representation of functions - Limit of a function - Continuity - Derivatives - Differentiation rules (sum, product, quotient, chain rules) - Implicit differentiation - Logarithmic differentiation - Applications : Maxima and Minima of functions of one variable.

UNIT III FUNCTIONS OF SEVERAL VARIABLES 9+3

Partial differentiation – Homogeneous functions and Euler's theorem – Total derivative – Change of variables – Jacobians – Partial differentiation of implicit functions – Taylor's series for functions of two variables – Applications : Maxima and minima of functions of two variables and Lagrange's method of undetermined multipliers.

UNIT IV INTEGRAL CALCULUS 9+3

Definite and Indefinite integrals - Substitution rule - Techniques of Integration : Integration by parts, Trigonometric integrals, Trigonometric substitutions, Integration of rational functions by partial fraction, Integration of irrational functions - Improper integrals - Applications : Hydrostatic force and pressure, moments and centres of mass.

UNIT V MULTIPLE INTEGRALS 9+3

Double integrals – Change of order of integration – Double integrals in polar coordinates – Area enclosed by plane curves – Triple integrals – Volume of solids – Change of variables in double and triple integrals – Applications : Moments and centres of mass, moment of inertia.

TEXT BOOKS : 1. Kreyszig.E, "Advanced Engineering Mathematics", John Wiley and Sons, 10th Edition, New Delhi, 2016. 2. Grewal.B.S., "Higher Engineering Mathematics", Khanna Publishers, New Delhi, 44th Edition, 2018. 3. James Stewart, "Calculus : Early Transcendentals", Cengage Learning, 8th Edition, New Delhi, 2015. [For Units II & IV - Sections 1.1, 2.2, 2.3, 2.5, 2.7 (Tangents problems only), 2.8, 3.1 to 3.6, 3.11, 4.1, 4.3, 5.1 (Area problems only), 5.2, 5.3, 5.4 (excluding net change theorem), 5.5, 7.1 - 7.4 and 7.8].

REFERENCES : 1. Anton. H, Bivens. I and Davis. S, " Calculus ", Wiley, 10th Edition, 2016 2. Bali. N., Goyal. M. and Watkins. C., "Advanced Engineering Mathematics", Firewall Media (An imprint of Lakshmi Publications Pvt., Ltd.,), New Delhi, 7th Edition, 2009. 3. Jain. R.K. and Iyengar. S.R.K., "Advanced Engineering Mathematics", Narosa Publications, New Delhi, 5th Edition, 2016. 4. Narayanan. S. and Manicavachagom Pillai. T. K.,"Calculus" Volume I and II, S. Viswanathan Publishers Pvt. Ltd., Chennai, 2009. 5. Ramana. B.V., "Higher Engineering Mathematics", McGraw Hill Education Pvt. Ltd, New Delhi, 2016. 6. Srimantha Pal and Bhunia. S.C, " Engineering Mathematics " Oxford University Press, 2015. 7. Thomas. G. B., Hass. J, and Weir. M.D, " Thomas Calculus ", 14th Edition, Pearson India, 2018

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UNIT-I- MATRICES

PART-A

1. **Define eigen values and eigen vectors of a matrix.**

   Soln:

   Let A be a square matrix of order n. Let I be the unit matrix of order n. Let λ be any scalar. If there exist a non- zero column vector X such that $AX = \lambda X$, then λ is an eigen value of A and X is an eigen vector corresponding to λ.

2. **What is characteristic equation of a matrix?**

   Soln:

   Let A be a square matrix of order n and let I be the unit matrix of order n. Then for any scalar λ, we can find a matrix (A - λI) of order n. The equation $|A - \lambda I| = 0$ is called the characteristic equation. This is a polynomial equation of degree n. The roots of this equation are the eigen values of A.

3. **Define trace of a square matrix.**

Soln: The trace of a square matrix A is defined as the sum of principal diagonal

elements of the matrix A.

4. **State any three properties of eigen values.**

   Sol:

   Any three properties of eigen values are as follows

   a. The eigen values of A and A^T are the same.
   b. The sum of the eigen values of the matrix A is equal to the trace of the matrix A
   c. The product of the eigen values is the determinant value of the matrix.

5. **Find the eigen values of A^{-1} if the two eigen values of the matrix**

   A = $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ **are equal to 1 each**

Sol: Sum of the eigen values = Sum of the diagonal elements

=2 + 3 + 2 = 7

Sum of two given eigen values = 1 + 1 = 2

$\therefore$ The third eigen value = 7 – 2 = 5

The eigen values of A are 1,1,5

$\therefore$ The eigen values of A^{-1} are $1, 1, \frac{1}{5}$.

6. If the sum of two eigen values and trace of a 3 X 3 matrix A are equal,find $|A|$.

Soln:

Let the eigen values be $\lambda_1, \lambda_2, \lambda_3$.

It is given that $\lambda_1 + \lambda_2 = \lambda_1 + \lambda_2 + \lambda_3$.

So, we have $\lambda_3 = 0$.

$|A| = \lambda_1 \lambda_2 \lambda_3 = 0$.

7. The product of two eigen values of the matrix

$$\mathbf{A} = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$ **is 16. Find the third eigen value.**

Soln: We know that the product of all the eigen values = the value of the determinant of the given matrix.

$$|A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 32$$

But it is given that the product of two eigen values = 16.

$\therefore$ The third eigen value = 32/16 = 2.

8. Find the sum and product of the eigen values of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

Soln:

We know that the sum of the eigen values = the sum of the principal diagonal elements = 2 + 2+ 2 = 6.

Also we know that the product of the eigen values = the value of the determinant of the matrix = 6.

9. Find the constants a and b such that the matrix $\begin{pmatrix} a & 4 \\ 1 & b \end{pmatrix}$ has 3 and -2 as its eigen values.

Soln: Let A = $\begin{pmatrix} a & 4 \\ 1 & b \end{pmatrix}$.

The sum of the eigen values =a+b,

$a+b = 3+(-2) = 1$……….. (1)

The product of eigen values is the determinant of the matrix,

so $ab-4 = 3(-2) = -6$.

$ab = -6 + 4 = -2$

$\therefore ab = -2$ ------------(2)

Now solving the equations (1) and (2), we get the values of a and b

$\therefore\ a = 1-b$

Substituting this in (2),we get $(1 - b)b = -2$

$\Rightarrow b - b^2 = -2.$

$\Rightarrow (b + 1)(b - 2) = 0.$

$\therefore b = -1$ and $b = 2$.

Now substituting $b = 2$ in (1), we get $a + b = 1$. That is $a + 2 = 1$

$\Rightarrow\ a = 1 - 2 = -1.$

$\therefore a = -1$ and $b = 2$.

10. If 2 and 3 are the eigen values of the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$$

find the eigen values of A^{-1} and A^3.

Soln:

Let λ_1, λ_2 and λ_3 be the eigen values of A. Then $\lambda_1 + \lambda_2 + \lambda_3 = 7$.

$\because\ \lambda_1 = 2$ and $\lambda_2 = 3 \Rightarrow \lambda_3 = 7 - 5 = 2$

$\therefore \lambda_3 = 2.$

The eigen values of A^{-1} are 1/2, 1/2 and 1/3 and the eigen values of A^3 are 2^3, 2^3 and 3^3.

11. If two of the eigen values of a 3×3 matrix, whose determinant equals 4 are -1 and 2, find the third eigen value.

Soln:

Let λ_1, λ_2 and λ_3 be the eigen values . Then $\lambda_1 \lambda_2 \lambda_3 = 4$

That is, $-1\times 2\times \lambda_3 = 4$.

$\therefore \lambda_3 = -2.$

12. If the matrix A is $\begin{pmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{pmatrix}$, find the eigen values of A^2.

Soln:

Since the given matrix is a triangular matrix its diagonal elements are its eigen values,

Hence the eigen values of A are -1,-3 and 2.

the eigen values of A^2 are 1^2, $(-3)^2$, 2^2.

That is, 1, 4, 9.

13. Find the eigen values of $3A^3 + 5A^2 - 6A + 2I$ if the matrix

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{pmatrix}.$$

Soln:

Since the given matrix is a triangular matrix its diagonal elements are its eigen values, the eigen values of A are 1,3,-2.

So the eigen values of A^3 are 1,27,-8.

Eigen values of A^2 are 1,9,4.

Eigen values of A are 1,3,-2.

Eigen values of I are 1,1,1

$\therefore$ The eigen values of $3A^3 + 5A^2 - 6A + 2I$

First eigen value $= 3(1)^3 + 5(1)^2 - 6(1) + 2(1) = 4$.

Second eigen value $= 3(27) + 5(9) - 6(3) + 2(1) = 110$.

Third eigen value $= 3(-8) + 5(4) - 6(-2) + 2(1) = 10$.

$\therefore$ The required eigen values are 4, 110, 10.

14. If two eigen values of $A = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{pmatrix}$ are equal and they are double the third, then find the eigen values of A^2 and $2A^{-1}$.

Soln:

By the given condition, the eigen values of A can be taken as $2\lambda, 2\lambda, \lambda$

We know that the sum of the eigen values = the sum of the principal diagonal

$$\therefore\ 2\lambda + 2\lambda + \lambda = 4 + 3 - 2$$

$$\Rightarrow \boxed{\lambda = 1}$$

The required eigen values are 2,2,1.

$\therefore$ The eigen values of A^2 are 4,4,1 and the eigen values of $2A^{-1}$ are 2(1/2) ,2(1/2) and 2(1/1).

i.e., 1, 1, 2.

15. State Cayley – Hamilton theorem.

Soln:

Every square matrix satisfies its own characteristic equation.

16. State any two uses of Cayley-Hamilton theorem.

Sol:

Cayley-Hamilton theorem can be used to find

(i). the inverse of the given matrix and

(ii). the higher powers of the given matrix.

17. If A is an orthogonal matrix, then show that A^{-1} is also orthogonal.

Sol:

For an orthogonal matrix, transpose will be the inverse.

$\therefore A^T = A^{-1}$ --------------(1)

Let $A^T = A^{-1} = B$ ---------(2)

Then $B^T = (A^T)^{-1} = (A^{-1})^T = B^{-1}$ using (2)
$\therefore\ B^T = B^{-1}$
$\Rightarrow$ The matrix B is orthogonal.
i.e., A^{-1} is also orthogonal.

18. Show that A = $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is orthogonal.

Sol:

Given $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and $\therefore\ A^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Now $AA^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

Since $AA^T = I$, A is orthogonal.

19. If A is an orthogonal matrix ,then prove that $|A| = \pm 1$.

Sol:

We know that , for an orthogonal matrix A, $AA^T = I$

$\therefore\ |A|\ |A^T| = 1$

$\therefore\ |A|^2 = 1.$

$\therefore\ |A| = \pm 1.$

20. Define quadratic form.

Sol:

A homogeneous polynomial of second degree in any number of variables is called a quadratic form.

Example:-

$x_1^2 + 5x_1x_2 + 2x_2^2$ is a quadratic form in two variables x_1 and x_2.

21. Write the matrix of the quadratic form $2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$.

Sol: Matrix of QF is $A=\begin{pmatrix} coef\left(x_1^2\right) & \frac{1}{2}coef\left(x_1x_2\right) & \frac{1}{2}coef\left(x_1x_3\right) \\ \frac{1}{2}coef\left(x_2x_1\right) & coef\left(x_2^2\right) & \frac{1}{2}coef\left(x_2x_3\right) \\ \frac{1}{2}coef\left(x_3x_1\right) & \frac{1}{2}coef\left(x_3x_2\right) & coef\left(x_3^2\right) \end{pmatrix}$

Hence the matrix of the quadratic form is $\begin{pmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{pmatrix}$.

22. Write the quadratic form corresponding to the given matrix $\begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 6 \\ 5 & 6 & 3 \end{pmatrix}$.

Sol:

The quadratic form to the matrix is $x_1^2 + 4x_2^2 + 3x_3^2 + 4x_1 x_2 + 10 x_1 x_3 + 12 x_2 x_3$.

23. Determine the nature of the quadratic form $x^2 + 2y^2 + 3z^2 + 2xy - 2xz + 2yz$.

Sol:

The matrix of the quadratic form is $\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{pmatrix}$

$D_1 = |1| = 1$; $D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$.

$D_3 = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{vmatrix} = -2$, D_1 and D2 are positive. But D_3 is negative.

$\therefore$ The quadratic form is indefinite.

24. A is a singular matrix of order 3. Two of its eigen values are 2 and 3. Find the third eigen value.

Sol:

Since A is singular, $|A| = 0$. $\therefore$ product of the eigen values = 0. Let λ be the third eigen value. Then $(2)(3)(\lambda) = 0$.

i.e., $6\lambda = 0$. $\Rightarrow \therefore \lambda = 0$.

25. If the matrix of the quadratic form $3x^2 + 3y^2 + 2axy$ has eigen values 2 and 4, find the value of a.

Sol:

The matrix of the quadratic form is $A = \begin{pmatrix} 3 & a \\ a & 3 \end{pmatrix}$.

The product of the eigen values = $|A|$.

$$(2)(4) = 9 - a^2$$

$$\text{i.e., } a^2 = 1. \ \therefore a = \pm 1.$$

UNIT-I-MATRICES

1. ***Find all the eigen values and eigen vectors of the matrix*** $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$

Solution : Given A = $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$

The characteristic equation of the matrix is

$$\lambda^3 - (sum\ of\ principal\ diagonal\ elements\ of\ A)\lambda^2$$

$$+(sum\ of\ minors\ of\ principal\ diagonal\ elements)\lambda - |A| = 0$$

$\lambda^3 - \lambda^2(2+1+1) + \lambda(-3+1+1) - [2(-3)-1(-1)-1(-1)] = 0$

$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$, which is the characteristic equation.

$$\begin{array}{c|cccc} 1 & 1 & -4 & -1 & 4 \\ & 0 & 1 & -3 & -4 \\ \hline & 1 & -3 & -4 & 0 \end{array}$$

$\lambda = 1$ is a root.

The other roots are $\lambda^2 - 3\lambda - 4 = 0$

$\Rightarrow (\lambda - 4)(\lambda + 1) = 0$

$\Rightarrow \lambda = 4 , -1$

Hence $\boldsymbol{\lambda = 1 , 4 , -4.}$

The eigen vectors of the matrix A is given by $(A - \lambda I)X = 0$

i.e. $\begin{pmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$$\left.\begin{array}{r} (2-\lambda)x_1 + x_2 - x_3 = 0 \\ x_1 + (1-\lambda)x_2 - 2x_3 = 0 \\ -x_1 - 2x_2 + (1-\lambda)x_3 = 0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

When $\lambda = 1$, equation (1) becomes

$x_1 + x_2 - x_3 = 0$

$x_1 + 0x_2 - 2x_3 = 0$

$-x_1 - 2x_2 + 0x_3 = 0$

Take first and second equation,

$x_1 + x_2 - x_3 = 0$

$x_1 + 0x_2 - 2x_3 = 0$

$$\Rightarrow \frac{x_1}{-2+0} = \frac{-x_2}{-2+1} = \frac{x_3}{0-1}$$

$$\Rightarrow \frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$\therefore$ $\mathbf{x_1} =$ $\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$

When $\lambda = -1$, equation (1) becomes

$3x_1 + x_2 - x_3 = 0$

$x_1 + 2x_2 - 2x_3 = 0$

$$\Rightarrow \frac{x_1}{-2+2} = \frac{-x_2}{-6+1} = \frac{x_3}{6-1}$$

$$\Rightarrow \frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

$\therefore$ **x_2**= $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

When $\lambda = 4$, equation (1) becomes

$-2x_1+x_2-x_3 = 0$

$x_1-3x_2-2x_3 = 0$

$$\Rightarrow \frac{x_1}{-2-3} = \frac{-x_2}{4+1} = \frac{x_3}{6-1}$$

$$\Rightarrow \frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$\therefore$ **x_3**= $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

Hence the required Eigen vectors are **x_1** = $\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$, **$x_2$**= $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ & **x_3**= $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

2. ***Find all the eigen values and eigen vectors of*** $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

Solution : Given A = $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

The characteristic equation of the matrix is

$\lambda^3 - \lambda^2(2+3+2) + \lambda(4+3+4) - [2(4)-2(1)+1(-1)] = 0$

$\lambda^3-7\lambda^2+11\lambda-5 = 0$, which is the characteristic equation.

1	1	-7	11	-5
	0	1	-6	5
	1	-6	5	0

$\lambda = 1$ is a root.

The other roots are $\lambda^2-6\lambda+5=0$

$\Rightarrow (\lambda -1)(\lambda -5) =0$

$\Rightarrow \lambda = 1 ,5$

Hence $\lambda = 1 , 1 , 5$.

The eigen vectors of the matrix A is given by $(A- \lambda I)X = 0$

i.e. $\begin{pmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$$\left.\begin{aligned} (2-\lambda)x_1 +2x_2+x_3 &= 0 \\ x_1+(3-\lambda)x_2+x_3 &= 0 \\ x_1+2x_2+(2-\lambda)x_3 &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(1)$$

When $\lambda = 1$, equation (1) becomes

$x_1+2x_2+x_3 = 0$

$x_1+2x_2+x_3 = 0$

$x_1+2x_2+x_3 = 0$

Here all the equations are same.

Put $x_3 = 0$, we get $x_1 + 2x_2 = 0$

$x_1 = -2x_2$

Now Put $x_2 = 1$

Then we have $x_1 =-2$

$\therefore$ **X₁** $= \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

For $\lambda = 1$, put $x_2 = 0$,we get

$x_1+x_3 = 0$

$x_1= -x_3$

Now Put $x_3 = 1$

Then we have $x_1 = -1$

$\therefore$ **X₂** $= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

When $\lambda = 5$, equation(1) becomes

$-3x_1+2x_2+x_3 = 0$

$x_1-2x_2+x_3 = 0$ (taking first and second equation)

$$\Rightarrow \frac{x_1}{2+2} = \frac{-x_2}{-3-1} = \frac{x_3}{6-2}$$

$$\Rightarrow \frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4} \qquad \therefore \mathbf{x_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Hence the required Eigen vectors are $\mathbf{x_1} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{x_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ & $\mathbf{x_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

3 .*Find the eigen values and eigen vectors of* $\begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

Solution : The characteristic equation of matrix A is

$$\lambda^3 - \lambda^2(1+2-1) + \lambda(-3-1+3) - [1(-3) - 1(1) - 2(-1)] = 0$$

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

2	1	-2	-1	2
	0	2	0	-2
	1	0	-1	0

$\lambda = 2$ is a root.

The other roots are

$\lambda^2 - 1 = 0$

$(\lambda - 1)(\lambda + 1) = 0$

$\lambda = 1, -1$

Hence $\boldsymbol{\lambda} = \mathbf{2 , 1 , -1}$

The eigen vectors of matrix A is given by

$(A - \lambda I)X = 0$

$$\left.\begin{aligned} (1-\lambda)x_1 + x_2 - 2x_3 &= 0 \\ -x_1 + (2-\lambda)x_2 + x_3 &= 0 \\ 0x_1 + x_2 + (-1-\lambda)x_3 &= 0 \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1)$$

When $\lambda = 1$,Equation (1) becomes

$0x_1 + x_2 - 2x_3 = 0$

$-x_1 + x_2 + x_3 = 0$

$$\Rightarrow \frac{x_1}{1+2} = \frac{-x_2}{0-2} = \frac{x_3}{0+1} \qquad \Rightarrow \boxed{\frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}}$$

$$\therefore X_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

When $\lambda = -1$,Equation (1) becomes

$x_2=0$

$2x_1-2x_3 = 0$

$x_1 = x_3$

$$X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

When $\lambda = 2$,

Equation (1) becomes

$-x_1+x_2-2x_3=0$

$-x_1+0x_2+x_3=0$(taking first and second equation)

$$\Rightarrow \frac{x_1}{1-0} = \frac{-x_2}{-1-2} = \frac{x_3}{0-1}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{3} = \frac{x_3}{1}$$

$$\therefore X_3 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

4.. *Find all the eigen values and eigen vectors of* $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Solution : Given A = $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Since the given matrix is a triangular matrix its diagonal elements are its eigen values,

$\lambda = 2 , 2 , 2.$

To find the eigen vectors :

The eigen vectors of the matrix A is given by $(A- \lambda I)X = 0$

i.e.

$$\left.\begin{aligned} (2-\lambda)x_1 + x_2 + 0x_3 &= 0 \\ 0x_1 + (2-\lambda)x_2 + x_3 &= 0 \\ 0x_1 + 0x_2 + (2-\lambda)x_3 &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots..(1)$$

When $\lambda = 2$, (1) becomes

$0x_1 + x_2 + 0x_3 = 0$

$0x_1 + 0x_2 + x_3 = 0$

$0x_1 + 0x_2 + 0x_3 = 0$

Taking the second and third equations and applying cross rule method ,

$$\Rightarrow \frac{x_1}{1-0} = \frac{-x_2}{0-0} = \frac{x_3}{0-0}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$\therefore \mathbf{x_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The second and the third eigen vectors are also the same as x_1.

These three eigen vectors are linearly dependent.

5. Find the eigen values of A and hence find A^n (n is a positive integer) given that $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

Solution : Given $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

The characteristic equation of the matrix is

$\lambda^2 - \lambda(1+3) + (3-8) = 0$
$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$
$\Rightarrow (\lambda - 5)(\lambda + 1) = 0$
$\Rightarrow \boldsymbol{\lambda = -1, 5}$
Which are the eigen values of A.
When λ^n is divided by $\lambda^2 - 4\lambda - 5$, let the Quotient be $Q(\lambda)$ and the remainder be $(a\lambda + b)$.
Then by division algorithm $\lambda^n = (\lambda^2 - 4\lambda - 5)Q(\lambda) + (a\lambda + b)$……………………(1)
Put $\lambda = -1$ in (1), we get $-a+b = (-1)^n$ …………………..(2)
Put $\lambda = 5$ in (1), we get $5a + b = 5^n$ ……………………(3)
(3) – (2) $\Rightarrow 6a = 5^n - (-1)^n$

$$\Rightarrow \quad a = \frac{5^n - (-1)^n}{6}$$

(2) x 5 +(3) $\Rightarrow 6b = 5(-1)^n + 5^n$

$$\Rightarrow \quad b = \frac{5(-1)^n + 5^n}{6}$$

Replacing λ by the matrix A in (1), we have
$A^n = (A^2 - 4A - 5I)\,Q(A) + (aA + bI)$
$= 0\,Q(A) + aA + bI$ (using Cayley Hamilton theorem)
$= aA + bI$

(i.e). $A^n = \left(\dfrac{5^n - (-1)^n}{6}\right)\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} + \left(\dfrac{5(-1)^n + 5^n}{6}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

7. If -1 , 1 , 4 are the eigen values of a matrix A of order 3 and $(0,1,1)^T$, $(2,-1,1)^T$, $(1,1,-1)^T$ are the corresponding eigen vectors , determine the matrix A.

Solution: Modal matrix = $\begin{pmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

Here the Eigen vectors X_1, X_2, X_3 are pair wise orthogonal.

Normalized modal matrix P = $\begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$

By orthogonal transformation , $D = P^T A P$

Hence $A = PD\,P^T$

$$= \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{4}{\sqrt{3}} & \frac{4}{\sqrt{3}} & \frac{-4}{\sqrt{3}} \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$$

8. Verify that the matrix $A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ ***satisfies its own characteristic equation and hence find A^4.***

Solution : ***The characteristic equation of matrix A is***

$\lambda^3 - \lambda^2(2+2+2) + \lambda(3+2+3) - [2(3)+1(-1)+2(-1)] = 0$

$\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$, which is the characteristic equation.

By Cayley – Hamilton theorem , we have to prove

$A^3 - 6A^2 + 8A - 3 = 0$

$$A^2 = A \times A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix}$$

$$A^3 = A^2 \times A = \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix}\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix}$$

$$A^3 - 6A^2 + 8A - 3I = \begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix} - 6\begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} + 8\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Hence Cayley – Hamilton theorem is verified.

To find A^4

We have $A^3 - 6A^2 + 8A - 3I = 0$

$A^3 = 6A^2 - 8A + 3I$

$A^4 = 6A^3 - 8A^2 + 3A$

$$= 6\begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix} - 8\begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} + 3\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$\mathbf{A^4} = \begin{pmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{pmatrix}$$

9. Verify Cayley – Hamilton theorem for the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ ***and hence find the inverse of A.***

Solution : The characteristic equation of matrix A is

$\lambda^3 - \lambda^2(1+4+6) + \lambda(-1-3+0) - [1(-1)-2(-3)+3(-2)] = 0$

$\lambda^3 - 11\lambda^2 - 4\lambda + 1 = 0$, which is the characteristic equation.

By Cayley – Hamilton theorem , we have to prove

$A^3 - 11A^2 - 4A + 1 = 0$

$$A^2 = A \times A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{pmatrix}$$

$$A^3 = A^2 \times A = \begin{pmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{pmatrix}$$

$$A^3-11A^2-4A+I = \begin{pmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{pmatrix} -11\begin{pmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{pmatrix} -4\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence the theorem is verified.

To find A^{-1}

We have $A^3-11A^2-4A+I = 0$

$I = -A^3+11A^2+4A$

$A^{-1} = -A^2-11A+4I$

$$= -\begin{pmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{pmatrix} -11\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} +4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A^{-1}} = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

10. If $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ ***then show that*** $A^n = A^{n-2} + A^2 - I$ ***for*** $n \geq 3$ ***using Cayley – Hamilton theorem .***

Solution : The characteristic equation of matrix A is

$\lambda^3 - \lambda^2(1+0+0) + \lambda(-1+0+0) - [1(-1)-0+0] = 0$

$\lambda^3-\lambda^2-\lambda+1 = 0$

By Cayley – Hamilton theorem , we have

$\mathbf{A^3-A^2-A+I = 0}$

$A^3-A^2 = A-I$

Pre multiplying both sides successively by A , we get

$A^3-A^2 = A-I$

$A^4-A^3 = A^2-A$

$A^5-A^4 = A^3-IA^2$

$A^6-A^5 = A^4-A^3$

…………………………………..

$A^{n-1}-A^{n-2} = A^{n-3}-A^{n-4}$

$A^n-A^{n-1} = A^{n-2}-A^{n-3}$

Adding all these equations , we get

$A^n-A^2 = A^{n-2}-I$

$A^n = A^2+A^{n-2}-I$, $n \geq 3$

11. Using Cayley- Hamilton theorem , evaluate the matrix

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 - 8A^2 + 2A - I \text{ if } A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

Solution : The characteristic equation of matrix A is

$\lambda^3 - \lambda^2(2+1+2) + \lambda(2+3+2) - [2(2)-1(0)+1(-1)] = 0$

$\boldsymbol{\lambda^3-5\lambda^2+7\lambda-3 = 0}$

we have to prove

$A^3-5A^2+7A-3I = 0$

$$A^2 = A \times A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix}$$

$$A^3 = A^2 \times A = \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{pmatrix}$$

$$A^3-5A^2+7A-3I = \begin{pmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{pmatrix} -5 \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix} +7 \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix} -3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Hence the theorem is verified.

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 - 8A^2 + 2A - I$$
$$= A^5(A^3-5A^2+7A-3I) + A(A^3-5A^2+7A-3I) - 15A^2 + 5A - I$$
$$= A^5(0) + A(0) - 15A^2 + 5A - I$$
$$= -15A^2 + 5A - I$$
$$= -15 \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix} +5 \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -66 & -55 & -55 \\ 0 & -11 & 0 \\ -55 & -55 & -66 \end{pmatrix}$$

12. ***Diagonalise the matrix*** $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ ***by means of an orthogonal transformation.***

Solution : The characteristic equation of matrix A is

$$\lambda^3 - \lambda^2(6+3+3) + \lambda(8+14+14) - [6(8)+2(-4)+2(-4)] = 0$$
$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

2	1	-12	36	-32
	0	2	-20	32
	1	-10	16	0

$\lambda=2$ is a root

the other roots are $\lambda^2 - 10\lambda + 16 = 0$

$(\lambda-2)(\lambda-8) = 0$

$\Rightarrow \lambda = 2, 8$

Hence $\lambda = 2, 2, 8$

The eigen vectors of matrix A is given by

$(A - \lambda I)X = 0$

$$\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

(i.e.)
$$(6-\lambda)x_1 - 2x_2 + 2x_3 = 0$$
$$-2x_1 + (3-\lambda)x_2 - x_3 = 0$$
$$2x_1 - x_2 + (3-\lambda)x_3 = 0$$

When $\lambda = 8$, equation(1) becomes

$$-2x_1 - 2x_2 + 2x_3 = 0$$
$$-2x_1 - 5x_2 - x_3 = 0$$
$$\Rightarrow \frac{x_1}{2+10} = \frac{-x_2}{2+4} = \frac{x_3}{10-4} \quad \Rightarrow \frac{x_1}{12} = \frac{-x_2}{6} = \frac{x_3}{6}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1} \qquad \therefore X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

When $\lambda = 2$, equation (1) becomes

$$4x_1 - 2x_2 + 2x_3 = 0$$
$$-2x_1 + x_2 - x_3 = 0$$
$$2x_1 - x_2 + x_3 = 0$$

Here all the equations are same.
Put $x_3 = 0$, we get

$$2x_1 - x_2 = 0$$
$$2x_1 = x_2$$

$$\frac{x_1}{1} = \frac{x_2}{2}$$

$$\therefore X_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

Let X3 = $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be third eigen vector.

$$X_1^T X_3 = 0 \Rightarrow 2a - b + c = 0$$
$$X_2^T X_3 = 0 \Rightarrow a + 2b + 0c = 0$$

Then $\Rightarrow \frac{a}{0-2} = \frac{-b}{0-1} = \frac{c}{4+1}$ Hence

$$\frac{a}{-2} = \frac{b}{1} = \frac{c}{5}$$

$$\therefore X_3 = \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$$

Hence the modal matrix is = $\begin{pmatrix} 2 & 1 & -2 \\ -1 & 2 & 1 \\ 1 & 0 & 5 \end{pmatrix}$

Here $X_1^T X_2 = X_2^T X_3 = X_3^T X_1 = 0$
So X_1 , X_2, X_3 are pairwise orthogonal.
The Normalised modal matrix is

$$P = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{pmatrix}$$

$$D = P^T AP = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{-2}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \frac{5}{\sqrt{30}} \end{pmatrix} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{30}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{-2}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \frac{5}{\sqrt{30}} \end{pmatrix} \begin{pmatrix} \frac{16}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{-4}{\sqrt{30}} \\ \frac{-8}{\sqrt{6}} & \frac{4}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ \frac{8}{\sqrt{6}} & 0 & \frac{10}{\sqrt{30}} \end{pmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

13. Reduce the Quadratic form** $3x_1^{\ 2} + 2x_2^{\ 2} + 3x_3^{\ 2} - 2x_1x_2 - 2x_2x_3$ **to the canonical form through orthogonal transformation and find its nature.

Solution : Quadratic form is X^TAX

The matrix A of Q.F. is $A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

The characteristic equation of matrix A is

$$\lambda^3 - \lambda^2(3+2+3) + \lambda(5+9+5) - [3(5) + 1(-3) + 0] = 0$$

$$\lambda^3 - 8\lambda^2 + 19\lambda - 12 = 0$$

1	1	-8	19	-12
	0	1	-7	12
	1	-7	12	0

$\lambda = 1$ is a root.

$$\lambda^2 - 7\lambda + 12 = 0$$

The other roots are $(\lambda - 3)(\lambda - 4) = 0$

$$\lambda = 3,4$$

Hence $\boldsymbol{\lambda = 1, 3, 4}$

The eigen vectors of matrix A is given by

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\left.\begin{aligned} (3-\lambda)x_1 - x_2 + 0x_3 &= 0 \\ -x_1 + (2-\lambda)x_2 - x_3 &= 0 \\ 0x_1 - x_2 + (3-\lambda)x_3 &= 0 \end{aligned}\right\} \quad \text{..............................(1)}$$

When $\lambda = 1$, Equation (1) becomes

$$2x_1 - x_2 + 0x_3 = 0$$

$$-x_1 + x_2 - x_3 = 0$$

$$\Rightarrow \frac{x_1}{1-0} = \frac{-x_2}{-2-0} = \frac{x_3}{2-1}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$\therefore X_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

When $\lambda = 3$, Equation (1) becomes

$$0x_1 - x_2 + 0x_3 = 0$$

$$-x_1 - x_2 - x_3 = 0$$

$$\Rightarrow \frac{x_1}{1-0} = \frac{-x_2}{0-0} = \frac{x_3}{0-1} \quad \Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1} \qquad \therefore X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

When $\lambda = 4$, Equation (1) becomes

$$-x_1 - x_2 + 0x_3 = 0$$
$$-x_1 - 2x_2 - x_3 = 0$$

$$\Rightarrow \frac{x_1}{1-0} = \frac{-x_2}{1-0} = \frac{x_3}{2-1} \Rightarrow \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1} \qquad \therefore X_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

Hence the modal matrix is = $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix}$

Here $X_1^T X_2 = X_2^T X_3 = X_3^T X_1 = 0$.

So X_1 , X_2 , X_3 are pairwise orthogonal.

The normalized modal matrix is

$$P = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$P^T AP = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Consider the orthogonal transformation X = PY

Substitute (2) in (1) we get

$(PY)^T A (PY) = Y^T P^T APY$

$$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= y_1^2 + 3y_2^2 + 4y_3^2$$

Which is the canonical form

Rank = No. of terms in the canonical form = 3

Index = No. of positive square terms in the canonical form = 3

Signature = (No. of positive square terms) – (No. of negative square terms) = 3

Nature = positive definite.

14. Reduce the Quadratic form $x^2+y^2+z^2+4xy+4yz+4zx$ into sum of squares form by an orthogonal transformation hence find the rank , index , signature and nature of Q. F.

Solution : Quadratic form is X^TAX

The matrix A of Q.F. is A = $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$

The characteristic equation of matrix A is

$$\lambda^3 - \lambda^2(1+1+1) + \lambda(-3-3-3) - [1(-3) - 2(-2) + 2(2)] = 0$$
$$\lambda^3 - 3\lambda^2 - 9\lambda - 5 = 0$$

-1	1	-3	-9	-5
	0	-1	4	5
	1	-4	-5	0

λ = -1 is a root.

$$\lambda^2 - 4\lambda - 5 = 0$$

The other roots are $(\lambda + 1)(\lambda - 5) = 0$

$$\lambda = -1, 5$$

Hence **λ = 5 , -1 , -1**

The eigen vectors of matrix A is given by

$(A - \lambda I)X = 0$

$$\begin{pmatrix} 2-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\left.\begin{aligned} (1-\lambda)x_1 + 2x_2 + 2x_3 &= 0 \\ 2x_1 + (1-\lambda)x_2 + 2x_3 &= 0 \\ 2x_1 + 2x_2 + (1-\lambda)x_3 &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

When λ = 5 ,Equation (1) becomes

$$-4x_1 + 2x_2 + 2x_3 = 0$$

$$2x_1 - 4x_2 + 2x_3 = 0$$

$$\Rightarrow \frac{x_1}{4+8} = \frac{-x_2}{-8-4} = \frac{x_3}{16-4}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$\therefore X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

When λ = -1,Equation (1) becomes

$$2x_1 + 2x_2 + 2x_3 = 0$$

$$2x_1 + 2x_2 + 2x_3 = 0$$

$$2x_1 + 2x_2 + 2x_3 = 0$$

$$Hence all the equations are same$$

$$Put x_3 = 0$$

$$2x_1 + 2x_2 = 0 \Rightarrow 2x_1 = -2x_2$$

$$\frac{x_1}{-2} = \frac{x_2}{2} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{1}$$

$$\therefore X_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be third eigen vector.

$$X_1^T X_3 = 0 \Rightarrow a + b + c = 0$$

$$X_2^T X_3 = 0 \Rightarrow -a + b + 0c = 0$$

Then $\Rightarrow \dfrac{a}{0-1} = \dfrac{-b}{0+1} = \dfrac{c}{1+1}$

$$\frac{a}{-1} = \frac{b}{-1} = \frac{c}{2}$$

$$\therefore X_3 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

Hence the modal matrix is = $\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$

Here $X_1^T X_2 = X_2^T X_3 = X_3^T X_1 = 0$.

So X_1 , X_2 , X_3 are pairwise orthogonal.

The normalized modal matrix is

$$P=\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}}\end{pmatrix}$$

$$P^T AP=\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}}\\ \frac{1}{\sqrt{6}} & 0 & \frac{2}{\sqrt{6}}\end{pmatrix}$$

$$=\begin{pmatrix}5&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$$

Consider the orthogonal transformation X = PY

Substitute (2) in (1) we get

$(PY)^T$ A (PY) = Y^TP^T APY

$$=\begin{bmatrix}y_1 & y_2 & y_3\end{bmatrix}\begin{pmatrix}5&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$$

$$=5y_1^{\,2}-y_2^{\,2}-y_3^{\,2}$$

Which is the canonical form

Rank = No. of terms in the canonical form = 3

Index = No. of positive square terms in the canonical form = 1

Signature = (No. of positive square terms) – (No. of negative square terms) = 1-2= -1

Nature = indefinite.

15 .*Reduce the Quadratic form* $2x_1^{\,2}+6x_2^{\,2}+2x_3^{\,2}+8x_1x_3$ ***to the canonical form through orthogonal transformation .***

Solution : Quadratic form is X^TAX

The matrix A of Q.F. is $A=\begin{pmatrix}2&0&4\\0&6&0\\4&0&2\end{pmatrix}$

The characteristic equation of matrix A is

$$\lambda^3-\lambda^2(2+6+2)+\lambda(12-12+12)-[2(12)-0+4(-24)]=0$$

$$\lambda^3-10\lambda^2+12\lambda+72=0$$

-2	1	-10	12	72
	0	-2	24	-72
	1	-12	36	0

λ = -2 is a root.

$$\lambda^2-12\lambda+36=0$$

The other roots are $(\lambda-6)(\lambda-6)=0$

$$\lambda=6,6$$

Hence **λ = -2, 6 , 6**

The eigen vectors of matrix A is given by

$(A-\lambda I)X=0$

$$\left.\begin{aligned}(2-\lambda)x_1+0x_2+4x_3&=0\\0x_1+(6-\lambda)x_2+0x_3&=0\\4x_1+0x_2+(2-\lambda)x_3&=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

When λ = -2,Equation (1) becomes

$$4x_1 + 0x_2 + 4x_3 = 0$$
$$0x_1 + 8x_2 + 0x_3 = 0$$

$$\Rightarrow \frac{x_1}{0-32} = \frac{-x_2}{0-0} = \frac{x_3}{32-0} \Rightarrow \frac{x_1}{-32} = \frac{x_2}{0} = \frac{x_3}{32} \quad \therefore X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

When $\lambda = 6$, Equation (1) becomes

$$4x_1 + 0x_2 + 4x_3 = 0$$
$$0x_1 + 0x_2 + 0x_3 = 0$$
$$4x_1 + 0x_2 - 4x_3 = 0$$

Hence all the equations are same

$Put\, x_3 = 0$

$$-4x_1 + 0x_2 = 0 \Rightarrow 4x_1 = 0x_2$$

$$\frac{x_1}{0} = \frac{x_2}{4} \Rightarrow \frac{x_1}{0} = \frac{x_2}{1}$$

$$\therefore X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be third eigen vector.

$$X_1^T X_3 = 0 \Rightarrow -a + 0b + c = 0$$
$$X_2^T X_3 = 0 \Rightarrow 0a + b + 0c = 0$$

Then $\Rightarrow \dfrac{a}{0-1} = \dfrac{-b}{0-0} = \dfrac{c}{-1-0}$

$$\frac{a}{-1} = \frac{b}{0} = \frac{c}{-1}$$

$$\therefore X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Hence the modal matrix is $= \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

Here $X_1^T X_2 = X_2^T X_3 = X_3^T X_1 = 0$.
So X_1, X_2, X_3 are pairwise orthogonal.
The normalized modal matrix is

$$P = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$P^T A P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Consider the orthogonal transformation X = PY
Substitute (2) in (1) we get
$(PY)^T A (PY) = Y^T P^T APY$

$$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= -2y_1^2 + 6y_2^2 + 6y_3^2$$

Which is the canonical form

16. Reduce the Quadratic form 10 $x_1^2 + 2x_2^2 + 5x_3^2 + 6x_2x_3 - 10x_3x_1 - 4x_1x_2$ ***to the canonical form through orthogonal transformation .Find a set of values of*** x_1, x_2, x_3 ***which will make the form vanish.***

Solution : Quadratic form is X^TAX

The matrix A of Q.F. is $A = \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix}$

The characteristic equation of matrix A is

$$\lambda^3 - \lambda^2(10+2+5) + \lambda(1+25+16) - [10(1) + 2(5) - 5(4)] = 0$$

$$\lambda^3 - 17\lambda^2 + 42\lambda = 0 \Rightarrow \lambda(\lambda^2 - 17\lambda + 42) = 0$$

$$\lambda = 0 or \lambda^2 - 17\lambda + 42 = 0$$

$$\lambda = 0 or (\lambda - 3)(\lambda - 14) = 0$$

$$\lambda = 3,14$$

Hence λ = 0 , 3 , 14.

The eigen vectors of matrix A is given by

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\left.\begin{aligned} (10-\lambda)x_1 - 2x_2 - 5x_3 &= 0 \\ -2x_1 + (2-\lambda)x_2 + 3x_3 &= 0 \\ -5x_1 + 3x_2 + (5-\lambda)x_3 &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

When λ = 0 ,Equation (1) becomes

$$10x_1 - 2x_2 - 5x_3 = 0$$

$$-2x_1 + 2x_2 + 3x_3 = 0$$

$$\Rightarrow \frac{x_1}{-6+10} = \frac{-x_2}{30-10} = \frac{x_3}{20-4}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{-5} = \frac{x_3}{4}$$

$$\therefore X_1 = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$$

When λ = 3 ,Equation (1) becomes

$$7x_1 - 2x_2 - 5x_3 = 0$$

$$-2x_1 - x_2 + 3x_3 = 0$$

$$\Rightarrow \frac{x_1}{-6-5} = \frac{-x_2}{21-10} = \frac{x_3}{-7-4}$$

$$\Rightarrow \frac{x_1}{-11} = \frac{x_2}{-11} = \frac{x_3}{-11}$$

$$\therefore X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

When λ = 14 , Equation (1) becomes

$$-4x_1-2x_2-5x_3=0$$
$$-2x_1-12x_2+5x_3=0$$

$$\Rightarrow \frac{x_1}{-6-60}=\frac{-x_2}{-12-10}=\frac{x_3}{48-4} \Rightarrow \frac{x_1}{-66}=\frac{x_2}{22}=\frac{x_3}{44} \quad \therefore X_3=\begin{pmatrix}-3\\1\\2\end{pmatrix}$$

Hence the modal matrix is $=\begin{pmatrix}1 & 1 & -3\\-5 & 1 & 1\\4 & 1 & 2\end{pmatrix}$

Here $X_1^TX_2 = X_2^TX_3 = X_3^TX_1 = 0$.
So X_1 , X_2 , X_3 are pairwise orthogonal.
The normalized modal matrix is

$$P=\begin{pmatrix}\frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}}\\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}}\\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}}\end{pmatrix}$$

$$P^TAP=\begin{pmatrix}\frac{1}{\sqrt{42}} & \frac{-5}{\sqrt{3}} & \frac{4}{\sqrt{14}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}}\end{pmatrix}\begin{pmatrix}10 & -2 & -5\\-2 & 2 & 3\\-5 & 3 & 5\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}}\\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}}\\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}}\end{pmatrix}$$

$$=\begin{pmatrix}0 & 0 & 0\\0 & 3 & 0\\0 & 0 & 14\end{pmatrix}$$

Consider the orthogonal transformation X = PY
Substitute (2) in (1) we get
$(PY)^T A (PY) = Y^TP^T APY$

$$=\begin{bmatrix}y_1 & y_2 & y_3\end{bmatrix}\begin{pmatrix}0 & 0 & 0\\0 & 3 & 0\\0 & 0 & 14\end{pmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$$

$$=0y_1^2+3y_2^2+14y_3^2$$

Which is the canonical form

<u>To find the set of non zero values of ***x₁,x₂, x₃*** which makes the QF zero</u>

From the orthogonal transformation X = PY we have

$$\left.\begin{aligned} x_1 &= \frac{y_1}{\sqrt{42}}+\frac{y_2}{\sqrt{3}}+\frac{-3y_3}{\sqrt{14}}\\ x_2 &= \frac{-5y_1}{\sqrt{42}}+\frac{y_2}{\sqrt{3}}+\frac{y_3}{\sqrt{3}}\\ x_3 &= \frac{4y_1}{\sqrt{42}}+\frac{y_2}{\sqrt{3}}+\frac{2y_3}{\sqrt{14}}\end{aligned}\right\} \qquad (*)$$

Clearly canonical form reduces to zero when $y_1=y_2=0$, using this in (*) we have

$$x_1=\frac{y_1}{\sqrt{42}}, x_2=\frac{-5y_1}{\sqrt{42}}, x_3=\frac{4y_1}{\sqrt{42}}$$

let $y_1=\sqrt{42}$ then required non zero values of x_1, x_2, x_3 which makes the QF zero is
$x_1=1, x_2=-5, x_3=4$

17. *Reduce the Quadratic form $x^2+3y^2+3z^2-2yz$ to the canonical form through orthogonal transformation and. hence find the nature of Q. F.*

Solution : Quadratic form is X^TAX

The matrix A of Q.F. is $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

The characteristic equation of matrix A is $\lambda^3 - \lambda^2(3+3+1) + \lambda(8+3+3) - [1(8)-0+0] = 0$

$\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$

$$\begin{array}{r|rrrr} 1 & 1 & -7 & 14 & -8 \\ & 0 & 1 & -6 & 8 \\ \hline & 1 & -6 & 8 & 0 \end{array}$$

$\lambda = 1$ is a root.

$\lambda^2 - 6\lambda + 8 = 0$

The other roots are $(\lambda-4)(\lambda-2) = 0$

$\lambda = 2,4$

Hence $\boldsymbol{\lambda = 1 , 2 , 4}$

The eigen vectors of matrix A is given by

$(A - \lambda I)X = 0$

$$\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\left.\begin{aligned} (1-\lambda)x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 + (3-\lambda)x_2 - x_3 &= 0 \\ 0x_1 - x_2 + (3-\lambda)x_3 &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

When $\lambda = 1$,Equation (1) becomes

$0x_1 + 2x_2 - x_3 = 0$

$0x_1 - x_2 + 2x_3 = 0$

$\Rightarrow \dfrac{x_1}{4-1} = \dfrac{-x_2}{0-0} = \dfrac{x_3}{0-0}$

$\Rightarrow \dfrac{x_1}{1} = \dfrac{x_2}{0} = \dfrac{x_3}{0}$

$\therefore X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

When $\lambda = 2$,Equation (1) becomes

$-x_1+0x_2+0x_3=0$

$0x_1+x_2-x_3=0$(taking first and second equation)

$\Rightarrow \dfrac{x_1}{0-0} = \dfrac{-x_2}{1-0} = \dfrac{x_3}{-1-0}$

$\Rightarrow \dfrac{x_1}{0} = \dfrac{-x_2}{1} = \dfrac{x_3}{1}$

$\therefore X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

When $\lambda = 4$,Equation (1) becomes

$-3x_1+0x_2+0x_3=0$

$0x_1-x_2-x_3=0$(taking first and second equation)

$\Rightarrow \dfrac{x_1}{0-0} = \dfrac{-x_2}{3-0} = \dfrac{x_3}{3-0} \Rightarrow \dfrac{x_1}{0} = \dfrac{-x_2}{-3} = \dfrac{x_3}{3} \qquad \therefore X_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

Hence the modal matrix is = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

Here $X_1^T X_2 = X_2^T X_3 = X_3^T X_1 = 0$.
So X_1 , X_2 , X_3 are pairwise orthogonal.
The normalized modal matrix is

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$P^T AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Consider the orthogonal transformation X = PY
Substitute (2) in (1) we get
$(PY)^T A (PY) = Y^T P^T APY$

$$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= y_1^2 + 2y_2^2 + 4y_3^2$$

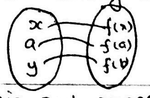
Which is the canonical form
Since all the eigen values are positive , the nature of Q.F. is positive definite.

UNIT - II DIFFERENTIAL CALCULUS

PART - A

1. Define a function with an example

A function f is a rule that assign to each element x in a set D exactly one element, called $f(x)$, in a set E



2) Find the domain and range of $f(x) = 2x-1$

Soln. Since $f(x) = 2x-1$ is defined for all real no's the domain of $f(x)$ is the set of all real no's and range of $f(x)$ is also the set of all real numbers

$\therefore$ The domain and range of f is R

3) Find the domain and range of $f(x) = x^2$

Soln For the function $f(x) = x^2$, the domain of f is $\mathbb{R}$ and the range is $[0, \infty]$

4) Find the domain of the function $f(x) = \frac{1}{x^2-x}$

Soln Let $f(x) = \frac{1}{x^2-x} = \frac{1}{x(x-1)}$, $x \neq 0$, $x \neq 1$

The domain of f is $(-\infty, 0) \cup (0,1) \cup (1, \infty)$

5) Determine whether the given function $f(x) = x^5 + x$ is even or odd ?

Soln Given $f(x) = x^5+x$

$f(-x) = (-x)^5 - x = -x^5 - x = -(x^5+x) = -f(x)$

$\Rightarrow f(x)$ is an odd.

6) Verify whether the given function $f(x) = 1-x^4$ is an odd or even

Soln Given $f(x) = 1-x^4$

$f(-x) = 1-(-x)^4 = 1-x^4 = f(x)$

$\Rightarrow f(x)$ is an even function.

7) Verify the given fn, $f(x) = 2x - x^2$ is an odd, even or neither even nor odd.

Soln $f(-x) = 2(-x) - (-x)^2$
$= -2x - x^2 = -(2x+x^2) \neq -f(x) (or) f(x)$

$f(x)$ is neither even nor odd.

8) Evaluate the difference quotient for the function $f(x) = x^3$, $\frac{f(a+h) - f(a)}{h}$

Soln. $\frac{f(a+h) - f(a)}{h} = \frac{(a+h)^3 - a^3}{h} = \frac{1}{h}[\cancel{a^3} + 3a^2h + 3ah^2 + h^3 - \cancel{a^3}]$

$= \frac{1}{h}[3a^2h + 3ah^2 + h^3]$

$= 3a^2 + 3ah + h^2$

9) Find the domain of the fn. $f(x) = \frac{x+4}{x^2-9}$

Soln Given $f(x) = \frac{x+4}{x^2-9} = \frac{x+4}{(x+3)(x-3)}$, $x \neq -3, 3$

The fn $f(x)$ is not defined at $x = \pm 3$

$\therefore$ The domain of f is

$(-\infty, 3) \cup (-3\ 3) \cup (3\ \infty)$

10) Determine the limit $\lim_{x \to 5} (2x^2 - 3x + 4)$

Soln $\lim_{x \to 5} (2x^2 - 3x + 4) = 2 \times 5^2 - 3 \times 5 + 4$
$= 2 \times 25 - 15 + 4$
$= 50 - 15 + 4$
$= 39$

11) State Squeeze or Sandwhich Theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near 'a' and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to c} g(x) = L$

12) Define a continuous function or Continuity of a function

A function f is continuous at a if

(a) $f(a)$ is defined

(b) $\lim_{x \to a} f(x)$ exists

(c) $\lim_{x \to a} f(x) = f(a)$

13) State intermediate value Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number b/w $f(a)$ & $f(b)$, where $f(a) \neq f(b)$. Then there exist no c in $[a, b]$ s.t $f(c) = N$

14) Show that the given fn is continuous at $a=-1$ for $f(x) = (x+2x^3)^4$

Soln $\lim_{x\to -1} f(x) = \lim_{x\to -1} (x+2x^3)^4$

$= [-1+2(-1)^3]^4 = (-1-2)^4$

$= (-3)^4 = 81$

By the defn of continuity f is continuous at $a=-1$

15) Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal

Soln Since the tangent line is horizontal

$\Rightarrow dy/dx = 0 \rightarrow ①$

Given $y = x^4 - 6x^2 + 4$

$dy/dx = 4x^3 - 12x$.

$\therefore$ by ① $4x^3 - 12x = 0$

$4x(x^2-3) = 0$

$x=0$ (or) $x^2 = 3 \Rightarrow x = \pm\sqrt{3}$

when $x=0$, $y = 0-0+4$

$\boxed{(0,4)}$

when $x = \pm\sqrt{3}$, $y = (\pm\sqrt{3})^4 - 6(\pm\sqrt{3})^2 + 4$

$= 9 - 18 + 4 = -5$

$\therefore$ $\boxed{(\pm\sqrt{3}, -5)}$ The points are $(0,4)$ $(\sqrt{3},-5)$ $(-\sqrt{3},\sqrt{5})$

16) The motion of a particle is given by $s = 2t^3 - 5t^2 + 3t + 4$ find the acceleration and what is the acceleration after 2 seconds

Soln Given $s = 2t^3 - 5t^2 + 3t + 4$

velocity $= v(t) = ds/dt = 6t^2 - 10t + 3$

acceleration $= a(t) = 12t - 10$

17. Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents I so where?

Ans Given $y = x^4 - 2x^2 + 2$

$dy/dx = 4x^3 - 4x$

Since horizontal tangents, $dy/dx = 0$

$\Rightarrow 4x^3 - 4x = 0$

$4x(x^2-1) = 0$

$\Rightarrow x = 0$, $x^2 - 1 = 0$

$\Rightarrow x = 0$, $x = 1$, $x = -1$

when $x = 0$, $y = 0 - 0 + 2$

$\Rightarrow x = 0, y = 2$ $(0,2)$

when $x = 1$, $y = 1 - 2 + 2 = 1$

$\Rightarrow x = 1, y = 1$ $(0,1)$

when $x = -1$ $y = (-1)^4 - 2(1)^2 + 2 = 1 - 2 + 2 = 1$

$\Rightarrow x = -1, y = 1$ $(-1,1)$

The points are $(0,2)$ $(1,1)$ $(-1,1)$

18) Find the derivative of $f(x) = \dfrac{1}{\sqrt[3]{x^2+x+1}}$

Soln Let $f(x) = \dfrac{1}{\sqrt[3]{x^2+x+1}} = (x^2+x+1)^{-1/3}$

$f(x) = -\frac{1}{3}(x^2+x+1)^{-1/3-1} \cdot (2x+1)$

$= -\frac{1}{3}(x^2+x+1)^{-4/3} \cdot (2x+1)$

19. Find the differentiation of $f(x) = \sqrt{\log x}$

Soln $f(x) = \frac{1}{2}(\log x)^{1/2-1} \frac{d}{dx}(\log x) = \frac{1}{2}(\log x)^{-1/2} \cdot \frac{1}{x} = \dfrac{1}{2x\sqrt{\log x}}$

20) Differentiate $f(x) = \log_{10}(2+\sin x)$

Soln Given $f(x) = \log_{10}(2+\sin x)$

$f'(x) = \dfrac{1}{2+\sin x} \cos x$

21) State the extreme value theorem

If f is continuous on a closed interval $[a,b]$ Then f attains absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some no's c and d in $[a,b]$

22) state Fermat's theorem

If f has a local maximum or minimum at c and if $f'(c)$ exists then $f'(c)=0$

23) what are the conditions for absolute maximum and minimum values.

Let c be a no. in the domain D of the fn f Then $f(c)$ is the

(i) absolute max value of f on D if $f(c) \geq f(x)$ $\forall x$ in D

(ii) Absolute min value of f on D if $f(c) \leq f(x)$ $\forall x$ in D. An absolute max or min is called global max(or) min. The max and min values are called extreme values of f.

24) Define the critical number in Max (or) min value

Soln If f has a local max or min at c, Then c is a critical number of f.

25) Explain the conditions for local maximum & local minimum value

Soln Let f be a fun, (i) If f' changes +ve to −ve then f has local maximum

(ii) If f' changes −ve to +ve then f has local minimum.

26 write down the steps of the first derivative test

Soln Suppose that c is a critical no of a continuous fn f.

(i) If f' changes from +ve to −ve at c, then f has a local maximum at c.

(ii) If f' changes from −ve to +ve at c, then f has a local minimum at c.

(iii) If f' does not changes sign at c, then f has no local maximum or minimum at c.

27) Write down the conditions for the concavity test (or) Write down the conditions for concave upward and downward in an interval

(i) If $f'(x) < 0$ $\forall x$ in I, Then the graph of f is Concave upward on I.

(ii) If $f''(x) > 0$ $\forall x \in I$ Then the graph of f is Concave down ward on I.

28. Define an inflexion point.

A point P on a curve $y=f(x)$ is called an inflexion Point if f is continuous and the curve changes from concave upward to concave downward ~~to~~ or from Concave downward to concave upward at P.

29). Explain the Second derivative test

Suppose f'' is continuous near c.

(i) If $f'(c)=0$ and $f''(c)>0$ then f has a local minimum at c

(ii) If $f'(c)=0$ and $f''(c)<0$ then f has a local maximum at c//.

30). If $f(1) = 10$, $f'(x) \geq 2$ for $1 \leq x \leq 4$. how small can $f(4)$ possibly be.

Soln By mean value thm

$f'(c) = \frac{f(b)-f(a)}{b-a}$ Here $b=4$, $a=1$

$\Rightarrow f'(c) = \frac{f(4)-f(1)}{4-1} \Rightarrow \frac{f'(4)-f(1)}{3} \geq 2$

$\Rightarrow \frac{f(4)-10}{3} \geq 2 \Rightarrow f(4) - 10 \geq 6$

$\Rightarrow f(4) \geq 16$ ∴ The min value is 16.

31) Find $d/dx\left[(\sin x)^{\cos x}\right]$

Soln. $d/dx\left[(\sin x)^{\cos x}\right] = d/dx\left[e^{\cos x \log \sin x}\right]$

$= e^{\cos x \log \sin x}\left[\cos x \cdot \frac{1}{\sin x}\cos x + \log \sin x(-\sin x)\right]$

$= (\sin x)^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x \log \sin x\right]$

32) limit of a function.

A function $f(x)$ said to have a limit 'l' if $|f(x)-l| < \epsilon$ whenever $|x-a| < \delta$

$\boxed{\text{ie } \lim_{x\to a} f(x) = l}$

PART-B

1. Guess the value of $\lim_{x\to 1} \frac{x^2-1}{x-1}$

Soln

Given $\lim_{x\to 1} \frac{x^2-1}{x-1}$.

Here $f(x) = \frac{x^2-1}{x-1} = \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x+1$

values of x below and above 1.

x / below 1	$f(x) = x+1$
0.9	1.9
0.99	1.99
0.999	1.999
0.9999	1.9999

x / above 1	$f(x) = x+1$
1.1	2.1
1.01	2.01
1.001	2.001
1.0001	2.0001

$f(x)$ tends to 2 as x tends to 1

$\Rightarrow \lim_{x\to 1} f(x) = 2$

$\Rightarrow \boxed{\lim_{x\to 1} \frac{x^2-1}{x-1} = 2}$

2) Guess the value of $\lim_{x\to 0} \frac{\sin x}{x}$.

Soln Given $\lim_{x\to 0} \frac{\sin x}{x}$, Here $f(x) = \frac{\sin x}{x}$.

0

Value of x below and above 0

x / below 0	$f(x) = \frac{\sin x}{x}$
-0.1	0.99833417
-.001	0.9998333
-0.001	0.99999983
-0.0001	0.99999999

x / above 0	$f(x) = \sin x/x$
0.1	0.99833417
0.01	0.99998333
0.001	0.999999583
0.0001	0.99934665

From the table $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Left hand limit of $f(x)$

The left hand limit of $f(x)$ as x tends to L

ie $\lim_{x \to a^-} f(x) = L$ Here a^- means $x<a$ (-ve)

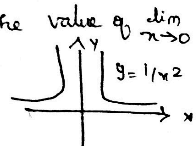
Infinite limits

Let. f be a function defined on both sides of a except possibly at 'a' it self.

(i) Then $\lim_{x \to a} f(x) = \infty$ (ii) $\lim_{x \to a} f(x) = -\infty$

3) Find the value of $\lim_{x \to 0} \frac{1}{x^2}$ if it exist.

Soln



As x close to 0 x^2 also becomes close to 0 but $1/x^2$ becomes Very large.

x	$y = 1/x^2$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.01	10,000
± 0.001	1,00,000

Thus the value of $f(x) = 1/x^2$ donot approach a number

So $\lim_{x \to 0} \frac{1}{x^2}$ does not exist

$$\Rightarrow \boxed{\lim_{x \to 0} \frac{1}{x^2} = \infty}$$

4. Sketch the graph of the function $f(x) = \begin{cases} 1+x, & x<-1 \\ x^2, & -1 \le x \le 1 \\ 2-x, & x \ge 1 \end{cases}$ and use it to determine the values of a for which $\lim_{x \to a} f(x)$ exists.

Soln. Given $f(x) = y = \begin{cases} 1+x, & x<-1 \\ x^2, & -1 \le x \le 1 \\ 2-x, & x \ge 1 \end{cases}$

$y = 1+x,\ x<-1$

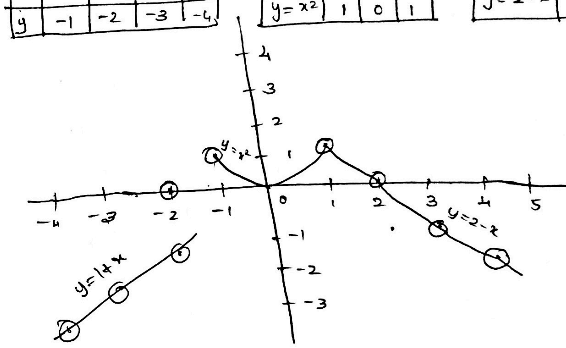
x	-2	-3	-4	-5
y	-1	-2	-3	-4

$y = x^2,\ -1 \le x \le 1$

x	-1	0	1
$y = x^2$	1	0	1

$2-x,\ x>1$

x	1	2	3	4
$y = 2-x$	1	0	-1	-2



From the graph $\lim_{x \to a} f(x)$ exist for all 'a' except at $a=-1$

$\Rightarrow$ right & left hand limits are different at $a=-1$

5) Find $\lim_{x \to 1} \frac{x^2+x-2}{x^2-x}$

Soln

$$\lim_{x \to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+2)}{x\cancel{(x-1)}} = \lim_{x \to 1} \frac{x+2}{x} = \frac{1+2}{1}$$

$= 3$

6. Find $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$

Soln $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} \times \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}$

$= \lim_{t\to 0} \frac{t^2+\cancel{9}-\cancel{9}}{t^2(\sqrt{t^2+9}+3)} = \lim_{t\to 0} \frac{\cancel{t^2}}{\cancel{t^2}\sqrt{t^2+9}+3}$

$= \frac{1}{\sqrt{0+9}+3} = \frac{1}{3+3} = \frac{1}{6}$ //

7. Find $\lim_{x\to\infty} \frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}$

Soln $\lim_{x\to 2} \frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}} = \lim_{x\to 2} \frac{(x^2-4)[\sqrt{3x-2}+\sqrt{x+2}]}{(\sqrt{3x-2}-\sqrt{x+2})(\sqrt{3x-2}-\sqrt{x+2})}$

$= \lim_{x\to 2} \frac{(x+2)(x-2)[\sqrt{3x-2}+\sqrt{x+2})}{(3x-2)-(x+2)}$

$= \lim_{x\to 2} \frac{(x+2)(x-2)[\sqrt{3x-2}+\sqrt{x+2}]}{2x-4}$

$= \lim_{x\to 2} \frac{(x+2)\cancel{(x-2)}[\sqrt{3x-2}+\sqrt{x+2}]}{2\cancel{(x-2)}}$

$= \frac{(2+2)[\sqrt{6-2}+\sqrt{4}]}{2}$

$= \frac{4[2+2]}{2} = \frac{4[4]}{2} = 8$ //

8. Find the limit if it exists? If the limit does not exist explain why?

$\lim_{x\to 1/2} \frac{2x-1}{|2x^3-x^2|}$

Soln. Here $f(x) = \frac{2x-1}{|2x^3-x^2|} = \frac{2x-1}{|x^2(2x-1)|}$

$= \frac{2x-1}{|x^2||2x-1|} = \frac{2x-1}{x^2|2x-1|}$

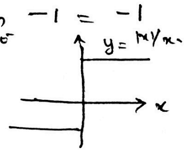
$\lim_{x\to 1/2^-} f(x) = \lim_{x\to 1/2^-} \frac{2\cancel{x-1}}{x^2(\cancel{2x-1})} = -1/(1/2)^2 = -1/(1/4) = -4$ //

9) prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist

Soln Let $f(x) = \frac{|x|}{x}$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} (x/x) = \lim_{x\to 0^+} 1 = 1$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} -1 = -1$



$\therefore \lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$

$\Rightarrow \lim_{x\to 0} \frac{|x|}{x}$ does not exist.

The Squeeze theorem :- (or) Sandwich Theorem.

$\boxed{\text{If } f(x) \le g(x) \le h(x) \text{ when } x \text{ is near 'a' and } \lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L \text{ Then } \lim_{x\to a} g(x) = L.}$

10) Show that $\lim_{x\to 0} t\sin(1/x) = 0$ ND 2016.

Soln we know that $-1 \le \sin 1/x \le 1$ $f(x) \le h(x) \le g(x)$

Multiply by x^2 $\Rightarrow$ $-x^2 \le x^2 \sin 1/x \le x^2 \to (1)$.

$\lim\limits_{x\to 0} x^2 = 0$ and $\lim\limits_{x\to 0} -x^2 = 0$

By Squeeze Theorem $\quad f(x) = -x^2 \quad g(x) = x^2$

$\lim\limits_{x\to 0} x^2 \sin 1/x = 0 \quad h(x) = x^2 \sin 1/x.$

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11). Evaluate $\lim\limits_{x\to\pi/2} \dfrac{1+\cos 2x}{(\pi-2x)^2}$. Jan(2016)

Soln

$$\lim_{x\to\pi/2} \frac{1+\cos 2x}{(\pi-2x)^2} = \lim_{x\to\pi/2} \frac{2\cos^2 x}{(\pi-2x)^2} = \lim_{x\to\pi/2} \frac{2\sin^2(\pi/2-x)}{2^2(\pi/2-x)^2}$$

$$\lim_{x\to\pi/2} \frac{1+\cos 2x}{(\pi-2x)^2} = \frac{1+\cos\pi}{(\pi-\pi)^2} = \frac{1-1}{0} = \frac{0}{0}$$

$$= \lim_{x\to\pi/2} \frac{(-\cancel{2})(\sin 2x)}{2(\pi-2x)(-\cancel{2})} = \frac{\sin\pi}{\pi-\pi} = \frac{0}{0}$$

$$= \lim_{x\to\pi/2} \frac{2\cos 2x}{2(2)} = \frac{\cos\pi}{-2} = \frac{-1}{-2} = 1/2.$$

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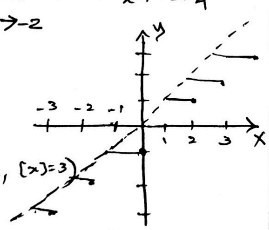
12) If $f(x) = [x]$ is the greatest integer function find the limit if it exists.

$\lim\limits_{x\to 3^+} [x]$, $\lim\limits_{x\to 3^-} [x]$, $\lim\limits_{x\to -2} [x]$, $\lim\limits_{x\to -2.4} [x]$

Soln Given $f(x) = [x]$

For any integer n,

$[x] = n$ if $n \le x < n+1$



$\lim\limits_{x\to 3^+} [x] = \lim\limits_{x\to 3^+} 3 = 3$ (If $3 \le x < 4$, $[x]=3$)

$\lim\limits_{x\to 3^-} [x] = \lim\limits_{x\to 3^-} 2 = 2$ (If $2 \le x < 3$, $[x]=2$)

$\lim\limits_{x\to 2^-} [x] = \lim\limits_{x\to 2^-} (-3) = -3$

$\lim\limits_{x\to 2^+} [x] = \lim\limits_{x\to -2^+} (-2) = -2$

$\therefore \lim\limits_{x\to 2^-} [x] \ne \lim\limits_{x\to -2^+} [x]$

$\Rightarrow \lim\limits_{x\to -2} [x]$ does not exist

$\lim\limits_{x\to -2.4} [x] \ne \lim\limits_{x\to -2.4} (-3) = -3$ //.

Continuity :-

A function f is Continuous at a point 'a' if $\lim\limits_{x\to 0} f(x) = f(a)$

[or]

(i) f is defined at a ie, $f(a)$ exists

(ii) $\lim\limits_{x\to a} f(x)$ exist . ie, $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x)$

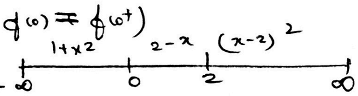
(iii) $\lim\limits_{x\to a} f(x) = f(a)$

1) Find the domain where the function f is continuous. Also find the numbers at which the function f is discontinuous where

$$f(x) = \begin{cases} 1+x^2 , & x \le 0 \\ 2-x , & 0 < x \le 2 \\ (x-2)^2 , & x > 2 \end{cases}$$

N/D 2016

Soln. To find $f(0^-) = f(0) = f(0^+)$



At $x=0$

$f(0) = \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} (1+x^2) = 1+0 = 1 \rightarrow (1)$

$f(0^-) = \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (1+x^2) = 1+0 = 1 \rightarrow (2)$

$f(0^+) = \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} (2-x) = 2-0 = 2 \rightarrow (3)$

From (1) (2) & (3)

$f(0^-) = f(0) \neq f(0^+)$

$\Rightarrow$ f is discontinuous at $x=0$

At $x=2$: $f(2) = \lim_{x\to 2} f(x) = \lim_{x\to 2} (2-x) = 2-2 = 0 \to (1)$

$f(2^-) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2-x) = 2-2 = 0 \to (2)$

$f(2^+) = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x-2)^2 = (2-2)^2 = 0 \to (3)$

From (1) (2) & (3)

$f(2^-) = f(2) = f(2^+)$

$\Rightarrow$ f is continuous at $x=2$

$\therefore$ The domain of f is $(-\infty, 0) \cup (0, \infty)$

2) Define the continuity of a function at any point x_0 in the ε-δ notation. If ~~xxx~~ $f(x) = 5x+6$ is a continuous function then find an interval around $x=1$ such that $|f(x) - f(1)| < 0.001$

Soln.: Defn:- A function f is said to be continuous at x_0 if $|f(x) - f(x_0)| < \varepsilon$ whenever $|x-x_0| < \delta$, $\varepsilon, \delta > 0$

Given $f(x)$ is continuous.

ie $f(x) = 5x+6$ is continuous

and $|f(x) - f(1)| < 0.001$ whenever $|x-1| < \delta$

$f(x) - f(1) = 5x+6-11$ $\quad \therefore f(1) = 5+6 = 11$.

$= 5x-5$

$= 5(x-1)$

when $x > 1$:

$|f(x) - f(1)| = |5(x-1)| < 0.0001$

$\Rightarrow 5|x-1| < 0.001$

$\Rightarrow |x-1| < \frac{0.001}{5}$

$\Rightarrow |x-1| < 0.0002$

$(x-1) < 0.0002 \to (1)$

when $x < 1$

$|f(x) - f(1)| = |5(x-1)| < 0.001$

$\Rightarrow -5(x-1) < 0.001$

$\Rightarrow -(x-1) < 0.0002$

$\Rightarrow x-1 > -0.0002$

From (1) & (2)

$-0.002 < x-1 < 0.0002$

Add 1, $\Rightarrow 1-0.002 < x < 1+0.0002$

$0.9998 < x < 1.0002$

$\boxed{\therefore x \in (0.9998, 1.0002)}$

Show that f is continuous on $(-\infty, \infty)$

$$f(x) = \begin{cases} \sin x, & x < \pi/4 \\ \cos x, & x \geq \pi/4 \end{cases}$$

Soln: At $x = \pi/4$: To prove $f(\pi/4^-) = f(\pi/4) = f(\pi/4^+)$

$f(\pi/4) = \lim_{x\to\pi/4} f(x) = \lim_{x\to\pi/4} \cos x = \cos\pi/4 = 1/\sqrt{2} \to (1)$

$$f(\pi/4^-) = \lim_{x\to\pi/4^-} f(x) = \lim_{x\to\pi/4^-} \sin x = \sin \pi/4 = \frac{1}{\sqrt{2}} \to (2)$$

$$f(\pi/4^+) = \lim_{x\to\pi/4^+} f(x) = \lim_{x\to\pi/4^+} \cos x = \cos \pi/4 = \frac{1}{\sqrt{2}} \to (3)$$

From (1), (2) & (3) we get

$$f(\pi/4^-) = f(\pi/4) = f(\pi/4^+)$$

$\Rightarrow f$ is continuous on $(-\infty, \infty)$

$$\Rightarrow f(x) = \begin{cases} \sin x, & x < \pi/4 \\ \cos x, & x \geq \pi/4 \end{cases} \text{ is continuous on } (-\infty, \infty)$$

Hence the proof.

---

3. If $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x < 2 \\ ax^2 - bx + 3, & 2 \leq x < 3 \\ 2x - a + b, & x \geq 3 \end{cases}$

is continuous for all real x, find the values of a & b.

Sol:

$\frac{x^2-4}{x-2} = x+2$ | ax^2-bx+3 | $2x-a+b$

$-\infty$ ——— 2 ——— 3 ——— ∞

At $x = 2$:

$$f(2) = \lim_{x\to 2} f(x) = \lim_{x\to 2} (ax^2 - bx + 3) = 4a - 2b + 3 \to (1)$$

$$f(2^-) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2-4}{x-2} = \lim_{x\to 2^-} x+2 = 4 \to (2)$$

$(1) \Rightarrow 4a - 2b + 3 = 4$

$4a - 2b = 1 \to (3)$

At $x = 3$:

Given $f(3^-) = f(3) = f(3^+) \to (A)$

$$f(3) = \lim_{x\to 3} f(x) = \lim_{x\to 3} (2x - a + b) = 6 - a + b \to (4)$$

$$f(3^-) = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (ax^2 - bx + 3) = 9a - 3b + 3 \to (5)$$

$(A) \Rightarrow 6 - a + b = 9a - 3b + 3$

$10a - 4b = 3 \to (6)$

Solving (3) & (5), we get

$(6) \Rightarrow 10a - 4b = 3$

$(3)\times 2 \Rightarrow \underset{(-)}{8a} \underset{(+)}{-} 4b \underset{(-)}{=} 2$

$2a = 1$

$$2a = 1$$

$$a = \frac{1}{2}$$

Sub $a = \frac{1}{2}$ in ③, we get

$$4\left(\tfrac{1}{2}\right) - 2b = 1$$

$$2 - 2b = 1$$

$$2b = 1$$

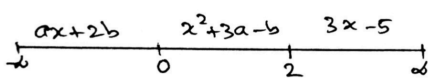
$$b = \frac{1}{2} \quad //$$

4. For what values of a&b is

$$g(x) = \begin{cases} ax + 2b, & x \le 0 \\ x^2 + 3a - b, & 0 < x \le 2 \\ 3x - 5, & x > 2 \end{cases}$$

continuos at every x?

soln :



At $x = 0$:

Given f is continuous at every x,

ie., $f(0^-) = f(0) = f(0^+) \longrightarrow$ Ⓐ

$$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} (ax + 2b) = 0 + 2b = 2b \rightarrow ①$$

$$f(0^+) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x^3 + 3a - b) = 3a - b \rightarrow ②$$

Ⓐ $\Rightarrow$ $2b = 3a - b$ $\quad \because ① = ②$

$$3a - 3b = 0$$

$$\Rightarrow a = b \rightarrow ③$$

At $x = 2$:

Given : $f(2^-) = f(2) = f(2^+)$

$$f(2) = \lim_{x\to 2} f(x) = \lim_{x\to 2} (x^2 + 3a - b) = 4 + 3a - b \rightarrow ④$$

$$f(2^+) = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3x - 5) = 6 - 5 = 1 \rightarrow ⑤$$

Ⓑ $\Rightarrow$ $4 + 3a - b = 1$ $\quad \because ④ = ⑤$

$$3a - b = -3$$

$$3a - a = -3 \quad \text{by ③}$$

$$2a = -3$$

$$a = -3/2$$

$$\therefore b = -3/2$$

Equation of tangent line is

$$y - y_1 = m(x - x_1) \quad \text{where } m = \frac{dy}{dx}$$

Equation of the normal is given by

$$y - y_1 = \frac{-1}{m}(x - x_1), \quad m = \frac{dy}{dx}$$

Problems:

1. For the function $r^2 = \cos\theta$, find the slope of the tangent line at $\theta = \pi/3$. Also find the points on the curve, where the tangent line is horizontal or vertical.

Sol:

Given $r^2 = \cos\theta \longrightarrow (1)$

$r = \sqrt{\cos\theta}$

Let $x = r\cos\theta$

$x = \sqrt{\cos\theta}\,\cos\theta$

$= (\cos\theta)^{1/2+1}$

$= \cos^{3/2}\theta$

$y = r\sin\theta$

$y = \sqrt{\cos\theta}\,\sin\theta$

$y = \cos^{1/2}\theta\,\sin\theta$

$$\frac{dx}{d\theta} = \frac{3}{2}\cos^{3/2-1}\theta\,(-\sin\theta)$$

$$= -\frac{3}{2}\sqrt{\cos\theta}\,\sin\theta$$

$$\frac{dy}{d\theta} = \cos^{1/2}\cos\theta + \sin\theta\,\tfrac{1}{2}\cos^{-1/2}(-\sin\theta)$$

$$= \cos^{3/2}\theta - \frac{1}{2}\sin^2\theta\cos^{-1/2}\theta$$

$$= \cos^{3/2}\theta - \frac{\sin^2\theta}{2\sqrt{\cos\theta}}$$

$$= \frac{2\cos^2\theta - \sin^2\theta}{2\sqrt{\cos\theta}}$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta}$$

$$= \frac{2\cos^2\theta - \sin^2\theta}{\cancel{2}\sqrt{\cos\theta}}\Big/ -\frac{3}{\cancel{2}}\sqrt{\cos\theta}\,\sin\theta$$

$$= \frac{2\cos^2\theta - \sin^2\theta}{-3\sqrt{\cos\theta}\sqrt{\cos\theta}\sin\theta}$$

$$= \frac{2\cos^2\theta - \sin^2\theta}{-3\cos\theta\sin\theta}$$

$$m = \left(\frac{dy}{dx}\right)_{\theta=\pi/3} = \frac{2\cos^2\pi/3 - \sin^2\pi/3}{-3\cos\pi/3\,\sin\pi/3} = \frac{2(1/2)^2 - \left(\frac{1}{\sqrt{3}}\right)^2}{-3(1/2)(1/\sqrt{3})}$$

$$= \frac{\frac{2}{4} - \frac{1}{3}}{-3/2\sqrt{3}} = \frac{\frac{6-4}{12}}{-3/2\sqrt{3}} = \frac{2/12}{-3/2\sqrt{3}} = \frac{2}{-12}\times\frac{2\sqrt{3}}{3}$$

$$m = \frac{1}{-3\sqrt{3}}$$

Case (i): The tangent line is horizontal.

$$\Rightarrow \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{d\theta} = 0$$

$$\frac{dy}{d\theta} = 0 \Rightarrow \frac{2\cos^2\theta - \sin^2\theta}{2\sqrt{\cos\theta}} = 0$$

$\Rightarrow 2\cos^2\theta - \sin^2\theta = 0$

$\Rightarrow 2r^4 - (1 - r^4) = 0$

$2r^4 - 1 + r^4 = 0$

$3r^4 = 1$

$r^4 = 1/3$

$\boxed{r = \frac{1}{3^{1/4}}}$

Given $r^2 = \cos\theta$

$r^4 = \cos^2\theta$

$\sin^2\theta = 1 - \cos^2\theta$

$= 1 - r^4$

Case (ii) The tangent line is vertical.

$$\text{i.e., } \frac{dy}{dx} = \infty \Rightarrow \frac{dy/d\theta}{dx/d\theta} = \infty \Rightarrow \frac{dx}{d\theta} = 0$$

$$\frac{dx}{d\theta} = 0 \Rightarrow -\frac{3}{2}\sqrt{\cos\theta}\,\sin\theta = 0$$

$\sqrt{\cos\theta} = r$

$\cos\theta = 0$ or $\sin^2\theta = 0$

$r^2 = 0$ $\qquad 1 - r^4 = 0$

$\boxed{r = 0}$ $\qquad r^4 = 1$

$\boxed{r = \pm 1}$

2. Find the equation of the tangent line to the curve at the given point $y = \sin(\sin x)$, $(\pi, 0)$

Sol:

Given $y = \sin(\sin x)$

$$\frac{dy}{dx} = \cos(\sin x)\cos x$$

$$m = \left(\frac{dy}{dx}\right)_{(\pi,0)} = \cos(\sin\pi)\cos\pi$$

$$= \cos(0)\cos(\pi)$$

$$= 1(-1) \qquad \because \sin\pi = 0, \cos 0 = 1, \cos\pi = -1$$

$$= -1$$

$\therefore$ The equation of tangent line at $(\pi, 0)$ is

$$y - y_1 = m(x - x_1)$$

$$y - 0 = (-1)(x - \pi)$$

$$y = -x + \pi$$

$$\boxed{x + y = \pi}$$

3. Find the tangent and normal at $(1,2)$ to the curve $y = x^3 + 1$

Sol: Given curve is $y = x^3 + 1$

$$\frac{dy}{dx} = 3x^2$$

$$m = \left(\frac{dy}{dx}\right)_{(1,2)} = 3(1)^2 = 3$$

Eq! of tangent is $y - y_1 = m(x - x_1)$

$$y - 2 = 3(x - 1)$$

$$y - 2 = 3x - 3$$

$$\boxed{3x - y - 1 = 0}$$

Eq! of normal is $y - y_1 = -\frac{1}{m}(x - x_1)$

$\Rightarrow y - 2 = -\frac{1}{3}(x - x_1) \Rightarrow 3y - 6 = -x + 1$ $\boxed{x + 3y - 7 = 0}$

4. Find the domain at which the function $f(x) = |x|$ is continuous and differentiable.

Sol: Let $f(x) = |x|$

$$f(x) = \begin{cases} -x, & x < 0 \\ x, & x > 0 \end{cases}$$

For $x > 0$ $\quad f(x) = |x| = x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{|x+h| - |f(x)|}{h}$$

$$= \lim_{h\to 0} \frac{x + h - x}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

f is differentiable for any $x > 0$.

For $x < 0$, $\quad f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{|x+h| - |x|}{h}$

$$= \lim_{h\to 0} \frac{-(x+h) - (-x)}{h}$$

In $x < 0$, $|x+h| = -(x+h)$, $|x| = -x$

$$= \lim_{h\to 0} \left(\frac{-x - h + x}{h}\right)$$

$$= \lim_{h\to 0} -1$$

$$= -1$$

$\Rightarrow$ f is differentiable for any $x < 0$.

For $x = 0$ $\quad f'(0) = \lim_{h\to 0} \frac{|0 + h| - |0|}{h} = \lim_{h\to 0} \frac{|h|}{h} = 1$

$$f'(0^-) = \lim_{h\to 0} -\left[\frac{h}{h}\right] = -1$$

$$f'(0^+) = \lim_{h\to 0^+} \left[\frac{h}{h}\right] = 1$$

Hence $f'(0^-) \neq f'(0^+) \Rightarrow f'(0)$ does not exists.

$\Rightarrow$ f is not differentiable at $x = 0$.

Hence f is differentiable at all x except $x=0$ $(\mathbb{R}-0)$ or $(-\infty,0)\cup(0,\infty)$

Rolle's theorem:

Let $f(x)$ be a real function defined in the closed interval $[a,b]$ such that

i) $f(a)=f(b)$

ii) f is continuous in the interval $[a,b]$.

iii) $f(x)$ is differentiable in the open interval (a,b). Then there is some point 'c' in the open interval (a,b) such that $f'(c)=0$.

1. Prove that the equation $x^3-15x+c=0$ has at most one real root in the interval $[-2,2]$

Soln:

Let $f(x) = x^3-15x+c$ for $x\in[-2,2]$.

$f'(x) = 3x^2-15$

clearly $f(x)$ is continuous and differentiable.

Now: suppose that f has two real roots a,b in $(-2,2)$ by Rolle's theorem,

There exists a no: r in (a,b) s.t $f'(r)=0$

ie $f'(r) = 3r^2-15=0$

$3(r^2-5)=0$

$r^2-5=0$ But $(a,b)\subseteq(-2,2)$

$\therefore r\in(-2,2)$ $\therefore |r|=2$ & $r^2=4$

$f'(r) = 3r^2-15 = 3(4)-15 = -3 < 0$

which is a contradiction for $f'(r)=0$ for $r\in(a,b)\subseteq(-2,2)$

2. State Rolle's theorem and using which prove that for every differentiable function f on $\mathbb{R}$ which has atleast 2 roots its derivative has atleast one root. Give a polynomial example to show the above result.

Sol:

"First write the statement of Rolle's theorem Here"

Given f is differentiable.

$\Rightarrow f$ is continuous on $\mathbb{R}$.

Let x_1 and x_2 be two roots of f.

$\therefore f(x_1)=f(x_2)=0$, $x_1,x_2\in\mathbb{R}$

By using Rolle's theorem, there exists a $c\in(x_1,x_2)$ such that $f'(c)=0$

$\Rightarrow f'$ has a root c in (x_1,x_2).

Polynomial example:

Let $f(x)=(x-a)(x-b)$, $a,b\in\mathbb{R}$

$f'(x) = (x-a).1+(x-b).1 = 2x-(a+b)$

$f'(x)=0$

$\Rightarrow 2x-(a+b)=0$

$2x=a+b$

$x = \frac{a+b}{2}$

3. If c is a real constant, show that the equation $x^3 - 12x + c = 0$ cannot have two distinct roots in the interval $[0, t]$.

Sol:

Let $f(x) = x^3 - 12x + c$

Clearly $f(x)$ is continuous & differentiable in $[0, 4]$.

$$f'(x) = 3x^2 - 12$$

$$f'(x) = 0$$

$$\Rightarrow 3x^2 - 12 = 0$$

$$3(x^2 - 4) = 0$$

$$x^2 - 4 = 0$$

$$x^2 = 4$$

$$x = \pm 2.$$

If $f(x)$ has two roots then $f'(x)$ has atleast the root in $(0, 4)$. But $f'(x)$ has root 2 or -2 in which -2 lies outside $[0, 4]$.

4. verify Rolle's theorem for $f(x) = 3x^4 - 4x^2 + 5$ in $[-1, 1]$.

Soln:

Let $f(x) = 3x^4 - 4x^2 + 5$

clearly $f(x)$ is continuous and differentiable in $[-1, 1]$.

Now $f(-1) = 3 - 4 + 5 = 4$

$$f(1) = 3 - 4 + 5 = 4$$

$$f(-1) = f(1)$$

$$f'(x) = 12x^3 - 8x$$

$$f'(x) = 0$$

$$\Rightarrow 12x^3 - 8x = 0$$

$$4x(3x^2 - 2) = 0$$

$$x = 0 \quad \text{or} \quad 3x^2 - 2 = 0$$

$$3x^2 = 2$$

$$x^2 = 2/3$$

$$x = \pm\sqrt{2/3}$$

$$\therefore \quad x = 0, \pm\sqrt{2/3}$$

$$f'(0) = 0$$

$$f'(\sqrt{2/3}) = 12(\sqrt{2/3})^3 - 8(\sqrt{2/3})$$

$$= 12 \times \frac{2\sqrt{2}}{3\sqrt{3}} - \frac{8\sqrt{2}}{\sqrt{3}}$$

$$= \frac{8\sqrt{2}}{\sqrt{3}} - \frac{8\sqrt{2}}{\sqrt{3}}$$

$$= 0$$

$$f'(-\sqrt{2/3}) = 12(-\sqrt{2/3})^3 - 8(-$$

$$= -12 \times \frac{2\sqrt{2}}{3\sqrt{3}} + \frac{8\sqrt{2}}{\sqrt{3}}$$

$$= \frac{-8\sqrt{2}}{\sqrt{3}} + \frac{8\sqrt{2}}{\sqrt{3}}$$

$$= 0$$

Hence $-1 < 0 < 1$; $-1 < \sqrt{2/3} < 1$, $-1 < -\sqrt{2/3} < 1$

Hence Rolles theorem is verified.

5. Verify the mean value theorem for $f(x) = x^3 - x$

Sol:

Given $f(x) = x^3 - x$

Clearly $f(x)$ is continuous & differentiable for all x.

$f'(x) = 3x^2 - 1$

Now, $f'(c) = \frac{f(b) - f(a)}{b - a}$, $a = 0, b = 2$

$3c^2 - 1 = \frac{f(2) - f(0)}{2 - 0}$

$= \frac{6 - 0}{2}$

$3c^2 - 1 = 3$

$3c^2 = 4$

$c^2 = 4/3$

$c = \pm 2/\sqrt{3}$

Here $c = 2/\sqrt{3}$ lies in $[0, 2]$

Hence Mean value theorem is verified.

Increasing and Decreasing functions:

Defn: A function $f(x)$ is an increasing function at $x = a$ if its derivative at $x = a$ is positive i.e, $f'(a) > 0$

A function $f(x)$ is an decreasing function at $x = a$ if its derivative at $x = a$ is negative. i.e $f'(a) < 0$.

Lagrange's Mean value theorem.

Let a function $f(x)$ satisfies the following conditions,

1. $f(x)$ is continuous in a closed interval $[a, b]$.
2. $f(x)$ is differentiable in the open interval (a, b).

Then there exists atleast one point 'c' in the open interval (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Test for concavity:

i) $f''(x) > 0 \Rightarrow f$ is concave upward

ii) $f''(x) < 0 \Rightarrow f$ is concave downward.

The First derivative test:

Suppose that c is a critical number of a continuous function f.

a) If f' changes from +ve to −ve at c, then f has a local maximum at c.

b) If f' changes from −ve to +ve at c, then f has a local minimum at c.

c) If f' does not change sign at c, then f has no local maximum or minimum at c.

The second derivative test:

Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at c.

Inflection point:

A point on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward (or) from concave downward to concave upward.

Absolute maximum & Absolute minimum:

A function f has or attains an absolute maximum at $x = x_1$ if $f(x_1) \geq f(x)$ for every x belonging to the domain of f.

A function f has or attains an absolute minimum at $x = x_1$ if $f(x_1) \leq f(x)$ for all x belonging to the domain of f.

1. Given $f(x) = \sin x + \cos x$, $0 \leq x \leq 2\pi$. find the intervals of increase and / or decrease, the local maximum and minimum values, intervals of concavity and the inflection points.

Sol: Given that $f(x) = \sin x + \cos x$, $0 \leq x \leq 2\pi$

$$f'(x) = \cos x - \sin x$$

For critical points, $f'(x) = 0$

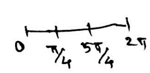
$$\Rightarrow \cos x - \sin x = 0$$

$$\Rightarrow \frac{\sin x}{\cos x} = 1 \Rightarrow \tan x = 1$$

$$\Rightarrow x = \tan^{-1}(1) = \pi/4, 5\pi/4$$

on $0 \leq x \leq \pi/4$, $f'(30) = \cos 30' - \sin 30'$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2}$$



$$= +ve$$

$\Rightarrow f(x)$ is Increasing.

on $\pi/4 \leq x < 5\pi/4$, $f'(90') = \cos 90' - \sin 90'$

$$= 0 - 1$$

$$= -ve$$

$\Rightarrow f(x)$ is decreasing.

on $5\pi/4 < x < 2\pi$, $f'(270') = \cos 270' - \sin 270$

$$= 0 - (-1)$$

$$= 1 \quad +ve$$

$\Rightarrow f(x)$ is Increasing.

Finding local maxima & local minima :

By 1^{st} derivative test,

$f(\pi/4) = \sin\pi/4 + \cos\pi/4 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} > 0$

$\Rightarrow f$ has local maximum.

$f(5\pi/4) = \sin 5\pi/4 + \cos 5\pi/4 = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2} < 0$

$\Rightarrow f$ has local minimum.

Testing of concavity :

$f''(x) = -\sin x - \cos x$

$f''(x) = 0$ (hyper critical number)

$\Rightarrow -\sin x - \cos x = 0$

$\sin x = -\cos x$

$\frac{\sin x}{\cos x} = -1$

$\tan x = -1$

$x = \tan^{-1}(-1)$

$x = \frac{3\pi}{4}, \frac{7\pi}{4}$

$0 \quad 3\pi/4 \quad 7\pi/4 \quad 2\pi$

On $0 < x < 3\pi/4$, $f''(90°) = -\sin 90 - \cos 90 = -1 < 0$

$\Rightarrow f$ is concave downwar

On $3\pi/4 < x < 7\pi/4$ $f''(270) = -\sin 270 - \cos 270$

$= -(-1) - 0$

$= 1 > 0$

$\Rightarrow f$ is concave upward.

On $7\pi/4 < x < 2\pi \Rightarrow f''(300°) = -\sin 300 - \cos 300$

$= \cos 30 - \sin 30$

$= \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}$

$= -ve$

$\Rightarrow f$ is concave downward

Point of inflexions are $[x_1, f(x_1)]$; $[x_2, f(x_2)]$

$(3\pi/4, 0), (7\pi/4, 0)$

Answer the following Question about the function where derivatives are given.

(a) what are the critical points of f?

(b) on what intervals is f increasing or decreasing?

(c) At what points is any does f assume local maximum and minimum value

(i) $f(x) = x^4 - 2x^2 + 3$ N/D 2016

(ii) $f(x) = 2 + \sin x$ $0 < x \leq 2\pi$

Problem (i) Given $f(x) = x^4 - 2x^2 + 3$

$f'(x) = 4x^3 - 4x = 4x(x^2 - 1)$

$= 4x(x+1)(x-1)$

(a) For critical points $f'(x)=0$

$\Rightarrow \cancel{f(x)}\ 4x(x-1)(x+1)=0$

$\Rightarrow x=0 \quad x=1, x=-1.$

$-\infty \quad -1 \quad 0 \quad 1 \quad \infty$

The critical points are $-1, 0, 1$

(b) $-\infty < x < -1$ $f'(-2) = 4(-2)(-3)(-1)$
$= -24 < 0$

$\Rightarrow f$ is decreasing

on $-1 < x < 0$ $f'(-0.5) = 4(-0.5)(-1.5)(0.5)$
$= +ve > 0$

$\Rightarrow f$ is increasing.

on $0 < x < 1$ $\Rightarrow f'(0.5) = 4(0.5)(-0.5)(1.5)$
$= -ve$

$\Rightarrow f$ is decreasing

on $1 < x < \infty$ $f'(2) = 4\cdot 2(1)(3) = +ve > 0$

$\Rightarrow f$ is increasing

c) Testing maxima and minima

x-critical point	Interval	value of f(x)	
$x=-1$	$x<-1$ $-1<x<0$	$-24<0$ $+ve>0$	local Min
$x=0$	$x<0$ $0<x<1$	$+ve$ ve	local max
$x=1$	$x<1$ $1<x<\infty$	ve $+ve$	local Min

local Min value at $x=0$, $f(0)=3$

local Max value at $x=1$, $f(x)=1-2+3=2$

local Max value at $x=-1$, $f(-1)=1-2+3=2$

d) Testing of concavity.

$-\infty \quad -1/\sqrt{3} \quad 1/\sqrt{3} \quad \infty$

$f''(x) = 12x^2-4$

$f''(x)=0 \Rightarrow 12x^2-4=0 \Rightarrow 4(3x^2-1)=0$

$\Rightarrow 3x^2=1$

$\Rightarrow x^2 = 1/3$

$\Rightarrow x^2 = \sqrt{1/3}$ $\quad x^2=\sqrt{1/3}$

On $-\infty < x < -1/\sqrt{3}$ $f''(x)=0 \Rightarrow f$ is concave up

on $-1/\sqrt{3} < x < 1/\sqrt{3}$, $f''(x) < 0 \Rightarrow f$ is concave down

on $+1/\sqrt{3} < x < \infty$, $f''(x) > 0 \Rightarrow f$ is concave up

e) Inflexion points are $(x_1, f(x_1))$

i.e. $\pm(1/\sqrt{3}, 22/9)$ since $f(\pm 1/\sqrt{3}) = 22/9$.

Problem (ii)

Given $f(x) = x + 2\sin x$, $0 < x < 2\pi$.

$f'(x) = 1 + 2\cos x$.

For critical points $f'(x)=0$

$\Rightarrow 1 + 2\cos x = 0$

$\cos x = -1/2$

$x = 2\pi/3, 4\pi/3$

(a) The critical points are $2\pi/3$, $4\pi/3$

b) $0 < x < 2\pi/3$ $f'(30) = \pi/6 + 2\sin\pi/6 = \pi/6 + 1 > 0$

$\Rightarrow f(x)$ is $\uparrow$

$0 \quad 2\pi/3 \quad 4\pi/3 \quad 2\pi$

$2\pi/3 < x < 4\pi/3$ $f'(180) = 1 + 2\cos\pi = -1 < 0$

$\Rightarrow f(x)$ is $\downarrow$

$4\pi/3 < x < 2\pi$, $f'(300) = 1 + 2\cos 300° = 1 - 2\sin 30 > 0$

$\Rightarrow f(x)$ is $\uparrow$

c) Local Maxima & Local Minima

$0 \quad 2\pi/3 \quad 4\pi/3 \quad 2\pi$

$x < 2\pi/3$, $f'(\pi/2) = 1 + 2\cos\pi/2 = 1 > 0$

$2\pi/3 < x < 4\pi/3$, $f'(180) = 1 + 2\cos\pi = 1 - 2 = -1 < 0$

$\Rightarrow f$ has local Maximum at $x = 2\pi/3$

$\therefore$ Maximum value $= f(2\pi/3) = 2\pi/3 + 2\sin\left(\frac{2\pi}{3}\right) = 3.83$

$x < 4\pi/3$, $f'(180) = -1 < 0$

$x > 4\pi/3$, $f'(300) = 1 + 2\cos 360° = 1 + 2 = 3 > 0$

$\Rightarrow f$ has local minimum at $x = 4\pi/3$

$\therefore$ Minimum value $= f(4\pi/3) = (4\pi/3) + 2\sin\left(\frac{4\pi}{3}\right)$

$= 2.48$//

d) Testing of Concavity

$f''(x) = -2\sin x$

$f''(x) = 0 \Rightarrow -2\sin x = 0$

$\Rightarrow \sin x = 0$

$\Rightarrow x = 0, \pi, 2\pi$

On $0 < x < \pi$, $f''(\pi/2) = -2\sin(\pi/2) = -2 < 0$

$\Rightarrow$ concave downward

$-\infty \quad 0 \quad \pi \quad 2\pi \quad \infty$

On $\pi < x < 2\pi$ $f''(\pi/2) = -2\sin(3\pi/2) = -2(-1) = 2 > 0 \Rightarrow$ Concave upward

(c) point of inflexion $\cdot (x, y)$

i.e. $(\pi, \pi + 2\sin x)$, i.e. $(\pi, \pi + 0)$

i.e. (π, π)

$\Rightarrow$ For the function $f(x) = 2x^3 + 3x^2 - 36x$

(i) Find the intervals on which it is increasing or decreasing

(ii) Find the local Maximum and minimum value of f.

(iii) Find the intervals of concavity and the inflexion points.

Soln Given $f(x) = 2x^3 + 3x^2 - 36x$

$f'(x) = 6x^2 + 6x - 36$

For critical points $f'(x) = 0$

$\Rightarrow 6x^2 + 6x - 36 = 0$

$x^2 + x - 6 = 0$

$(x-2)(x+3) = 0$

$x = 2, -3$//

Critical points are $-3, 2$

(a) Increasing or decreasing

On $x < -3$ $f'(-4) = 6(-4)^2 + 6(-4) - 36 = 6 \times 16 - 24 - 36$

$= 96 - 60 = 36 > 0$

$\Rightarrow f(x)$ is increasing

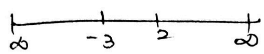
on $-3<x<2$ $f'(0) = -36<0$

$\Rightarrow f(x)$ is decreasing

On $x>2$, $f'(3) = 6(3^2)+6(3)-36$

$= 6\times 9+18-36 = 54-18>0$

$\Rightarrow f(x)$ is increasing



(ii) Local Maxima and Local Minima

$f'(x) = 6x^2+6x-36$

$f''(x) = 12x+6$

At $x=-3$ f' changes from +ve to -ve

$\Rightarrow f$ has local maximum

local Maximum value $= f(-3)$

$= 2(-3)^3+3(-3)^2-36(-3)$

$= -54+27+108$

$= 81$

At $x=2$ f' changes from -ve to +ve

$\Rightarrow f$ has local minimum.

$\therefore$ local minimum value $= f(2)$

$= 2(2^3)+3(2^2)-36(2)$

$= 16+12-72$

$= 28-72$

$= -44$.

(iii) Testing Concavity

$f''(x) = 12x+6$

$f''(x) = 0$ (hyper critical number)

$\Rightarrow 12x+6=0$

$\Rightarrow 6(2x+1)=0$

$\Rightarrow \boxed{x=-1/2}$

on $-\infty<x<-1/2$ $f''(-1) = 12(-1)+6$

$= -12+6$

$= -6<0$

$\Rightarrow f$ is concave downward

$-1/2<x<\infty$, $f''(1) = 12(1)+6$

$= 18>0$

$\Rightarrow f$ is concave upward.

Inflexion point $(-1/2, f(-1/2)) = (-1/2, 37/2)$

$\therefore f(-1/2) = 2(-1/2)^3+3(-1/2)^2-36(-1/2)$

$= -2/8+3/4+36/2$

$= -1/4+3/4+36/2$

$= 2/4+36/2$

$= \frac{37}{2}$

4) Find the local maximum and minimum values of f using both the first and second derivative test for $y = x^5-5x+3$

Soln Given $y = x^5-5x+3$ Here $y = f(x)$

$f'(x) = 5x^4-5$

For critical points $f(x)=0$

$\Rightarrow 5x^4-5=0$

$5(x^4-1)=0$

$\Rightarrow x^4=1$

$\Rightarrow x^2=\pm 1$ (-1 is not possible)

$\Rightarrow x=\pm 1$

The critical points are $1, -1$

Increasing & decreasing

$$\underset{-\infty}{\vdash}\overset{+ve}{\text{---}}\underset{-1}{|}\overset{-ve}{\text{---}}\underset{1}{|}\overset{+ve}{\text{---}}\underset{\infty}{\dashv}$$

on $-\infty < x < -1$ $x = -2$

$f'(-2) = 5(-2)^4 - 5 = 5 \times 16 - 5 = 75 > 0$

$\Rightarrow f$ is increasing

on $-1 < x < 1$ $x = 0$

$f'(0) = 5(0^4) - 5 = -5 < 0$

$\Rightarrow f$ is decreasing

on $1 < x < \infty$ $x = 2$

$f'(2) = 5(2)^4 - 5 = 80 - 5 = 75 > 0$.

$\Rightarrow f$ is increasing

Local Maximum & Local Minimum (First derivative Test)

At $x = -1$ f' changes +ve to −ve

$\Rightarrow f$ has local Maximum

max value $= f(-1) = (-1)^5 - 5(-1) + 3$

$= -1 + 5 + 3 = 7$

At $x = 1$ f' changes −ve to +ve

$\Rightarrow f$ has local minimum

min value $= f(1) = 1^5 - 5(1) + 3 \Rightarrow 4 - 5 = -1$

By IInd derivative test.

$f'(x) = 5x^4 - 5$

$f''(x) = 20x^3$

$\Rightarrow f''(x) = 0 \Rightarrow 20x^3 = 0$

$\Rightarrow x = 0$

on $-\infty < x < 0$, $x = -1$.

$$\underset{\infty}{\vdash}\overset{-ve}{\text{---}}\underset{0}{|}\overset{+ve}{\text{---}}\underset{\infty}{\dashv}$$

$f''(-1) = 20(-1)^3 = -20 < 0$

$\Rightarrow f$ is concave downward

on $0 < x < \infty$, $x = 1$

$f''(1) = 20(1^3) = 20 > 0 \Rightarrow$ f is Concave upward

Local Maxima & minima : by IInd derivative

At $x = 1$ $f'(1) = 0 \Rightarrow f''(1) = 20 > 1$

$\Rightarrow f(1) = -1$ is a local minimum

At $x = -1$ $f'(-1) = 0$ $f'(-1) = -20$

$\Rightarrow f(-1) = -1$ is a local maximum.

5) Find maxima and minima for the fun $f(x) = x \log x$, $x > 0$

Solo $y = x \log x$.

$dy/dx = d/dx\,(x \log x) = x \cdot \frac{1}{x} + \log x \cdot 1$

$= 1 + \log x$

$dy/dx = 0$

$\Rightarrow 1 + \log x = 0$

$\log x = -1$

$e^{\log x} = e^{-1}$

$x = e^{-1} = 1/e$

$\boxed{x = 1/e}$

To find d^2y/dx^2 $\quad d^2y/dx^2 = d/dx\,(1 + \log x) = 1/x$.

$\left(d^2y/dx^2\right)_{at\ x = 1/e} = \frac{1}{1/e} = e = 2.71$ +ve.

The given fn is minimum at $x = 1/e$

The min value is $f(1/e) = 1/e \times \log(1/e)$

$= 1/e \times (\log 1 - \log e)$

$= 1/e(-1)$

$= -1/e$.

$\left\{ \begin{array}{l} \log 1 = 0 \\ \log e = 1 \end{array} \right\}$

6) Find the Maximum and Minimum value of the function $y = (x-2)^2(x-3)$ (or)

$f(x) = (x-2)^2(x-3)$

Soln Given $y = (x-2)^2(x-3)$

$dy/dx = (x-2)^2 \cdot 1 + (x-3)\, 2(x-2)$

$= (x-2)[(x-2) + 2(x-3)]$

$= (x-2)[x-2+2x-6]$

$= (x-2)[3x-8]$

$dy/dx = 0$

$\Rightarrow (x-2)(3x-8) = 0$

$\Rightarrow x-2 = 0$ (or) $3x-8 = 0$

$\boxed{x = 2}$ $\quad 3x = 8$

$\boxed{x = 8/3}$

To find d^2y/dx^2

$dy/dx = (x-2)(3x-8)$

$d^2y/dx^2 = (x-2)\cdot 3 + (3x-8)\cdot 1$

$= 3x-6+3x-8$

$= 6x-14$

$\left(d^2y/dx^2\right)_{at\ x=2} = 6(2)-14 = 12-14$

$= -2 < 0$

$\Rightarrow f(x)$ has maximum

Max value is $y = (x-2)^2(x-3)$

$= (2-2)^2(x-3)$

$\boxed{y = 0}$

$\left(\frac{d^2y}{dx^2}\right)_{at\ x = 8/3} = \cancel{6}^{2}(8/\cancel{3}) - 14$

$= 16-12 = 4 > 0$

$\Rightarrow f$ has minimum.

$\therefore$ Min value is $f(8/3) = (8/3-2)^2(8/3-3)$.

$= \left(\frac{8-6}{3}\right)^2\left(\frac{8-9}{3}\right)$

$= (2/3)^2(-1/3)$

$= 4/9(-1/3)$

$= -4/27$.

FUNCTIONS OF SEVERAL VARIABLES

UNIT-III

Homogeneous Function

A function $f(x,y)$ is said to be a homogeneous function of degree n if $f(tx,ty)=t^n f(x,y)$.

Example:

$$f(x,y)=\frac{x^6+y^6}{x^4-y^4}$$

$$f(tx,ty)=t^2 f(x,y)$$

Euler's Theorem

If f be a homogeneous function of degree n in x and y, then $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$ and $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=nf$

1) If $u=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$ **then show that** $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$

Solution:

Given $u(x,y,z)=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$.

$$u(tx,ty,tz)=\frac{tx}{ty}+\frac{ty}{tz}+\frac{tz}{tx}=\frac{t}{t}(\frac{x}{y}+\frac{y}{z}+\frac{z}{x})=t^0u(x,y,z)$$

$\Rightarrow u$ is a homogenous function of degree $n=0$

Hence by Euler's theorem, we have $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu=0\times u=0$

Hence proved.

2) If $u=\frac{x}{y}$ **prove that** $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+=0$

Solution:

Given $u(x,y)=\frac{x}{y}$.

$$u(tx,ty)=\frac{tx}{ty}=\frac{t}{t}(\frac{x}{y})=t^0u(x,y)$$

$\Rightarrow u$ is a homogenous function of degree $n=0$

Hence by Euler's theorem, we have $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu=0\times u=0$

Hence proved.

3) Verify Euler's theorem for the function $u=x^2+y^2+2xy$

Solution

Given $u=x^2+y^2+2xy$. This is a homogenous function of degree 2.

$$\frac{\partial u}{\partial x}=2x+2y$$

$$x\frac{\partial u}{\partial x}=2x^2+2xy\rightarrow(1)$$

$$\frac{\partial u}{\partial y}=2y+2x$$

$$y\frac{\partial u}{\partial y}=2y^2+2xy\rightarrow(2)$$

Adding equation (1) and (2), we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2[x^2+y^2+2xy]$$
$$=2u$$

Hence Euler's theorem is verified

4) Using Euler's theorem given *u(x,y)* is a homogenous function of degree *n* prove that $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=n(n-1)u$

Solution

Since *u(x,y)* is homogenous function of degree *n* ,by Euler's theorem we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \rightarrow (1)$$

Diff (1) p.w.r.to *x*,

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x}\rightarrow (2)$$

Diff (1) p.w.r.to *y*,

$$y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}+x\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial y}\rightarrow (3)$$

$$(2)x+(3)y\Rightarrow[\{\ x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\partial y}\ \}=\{\ nx\frac{\partial u}{\partial x}\ \}+\{\ ny\frac{\partial u}{\partial y}\ \}$$
$$+\{\ y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}+xy\frac{\partial^2 u}{\partial x\partial y}\ \}]$$
$$\Rightarrow x^2u_{xx}+y^2u_{yy}+2xyu_{xy}+nu=n^2u$$
$$\Rightarrow x^2u_{xx}+y^2u_{yy}+2xyu_{xy}=nu(n-1)$$

Total Derivative

If $u=f(x,y)\ where\ x=\varphi(t),y=\Psi(t)\ then\ \frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$ is called the total differential of $u\ w.r.to\ t$

In general

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$

Note:

1. $u=f(x,y,z)\ where\ x,y,z\ are\ functions\ of\ t\ then\ \frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt}$
2. $If\ \ z=f(x,y)\ where\ x=f(u,v),\ y=g(u,v)\ then$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

3. $If\ u=f(x,y)\ and\ y=\varphi(x)\ then\ \frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}$

1) If $u=e^x yz^2$ find *du*

Solution:

We know that $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz\quad =e^x yz^2dx+e^xz^2dy+2ze^xydz$

2) If $u = f(x-y, y-z, z-x)$ show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Solution:

Put $\left.\begin{array}{l} x-y = x_1 \\ y-z = x_2 \\ z-x = x_3 \end{array}\right\} \rightarrow (A)$

Now from (A), we get

$$\therefore u = f(x-y, y-z, z-x)$$
$$= f(x_1, x_2, x_3)$$
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial x} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x_3}\cdot\frac{\partial x_3}{\partial x}$$
$$\frac{\partial x_1}{\partial x} = 1, \frac{\partial x_2}{\partial x} = 0, \frac{\partial x_3}{\partial x} = -1$$
$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_3} \rightarrow (1)$$

Similarly $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial y} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial y} + \frac{\partial u}{\partial x_3}\cdot\frac{\partial x_3}{\partial y}$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial y} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial y} + \frac{\partial u}{\partial x_3}\cdot\frac{\partial x_3}{\partial y}$$
$$= \frac{\partial u}{\partial x_1}(-1) + \frac{\partial u}{\partial x_2}(1) + \frac{\partial u}{\partial x_3}(0)$$
$$= -\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} \rightarrow (2)$$
$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial z} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial z} + \frac{\partial u}{\partial x_3}\cdot\frac{\partial x_3}{\partial z}$$
$$= \frac{\partial u}{\partial x_1}(0) + \frac{\partial u}{\partial x_2}(-1) + \frac{\partial u}{\partial x_3}(1)$$
$$= -\frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_3} \rightarrow (3)$$

Adding (1), (2), and (3), we get $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

3) Find $\frac{du}{dt}$ if $u = x^3y^2 + x^2y^3$ where $x = at^2$ & $y = 2at$ using partial derivative

Solution:

We know that

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$
$$= (3x^2y^2 + 2xy^3)\times 2at + (2yx^3 + 3x^2y^2)\times 2a$$
$$= 8a^5t^6(3t+4) + 8a^5t^6(t+3)$$
$$= 8a^5t^6(4t+7)$$

4) Find du/dt if $u = x^3\ y^4$ where $x = t^3 and y = t^2$

Solution:

We know that

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= 3x^2y^4\cdot 3t^2 + 4x^3y^3\cdot 2t$$

$$= 3t^6t^8 3t^2 + 4t^9t^6 2t$$

$$= 9t^{16} + 8t^{16}$$

$$= 17t^{16}$$

5) Using the definition of total derivative find the value of $\frac{du}{dt}$ given

$u = y^2 - 4ax, x = at^2, y = 2at$

Solution:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$
$$= -4a \times 2at + 2y \times 2a$$
$$= -8a^2t + 8a^2t$$
$$= 0$$

6) If sin zy = cos zx compute $\frac{\partial z}{\partial x}$ when $z = \pi, x = \frac{1}{3}$ & $y = \frac{1}{6}$

Solution:

Given $\sin zy = \cos zx$

Diff p.w.r to x on both sides,we get

$$\cos zy \times \left(y\frac{\partial z}{\partial x} \right) = -\sin zx \times \left(x\frac{\partial z}{\partial x} + z \right)$$

when $z = \pi, x = \frac{1}{3}$ & $y = \frac{1}{6}$ we get

$$\cos\frac{\pi}{6}\left(\frac{1}{6} \times \frac{\partial z}{\partial x} \right) = -\sin\frac{\pi}{3}\left(\frac{1}{3} \times \frac{\partial z}{\partial x} + \pi \right)$$

$$\Rightarrow \frac{\partial z}{\partial x}\left(\frac{\sqrt{3}}{12} + \frac{\sqrt{3}}{6} \right) = -\frac{\pi\sqrt{3}}{2}$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{\pi \times \sqrt{3} \times 12}{2 \times \sqrt{3} \times 3} = -2\pi$$

7)If z be a function of x and y and u and v are other two variables, such that

$u = lx + my,\ v = ly - mx$ **show that** $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \right)$

Solution:

z is a function of u and v

u and v are functions of x and y

$$u = lx + my, \qquad v = ly - mx$$

$$\frac{\partial u}{\partial x} = l \qquad \frac{\partial v}{\partial x} = -m$$

$$\frac{\partial u}{\partial y} = m \qquad \frac{\partial v}{\partial y} = l$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\ l + \frac{\partial z}{\partial v}\ (-m)$$

$$\frac{\partial}{\partial x} = l\frac{\partial}{\partial u} - m\frac{\partial}{\partial v}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x} \right) = \left(l\frac{\partial}{\partial u} - m\frac{\partial}{\partial v} \right)\left(l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v} \right)$$

$$\frac{\partial^2 z}{\partial x^2} = l^2\frac{\partial^2 z}{\partial u^2} - lm\frac{\partial^2 z}{\partial u \partial v} - lm\frac{\partial^2 z}{\partial u \partial v} + m^2\frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial x^2} = l^2\frac{\partial^2 z}{\partial u^2} - 2lm\frac{\partial^2 z}{\partial u \partial v} + m^2\frac{\partial^2 z}{\partial v^2} \qquad -------(1)$$

Similarly

$$\frac{\partial^2 z}{\partial y^2} = m^2\frac{\partial^2 z}{\partial u^2} + 2lm\frac{\partial^2 z}{\partial u\partial v} + l^2\frac{\partial^2 z}{\partial v^2} \qquad -------(2)$$

$(1)+(2) \Rightarrow$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \left(l^2 + m^2\right)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

8) If $u = x^2 - y^2,\ \ v = 2xy,\ \ f(x,y) = \varphi(u,v)$ *show that* $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 4\left(x^2 + y^2\right)\left(\dfrac{\partial^2 \varphi}{\partial u^2} + \dfrac{\partial^2 \varphi}{\partial v^2}\right)$

Solution:

$$u = x^2 - y^2 \qquad v = 2xy$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial f}{\partial x} = \frac{\partial \varphi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial x} = \frac{\partial \varphi}{\partial u}.\,(2x) + \frac{\partial \varphi}{\partial v}\,(2y)$$

$$\frac{\partial}{\partial x} = 2x\frac{\partial}{\partial u} + 2y\frac{\partial}{\partial v}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \left(2x\frac{\partial}{\partial u} + 2y\frac{\partial}{\partial v}\right)\left(2x\frac{\partial \varphi}{\partial u} + 2y\frac{\partial \varphi}{\partial v}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = 4x^2\frac{\partial^2 \varphi}{\partial u^2} + 4xy\frac{\partial^2 \varphi}{\partial u\partial v} + 4xy\frac{\partial^2 \varphi}{\partial u\partial v} + 4y^2\frac{\partial^2 \varphi}{\partial v^2}$$

$$\frac{\partial^2 f}{\partial x^2} = 4x^2\frac{\partial^2 \varphi}{\partial u^2} + 8xy\frac{\partial^2 \varphi}{\partial u\partial v} + 4y^2\frac{\partial^2 \varphi}{\partial v^2} \qquad -------(1)$$

Similarly

$$\frac{\partial^2 f}{\partial y^2} = 4y^2\frac{\partial^2 \varphi}{\partial u^2} + 8xy\frac{\partial^2 \varphi}{\partial u\partial v} + 4x^2\frac{\partial^2 \varphi}{\partial v^2} \qquad -------(2)$$

$(1)+(2) \Rightarrow$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4\left(x^2 + y^2\right)\left(\frac{\partial^2 \varphi}{\partial u^2} + \frac{\partial^2 \varphi}{\partial v^2}\right)$$

Differentiation of implicit functions

If $f(x,y) = c$ be an implicit relation between x *and* y which defines as a differentiable function of x, then $\dfrac{du}{dx} = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dx}$ becomes $0 = \dfrac{df}{dx} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dx}$

This gives the important formula $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y}$

1) If $e^y - e^x + xy = 0$ **find** $\dfrac{dy}{dx}$

Solution:

We know that $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y}$

Let $f(x,y) = e^y - e^x + xy;\ \dfrac{\partial f}{\partial x} = -e^x + y$ and $\dfrac{\partial f}{\partial y} = e^y + x$

$$\frac{dy}{dx} = -\frac{(-e^x + y)}{(e^y + x)} = \frac{e^x - y}{e^y + x}$$

2) Find $\frac{dy}{dx}$ **when** $f(x,y)=\log(x^2+y^2)+\tan^{-1}\frac{y}{x}$

Solution:

We know that $\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}$ **;** $\frac{\partial f}{\partial x}=\frac{2x}{x^2+y^2}-\frac{y}{x^2+y^2}$ **;** $\frac{\partial f}{\partial y}=\frac{2y}{x^2+y^2}+\frac{x}{x^2+y^2}$

$$\frac{dy}{dx}=-\frac{\frac{2x-y}{x^2+y^2}}{\frac{2y+x}{x^2+y^2}}=\frac{y-2x}{2y+x}$$

3) Find $\frac{dy}{dx}$ **if** $x^3+y^3=3ax^2y$

Solution:

Let $f(x,y)=x^3+y^3-3ax^2y$ **;** $\frac{\partial f}{\partial x}=3x^2-6axy$ **;** $\frac{\partial f}{\partial y}=3y^2-3ax^2$

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}=-\frac{3(x^2-2axy)}{3(y^2-ax^2)}=\frac{x^2-2axy}{y^2-ax^2}$$

4) Find $\frac{dy}{dx}$**, if** $x^3+y^3=3axy$

Solution:

Given $f(x,y)=x^3+y^3-3axy$. Then

$$\frac{\partial f}{\partial x}=3x^2-3ay$$

$$\frac{\partial f}{\partial y}=3y^2-3ax$$

$$\frac{dy}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}=-\frac{(3x^2-3ay)}{(3y^2-3ax)}=\frac{ay-x^2}{y^2-ax}$$

Taylor's series expansion of a function of two variables

$$f(x,y)=f(a,b)+\frac{1}{1!}\left[(x-a)f_x(a,b)+(y-b)f_y(a,b)\right]+$$
$$\frac{1}{2!}\left[(x-a)^2f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2f_{yy}(a,b)\right]+$$
$$\frac{1}{3!}\left[(x-a)^3f_{xxx}(a,b)+3(x-a)^2(y-b)f_{xxy}(a,b)+3(x-a)(y-b)^2f_{xyy}(a,b)+(y-b)^3f_{yyy}(a,b)\right]+\ldots$$

This is called the Taylor's series of $f(x,y)$ at the point (a, b)

1) Find the Taylor's series expansion of $e^x\sin y$ **near the point** $\left(-1,\frac{\pi}{4}\right)$ **up to the first degree terms.**

Solution:

By Taylor's theorem, we have

$$f(x,y)=f(-1,\frac{\pi}{4})+(x+1)f_x(-1,\frac{\pi}{4})+(y-\frac{\pi}{4})f_y(-1,\frac{\pi}{4})+\frac{(x+1)^2}{2}f_{xx}(-1,\frac{\pi}{4})+$$

$$\frac{2(x+1)(y-\frac{\pi}{4})}{2}f_{xy}(-1,\frac{\pi}{4})+\frac{(y-\frac{\pi}{4})^2}{2}f_{yy}(-1,\frac{\pi}{4})+\ldots\rightarrow(1)$$

$$f(x,y)=e^{x}\sin y; f(-1,\frac{\pi}{4})=\frac{1}{e\sqrt{2}}$$
$$f_x=e^{x}\sin y; f_x(-1,\frac{\pi}{4})=\frac{1}{e\sqrt{2}}$$
$$f_y=e^{x}\cos y; f_y(-1,\frac{\pi}{4})=\frac{1}{e\sqrt{2}}$$

Using these values in (1) we get

$$f(x,y)=e^{x}\sin y=\frac{1}{e\sqrt{2}}\left(1+\left\{(x+1)+(y+\frac{\pi}{4})\right\}\right)+...$$

2) Find the Taylor's series expansion of $e^{x}\sin y$ in powers of x & y as far as terms of the third degree.

Solution:

By Taylor's theorem, we have

$$f(x,y)=f(0,0)+xf_x(0,0)+yf_y(0,0)+\frac{1}{2!}\left[x^2 f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2 f_{yy}(0,0)\right]+$$
$$\frac{1}{3!}\left[x^3 f_{xxx}(0,0)+3x^2 yf_{xxy}(0,0)+3xy^2 f_{xyy}(0,0)+y^3 f_{yyy}(0,0)\right]+...\to(1)$$

Given

$$f(x,y)=e^{x}\sin y; f(0,0)=0$$
$$f_x=e^{x}\sin y; f_x(0,0)=0$$
$$f_y=e^{x}\cos y; f_y(0,0)=1$$
$$f_{xx}=e^{x}\sin y; f_{xx}(0,0)=0$$
$$f_{xy}=e^{x}\cos y; f_{xy}(0,0)=1$$
$$f_{xxx}=e^{x}\sin y; f_{xxx}(0,0)=0$$
$$f_{xxy}=e^{x}\cos y; f_{xxy}(0,0)=1$$
$$f_{xyy}=-e^{x}\sin y; f_{xyy}(0,0)=0$$
$$f_{yyy}=-e^{x}\cos y; f_{yyy}(0,0)=-1$$

Using these values in (1) we get

$$f(x,y)=e^{x}\sin y=0+x\times 0+y\times 1+\frac{x^2}{2}\times 0+xy\times 1+\frac{1}{3!}\left[x^3(0)+3x^2 y(1)+3xy^2(0)+y^3(-1)\right]+...$$
$$=y+xy+\frac{1}{2}x^2 y-\frac{y^3}{6}+...$$

3) Expand $e^{x}\cos y$ in powers of x and y as far as terms of the first degree.

Solution:

Given $f(x,y)=e^{x}\cos y$ then

$f(x,y)=e^{x}\cos y, f(0,0)=1$

$f_x(x,y)=e^{x}\cos y, f_x(0,0)=1$

$$f_y(x,y)=-e^{x}\sin y, f_y(0,0)=0$$
$$f_{xx}(x,y)=e^{x}\cos y, f_{xx}(0,0)=1$$
$$f_{xy}(x,y)=-e^{x}\sin y, f_{xy}(0,0)=0$$

By Taylor's theorem, we have

$$f(x,y)=f(0,0)+xf_x(0,0)+yf_y(0,0)+\frac{x^2}{2}f_{xx}(0,0)+\frac{2xy}{2}f_{xy}(0,0)+\frac{y^2}{2}f_{yy}(0,0)+...$$

$$=1+(x\times 1+y\times 0)+....=1+x+...$$

4) Find Taylor's series expansion of x^{y} near the point (1,1) upto the first degree terms.

Solution:

Taylor's series of $f(x,y)$ near the point (1,1) is

$$f(x,y)=f(1,1)+(x-1)f_x(1,1)+(y-1)f_y(1,1)+\frac{(x-1)^2}{2}f_{xx}(1,1)+\frac{2(x-1)(y-1)}{2}f_{xy}(1,1)+\frac{(y-1)^2}{2}f_{yy}(1,1)+\ldots\to(1)$$

$$f(x,y)=x^y, f(1,1)=1$$
$$f_x(x,y)=yx^{y-1}, f_x(1,1)=1$$
$$f_{xx}(x,y)=y(y-1)x^{y-2}, f_{xx}(1,1))=0$$

Let $f(x,y)=x^y$

Taking log on both sides, we get

$$\log f(x,y)=y\log x$$

Diff p.w.r.to y, we get

$$\frac{1}{f}f_y=\log x$$

$$f_y=f\log x=x^y\log x; f_y(1,1)=0$$

Using all these values in (1), we get

$$f(x,y)=x^y=1+(x-1)+\ldots$$

5) Expand $e^x\log(1+y)$ in powers of *x* and *y* upto third degree.

Solution:

Given

$$f(x,y)=e^x\log(1+y); f(0,0)=0$$
$$f_x=e^x\log(1+y); f_x(0,0)=0$$
$$f_y=e^x\times\frac{1}{1+y}; f_y(0,0)=1$$
$$f_{xx}=e^x\log(1+y); f_{xx}(0,0)=0$$
$$f_{xy}=e^x\times\frac{1}{1+y}; f_{xy}(0,0)=1$$
$$f_{yy}=\frac{-e^x}{1+y}; f_{yy}(0,0)-1$$
$$f_{xxx}=e^x\log(1+y); f_{xxx}(0,0)=0$$
$$f_{xxy}=e^x\frac{1}{1+y}; f_{xxy}(0,0)=1$$
$$f_{xyy}=-e^x\frac{1}{(1+y)^2}; f_{xyy}(0,0)=-1$$
$$f_{yyy}=2e^x\frac{1}{(1+y)^3}; f_{yyy}(0,0)=2$$

By Taylor's theorem, we have

$$f(x,y)=f(0,0)+xf_x(0,0)+yf_y(0,0)+\frac{1}{2!}\left[x^2f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2f_{yy}(0,0)\right]+$$
$$\frac{1}{3!}\left[x^3f_{xxx}(0,0)+3x^2yf_{xxy}(0,0)+3xy^2f_{xyy}(0,0)+y^3f_{yyy}(0,0)\right]+\ldots\to(1)$$
$$=y+xy-\frac{y^2}{2}+\frac{\left(x^2y-xy^2\right)}{2}+\frac{y^3}{3}+\ldots$$

6) Expand $xy+2x-3y+2$ in powers of $x-1$ and $y+2$ using Taylor's theorem upto first degree.

Solution:

Let $f(x,y)=xy+2x-3y+2$

$$f_x = y+2; f_x(1,-2) = -2+2 = 0$$
$$f_y = x-3; f_y(1,-2) = 1-3 = -2$$

By Taylor's theorem, we have

$$f(x,y) = f(1,-2)+(x-1)f_x(1,-2)+(y+2)f_y(1,-2)+\frac{(x-1)^2}{2}f_{xx}(1,-2)+\frac{2(x-1)(y+2)}{2}f_{xy}(1,-2)$$
$$+\frac{(y+2)^2}{2}f_{yy}(1,-2)+...$$

$$=8+((x-1)\times 0+(y+2)\times -2)+......$$
$$=8-2y-4+....$$

Jacobian in two dimension (or) functional determinant in two dimension

If u,v are functions of two independent variables x and y then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called the ***Jacobian*** or ***functional determinant*** of u,v with respect to x and y and is written as $\frac{\partial(u,v)}{\partial(x,y)} or J\left(\frac{u,v}{x,y}\right)$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Properties of Jacobians

1. *If* $J = \frac{\partial(u,v)}{\partial(x,y)}$ *and* $J' = \frac{\partial(x,y)}{\partial(u,v)}$ *then* $J \bullet J' = 1$
2. If u *and* v *are functions of* r,s *and* r,s *are functions of* x,y *then* $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$
3. If u,v,w are functionaly dependent functions of three independent variables x,y,z then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$.

1) If $x = u(1+v)$ and $y = v(1+u)$ find $\frac{\partial(x,y)}{\partial(u,v)}$

Solution:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = (1+v)(1+u) - uv = 1+u+v$$

2) Find the Jacobian of $\frac{\partial(r,\theta)}{\partial(x,y)}$ **if** $x = r\cos\theta, y = r\sin\theta$

Solution:

Let $J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$ and $J' = \frac{\partial(r,\theta)}{\partial(x,y)}$

But $JJ' = 1 \Rightarrow J' = \frac{1}{r}$

3) If $u=\dfrac{y^2}{x}, v=\dfrac{x^2}{y}$ **then find** $\dfrac{\partial(x,y)}{\partial(u,v)}$

Solution:

Let $J=\dfrac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\dfrac{-y^2}{x^2} & \dfrac{2y}{x}\\ \dfrac{2x}{y} & \dfrac{-x^2}{y^2}\end{vmatrix}=1-4=-3$ and $J'=\dfrac{\partial(u,v)}{\partial(x,y)}$

But $JJ'=1 \Rightarrow J'=\dfrac{1}{J}=\dfrac{1}{-3}$

4) If $u=xy, v=x^2$ **evaluate** $\dfrac{\partial(u,v)}{\partial(x,y)}$

Solution:

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}y & x\\ 2x & 0\end{vmatrix}=-2x^2$$

5) If $x=r\cos\theta, y=r\sin\theta$, **prove that** $\dfrac{\partial r}{\partial x}=\dfrac{\partial x}{\partial r}$

Solution:

We know that,

$$x^2+y^2=r^2, \theta=\tan^{-1}\frac{y}{x}$$

Diff above equation p.w.r. to x, we get

$$2r\frac{\partial r}{\partial x}=2x \Rightarrow \frac{\partial r}{\partial x}=\frac{x}{r}=\cos\theta \rightarrow (1)$$

$$x=r\cos\theta, \frac{\partial x}{\partial r}=\cos\theta \rightarrow (2)$$

$$(1)=(2)$$

Hence proved.

6) If $u=x-y, v=y-z \;\&\; w=z-x$ **find** $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$

Solution:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}u_x & u_y & u_z\\ v_x & v_y & v_z\\ w_x & w_y & w_z\end{vmatrix}=\begin{vmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ -1 & 0 & 1\end{vmatrix}=1-1=0$$

7) If $x=r\cos\theta, y=r\sin\theta, z=z$ **then find the Jacobian of *x,y,z* interms of** r,θ,z

Solution:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)}=\begin{vmatrix}x_r & u_\theta & x_z\\ y_r & y_\theta & y_z\\ z_r & z_\theta & z_z\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}=r\cos\theta\cos\theta+r\sin\theta\sin\theta=r\left(\cos^2\theta+\sin^2\theta\right)=r$$

8) If $u=\dfrac{x+y}{1-xy}, v=\tan^{-1}x+\tan^{-1}y$ **then prove that *u* and *v* are functionally related**

Solution:

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}\frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2}\\ \frac{1}{1+x^2} & \frac{1}{1+y^2}\end{vmatrix}=\frac{1}{(1-xy)^2}-\frac{1}{(1-xy)^2}=0$$

Hence u & v are functionally related.

9) Find $\frac{\partial(u,v)}{\partial(r,\theta)}$ **if** $u=2xy, v=x^2-y^2, x=r\cos\theta.y=r\sin\theta.$

Solution

$$\frac{\partial(u,v)}{\partial(r,\theta)}=\frac{\partial(u,v)}{\partial(x,y)}\times\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}u_x & u_y\\ v_x & v_y\end{vmatrix}\times\begin{vmatrix}x_r & x_\theta\\ y_r & y_\theta\end{vmatrix}$$

$$=\begin{vmatrix}2y & 2x\\ 2x & -2y\end{vmatrix}\times\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}$$

$$=(-4y^2-4x^2)(r\cos^2\theta+r\sin^2\theta)$$

$$=-4(x^2+y^2)\cdot r$$

$$=-4r^3$$

10)If $u=x+y, y=uv$ **find the Jacobian of (** x,y**) w.r.to (**u,v**)**

Solution:

Given $u=x+y, y=uv\Rightarrow u=x+uv\Rightarrow x=u-uv$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}1-v & -u\\ v & 1+u\end{vmatrix}=(1-v)u+uv=u$$

11)If $x=r\cos\theta$, $y=r\sin\theta$, **prove that the Jacobian** $J=\frac{\partial(x,y)}{\partial(r,\theta)}=r\ and\ J'=\frac{\partial(r,\theta)}{\partial(x,y)}=\frac{1}{r}$

Solution:

$x=r\cos\theta$

$\frac{\partial x}{\partial r}=\cos\theta\ and\ \frac{\partial x}{\partial\theta}=-r\sin\theta$

$y=r\sin\theta$

$\frac{\partial y}{\partial r}=\sin\theta\ and\ \frac{\partial y}{\partial\theta}=r\cos\theta$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r(\cos^2\theta+\sin^2\theta)=r$$

To find J'

$x = r\cos\theta \quad , \; y = r\sin\theta,$

$x^2 + y^2 = r^2\left(\cos^2\theta + \sin^2\theta\right)$

$\Rightarrow x^2 + y^2 = r^2$

$and \; \frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \;\Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\therefore r^2 = x^2 + y^2 and \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$Diff.\; 2r\frac{\partial r}{\partial x} = 2x \;\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

$Similarly \; \frac{\partial r}{\partial y} = \frac{y}{r}$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = -\frac{y}{x^2+y^2} = \frac{-r\sin\theta}{r^2} = \frac{-\sin\theta}{r}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x^2}\right) = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

$$J' = \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{r} & -\frac{\sin\theta}{r} \\ \frac{y}{r} & \frac{\cos\theta}{r} \end{vmatrix} = \frac{x\cos\theta}{r^2} + \frac{y\sin\theta}{r^2}$$

$$= \frac{r\cos^2\theta}{r^2} + \frac{r\sin^2\theta}{r^2} = \frac{1}{r}\left(\cos^2\theta + \sin^2\theta\right) = \frac{1}{r}$$

Hence $J\;J' = 1$

12) If $u = \frac{yz}{x},\; v = \frac{zx}{y},\;\; w = \frac{xy}{z}$ **find the Jacobian of** $u, v, w \quad w.r.t \quad x, y, z.$

Solution:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix} = -\frac{yz}{x^2}\left(\frac{x^2}{yz} - \frac{x^2}{yz}\right) - \frac{z}{x}\left(-\frac{x}{z} - \frac{x}{z}\right) + \frac{y}{x}\left(\frac{x}{y} + \frac{x}{y}\right)$$

$$= 2 + 2 = 4$$

13) Prove $u = x + y + z,\;\; v = xy + yz + zx,\;\; w = x^2 + y^2 + z^2$ **are functionally dependent. Find the relationship between them.**

Solution:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ y+z & z+x & x+y \\ 2x & 2y & 2z \end{vmatrix}$$

$$= 1[2z(z+x) - 2y(x+y)] - 1[2z(y+z) - 2x(x+y)]$$
$$+ 1[2y(y+z) - 2x(z+x)]$$
$$= 2z^2 + 2xz - 2xy - 2y^2 - 2yz - 2z^2 + 2x^2 + 2xy$$
$$+ 2y^2 + 2yz - 2xz - 2x^2$$
$$= 0$$

$\therefore u, v \ and \ w$ are functionally dependent. The relation between them is given by the Formula

$$(x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$$
$$u^2 = w + 2v$$

Maxima and Minima for functions of Two variables

Working rule to find the maximum or minimum values of $f(x, y)$

Let $f(x, y)$ be the given function

- f_x, f_y, f_{xy}, f_{yy} & f_{xx} should exist.
- Substitute $f_x = 0$ & $f_y = 0$. Solving these equation will give the points at which maxima & minima exists. Let the points be (*a*,*b*).
- Let $r = f_{xx}, s = f_{xy}, t = f_{yy}$ & $\Delta = rt - s^2$.
- If $\Delta \succ 0$ & $r(or\, t) \prec 0$ for the solution (*a*,*b*) then *f(x,y)* has a maximum value at (a,b).
- If $\Delta \succ 0$ & $r(or\ t) \succ 0$ for the solution (*a*,*b*) then *f(x,y)* has a minimum value at (*a*,*b*).
- If $\Delta \prec 0$ for the solution (*a*,*b*) then *f(x,y)* has neither a maximum nor minimum value at *(a,b).* In this case the point (*a*,*b*) is called a saddle point of the function *f(x,y)*
- If $\Delta = 0 \ or \ \ r = 0$ then further investigations is needed.

1) Define saddle point

Solution:

Let $r = \dfrac{\partial^2 f}{\partial x^2}, s = \dfrac{\partial^2 f}{\partial x \partial y}, t = \dfrac{\partial^2 f}{\partial y^2}$

If $rt - s^2$<0 for certain point *(x,y)* then the function is neither maximum nor minimum at that point. This point is known as ***saddle point***

2) Find the maximum and minimum values of $x^2 - xy + y^2 - 2x + y$

Solution:

$$f(x, y) = x^2 - xy + y^2 - 2x + y$$
$$p = f_x = 2x - y - 2$$
$$q = f_y = -x + 2y + 1$$
$$r = f_{xx} = 2$$
$$s = f_{xy} = -1$$
$$t = f_{yy} = 2$$

At maximum and minimum point: $p = q = 0$, (1,0) may be a maximum point or minimum point .

At (1,0)**:** $\left.\begin{array}{l} rt - s^2 = 4 - 1 = 3 \\ r = 2 \end{array}\right\} >0$

$\Rightarrow$ (1,0) is the minimum point

Therefore the minimum value is 1

3) Find the maximum and minimum value of $x^2 + y^2 - xy - 2x + y$

Solution:

$$f(x,y)=x^2-xy+y^2-2x+y$$
$$f_x=2x-y-2$$
$$f_y=-x+2y+1$$
$$f_x=0 \Rightarrow 2x-y-2=0 \rightarrow (1)$$
$$f_y=0 \Rightarrow -x+2y+1=0 \rightarrow (2)$$

Solving (1) & (2) we get $x = 1$ and $y = 0$

The stationary point is (1,0).
At (1,0)

$$rt-s^2=4-1=3>0$$
$$r=2>0$$

$\therefore$ (0,1) is a minimum point.
$\therefore$ minimum value = $f(1,0)=-1$

4) Find the stationary point of $f(x,y)=x^3+y^3-12xy$
Solution:

$$f(x,y)=x^3+y^3-12xy$$

$$f_x=3x^2-12y$$
$$f_y=3y^2-12x$$
$$f_x=0 \Rightarrow 3x^2-12y=0 \rightarrow (1)$$
$$f_y=0 \Rightarrow 3y^2-12x=0 \rightarrow (2)$$

Solving (1) & (2) we get
$$y(y^3-64)=0$$
$$y=0,\ y=4$$
Sub in (2) we get $x=0, x=4$. The Stationary points are (0,0) and (4,4)

5) Find the extreme values of the function $f(x,y)=x^3+y^3-12x-3y+20$

Solution:

$$f(x,y)=x^3+y^3-12x-3y+20$$
$$p=f_x=3x^2-12;\quad q=f_y=3y^2-3;\quad r=f_{xx}=6x$$
$$s=f_{xy}=0\ ;\quad t=f_{yy}=6y$$

To find stationary points:

$$\text{p}=0=\text{q}$$
$$3x^2-12=0,\ 3y^2-3=0$$

$$x^2=4 \qquad y^2=1$$
$$x=\pm 2 \qquad y=\pm 1$$

$\therefore$ $(2,1),(2,-1),(-2,1),(-2,-1)$ are stationary points.

	(2,1)	(2,-1)	(-2,1)	(-2,-1)
r	12	12	-12	-12
s	0	0	0	0
t	6	-6	6	-6
$rt-s^2$	72	-72	-72	72

	Extreme point	Saddle point	Saddle point	Extreme point

At (2,1) r = 12 > 0 , minimum point
At (-2,-1) r = -12 < 0 maximum point

Minimum value = $f(2,1)$

$$= 2^3 + 1^3 - 12(2) - 3(1) + 20 = 2$$

Maximum value = $f(-2,-1)$

$$= (-2)^3 + (-1)^3 - 12(-2) - 3(-1) + 20 = 38$$

6) Find the maximum and minimum of $f(x,y) = \sin x + \sin y + \sin(x+y) \quad , 0 \le x, y \le \dfrac{\pi}{2}$

Solution:

p = cos x cos (x+y) q = cos y cos (x+y)

r = -sinx –sin(x+y) s = –sin(x+y) t = = -sinx –sin(x+y)

To find stationary points:

p = 0 = q

cos x cos (x+y) = 0 , cos y cos (x+y) = 0

$$\Rightarrow 2\cos\left(\frac{2x+y}{2}\right)\cos\left(\frac{y}{2}\right) = 0 \qquad 2\cos\left(\frac{x+2y}{2}\right)\cos\left(\frac{x}{2}\right) = 0$$

$$\Rightarrow \frac{2x+y}{2} = \frac{\pi}{2} \quad and \quad \frac{x+2y}{2} = \frac{\pi}{2} \;\because if\; \frac{y}{2} = \frac{\pi}{2} then\; y = \pi\; lies\, outside\;\; the\, range\, similarly\; for\, x$$

$$2x + y = \pi \quad and\; x + 2y = \pi$$

$$\Rightarrow x = \frac{\pi}{3} \quad , \quad y = \frac{\pi}{3}$$

$$\therefore \left(\frac{\pi}{3}, \frac{\pi}{3}\right) \; is \;\; a \;\; stationary\; po\text{int}$$

$$r = -\sin\frac{\pi}{3} - \sin\frac{2\pi}{3} = \frac{-\sqrt{3}}{2} - \frac{1}{2} = -\frac{\sqrt{3}+1}{2}$$

$$s = -\sin\frac{2\pi}{3} = -\frac{1}{2}$$

$$t = -\sin\frac{\pi}{3} - -\sin\frac{2\pi}{3} = -\frac{\sqrt{3}+1}{2}$$

$$rt - s^2 = \frac{\left(\sqrt{3}+1\right)^2}{4} - \frac{1}{4} > 0 \quad \therefore extreme\; po\text{int}$$

$$and\; r = -\frac{\sqrt{3}+1}{2} < 0 \quad \max imum\; po\text{int}$$

$$f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\frac{2\pi}{3} = \sqrt{3} + \frac{1}{2} \;\; is \; \max imum \;\; value$$

7)Find the maximum and minimum of $f(x,y) = x^4 + x^2 y + y^2$

Solution:

$$p = 4x^3 + 2xy \qquad q = x^2 + 2y \qquad r = 12x^2 + 2y \qquad s = 2x \qquad t = 2$$

To find stationary points:

p = 0 = q

$4x^3+2xy=0 \quad and \quad x^2+2y=0$

$2x\left(2x^2+y\right)=0 \quad x^2+2y=0$

$\Rightarrow 2x^2+y=0 \quad x^2+2y=0$

$\Rightarrow 2x^2=-y \qquad x^2=-2y$

$\Rightarrow 2(-2y)=-y$

$\Rightarrow -4y=-y \Rightarrow -4y+y=0 \Rightarrow -3y=0 \Rightarrow y=0$

$\therefore x=0$

$\therefore$ (0,0) is a stationary point.

r = 0 , s = 0 , t = 2

$\therefore$ rt - $s^2=0$

further investigation is necessary.

8) Examine the function $f(x,y)=x^3y^2(12-x-y)$ for extreme values.

Solution:

$f(x,y)=12x^3y^2-x^4y^2-x^3y^3$

$f_x=36x^2y^2-4x^3y^2-3x^2y^3$

$f_y=24x^3y-2x^4y-3x^3y^2$

$f_{xx}=72xy^2-12x^2y^2-6xy^3$

$f_{xy}=72x^3y-8x^2y-9x^2y^2$

$f_{yy}=24x^3-2x^4-6x^3$

The stationary points are given by

$f_x=0 \ ; f_y=0$

$x^2y^2(36-4x-3y)=0 \qquad ---------(1)$

$x^3y(24-2x-3y)=0 \qquad --------(2)$

$(1) \Rightarrow \ 4x+3y=36 \qquad ---------(3)$

$(2) \Rightarrow \ 2x+3y=24 \qquad --------(4)$

Solve (3) and (4), the stationary points are (0,0),(0,8),(0,12),(12,0),(9,0) and (6,4).

For the first five points, $rt-s^2=0$

Further investigation is required

At (6,4)

$r=-2304, \quad s=-1728, \quad t=-2592$

$\therefore rt-s^2>0 \ and \ r>0$

$\therefore f$ has a maximum at (6,4)

Maximum value of $f(x,y)=6912$.

Lagrange's Method of undetermined multipliers

Suppose we require to find the maximum and minimum values $f(x,y,z)$ where x,y,z Are subject to a constraint equation $\varphi(x,y,z)=0$.

We define a function $L=f+\lambda\varphi$ where λ is called Lagrange multipliers which is independent of x,y,z.

The stationary points of L are given by $L_x=0, L_y=0, L_z=0, L_\lambda=0$

1)Find the minimum value of $x^2+y^2+z^2$ *, subject to* $ax+by+cz=p$

Solution:

$L(x,y,z)=x^2+y^2+z^2+\lambda(ax+by+cz-p)$

Differentiate w.r.t.x,y,z& λ partially

$L_x=2x+\lambda y=0-----(1)$

$L_y=2y+\lambda b=0-----(2)$

$L_z=2z+\lambda c=0-----(3)$

$L_\lambda=ax+by+cz-p=0-----(4)$

(1) b-(2) a $\Rightarrow 2bx+\lambda by-2ay-\lambda ab=0$

$$\Rightarrow bx = ay \Rightarrow \frac{x}{y} = \frac{a}{b}$$

(2)c-(3)b $\Rightarrow 2cyx + \lambda bc - 2bz - \lambda bc = 0$

$$\Rightarrow cy = bz \Rightarrow \frac{y}{z} = \frac{b}{c}$$

$$x = \alpha(assume),\ y = \frac{b}{a}x = \frac{b}{a}\alpha$$

$$z = \frac{c}{b}y = \frac{c}{b}\frac{b}{a}\alpha \Rightarrow z = \frac{c}{a}\alpha$$

substitute in (4)

$$a\alpha + \frac{b^2}{a}\alpha + \frac{c^2}{a}\alpha - p = 0$$

$$\left(a^2 + b^2 + c^2\right)\alpha - ap = 0$$

$$\alpha = \frac{ap}{a^2 + b^2 + c^2}$$

$$x = \frac{ap}{a^2 + b^2 + c^2},\ y = \frac{b}{a} \times \frac{ap}{a^2 + b^2 + c^2} = \alpha = \frac{bp}{a^2 + b^2 + c^2}$$

$$z = \frac{c}{a}\alpha = \alpha = \frac{cp}{a^2 + b^2 + c^2}$$

and minimum value $f(x, y, z) = x^2 + y^2 + z^2$

$$= \frac{a^2 p^2}{\left(a^2 + b^2 + c^2\right)^2} + \frac{b^2 p^2}{\left(a^2 + b^2 + c^2\right)^2} + \frac{c^2 p^2}{\left(a^2 + b^2 + c^2\right)^2} = \frac{p^2}{\left(a^2 + b^2 + c^2\right)}$$

2)The temperature at any point (x,y,z)in space is given by $T = kxyz^2$, where k is a constant . Find the highest temperature on the surface of the sphere $x^2 + y^2 + z^2 = a^2$

Solution:

$$T = kxyz^2 \qquad \phi: x^2 + y^2 + z^2 - a^2 = 0$$

$$L(x, y, z) = kxyz^2 + \lambda\left(x^2 + y^2 + z^2 - a^2\right)$$

Differentiate w.r.t. x, y, z & λ partially

$$L_x = kyz^2 + \lambda 2x = 0 \quad ------------(1)$$

$$L_y = kxz^2 + \lambda 2y = 0 \quad ------------(2)$$

$$L_z = k2xyz + \lambda 2z = 0 \quad ------------(3)$$

$$L_\lambda = x^2 + y^2 + z^2 - a^2 = 0 \quad ------------(4)$$

$$(1)y - (2)x \Rightarrow kz^2\left(y^2 - x^2\right) = 0 \Rightarrow y^2 - x^2 = 0$$

$$\Rightarrow y = \pm x$$

$$(1)z - (3)x \Rightarrow kyz\ \left(z^2 - 2x^2\right) = 0 \Rightarrow z^2 - 2x^2 = 0$$

$$\Rightarrow z = \pm\sqrt{2}x$$

Assume $x = \alpha$, $y = \pm\alpha \quad z = \pm\sqrt{2}\alpha$

$$\alpha^2 + \alpha^2 + 2\alpha^2 = a^2$$

Substitute in (4), $4\alpha^2 = a^2 \Rightarrow \alpha = \pm\frac{a}{2}$

$$\therefore x = \frac{a}{2} \quad y = \frac{a}{2} \quad z = \frac{\sqrt{2}a}{2} = \frac{a}{\sqrt{2}}$$

$\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{\sqrt{2}}\right)$ *is the point*

$$\therefore Max \ \ temp = kxyz^2 = \frac{ka^4}{8}$$

3)Show that of all rectangular parallelepiped with given surface area ,cube has the greatest volume.

Solution:

Assume surface area = S = $2(xy+yz+zx)$

Max $V = xyz$ subject to $2(xy+yz+zx) = S$

$$L(x,y,z) = xyz + \lambda(2xy + 2yz + 2zx - S)$$

$$L_x = yz + \lambda(2y + 2z) = 0 ------(1)$$
$$L_y = xz + \lambda(2x + 2z) = 0 ------(2)$$
$$L_z = xy + \lambda(2x + 2y) = 0 ------(3)$$
$$L_\lambda = 2xy + 2xz + 2yz - S = 0 ------(4)$$

(1)x-(2) y $\Rightarrow \lambda 2z(x-y) = 0 \Rightarrow x = y$

similarly (1)z-(2) x $\Rightarrow x = z$

$$\Rightarrow x = y = z \ \ is \ a \ \ cube$$

4) Find the volume of the largest rectangular parallelepiped that can be inscribed in the Ellipsoid.

Solution:

The given ellipsoid is

$$\phi(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

The volume of the parallelepiped is $f(x,y,z) = 8xyz$

$$L(x,y,z) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$$L_x = 8yz + \lambda\left(\frac{2x}{a^2}\right) = 0 ------(1)$$

$$L_y = 8xz + \lambda\left(\frac{2y}{b^2}\right) = 0 ------(2)$$

$$L_z = 8xy + \lambda\left(\frac{2z}{c^2}\right) = 0 ------(3)$$

$$L_\lambda = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 ------(4)$$

Solve the equations

$(1)x + (2)y + (3)z \Rightarrow$

$$24xyz + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0$$

$$\lambda = -12xyz$$

$$put \ \ in \ \ (1) \Rightarrow \ 8yz - 12xyz\left(\frac{2x}{a^2}\right) = 0$$

$$8yz\left(1 - \frac{3x^2}{a^2}\right) = 0$$

$$\frac{3x^2}{a^2} = 1 \Rightarrow x = \frac{a}{\sqrt{3}}$$

$$similarly, \qquad y = \frac{b}{\sqrt{3}}, \ \ z = \frac{c}{\sqrt{3}}$$

$$\therefore Max \ \ Volume = \frac{8abc}{3\sqrt{3}}$$

MOHAMED SATHAK A.J.COLLEGE OF ENGINEERING

UNIT-IV INTEGRAL CALCULUS

1. Define definite integral.

The definite integral is $\int_a^b f(x)dx = \lim_{x\to 0} \sum_{i=1}^{n} f(x_i)\Delta x$.

2. write down the midpoint rule in definite integral.

$$\int_a^b f(x)dx = \sum_{i=1}^{n} f(\overline{x_i})\Delta x = \Delta x\left[f(\overline{x_1}) + f(\overline{x_2}) + \cdots + f(\overline{x_n})\right]$$

where $\Delta x = \frac{b-a}{n}$ and $\overline{x_i} = \frac{1}{2}(x_{i-1}+x_i)$ = midpoint of $[x_{i-1}, x_i]$.

3. write down the properties of definite integrals.

(1) $\int_a^b f(x)dx = -\int_b^a f(x)dx$ (2) $\int_a^b c\,dx = c\int_a^b dx = c(b-a)$, c is constant

(3) $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

(4) $\int_a^b c\, f(x)dx = c\int_a^b f(x)dx$ where c is a constant.

(5) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

(6) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

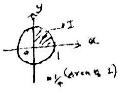
4. Evaluate the integral $\int_0^1 \sqrt{1-x^2}\,dx$ by intrepreting each interms of area.

Soln: Let $f(x) = \sqrt{1-x^2}$ $\Rightarrow y = \sqrt{1-x^2}$ $\Rightarrow y^2 = (\sqrt{1-x^2})^2$

$\Rightarrow x^2+y^2 = 1$

which is a circle with radius 1.

$\therefore \int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(1)^2 = \frac{\pi}{4}$



5. State the fundamental theorem of calculus part-I

If f is continuous on [a,b], then the function g is defined by $g(x) = \int_a^x f(t)dt$; $a \le x \le b$ is continuous on [a,b] and differentiable on (a,b), and $g'(x) = f(x)$.

6. State the fundamental theorem of calculus part-II

If f is continuous on [a,b], then $\int_a^b f(x)dx = F(b) - F(a)$ where F is any anti-derivative of f, ie, $F' = f$.

7. state the fundamental theorem of calculus.

suppose f is continuous on [a,b]

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$.
2. $\int_a^b f(x)dx = F(b) - F(a)$, where F is any anti-derivative of f. ie, $F' = f$.

8. Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\,dt$.

Here $f(t) = \sqrt{1+t^2}$ is continuous,

$g(x) = \int_0^x \sqrt{1+t^2}\,dt$

$\boxed{g'(x) = \sqrt{1+x^2}}$

9. what is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2}dx = \left[\frac{x^{-1}}{-1}\right]_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}$$

Here $f(x) = \frac{1}{x^2}$ is not continuous on [-1,3]

f(x) has infinite discontinuity at x=0

$\therefore \int_{-1}^{3} \frac{1}{x^2}dx$ does not exists.

10. Prove that following integral by interpreting each in terms of area

$$\int_a^b x\,dx = \frac{b^2-a^2}{2}$$

(TU 2016)

soln:

Given $\int_a^b x\,dx$, Here $f(x) = x$;

$\int f(x)\,dx = \frac{x^2}{2}$.

$\therefore$ Area $= F(b) - F(a) = \frac{b^2}{2} - \frac{a^2}{2} = \frac{b^2-a^2}{2}$ by FTC2

11. Evaluate $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$

soln:

Let $I = \int \frac{\cos\theta}{\sin^2\theta}\,d\theta$

$= \int \frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\sin\theta}\,d\theta$

$= \int \cot\theta \,\mathrm{cosec}\,\theta\,d\theta$

$= -\mathrm{cosec}\,\theta + c$

12. Evaluate $\int \tanh x\,dx$

soln:

put $y = \cosh x$

$dy = \sinh x$

$\int \tanh x\,dx = \int \frac{\sinh x}{\cosh x}\,dx$

$= \int \frac{dy}{y} = \log y + c$

$= \log(\cosh x) + c$ //

13. Evaluate $\int \tan^3 x\,dx$.

soln:

$\int \tan^3 x\,dx = \int \tan^2 x \cdot \tan x\,dx$

$= \int (\sec^2 x - 1)\tan x\,dx$

$= \int \sec^2 x \tan x\,dx - \int \tan x\,dx$

$= \frac{\tan^2 x}{2} - \log \sec x + c$ //

14. Evaluate $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$

soln:

Let $x = a\sin\theta \Rightarrow \sin\theta = \frac{x}{a} \Rightarrow \theta = \sin^{-1}(x/a)$

$dx = a\cos\theta\,d\theta$

$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta}{\sqrt{a^2-a^2\sin^2\theta}}\,d\theta = \int \frac{\cancel{a}\cos\theta}{\cancel{a}\sqrt{\cos^2\theta}}\,d\theta$

$= \int \frac{\cos\theta}{\cos\theta}\,d\theta$

$= \int d\theta$

$= \theta + c$

$= \sin^{-1}(x/a) + c$ //

15. Evaluate $\int \frac{1}{\sqrt{x^2-a^2}}\,dx$

soln:

Let $x = a\cosh\theta \Rightarrow \cosh\theta = x/a \Rightarrow \theta = \cosh^{-1}(x/a)$

$dx = a\sinh\theta\,d\theta$

$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{1}{\sqrt{a^2\cosh^2\theta - a^2}}(a\sinh\theta)\,d\theta$

$= \int \frac{\cancel{a}\cancel{\sinh\theta}}{\cancel{a}\cancel{\sinh\theta}}\,d\theta = \int d\theta = \theta = \cosh^{-1}(x/a) + c.$

(or) $\log(x + \sqrt{x^2-a^2}) + c$

16. Evaluate $\int \sqrt{1+\sin 2x}\,dx$

soln:

$\int \sqrt{1+\sin 2x}\,dx = \int \sqrt{\sin^2 x + \cos^2 x + 2\sin x\cos x}\,dx$

$= \int \sqrt{(\sin x + \cos x)^2}\,dx$

$= \int (\sin x + \cos x)\,dx$

$= -\cos x + \sin x + c$ //

17. Evaluate $\int_1^{\infty} \frac{1}{x} dx$ and determine whether the integral is convergent or divergent

Soln:

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{t\to\infty} \int_1^t \frac{1}{x} dx = \lim_{t\to\infty} [\log x]_1^{\infty}$$

$$= \lim_{t\to\infty} (\log t - \cancel{\log 1}^{\,0}) = \lim_{t\to\infty} t = \log \infty = \infty$$

$$\Rightarrow \int_1^{\infty} \frac{1}{x} dx \text{ is divergent.}$$

18. Evaluate $\int_4^{\infty} \frac{1}{\sqrt{x}} dx$

Soln:

$$\int_4^{\infty} \frac{1}{\sqrt{x}} dx = \lim_{t\to\infty} \int_4^t \frac{1}{\sqrt{x}} dx$$

$$= \lim_{t\to\infty} \left[2\sqrt{x}\right]_4^t$$

$$= \lim_{t\to\infty} (2\sqrt{t} - 2\sqrt{4})$$

$$= \infty.$$

$$\Rightarrow \int_4^{\infty} \frac{1}{\sqrt{x}} dx \text{ is divergent.}$$

19. Evaluate $\int_1^2 \frac{dx}{1-x} dx$

$$\int_1^2 \frac{dx}{1-x} = \left[\frac{\log(1-x)}{-1}\right]_1^2 = -\left[\log(1-x)\right]_1^2$$

$$= -\left[\log 0 - \log(-1)\right]$$

$$= -\infty$$

$$\Rightarrow \int_1^2 \frac{dx}{1-x} \text{ is divergent.}$$

20. Evaluate $\int_{-1}^1 \frac{dx}{x}$

$$\int_{-1}^1 \frac{dx}{x} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x} = \lim_{t\to 0^+} \log x = \infty$$

$$\Rightarrow \int_{-1}^1 \frac{dx}{x} \text{ is divergent}$$

21) Evaluate $\int_1^{\infty} \frac{1}{e^x + x^2} dx$

Soln:

Here $\frac{1}{x^2} > \frac{1}{e^x + x^2}$ and $\int_1^{\infty} \frac{dx}{x^2}$ is convergent by P-Test

$$\Rightarrow \int_1^{\infty} \frac{dx}{e^x + x^2} \text{ is convergent.}$$

22. Evaluate $\int_0^1 \tan^{-1} x\, dx$ (Jan 2010)

$$\int_0^1 \tan^{-1} x\, dx = \left[\tan^{-1} x \cdot x\right]_0^1 - \int_0^1 x\, d(\tan^{-1} x)$$

$$= \pi/4 - 1/2 \int_0^1 \frac{2x\,dx}{1+x^2}$$

$$= \pi/4 - \frac{1}{2}\left[\log(1+x^2)\right]_0^1$$

$$= \pi/4 - 1/2 \log 2.$$

23. Evaluate $\int \frac{x^3}{\sqrt{4+x^2}} dx$.

Let $t = x^2 + 4 \Rightarrow x^2 = t - 4$

$dt = 2x\,dx$.

$$\therefore \int \frac{x^3}{\sqrt{4+x^2}} = \int \frac{x^2}{\sqrt{x^2+4}} x\,dx$$

$$= \int \frac{t-4}{\sqrt{t}} \frac{dt}{2}$$

$$= \int (t^{1/2} - 4t^{-1/2}) \frac{dt}{2}$$

$$= \frac{1}{2}\left[\frac{t^{3/2}}{3/2} - 4\frac{t^{1/2}}{1/2}\right]$$

$$= \frac{t^{3/2}}{3} - 4t^{1/2} + C.$$

$$= 1/3 (x^2+4)^{3/2} - 4\sqrt{x^2+4} + C$$

24. Evaluate $\int (\log x)^2 dx$

Soln:

Let $u = (\log x)^2$, $dv = dx$

$du = 2\log x \cdot \frac{1}{x} dx$ $\quad v = x$.

$$I = x(\log x)^2 - \int \cancel{x} \cdot 2\log x \cdot \frac{1}{\cancel{x}} dx = x(\log x)^2 - 2(x\log x - x) + C$$

25. Evaluate $\int \frac{x}{1+\cos x}\,dx$.

soln:

$$\int \frac{x}{1+\cos x}\,dx = \int \frac{x}{2\cos^2 x/2}\,dx \qquad \because 1+\cos x = 2\cos^2 x/2$$

$$= \frac{1}{2}\int x\sec^2 x/2\,dx \quad \text{--- (1)}$$

Let $u = x$, $du = dx$

$dv = \sec^2 x/2\,dx$

$v = \int \sec^2 x/2\,dx$

$= \frac{\tan x/2}{1/2}$

$v = 2\tan x/2$

$$\int u\,dv = uv - \int v\,du$$

$$\int x\sec^2 x/2\,dx = x\cdot 2\tan x/2 - \int 2\tan x/2\,dx.$$

$$= 2x\tan x/2 - \frac{\log\sec(x/2)}{1/2} + C$$

$$= 2x\tan x/2 - 2\log(\sec x/2) + C.$$

$$(1) \Rightarrow \int \frac{x}{1+\cos x}\,dx = \frac{1}{2}\, 2\left(x\tan x/2 - \log(\sec x/2)\right) + C$$

$$= x\tan x/2 - \log\sec(x/2) + C \;//$$

26. Evaluate $\int \frac{1}{3+2x+x^2}\,dx$.

soln:

$$\int \frac{1}{x^2+2x+3}\,dx = \int \frac{dx}{x^2+2x+3}$$

$2ab = 2x$, $a = x$

$\cancel{2x}b = \cancel{2x}$

$b = 1$

$$= \int \frac{dx}{x^2+2x+1-1+3}$$

$$= \int \frac{dx}{(x+1)^2+2} = \int \frac{dx}{(x+1)^2+(\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C \qquad \because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

29). Determine whether the integral $\int_2^3 \frac{1}{\sqrt{3-x}}\,dx$ is cgt or dgt

soln.

Here, infinite discontinuity occur at $x=3$

$$\int_2^3 \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3}\int_2^t \frac{1}{\sqrt{3-x}}\,dx$$

$$= \lim_{t\to 3}\int_2^t (3-x)^{-1/2}\,dx$$

$$= \lim_{t\to 3}\left[\frac{(3-x)^{-1/2+1}}{-(-1/2+1)}\right]_2^t$$

$$= \lim_{t\to 3}\left[\frac{(3-x)^{1/2}}{-1/2}\right]_2^t$$

$$= \lim_{t\to 3} -2\left[(3-x)^{1/2}\right]_2^t$$

$$= -2\lim_{t\to 3}\left[(3-t)^{1/2} - (3-2)^{1/2}\right]$$

$$= -2\lim_{t\to 3}\left[\sqrt{3-t} - 1\right]$$

$$= -2\,(0-1)$$

$$= 2 \text{ (finite)}$$

$\int_2^3 \frac{1}{\sqrt{3-x}}\,dx$ is convergent.

30) Evaluate $\int \frac{1}{1+\sin x}\,dx$

soln:

$$\int \frac{1}{1+\sin x}\,dx = \int \frac{1}{1+\sin x}\times\frac{1-\sin x}{1-\sin x}\,dx$$

$$= \int \frac{1-\sin x}{1-\sin^2 x}\,dx$$

$$= \int \frac{1-\sin x}{\cos^2 x}\,dx = \int \frac{1}{\cos^2 x}\,dx - \int \frac{\sin x}{\cos^2 x}\,dx$$

$$= \int \sec^2 x\,dx - \int \sec x\tan x\,dx$$

$$= \tan x - \sec x + C \;//$$

PART-B

1. Evaluate $\int_0^3 (x^2-2x)dx$ by using Riemann sum by taking Right end points at the sample points (N/D 2015)

Sol:

Riemann sum is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

Given $\int_0^3 (x^2-2x)\,dx$, $\Delta x = \frac{3-0}{n} = \frac{3}{n}$

$x_0 = 0,\ x_1 = \frac{3}{n},\ x_2 = \frac{3}{n}(2), \ldots\ x_i = \frac{3}{n}i$

$$\int_0^3 (x^2-2x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right) \qquad \because f(x) = x^2-2x$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^2 - 2\left(\frac{3i}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \left\{ \frac{9}{n^2}\sum_{i=1}^{n} i^2 - \frac{6}{n}\sum_{i=1}^{n} i \right\}$$

$$= \lim_{n\to\infty} \frac{27}{n^3} \sum_{i=1}^{n} i^2 - \lim_{n\to\infty} \frac{18}{n^2} \sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \frac{\cancel{27}^{\,9}}{n^3} \frac{n(n+1)(2n+1)}{\cancel{6}_{\,2}} - \lim_{n\to\infty} \frac{\cancel{18}^{\,9}}{n^2} \frac{n(n+1)}{\cancel{2}}$$

$$= \frac{9}{2} \lim_{n\to\infty} \frac{1}{n^3}\, n \cdot n\left(1+\frac{1}{n}\right) n\left(2+\frac{1}{n}\right) - 9 \lim_{n\to\infty} \frac{1}{\cancel{n^2}}\, \cancel{n} \cdot n\left(1+\frac{1}{n}\right)$$

$= \frac{9}{2}\, 2 - 9$

$= 9 - 9$

$= 0$ //

2) a) Evaluate the Riemann sum for $f(x) = x^3 - 6x$ taking the sample points to be right end points and $a=0$ $b=3$ and $n=6$

(b) Evaluate $\int_0^3 (x^3-6x)dx$.

Sol:-

(a) Here $n=6$, $\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2} = 0.5$

$x_0 = 0,\ x_1 = 0.5,\ x_2 = 1,\ x_3 = 1.5,\ x_4 = 2,\ x_5 = 2.5\ \&\ x_6 = 3.$

The Riemann sum is

$$R_6 = \sum_{i=1}^{6} f(x_i)\,\Delta x$$

$$= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x$$

$$= [f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3)]\,\Delta x$$

$$= [-2.875 - 5 - 5.625 - 4 + 0.625 + 9]\,\frac{1}{2}$$

$$= -3.9375.$$

(b) Here $\Delta x = \frac{3-0}{n} = \frac{3}{n}$

$\Rightarrow x_0 = 0,\ x_1 = \frac{3}{n},\ x_2 = \frac{6}{n},\ x_3 = \frac{9}{n} \ldots\ x_i = \frac{3i}{n}.$

$$\therefore \int_0^3 (x^3-6x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{27}{n^3} i^3 - \frac{18}{n} i\right]$$

$$= \lim_{n\to\infty} \frac{81}{n^4} \sum_{i=1}^{n} i^3 - \lim_{n\to\infty} \frac{54}{n^2} \sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \frac{81}{n^4} \left[\frac{n(n+1)}{2}\right]^2 - \lim_{n\to\infty} \frac{54}{n^2} \frac{n(n+1)}{\cancel{2}}$$

$$= \lim_{n\to\infty} \frac{81}{\cancel{n^4}} \frac{\cancel{n^4}\left(1+\frac{1}{n}\right)^2}{4} - \lim_{n\to\infty} \frac{27}{\cancel{n^2}} \cancel{n^2}\left(1+\frac{1}{n}\right)$$

$= \frac{81}{4} - 27$

$= \frac{-27}{4}$

$= -6.75$ //

3. Evaluate $\int e^{\tan^{-1}x}\left[\frac{1+x+x^2}{1+x^2}\right]dx$ (Jan 2016)

Soln:-

put $u = \tan^{-1}x \Rightarrow \tan u = x$

$du = \frac{1}{1+x^2}dx$

Let $I = \int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx = \int e^u\left(1+\tan u+\tan^2 u\right)\frac{dx}{1+x^2}$

$= \int e^u(\tan u+\sec^2 u)\,du$

put $t = e^u \tan u$

$dt = [e^u \sec^2 u + e^u \tan u]\,du$

$\therefore \quad I = \int dt$

$= t + c$

$= e^u \tan u + c$

$= e^{\tan^{-1}x}\cdot x + c$

$= x e^{\tan^{-1}x} + c$ //

Integration by parts:

1. Evaluate $\int (\log x)^2 dx$

Soln:-

Let $u = (\log x)^2$ $\quad dv = dx$

$du = 2\log x\,\frac{1}{x}dx$ $\quad v = x.$

$\int u\,dv = uv - \int v\,du$

$\int (\log x)^2 dx = (\log x)^2 x - \int x\cdot 2\log x\cdot\frac{1}{x}dx.$

$= (\log x)^2 x - 2\int \log x\,dx$ —— ①

Find $\int \log x\,dx.$

put $u = \log x$ $\quad dv = dx$

$du = \frac{1}{x}dx$ $\quad v = x$

$\therefore \int u\,dv = uv - \int v\,du$

$\int \log x\,dx = \log x\cdot x - \int x\cdot\frac{1}{x}dx$

$= x\log x - \int dx$

$= x\log x - x$

$\therefore ① \Rightarrow \int (\log x)^2 dx = x(\log x)^2 - 2[x\log x - x] + c$ //

2. Evaluate $\int \frac{x}{1+\cos x}dx$

Soln:

$\int \frac{x}{1+\cos x}dx = \int \frac{x}{2\cos^2 x/2}dx$ $\quad \because 1+\cos x = 2\cos^2 x/2$

$= \frac{1}{2}\int x\sec^2 x/2\,dx$ —①

put $u = x$ $\quad dv = \sec^2 x/2\,dx$

$du = dx$ $\quad v = \int \sec^2 x/2\,dx = \frac{\tan x/2}{1/2} = 2\tan x/2$

$\therefore \int u\,dv = uv - \int v\,du$

$\int \frac{x}{1+\cos x}dx = \frac{1}{2}\left[x\cdot 2\tan\frac{x}{2} - \int 2\tan x/2\,dx\right]$

$= x\tan(x/2) - \frac{\log\sec(x/2)}{1/2} + c$

$= x\tan x/2 - 2\log[\sec(x/2)] + c$ //

3. Evaluate $\int \frac{x}{1+\sin x}\,dx$

Soln:

$$\int \frac{x}{1+\sin x}\,dx = \int \frac{x(1-\sin x)}{(1+\sin x)(1-\sin x)}\,dx$$

$$= \int \frac{x - x\sin x}{1-\sin^2 x}\,dx$$

$$= \int \frac{x - x\sin x}{\cos^2 x}\,dx$$

$$= \int \frac{x}{\cos^2 x}\,dx - \int \frac{x\sin x}{\cos^2 x}\,dx$$

$$= \int x\sec^2 x\,dx - \int x\sec x\tan x\,dx.$$

$$= I_1 - I_2 \quad \text{—— (1)}$$

$I_1 = \int x\sec^2 x\,dx$

$u = x \quad | \quad dv = \sec^2 x\,dx$

$du = dx \quad | \quad v = \int \sec^2 x\,dx = \tan x$

$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx$

$= x\tan x - \log\sec x$

$I_2 = \int x\sec x\tan x\,dx.$

$u = x \quad dv = \sec x\tan x\,dx$

$du = dx \quad v = \int \sec x\tan x\,dx = \sec x$

$\therefore \int x\sec x\tan x\,dx = x\sec x - \int \sec x\,dx$

$= x\sec x - \log(\sec x + \tan x)$

$$\therefore (1) \Rightarrow \int \frac{x}{1+\sin x}\,dx = x\tan x - \log\sec x - x\sec x + \log(\sec x + \tan x) + C.$$

4. Evaluate $\int \tan^{-1}x\,dx$. Also find $\int_0^1 \tan^{-1}x\,dx$ (N/o 2010)

Let $u = \tan^{-1}x \quad | \quad dv = dx$

$du = \frac{1}{1+x^2}\,dx \quad | \quad v = x.$

$\int u\,dv = uv - \int v\,du$

$$\Rightarrow \int \tan^{-1}x\,dx = \tan^{-1}x \cdot x - \int x \cdot \frac{1}{1+x^2}\,dx.$$

$$= x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx \quad \text{—— (1)}$$

Evaluate $\int \frac{x}{1+x^2}\,dx.$

put $t = 1+x^2$

$dt = 2x\,dx$

$x\,dx = \frac{dt}{2}$

$$\therefore \int \frac{x}{1+x^2}\,dx = \int \frac{\frac{dt}{2}}{t}$$

$$= \frac{1}{2}\int \frac{dt}{t}$$

$$= \frac{1}{2}\log t + c$$

$$= \frac{1}{2}\log(1+x^2) + c.$$

$$\therefore (1) \Rightarrow \int \tan^{-1}x\,dx = x\tan^{-1}x - \frac{1}{2}\log(1+x^2) + c.$$

Now,

$$\int_0^1 \tan^{-1}x\,dx = \left[x\tan^{-1}x\right]_0^1 - \frac{1}{2}\left[\log(1+x^2)\right]_0^1 + c$$

$$= (\tan^{-1}(1) - 0) - \frac{1}{2}(\log 2 - \log 1)$$

$$= 1 \cdot \pi/4 - \tfrac{1}{2}\log 2$$

$$= \pi/4 - \tfrac{1}{2}\log 2.$$

Assignment:

5. Evaluate $\int_0^{1/2} \cos^{-1}x\,dx$

6. Evaluate $\int \sin^{-1}x\,dx$

Reduction formula:-

1) Find the reduction formula for $\int_0^{\pi/2} \sin^n x\,dx$ (N/D 2016)

Soln:

Let $I_n = \int \sin^n(x)\,dx$ —— (1)

$= \int \sin^{n-1}x \sin x\,dx$

$u = \sin^{n-1}x$ $\quad$ $dv = \sin x\,dx$

$\quad$ $v = \int \sin x\,dx$

$du = (n-1)\sin^{n-2}x \cos x\,dx$ $\quad$ $v = -\cos x$

$\therefore \int u\,dv = uv - \int v\,du$

$I_n = \sin^{n-1}x(-\cos x) - \int -\cos x\,(n-1)\sin^{n-2}x \cos x\,dx$

$= -\cos x \sin^{n-1}x + \int (n-1)\sin^{n-2}x \cos^2 x\,dx$

$= -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,(1-\sin^2 x)\,dx$

$= -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^{n-2}x \sin^2 x\,dx$

$= -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$

$= -\cos x \sin^{n-1}x + (n-1)I_{n-2} - (n-1)I_n$

$I_n + (n-1)I_n = -\cos x \sin^{n-1}x + (n-1)I_{n-2}$

$\cancel{I_n} + nI_n - \cancel{I_n} = -\cos x \sin^{n-1}x + (n-1)I_{n-2}$

$nI_n = -\cos x \sin^{n-1}x + (n-1)I_{n-2}$

$I_n = -\frac{1}{n}\cos x \sin^{n-1}x + \frac{n-1}{n}I_{n-2}$ —— (2)

ie, $\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$ —— (3)

To find $\int_0^{\pi/2} \sin^n x\,dx$

(3) $\Rightarrow$ $\int_0^{\pi/2} \sin^n x\,dx = \left[-\frac{1}{n}\cos x \sin^{n-1}x\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}x\,dx$

$I_n = (0-0) + \frac{n-1}{n}I_{n-2}$

$I_n = \frac{n-1}{n}I_{n-2}$ —— (3')

$I_{n-2} = \frac{n-3}{n-2}I_{n-4}$

$I_{n-4} = \frac{n-5}{n-4}I_{n-6}$

(3') $\Rightarrow$ $I_n = \frac{n-1}{n}\,\frac{n-3}{n-2}\,I_{n-4}$

$= \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\,I_{n-6}$

$$I_n = \begin{cases} \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\frac{1}{2}I_0 & \text{if } n \text{ is even} \\ \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\frac{2}{3}I_1 & \text{if } n \text{ is odd} \end{cases} \quad \text{—— (4)}$$

put $n=0$ in (1)

$I_0 = \int_0^{\pi/2} \sin^0 x\,dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}$

put $n=1$ in (1),

$I_1 = \int_0^{\pi/2} \sin x\,dx = [-\cos x]_0^{\pi/2} = -[\cos\frac{\pi}{2} - \cos 0] = -[0-1] = 1$

(4) $\Rightarrow$

$$\therefore I_n = \begin{cases} \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\frac{1}{2}\,\frac{\pi}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\frac{2}{3}\cdot 1 & \text{if } n \text{ is odd} \end{cases}$$

2. Evaluate $\int \sin^n x\,dx$.

write upto equation (3) in previous problem.

3. Find the reduction formula for $\int_0^{\pi/2} \cos^n x\,dx$

Soln:

Let $I_n = \int \cos^n x\,dx$ ——— (1)

$= \int \cos^{n-1} x \cos x\,dx$

$u = \cos^{n-1} x$	$dv = \cos x$
	$v = \int \cos x\,dx$
$du = (n-1)\cos^{n-2} x\,(-\sin x)\,dx$	$v = \sin x$

$\int u\,dv = uv - \int v\,du$

$I_n = \cos^{n-1} x \sin x - \int \sin x\,(n-1)\cos^{n-2} x\,(-\sin x)\,dx$

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\,dx$

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,(1-\cos^2 x)\,dx$

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^{n-2} x \cos^2 x\,dx$

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$

$I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$.

$\Rightarrow I_n + (n-1) I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$

$\cancel{I_n} + n I_n - \cancel{I_n} = \cos^{n-1} x \sin x + (n-1) I_{n-2}$

$n I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$

$I_n = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}$ ——— (2)

$\Rightarrow \int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$ ——— (3)

where $I_0 = \int dx = x + c$ [put $n=0$ in (1)]

$I_1 = \int \cos x\,dx = \sin x + c$ [put $n=1$ in (1)]

To find $\int_0^{\pi/2} \cos^n x\,dx$:

(2) $\Rightarrow$

$I_n = \left[\frac{1}{n}\cos^{n-1} x \sin x\right]_0^{\pi/2} + \frac{n-1}{n} I_{n-2}$

$I_n = (0-0) + \frac{n-1}{n} I_{n-2}$

$\Rightarrow I_n = \frac{n-1}{n} I_{n-2}$

$= \frac{n-1}{n}\,\frac{n-3}{n-2}\, I_{n-4}$ $\quad \| \; I_{n-2} = \frac{n-3}{n-2} I_{n-4}$

$= \frac{n-1}{n}\,\frac{n-5}{n-4}\, I_{n-6}$

$\vdots$

$$= \begin{cases} \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\frac{1}{2} I_0 & \text{if } n \text{ is even} \\ \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\frac{2}{3} I_1 & \text{if } n \text{ is odd.} \end{cases} \quad \text{——— (4)}$$

put $n=0$ in (1) $\quad I_0 = \int_0^{\pi/2} \cos^0 x\,dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \pi/2$

put $n=1$ in (1)

$I_1 = \int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1$

$$\therefore (4) \Rightarrow \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\frac{1}{2}\,\frac{\pi}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\frac{1}{2}\cdot 1 & \text{if } n \text{ is odd.} \end{cases}$$

4. Evaluate $\int_0 \cos^n x\,dx$ (or) Find the reduction formula for $\int \cos^n x\,dx$

Soln:

writ upto equation (3) in problem (3).

5. Find the reduction formula for $\int \sec^n x\,dx$, $n \geq 2$ is an integer.

<u>Soln:</u>

Let $I_n = \int \sec^n x\,dx$ ——— ①

$= \int \sec^{n-2} x \sec^2 x\,dx$

$u = \sec^{n-2} x$ | $dv = \sec^2 x\,dx$

$du = (n-2)\sec^{n-3} x(\sec x \tan x)\,dx$ | $v = \int \sec^2 x\,dx$

$= \tan x$

$\int u\,dv = uv - \int v\,du$

$I_n = \sec^{n-2} x \tan x - \int \tan x\,(n-2)\sec^{n-3} x \sec x \tan x\,dx$

$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\,dx$

$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)\,dx$

$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx.$

$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$

$I_n + (n-2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

$I_n + nI_n - 2I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

$nI_n - I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

$\Rightarrow (n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

$\Rightarrow I_n = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1} I_{n-2}.$

where $I_0 = \int \sec^0 x\,dx = \int dx = x + c$ (put $n=0$ in ①)

$I_1 = \int \sec x\,dx = \log(\sec x + \tan x) + c$ (put $n=1$ in ①)

6. Find the reduction formula for $\int \tan^n x\,dx$, $n \neq 1$

<u>Soln:</u>

Let $I_n = \int \tan^n x\,dx$ ——— ①

$= \int \tan^{n-2} x \tan^2 x\,dx$

$= \int \tan^{n-2} x(\sec^2 x - 1)\,dx$

$= \int \tan^{n-2} x \sec^2 x - \int \tan^{n-2} x\,dx$

$= \int \tan^{n-2} x\,d(\tan x) - I_{n-2}$

$= \frac{(\tan x)^{n-1}}{n-1} - I_{n-2}$ $\quad \| \because \int t^{n-2}\,dt = \frac{t^{n-1}}{n-1}$

$= \frac{\tan^{n-1} x}{n-1} - I_{n-2}$

where $I_0 = \int \tan^0 x\,dx = \int dx = x + c$ (put $n=0$ in ①)

$I_1 = \int \tan^1 x\,dx = \log \sec x + c$ (put $n=1$ in ①)

Note:

1) $$\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \begin{cases} \frac{m-1}{m+n}\,\frac{m-3}{m+n-2}\cdots\frac{2}{3+n}\,\frac{1}{1+n} & \text{if } m \text{ is odd, } n \text{ even or odd.} \\ \frac{m-1}{m+n}\,\frac{m-3}{m+n-2}\cdots\frac{1}{2+n}\,\frac{n-1}{n}\,\frac{n-3}{n-2}\cdots\frac{2}{3} & \text{if } m \text{ is even, } n \text{ is odd.} \\ \frac{m-1}{m+n}\,\frac{m-3}{m+n-2}\cdots\frac{1}{2+n}\,\frac{n-1}{n}\,\frac{n-3}{n-2}\cdots\frac{1}{2}\frac{\pi}{2} & \text{if } m \text{ is even, } n \text{ is even.} \end{cases}$$

2) Evaluate $\int_0^{\pi/2} \sin^7\theta \cos^5\theta\,d\theta$.

$I_{m,n} = \frac{m-1}{m+n}\,\frac{m-3}{m+n-2}\cdots\frac{2}{3+n}\,\frac{1}{1+n}$ if m is odd, n even or odd.

$I_{7,5} = \frac{7-1}{7+5}\,\frac{7-3}{7+6-2}\,\frac{2}{3+5}\,\frac{1}{1+5} = \frac{6}{12}\cdot\frac{4}{10}\cdot\frac{2}{8}\,\frac{1}{6}$

$= \frac{1}{120}$

Trignometric substitution :-

The following table gives the trigonometric substitutions for some expressions with trigonometric identities to evaluate the given integral.

Expression	substitution	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\sinh\theta$	$1+\sinh^2\theta = \cosh^2\theta$
$\sqrt{x^2-a^2}$	$x = a\cosh\theta$	$\cosh^2\theta - 1 = \sinh^2\theta$

1. Evaluate $\int \frac{1}{\sqrt{a^2-x^2}}dx$

Soln:-

Let $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$ $\Rightarrow \sin\theta = \frac{x}{a}$ $\Rightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right)$ —— ①

$$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2\sin^2\theta}}$$

$$= \int \frac{a\cos\theta}{a\sqrt{1-\sin^2\theta}}\, d\theta$$

$$= \int \frac{a\cos\theta}{a\sqrt{\cos^2\theta}}\, d\theta$$

$$= \int \frac{\cancel{a\cos\theta}}{\cancel{a\cos\theta}}\, d\theta$$

$$= \int d\theta$$

$$= \theta + C$$

$$= \sin^{-1}\left(\frac{x}{a}\right) + C \quad \text{by ①}$$

$$\therefore \boxed{\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a) + C}$$ //

2. Evaluate $\int \frac{1}{\sqrt{a^2+x^2}}dx$

Soln:-

Let $x = a\sinh\theta$, $dx = a\cosh\theta\, d\theta$ $\Rightarrow \sinh\theta = \frac{x}{a}$ $\Rightarrow \theta = \sinh^{-1}\left(\frac{x}{a}\right)$ —— ①

$$\therefore \int \frac{dx}{\sqrt{a^2+x^2}} = \int \frac{a\cosh\theta\, d\theta}{\sqrt{a^2+a^2\sinh^2\theta}}$$

$$= \int \frac{a\cosh\theta}{a\sqrt{1+\sinh^2\theta}}\, d\theta$$

$$= \int \frac{a\cosh\theta}{a\sqrt{\cosh^2\theta}}\, d\theta$$

$$= \int \frac{\cancel{a\cosh\theta}}{\cancel{a\cosh\theta}}\, d\theta$$

$$= \int d\theta$$

$$= \theta + C$$

$$= \sinh^{-1}\left(\frac{x}{a}\right) + C \quad \text{by ①}$$

$$\therefore \boxed{\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C}$$ //

3. Evaluate $\int \frac{1}{\sqrt{x^2-a^2}}dx$

Soln:-

Let $x = a\cosh\theta$, $dx = a\sinh\theta\, d\theta$ $\Rightarrow \cosh\theta = \frac{x}{a}$ $\Rightarrow \theta = \cosh^{-1}(x/a)$ —— ①

$$\therefore \int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sinh\theta\, d\theta}{\sqrt{a^2\cosh^2\theta - a^2}} = \int \frac{a\sinh\theta}{a\sqrt{\cosh^2\theta-1}}\, d\theta$$

$$= \int \frac{a\sinh\theta}{a\sqrt{\sinh^2\theta}}\, d\theta = \int \frac{\cancel{a\sinh\theta}}{\cancel{a\sinh\theta}}\, d\theta = \int d\theta$$

$$= \theta + C$$

$$\boxed{\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C} \quad \text{by ①}$$ //

4. Evaluate $\int \sqrt{a^2-x^2}\,dx$

Soln :-

Put $x = a\sin\theta \;\Big|\; \Rightarrow \sin\theta = \dfrac{x}{a}$ ——— ①

$dx = a\cos\theta\,d\theta \;\Big|\; \Rightarrow \theta = \sin^{-1}\left(\dfrac{x}{a}\right)$ ——— ②

$$\therefore \int \sqrt{a^2-x^2}\,dx = \int \sqrt{a^2-a^2\sin^2\theta}\; a\cos\theta\,d\theta$$
$$= \int a\sqrt{1-\sin^2\theta}\; a\cos\theta\,d\theta$$
$$= a^2\int \sqrt{\cos^2\theta}\;\cos\theta\,d\theta$$
$$= a^2\int \cos\theta\cdot\cos\theta\,d\theta$$
$$= a^2\int \cos^2\theta\,d\theta$$
$$= a^2\int \frac{1+\cos2\theta}{2}\,d\theta$$
$$= \frac{a^2}{2}\int (1+\cos2\theta)\,d\theta$$
$$= \frac{a^2}{2}\left(\theta + \frac{\sin2\theta}{2}\right) + c, \quad \because \sin2\theta = 2\sin\theta\cos\theta.$$
$$= \frac{a^2}{2}\left(\theta + \sin\theta\cos\theta\right) + c$$
$$= \frac{a^2}{2}\Bigg[\underbrace{\sin^{-1}\left(\frac{x}{a}\right)}_{\text{by ①}} + \underbrace{\frac{x}{a}}_{\text{by ②}}\left(\sqrt{1-\frac{x^2}{a^2}}\right)\Bigg]$$

$\cos^2\theta = 1-\sin^2\theta$

$= 1 - \dfrac{x^2}{a^2}$

$\cos\theta = \sqrt{1-\dfrac{x^2}{a^2}}$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{\cancel{a^2}}{2}\,\frac{x}{\cancel{a}}\,\frac{\sqrt{a^2-x^2}}{\cancel{a}}$$
$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + c$$

Hence $\boxed{\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + c}$ //.

5. Evaluate $\int \sqrt{a^2+x^2}\,dx$

Soln :-

Put $x = a\sinh\theta \;\Big|\; \sinh\theta = \dfrac{x}{a}$ ——— ①

$dx = a\cosh\theta\,d\theta \;\Big|\; \theta = \sinh^{-1}\left(\dfrac{x}{a}\right)$ ——— ②

$$\therefore \int \sqrt{a^2+x^2}\,dx = \int \sqrt{a^2+a^2\sinh^2\theta}\cdot a\cosh\theta\,d\theta$$
$$= \int a\sqrt{1+\sinh^2\theta}\cdot a\cosh\theta\,d\theta$$
$$= a^2\int \cosh^2\theta\,d\theta$$
$$= a^2\int \left(\frac{1+\cosh2\theta}{2}\right)d\theta$$
$$= \frac{a^2}{2}\int (1+\cosh2\theta)\,d\theta$$
$$= \frac{a^2}{2}\left(\theta + \frac{\sinh2\theta}{2}\right)$$
$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\,\frac{\sinh2\theta}{2}$$
$$= \frac{a^2}{2}\underbrace{\sinh^{-1}\left(\frac{x}{a}\right)}_{\text{by ②}} + \frac{a^2}{2}\sinh\theta\cosh\theta$$
$$= \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{\cancel{a^2}}{2}\cdot\underbrace{\frac{x}{\cancel{a}}}_{\text{by ①}}\,\frac{\sqrt{a^2+x^2}}{\cancel{a}}$$

$\cosh^2\theta = 1+\sinh^2\theta$

$= 1+\left(\dfrac{x}{a}\right)^2$

$= \dfrac{a^2+x^2}{a^2}$

$\cosh\theta = \dfrac{\sqrt{a^2+x^2}}{a}$

$$= \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2+x^2}$$

Hence $\boxed{\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)}$ //

6. Evaluate $\int \sqrt{x^2-a^2}\,dx$

Soln:-

Let $x = a\cosh\theta$ $\;\Big\|\;$ $\cosh\theta = \frac{x}{a}$ —— ①

$dx = a\sinh\theta\, d\theta$ $\;\Big\|\;$ $\Rightarrow \theta = \cosh^{-1}\left(\frac{x}{a}\right)$ —— ②

$$\therefore \int \sqrt{x^2-a^2}\,dx = \int \sqrt{a^2\cosh^2\theta - a^2}\,.\, a\sinh\theta\, d\theta$$

$$= \int \sqrt{a^2\sinh^2\theta}\,.\, a\sinh\theta\, d\theta$$

$$= \int a^2 \sinh\theta \,.\, \sinh\theta\, d\theta$$

$$= a^2 \int \sinh^2\theta\, d\theta$$

$$= a^2 \int \frac{\cosh 2\theta - 1}{2}\, d\theta \qquad \because \sinh^2\theta = \frac{\cosh 2\theta - 1}{2}$$

$$= \frac{a^2}{2} \int (\cosh 2\theta - 1)\, d\theta$$

$$= \frac{a^2}{2}\left(\frac{\sinh 2\theta}{2} - \theta\right) + c$$

$$= \frac{a^2}{2}\sinh\theta\cosh\theta - \frac{a^2}{2}\theta + C \qquad \because \sinh 2\theta = \sinh\theta\cosh\theta$$

$$= \frac{\cancel{a^2}}{2}\,\frac{\sqrt{x^2-a^2}}{\cancel{a}}\cdot\frac{x}{\cancel{a}} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right) + C$$

$$= \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right) + C.$$

$$\sinh^2\theta = \cosh^2\theta - 1 = \frac{x^2}{a^2} - 1 = \frac{x^2-a^2}{a^2}$$

$$\sinh\theta = \frac{\sqrt{x^2-a^2}}{a}$$

Hence $\boxed{\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right) + C}$ //.

7. Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

Soln:-

Let $x = 3\sin\theta \;\Rightarrow\; \sin\theta = \frac{x}{3}$

$dx = 3\cos\theta\, d\theta$ $\qquad \theta = \sin^{-1}\left(\frac{x}{3}\right)$

$$\therefore \int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{\sqrt{9-3^2\sin^2\theta}}{9\sin^2\theta}\, 3\cos\theta\, d\theta$$

$$= \int \frac{\cancel{3}\sqrt{1-\sin^2\theta}}{\cancel{9}\sin^2\theta}\, \cancel{3}\cos\theta\, d\theta$$

$$= \int \frac{\sqrt{\cos^2\theta}}{\sin^2\theta}\cos\theta\, d\theta$$

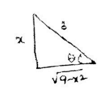
$$= \int \frac{\cos\theta}{\sin^2\theta}\cos\theta\, d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta.$$

$$= \int \cot^2\theta\, d\theta$$

$$= \int (\operatorname{cosec}^2\theta - 1)\, d\theta$$

$$= -\cot\theta - \theta + C.$$



$$\sin\theta = \frac{opp}{hyp} = \frac{x}{3}$$

$$\cos\theta = \frac{adj}{hyp} = \frac{\sqrt{9-x^2}}{3}$$

$$= -\frac{\cos\theta}{\sin\theta} - \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$= -\frac{\frac{\sqrt{9-x^2}}{\cancel{3}}}{\frac{x}{\cancel{3}}} - \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{a}\right) + C. //$$

8. Evaluate $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

Soln:

put $x = 2\tan\theta$ $\;\Big|\;$ $\Rightarrow \tan\theta = \frac{x}{2}$

$dx = 2\sec^2\theta\, d\theta$ $\;\Big|\;$ $\Rightarrow \theta = \tan^{-1}\left(\frac{x}{2}\right)$

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \frac{1}{4\tan^2\theta\sqrt{4+4\tan^2\theta}}\,2\sec^2\theta\,d\theta$$

$$= \int \frac{1}{4\tan^2\theta \cdot \cancel{2}\sqrt{1+\tan^2\theta}}\,\cancel{2}\sec^2\theta\,d\theta$$

$$= \frac{1}{4}\int \frac{1}{\tan^2\theta}\,\frac{1}{\cancel{\sec\theta}}\,\sec^{\cancel{2}}\theta\,d\theta \qquad \because \sqrt{1+\tan^2\theta} = \sec\theta.$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\,d\theta$$

$$= \frac{1}{4}\int \frac{1}{\cancel{\cos\theta}} \times \frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

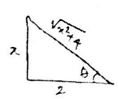
Let $u = \sin\theta$

$du = \cos\theta\,d\theta$

$$= \frac{1}{4}\int \frac{du}{u^2}$$

$$= \frac{1}{4}\left(\frac{-1}{u}\right) + C$$

$$= \frac{-1}{4\sin\theta} + C.$$



$\tan\theta = \frac{x}{2} = \frac{opp}{adj}$

$\cot\theta = \frac{2}{x}.$

$\operatorname{cosec}^2\theta = 1 + \cot^2\theta$

$= 1 + \left(\frac{2}{x}\right)^2$

$= \frac{x^2+4}{x^2}$

$\operatorname{cosec}\theta = \frac{\sqrt{x^2+4}}{x}$

$$= -\frac{1}{4}\operatorname{cosec}\theta + C.$$

$$= -\frac{1}{4}\,\frac{\sqrt{x^2+4}}{x} + C.$$

$$\therefore \boxed{\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = -\frac{\sqrt{x^2+4}}{4x} + C}$$ //

9. Evaluate $\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$

Soln :-

Put $x = \frac{3}{2}\tan\theta$

$dx = \frac{3}{2}\sec^2\theta\,d\theta$

when $x = 0 \Rightarrow \frac{3}{2}\tan\theta = 0 \Rightarrow \tan\theta = 0$

$\Rightarrow \theta = \tan^{-1}(0)$

$\boxed{\theta = 0}$

when $x = \frac{3\sqrt{3}}{2} \Rightarrow \frac{\cancel{3}}{\cancel{2}}\tan\theta = \frac{\cancel{3}\sqrt{3}}{\cancel{2}}$

$\theta = \tan^{-1}(\sqrt{3})$

$\boxed{\theta = \pi/3}$

$\therefore \theta : 0 \to \pi/3$

$$\therefore \int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{3/2}}\,dx = \int_0^{\pi/3} \frac{\left(\frac{3}{2}\right)^3\tan^3\theta}{\left[4\cdot\left(\frac{3}{2}\right)^2\tan^2\theta+9\right]^{3/2}}\cdot\frac{3}{2}\sec^2\theta\,d\theta$$

$$= \int_0^{\pi/3} \frac{\frac{27}{8}\tan^3\theta}{\left[\cancel{4}\cdot\frac{9}{\cancel{4}}\tan^2\theta+9\right]^{3/2}}\cdot\frac{3}{2}\sec^2\theta\,d\theta$$

$$= \int_0^{\pi/3} \frac{\frac{27}{8}\tan^3\theta}{9^{3/2}\left[1+\tan^2\theta\right]^{3/2}}\cdot\frac{3}{2}\sec^2\theta\,d\theta$$

$$= \int_0^{\pi/3} \frac{27}{8}\cdot\frac{3}{2}\cdot\frac{1}{9\sqrt{9}}\;\frac{\tan^3\theta}{(\sec^2\theta)^{3/2}}\sec^2\theta\,d\theta$$

$$= \int_0^{\pi/3} \frac{\cancel{27}}{8}\cdot\frac{3}{2}\cdot\frac{1}{\cancel{9}\,\cancel{3}}\;\frac{\tan^3\theta}{\sec^3\theta}\sec^2\theta\,d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}\,d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{\frac{\sin^3\theta}{\cos^3\theta}}{\frac{1}{\cos\theta}}\,d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^3\theta}\cdot\cos\theta \, d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta} \, d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{\sin^2\theta}{\cos^2\theta} \sin\theta \, d\theta$$

$$= \frac{3}{16}\int_0^{\pi/3} \frac{1-\cos^2\theta}{\cos^2\theta} \sin\theta \, d\theta$$

Put $u = \cos\theta$, $du = -\sin\theta \, d\theta$

when $\theta = 0$ $\Rightarrow u = \cos 0 = 1$, $\boxed{u=1}$

when $\theta = \pi/3$ $\Rightarrow u = \cos\pi/3 = 1/2$, $\boxed{u = 1/2}$

$\therefore u : 1 \to 1/2$.

$$= \frac{3}{16}\int_1^{1/2} \frac{1-u^2}{u^2} (-du)$$

$$= \frac{3}{16}\int_1^{1/2} \left(\frac{1}{u^2} - 1\right)(-du)$$

$$= \frac{3}{16}\int_1^{1/2} \left(1 - \frac{1}{u^2}\right) du$$

$$= \frac{3}{16}\left(u + \frac{1}{u}\right)_1^{1/2}$$

$$= \frac{3}{16}\left[\left(\frac{1}{2}+2\right) - (1+1)\right]$$

$$= \frac{3}{16}\left[\frac{1}{2} + \cancel{2} - \cancel{2}\right]$$

$$= \frac{3}{32}$$

$$\boxed{\int_0^{\frac{3\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{3/2}} \, dx = \frac{3}{32}}$$

//

* Integrals of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$ (or) $\int (px+q)\sqrt{ax^2+bx+c}\,dx$ (or) $\int \frac{px+q}{ax^2+bx+c}\,dx$

$$px+q = A\frac{d}{dx}(ax^2+bx+c) + B$$

* Integrals of the form $\int \frac{dx}{(x-k)\sqrt{ax^2+bx+c}}$

Take $x - k = \frac{1}{t}$

* Integrals of the form $\int \sqrt{(x-\alpha)(x-\beta)}\,dx$, $\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$, $\int \sqrt{\frac{x-\alpha}{x-\beta}}\,dx$, $\beta > \alpha$.

Take $x = \alpha\cos^2\theta + \beta\sin^2\theta$

Important formulas :-

1. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$, $x > a$
2. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$, $x < a$
3. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
4. $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right)$ (or) $\log\left(x+\sqrt{x^2-a^2}\right)$

* 5. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

6. $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right)$ (or) $\log\left(x+\sqrt{x^2+a^2}\right)$
7. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)$

* 8. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

9. $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)$

1) Evaluate $\int \frac{1}{\sqrt{a^2-x^2}}dx$ by using trignometric substitution. Hence use it in evaluation $\int \frac{6x+5}{\sqrt{6+x-2x^2}}dx$

Soln:

(i) Let $x = a\sin\theta$ $\qquad \sin\theta = \frac{x}{a}$

$dx = a\cos\theta\, d\theta \qquad \theta = \sin^{-1}\left(\frac{x}{a}\right)$

$$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2\sin^2\theta}}$$

$$= \int \frac{a\cos\theta}{a\sqrt{1-\sin^2\theta}}d\theta$$

$$= \int \frac{a\cos\theta}{a\cos\theta}d\theta$$

$$= \int d\theta$$

$$= \theta + C$$

$$= \sin^{-1}\left(\frac{x}{a}\right) + C.$$

Hence $\boxed{\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C}$

(ii) $\int \frac{6x+5}{\sqrt{6+x-2x^2}}dx$

$6x+5 = A\frac{d}{dx}(6+x-2x^2) + B$ —— ①

$6x+5 = A(1-4x) + B$

Equating x,	put $x=0$
$6 = -4A$	$5 = A+B$
$A = -\frac{6}{4}$	$B = 5-A$
$\boxed{A = -\frac{3}{2}}$	$= 5+\frac{3}{2}$
	$\boxed{B = \frac{13}{2}}$

$\therefore$ ① $\Rightarrow \quad 6x+5 = -\frac{3}{2}\frac{d}{dx}(6+x-2x^2) + \frac{13}{2}$

$\div$ by $\sqrt{6+x-2x^2}$

$$\frac{6x+5}{\sqrt{6+x-2x^2}} = -\frac{3}{2}\frac{\frac{d}{dx}(6+x-2x^2)}{\sqrt{6+x-2x^2}} + \frac{13}{2}\frac{1}{\sqrt{6+x-2x^2}}$$

$$\int \frac{6x+5}{\sqrt{6+x-2x^2}}dx = -\frac{3}{2}\int \frac{\frac{d}{dx}(6+x-2x^2)}{\sqrt{6+x-2x^2}}dx + \frac{13}{2}\int \frac{dx}{\sqrt{-2x^2+x+6}}$$

$$= -\frac{3}{\cancel{2}}\cancel{2}\sqrt{6+x-2x^2} + \frac{13}{2}\int \frac{dx}{\sqrt{2}\sqrt{-(x^2-\frac{x}{2}-3)}}$$

$\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + C$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}}\int \frac{dx}{\sqrt{-(x^2-\frac{x}{2}-3)}}$$

$2xb = \frac{x}{2}$

$2xb = \frac{x}{2}$

$b = \frac{1}{4}$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}}\int \frac{dx}{\sqrt{-\left[x^2-\frac{x}{2}+\left(\frac{1}{4}\right)^2-\frac{1}{16}-3\right]}}$$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}}\int \frac{dx}{\sqrt{-\left[\left(x-\frac{1}{4}\right)^2-\left(\frac{7}{4}\right)^2\right]}}$$

$-\frac{1}{16}-3$

$= \frac{-1-48}{16}$

$= -\frac{49}{16}$

$= -\left(\frac{7}{4}\right)^2$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}}\int \frac{dx}{\sqrt{\left(\frac{7}{4}\right)^2-\left(x-\frac{1}{4}\right)^2}}$$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}}\sin^{-1}\left(\frac{x-\frac{1}{4}}{\frac{7}{4}}\right) + C$$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}}\sin^{-1}\left(\frac{\frac{4x-1}{4}}{\frac{7}{4}}\right) + C$$

$$= -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right) + C.$$

Hence $\int \frac{6x+5}{\sqrt{6+x-2x^2}}dx = -3\sqrt{6+x-2x^2} + \frac{13}{2\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right) + C$ //

2. Evaluate $\int \frac{1}{\sqrt{x^2-a^2}}\,dx$ by using trignomatric substitution. Hence use it in evaluating $\int \frac{3x-2}{\sqrt{4x^2-4x-5}}\,dx$.

Sol:-

(i) Put $x = a\cosh\theta \Rightarrow \cosh\theta = \frac{x}{a}$

$dx = a\sinh\theta\, d\theta \Rightarrow \theta = \cosh^{-1}\left(\frac{x}{a}\right)$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sinh\theta\, d\theta}{\sqrt{a^2\cosh^2\theta - a^2}}$$

$$= \int \frac{a\sinh\theta}{a\sqrt{\cosh^2\theta-1}}\,d\theta$$

$$= \int \frac{\cancel{a}\,\cancel{\sinh\theta}}{\cancel{a}\,\cancel{\sinh\theta}}\,d\theta$$

$$= \int d\theta$$

$$= \theta + c$$

$$\boxed{\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}(x/a) + c}$$ //

(ii) $$\int \frac{3x-2}{\sqrt{4x^2-4x-5}}\,dx = \int \frac{3x-2}{\sqrt{4}\sqrt{x^2-x-5/4}}\,dx$$

$$3x-2 = A\frac{d}{dx}(4x^2-4x-5) + B \quad \text{——— (1)}$$

$$3x-2 = A(8x-4) + B$$

Equating x,

$3 = 8A$

$\boxed{A = 3/8}$

Put $x = 0$

$-2 = -4A + B$

$B = -2 + 4A$

$= -2 + \cancel{4}\cdot 3/\cancel{8}_2$

$= -2 + 3/2$

$= \frac{-4+3}{2}$

$\boxed{B = \frac{-1}{2}}$

$\therefore$ (1) $\Rightarrow$ $3x-2 = \frac{3}{8}\frac{d}{dx}(4x^2-4x-5) - \frac{1}{2}$

$\div$ by $\sqrt{4x^2-4x-5}$

$$\frac{3x-2}{\sqrt{4x^2-4x-5}} = \frac{3}{8}\frac{\frac{d}{dx}(4x^2-4x-5)}{\sqrt{4x^2-4x-5}} - \frac{1}{2}\frac{1}{\sqrt{4x^2-4x-5}}$$

$$\int \frac{3x-2}{\sqrt{4x^2-4x-5}}\,dx = \frac{3}{8}\int \frac{\frac{d}{dx}(4x^2-4x-5)}{\sqrt{4x^2-4x-5}} - \frac{1}{2}\int \frac{1}{\sqrt{4x^2-4x-5}}\,dx$$

$$= \frac{3}{\cancel{8}_4}\cdot\cancel{2}\sqrt{4x^2-4x-5} - \frac{1}{2}\int \frac{dx}{\sqrt{4}\sqrt{x^2-x-5/4}} \quad \Big|\ \because \int \frac{d(f(x))}{\sqrt{f(x)}} = 2\sqrt{f(x)}$$

$$= \frac{3}{4}\sqrt{4x^2-4x-5} - \frac{1}{4}\int \frac{dx}{\sqrt{x^2-x-5/4}}$$

$2ab = x$, $a = x$

$2\cancel{x}b = \cancel{x}$

$b = 1/2$

$b^2 = 1/4$

$$= \frac{3}{4}\sqrt{4x^2-4x-5} - \frac{1}{4}\int \frac{dx}{\sqrt{x^2-x+(\tfrac{1}{2})^2-1/4-5/4}}$$

$$= \frac{3}{4}\sqrt{4x^2-4x-5} - \frac{1}{4}\int \frac{dx}{\sqrt{(x-\frac{1}{2})^2-6/4}}$$

$$= \frac{3}{4}\sqrt{4x^2-4x-5} - \frac{1}{4}\int \frac{dx}{\sqrt{(x-\frac{1}{2})^2-(\frac{3}{2})^2}}$$

$$= \frac{3}{4}\sqrt{4x^2-4x-5} - \frac{1}{4}\cosh^{-1}\left(\frac{x-\frac{1}{2}}{3/2}\right) + c$$

$$= \frac{3}{4}\sqrt{4x^2-4x-5} - \frac{1}{4}\cosh^{-1}\left(\frac{2x-1/\cancel{2}}{3/\cancel{2}}\right) + c$$

$$= \frac{3}{4}\sqrt{4x^2-4x-5} - \frac{1}{4}\cosh^{-1}\left(\frac{2x-1}{3}\right) + c.$$

$$\therefore \boxed{\int \frac{3x-2}{\sqrt{4x^2-4x-5}}\,dx = \frac{3}{4}\sqrt{4x^2-4x-5} - \frac{1}{4}\cosh^{-1}\left(\frac{2x-1}{3}\right) + c}$$ //

Evaluate the integral $\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$

Soln:-

Put $x+1 = \frac{1}{t} \Rightarrow t = \frac{1}{x+1}$

$dx = \frac{-1}{t^2}\,dt$

$$\int \frac{dx}{(x+1)\sqrt{x^2+x+1}} = \int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{\frac{1}{t^2}-\frac{1}{t}+1}}$$

$$= -\int \frac{\frac{dt}{t}}{\frac{\sqrt{1-t+t^2}}{t}}$$

$$= -\int \frac{dt}{\sqrt{t^2-t+1}}$$

$2ab = t$

$2 \times b = 1$

$b = \frac{1}{2}$

$$= -\int \frac{dt}{\sqrt{t^2-t+(\frac{1}{2})^2-\frac{1}{4}+1}}$$

$$= -\int \frac{dt}{\sqrt{(t-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}}$$

$-\frac{1}{4}+1 = \frac{-1+4}{4} = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2$

$$= -\left[\sinh^{-1}\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]+c$$

$\therefore \int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}(x/a)$

$$= -\sinh^{-1}\left(\frac{\frac{2t-1}{2}}{\frac{\sqrt{3}}{2}}\right)+c$$

$$= -\sinh^{-1}\left(\frac{2t-1}{\sqrt{3}}\right)+c$$

$$= -\sinh^{-1}\left(\frac{\frac{2}{x+1}-1}{\sqrt{3}}\right)+c$$

$$= -\sinh^{-1}\left(\frac{2-x-1}{\sqrt{3}\,(x+1)}\right)+c$$

$$= -\sinh^{-1}\left[\frac{1-x}{\sqrt{3}(1+x)}\right]+c$$

\# Evaluate $\int \frac{dx}{(x+1)\sqrt{x^2+1}}$

Soln:-

Put $x+1 = \frac{1}{t}$

$dx = \frac{-1}{t^2}\,dt$

$$\therefore \int \frac{dx}{(x+1)\sqrt{x^2+1}} = \int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{(\frac{1}{t}-1)^2+1}}$$

$$= \int \frac{-\frac{dt}{t^2}}{\frac{1}{t}\sqrt{(\frac{1-t}{t})^2+1}}$$

$$= \int \frac{-\frac{dt}{t^2}}{\frac{1}{t}\frac{\sqrt{(1-t)^2+t^2}}{t}}$$

$$= -\int \frac{dt}{\sqrt{1-2t+t^2+t^2}}$$

$$= -\int \frac{dt}{\sqrt{2t^2-2t+1}}$$

$$= \frac{-1}{\sqrt{2}}\int \frac{dt}{\sqrt{t^2-t+\frac{1}{2}}}$$

$2ab = t$

$2 \times b = 1$

$b = \frac{1}{2}$

$$= \frac{-1}{\sqrt{2}}\int \frac{dt}{\sqrt{t^2-t+(\frac{1}{2})^2-\frac{1}{4}+\frac{1}{2}}}$$

$$= \frac{-1}{\sqrt{2}}\int \frac{dt}{\sqrt{(t-\frac{1}{2})^2+(\frac{1}{2})^2}}$$

$\frac{-1}{4}+\frac{1}{2} = \frac{-1+2}{4} = \frac{1}{4} = (\frac{1}{2})^2$

$$= \frac{-1}{\sqrt{2}}\sinh^{-1}\left(\frac{t-\frac{1}{2}}{\frac{1}{2}}\right)+c$$

$$= \frac{-1}{\sqrt{2}}\sinh^{-1}\left(\frac{\frac{2t-1}{2}}{\frac{1}{2}}\right)+c$$

$$= \frac{-1}{\sqrt{2}}\sinh^{-1}(2t-1)+c = \frac{-1}{\sqrt{2}}\sinh^{-1}\left[2\left(\frac{1}{1+x}\right)-1\right]+c$$

$$= \frac{-1}{\sqrt{2}}\sinh^{-1}\left(\frac{2-1-x}{1+x}\right) = \frac{-1}{\sqrt{2}}\sinh^{-1}\left(\frac{1-x}{1+x}\right)+c$$

Integration of rational functions by partial fraction

1. Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

soln:

Take $\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{x^2+2x-1}{2x^3+4x^2-x^2-2x}$

$$= \frac{x^2+2x-1}{2x^2(x+2)-x(x+2)}$$

$$= \frac{x^2+2x-1}{(2x^2-x)(x+2)}$$

$$= \frac{x^2+2x-1}{x(2x-1)(x+2)} \quad \text{—— (1)}$$

Now,

$$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2} \quad \text{—— (2)}$$

x'ly by $x(2x-1)(x+2)$ on both sides, we get

$$x^2+2x-1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

Put $x=0$

$-1 = A(-1)(2)$

$-1 = -2A$

$\boxed{A = 1/2}$

Put $x=1/2$

$(1/2)^2+2(1/2)-1 = A(2\cdot 1/2-1)(1/2+2) + B\,1/2(1/2+2) + C\,1/2(2\cdot 1/2-1)$

$1/4+\cancel{1}-\cancel{1} = A(1-1)(5/2) + B/2(5/2) + C/2(1-1)$

$1/4 = \frac{5}{4}B \Rightarrow B = \frac{1}{\cancel{4}} \times \frac{\cancel{4}}{5} \Rightarrow \boxed{B = 1/5}$

Put $x=-2$

$(-2)^2+2(-2)-1 = A(-4-1)(-2+2) + B(-2)(-2+2) + C(-2)(-4-1)$

$4-4-1 = A(0)+B(0)+C(-2)(-5)$

$-1 = 10C$

$\boxed{C = -1/10}$

(2) becomes

$$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{1/2}{x} + \frac{1/5}{2x-1} + \frac{-1/10}{x+2}$$

$$\int \frac{x^2+2x-1}{x(2x-1)(x+2)}\,dx = \frac{1}{2}\int \frac{1}{x}\,dx + \frac{1}{5}\int \frac{1}{2x-1}\,dx - \frac{1}{10}\int \frac{1}{x+2}\,dx$$

$$= \frac{1}{2}\log x + \frac{1}{5}\,\frac{\log(2x-1)}{2} - \frac{1}{10}\log(x+2) + c$$

$$= \frac{1}{2}\log x + \frac{1}{10}\log(2x-1) - \frac{1}{10}\log(x+2) + c$$

$$= \frac{1}{2}\log x + \frac{1}{10}\left[\log(2x-1) - \log(x+2)\right] + c$$

$$= \frac{1}{2}\log x + \frac{1}{10}\log\left(\frac{2x-1}{x+2}\right) + c \qquad \because \log a - \log b = \log\frac{a}{b}$$

2. Evaluate $\int \frac{\sec^2 x}{\tan^2 x+3\tan x+2}\,dx$ (Jan 2016)

soln:

Put $u = \tan x$

$du = \sec^2 x\,dx$

$$\therefore \int \frac{\sec^2 x}{\tan^2 x+3\tan x+2}\,dx = \int \frac{du}{u^2+3u+2}$$

Take $\frac{1}{u^2+3u+2} = \frac{1}{(u+1)(u+2)}$

$$= \frac{A}{u+1} + \frac{B}{u+2} \quad \text{—— (1)}$$

x'ly by $(u+1)(u+2)$ on both sides, we get

$$1 = A(u+2) + B(u+1) \quad \text{—— (2)}$$

Put $u=-2$ in (2)

$1 = B(-2+1) \Rightarrow \boxed{B=-1}$

Put $u=-1$ in (2)

$1 = A(-1+2) + B(-1+1)$

$\Rightarrow \boxed{A=1}$

① becomes,

$$\frac{1}{u^2+3u+2} = \frac{1}{u+1} + \frac{-1}{u+2}$$

$$\int \frac{du}{u^2+3u+2} = \int \frac{du}{u+1} - \int \frac{du}{u+2}$$

$$= \log(u+1) - \log(u+2)$$

$$= \log\left(\frac{u+1}{u+2}\right) + c$$

$$\therefore \boxed{\int \frac{\sec^2 x\, dx}{\tan^2 x + 3\tan x + 2} = \log\left(\frac{\tan x + 1}{\tan x + 2}\right) + c} \quad , \because u = \tan x .$$

//

3. Evaluate $\int \frac{x^2}{(x-1)^3(x-2)}dx$

Soln:

Take $\frac{x^2}{(x-1)^3(x-2)} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$ ——— ①

x'ly by $(x-1)^3(x-2)$ on both sides,

$x^2 = A(x-1)^3 + B(x-1)^2(x-2) + C(x-1)(x-2) + D(x-2)$ ——— ②

put $x=2$	Equating the co-eff of x^3 on both sides	Equating co-eff of x^2 on both sides (or) Put $x=1$	put $x=0$
$4 = A(1)^2$	$0 = A+B$	$1 = D(-1)$	$0 = -A-2B+2C-2D$
$4 = 4A$	$B = -A$	$\boxed{D=-1}$	$0 = -4+8+2C+2$
$\boxed{A=4}$	$\Rightarrow \boxed{B=-4}$		$\Rightarrow 0 = 6+2C$
			$\Rightarrow 2C = -6$
			$\boxed{C=-3}$

$$\therefore ① \Rightarrow \frac{x^2}{(x-1)^3(x-2)} = \frac{4}{x-2} + \frac{-4}{x-1} + \frac{-3}{(x-1)^2} + \frac{-1}{(x-1)^3}$$

$$\int \frac{x^2}{(x-1)^3(x-2)}dx = 4\int\frac{dx}{x-2} - 4\int\frac{dx}{x-1} - 3\int\frac{dx}{(x-1)^2} - \int\frac{dx}{(x-1)^3}$$

$$= 4\log(x-2) - 4\log(x-1) + 3\left[\frac{1}{x-1}\right] + \frac{1}{2(x-1)^2} + c.$$

$$= 4\log\left(\frac{x-2}{x-1}\right) + \frac{3}{x-1} + \frac{1}{2(x-1)^2} + c$$ //

4. Evaluate $\int \frac{10}{(x-1)(x^2+9)}dx$ (Jan 2015)

Soln:

Take $\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$ ——— ①

x'ly by $(x-1)(x^2+9)$ on both sides,

$10 = A(x^2+9) + (Bx+C)(x-1)$

put $x=1$	Equating co-eff of x^2 on both sides.	Equating co-eff of x on both sides,
$10 = A(10)$	$0 = A+B$	$0 = -B+C$
$\boxed{A=1}$	$\Rightarrow B = -A$	$C = B \Rightarrow \boxed{C=-1}$
	$\boxed{B=-1}$	

$$\therefore ① \Rightarrow \frac{10}{(x-1)(x^2+9)} = \frac{1}{x-1} + \frac{-x-1}{x^2+9}$$

$$\int \frac{10}{(x-1)(x^2+9)}dx = \int\frac{1}{x-1}dx - \int\frac{x+1}{x^2+9}dx$$

$$= \int\frac{1}{x-1}dx - \int\frac{x}{x^2+9}dx - \int\frac{dx}{x^2+9}$$

$$= \frac{1}{2}\int\frac{dx}{x-1} - \frac{1}{2}\int\frac{2x}{x^2+9}dx - \int\frac{dx}{x^2+9}$$

$$= \frac{1}{2}\log(x-1) - \frac{1}{2}\log(x^2+9) - \frac{1}{3}\tan^{-1}(x/3) + C \quad \because \int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(x/a)$$ //

5. Evaluate $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$ (N/D 2016)

Soln:

$\begin{Bmatrix}\text{Highest power} \\ \text{of } x \text{ in Nr}\end{Bmatrix} > \begin{Bmatrix}\text{Highest power} \\ \text{of } x \text{ in Dr}\end{Bmatrix}$ ⇒ use following Method.

$$\begin{array}{r|l} & x+1 \\ \hline x^3-x^2-x+1 & x^4-0x^3-2x^2+4x+1 \\ & x^4-x^3-x^2+x \\ & (-)\quad(+)\quad(+)\quad(-) \\ \hline & x^3-x^2+5x+1 \\ & x^3-x^2-x+1 \\ & (-)\quad(+)\quad(+)\quad(-) \\ \hline & 4x \end{array}$$

$\therefore \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = (x+1) + \frac{4x}{x^3-x^2-x+1}$ —— (1)

$\frac{20}{3} = ?$

$$\begin{array}{r|l} & 6 \\ \hline 3 & 20 \\ & 18 \\ \hline & 2 \end{array}$$

$\Rightarrow \frac{20}{3} = 6 + \frac{2}{3}$

Take $\frac{4x}{x^3-x^2-x+1} = \frac{4x}{x^2(x-1)-(x-1)}$

$= \frac{4x}{(x^2-1)(x-1)}$

$= \frac{4x}{(x+1)(x-1)(x-1)}$ $\because x^2-1=(x+1)(x-1)$

$= \frac{4x}{(x+1)(x-1)^2}$

$\Rightarrow \frac{4x}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$ —— (2)

x'ly by $(x+1)(x-1)^2$ on both sides,

$4x = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$ —— (2)

put $x=-1$	put $x=1$	Equating co-eff of x^2 on both sides of (2)
$-4 = A(-2)^2$	$4 = C(2)$	$0 = A+B$
$-4 = 4A$	$\boxed{C=2}$	$B = -A$
$\boxed{A=-1}$		$\Rightarrow \boxed{B=1}$

$\therefore (1) \Rightarrow \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1 + \frac{-1}{x+1} + \frac{1}{x-1} + \frac{2}{(x-1)^2}$.

$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx = \int (x+1)\,dx - \int \frac{dx}{x+1} + \int \frac{dx}{x-1} + 2\int \frac{dx}{(x-1)^2}$

$= \frac{x^2}{2} + x - \log(x+1) + \log(x-1) + 2\left(\frac{-1}{x-1}\right) + C$

$= \frac{x^2}{2} + x + \log\left(\frac{x-1}{x+1}\right) - 2\frac{1}{x-1} + c$

$= \frac{x^2}{2} + x - \frac{2}{x-1} + \log\left(\frac{x-1}{x+1}\right) + c$ //.

6. Evaluate $\int \frac{2x^2-x+4}{x^3+4x}\,dx$

Soln:

Take $\frac{2x^2-x+4}{x^3+4x} = \frac{2x^3-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$ ——→ (1)

$\Rightarrow \frac{2x^3-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$

x'ly by $x(x^2+4)$ on both sides,

$2x^2-x+4 = A(x^2+4) + (Bx+C)x$

put $x=0$	Equating x^2	Equating x
$\boxed{4=A4}$	$2 = A+B$	$\boxed{-1=C}$
$\Rightarrow \boxed{A=1}$	$\boxed{B=+1}$	

$$(1) \Rightarrow \frac{2x^2-x+4}{x^2+4x} = \frac{1}{x} + \frac{x-1}{x^2+4}$$

$$\int \frac{2x^2-x+4}{x^2+4x}\,dx = \int \frac{1}{x}\,dx + \int \frac{x-1}{x^2+4}\,dx$$

$$= \int \frac{1}{x}\,dx + \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx$$

$$= \log x + \frac{1}{2}\int \frac{2x}{x^2+4}\,dx - \int \frac{dx}{x^2+4} \qquad \because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$= \log x + \tfrac{1}{2}\log(x^2+4) - \frac{1}{2}\tan^{-1}(x/2) + c$$

$$= \log x + \log\sqrt{x^2+4} - \tfrac{1}{2}\tan^{-1}(x/2) + c$$

7. Evaluate $\int \frac{x^2+1}{(x^2-1)(2x+1)}\,dx$

Sol.

Take $\frac{x^2+1}{(x^2-1)(2x+1)} = \frac{x^2+1}{(x+1)(x-1)(2x+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{2x+1}$ —— (1)

$$\Rightarrow \frac{x^2+1}{(x+1)(x-1)(2x+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{2x+1}$$

x'ly by $(x+1)(x-1)(2x+1)$,

$$x^2+1 = A(x-1)(2x+1) + B(x+1)(2x+1) + C(x+1)(x-1)$$

put $x=-1$	put $x=1$	put $x=-\frac{1}{2}$
$2 = A(-2)(-1)$	$\cancel{2} = B(\cancel{2})(3)$	$(-\frac{1}{2})^2+1 = C(-\frac{1}{2}+1)(-\frac{1}{2}-1)$
$\boxed{A=1}$	$\boxed{B=\frac{1}{3}}$	$\frac{1}{4}+1 = C(\frac{1}{2})(-\frac{3}{2})$
		$\frac{5}{\cancel{4}} = -\frac{3C}{\cancel{4}}$
		$\boxed{C=-\frac{5}{3}}$

$$(1) \Rightarrow \frac{x^2+1}{(x^2-1)(2x+1)} = \frac{1}{x+1} + \frac{1/3}{x-1} + \frac{-5/3}{2x+1}$$

$$\frac{x^2+1}{(x^2-1)(2x+1)} = \frac{1}{x+1} + \frac{1}{3}\,\frac{1}{x-1} - \frac{5}{3}\,\frac{1}{2x+1}$$

$$\int \frac{x^2+1}{(x^2-1)(2x+1)}\,dx = \int \frac{dx}{x+1} + \frac{1}{3}\int \frac{dx}{x-1} - \frac{5}{3}\int \frac{dx}{2x+1}$$

$$= \log(x+1) + \tfrac{1}{3}\log(x-1) - \frac{5}{3}\,\frac{\log(2x+1)}{2} + c$$

$$= \log(x+1) + \tfrac{1}{3}\log(x-1) - \tfrac{5}{6}\log(2x+1) + c \;//$$

Improper Integrals :-

Definition :

The improper integrals, $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called convergent if the corresponding limit exists and divergent if the limit does not exists.

1. Determine whether the integral $\int_1^\infty \frac{\log x}{x^2}dx$ is convergent or divergent.

Soln:

Take $\int \frac{\log x}{x^2}dx$

$u = \log x$, $du = \frac{1}{x}dx$ | $dv = \frac{dx}{x^2}$, $v = \int \frac{dx}{x^2} = \frac{-1}{x}$

$$\int u\,dv = uv - \int v\,du$$

$$= \log x\left(\frac{-1}{x}\right) - \int \frac{-1}{x}\frac{1}{x}dx$$

$$= -\frac{\log x}{x} + \int \frac{1}{x^2}dx$$

$$= -\frac{\log x}{x} - \frac{1}{x} + C.$$

$$\int_1^\infty \frac{\log x}{x^2}dx = \lim_{t\to\infty}\int_1^t \frac{\log x}{x^2}dx = \lim_{t\to\infty}\left[\frac{-\log x}{x} - \frac{1}{x}\right]_1^t$$

$$= \lim_{t\to\infty}\left[\left(\frac{-\log t}{t} - \frac{1}{t}\right) - \left(\frac{-\log 1}{1} - 1\right)\right]$$

$$= \lim_{t\to\infty}\left(\frac{-\log t}{t} - \frac{1}{t} + 1\right)$$

$$= \lim_{t\to\infty}\left(\frac{-\log t}{t}\right) - \frac{1}{\infty} + 1$$

$$= -\lim_{t\to\infty}\frac{\log t}{t} + 1$$

$$= \frac{-\infty}{\infty} + 1$$

$$= -\lim_{t\to\infty}\left(\frac{1/t}{1}\right) + 1$$

$$= -\frac{1}{\infty} + 1$$

$$= 1. \text{ (finite)}$$

$\because \lim_{t\to\infty}\frac{\log t}{t} = \frac{\infty}{\infty}$ [Form]

$\therefore \lim_{t\to\infty}\frac{d[\log t]}{d(t)} = \lim_{t\to\infty}\frac{1/t}{1}$

$= \frac{1}{\infty}$

$= 0.$

$\therefore \int_1^\infty \frac{\log x}{x^2}dx$ is convergent //.

2. Evaluate $\int_{-\infty}^\infty \frac{1}{x^2+a^2}dx$

Soln:-

Given $\int_{-\infty}^\infty \frac{dx}{x^2+a^2} = \int_{-\infty}^0 \frac{dx}{x^2+a^2} + \int_0^\infty \frac{dx}{x^2+a^2}$

$$= \lim_{t\to-\infty}\int_t^0 \frac{dx}{x^2+a^2} + \lim_{t\to\infty}\int_0^t \frac{dx}{x^2+a^2}$$

$$= \lim_{t\to-\infty}\left[\tan^{-1}(x/a)\right]_t^0 + \lim_{t\to\infty}\left[\tan^{-1}(x/a)\right]_0^t$$

$$= \lim_{t\to-\infty}\left[\cancel{\tan^{-1}0} - \frac{1}{a}\tan^{-1}(t/a)\right] + \lim_{t\to\infty}\left(\frac{1}{a}\tan^{-1}(t/a) - \frac{1}{a}\cancel{\tan^{-1}(0)}\right)$$

$$= \frac{-1}{a}\lim_{t\to-\infty}\tan^{-1}(t/a) + \frac{1}{a}\lim_{t\to\infty}\tan^{-1}(t/a)$$

$$= \frac{-1}{a}\tan^{-1}(-\infty) + \frac{1}{a}\tan^{-1}(\infty)$$

$$= \frac{-1}{a}(-\pi/2) + 1/a(\pi/2)$$

$\because \tan^{-1}(-\infty) = -\pi/2$

$\tan^{-1}(\infty) = \pi/2$

$$= \frac{\pi}{2a} + \frac{\pi}{2a}$$

$$= \frac{2\pi}{2a}$$

$$= \frac{\pi}{a} \text{ //.}$$

3. Evaluate $\int_{-\infty}^{\infty} x e^{-x^2} dx$

Soln:

Take $\int x e^{-x^2} dx$

$u = x^2$

$du = 2x\,dx \Rightarrow x\,dx = \frac{du}{2}$

$\therefore \int x e^{-x^2} dx = \int e^{-u} \frac{du}{2}$

$= \frac{1}{2} \int e^{-u} du$

$= \frac{1}{2} \left(\frac{e^{-u}}{-1} \right)$

$= \frac{-1}{2} e^{-u}$

$= \frac{-1}{2} e^{-x^2}$ —— ①

Now,

$\int_{-\infty}^{\infty} x e^{-x^2} dx = \int_{-\infty}^{0} x e^{-x^2} dx + \int_{0}^{\infty} x e^{-x^2} dx$

$= \lim_{t \to -\infty} \int_{t}^{0} x e^{-x^2} dx + \lim_{t \to \infty} \int_{0}^{t} x e^{-x^2} dx$

$= \lim_{t \to -\infty} \left[\frac{-1}{2} e^{-x^2} \right]_t^0 + \lim_{t \to \infty} \left[\frac{-1}{2} e^{-x^2} \right]_0^t$

$= \lim_{t \to -\infty} \left\{ \frac{-1}{2} \left(e^{-0} - e^{-t^2} \right) \right\} + \lim_{t \to \infty} \left\{ \frac{-1}{2} \left(e^{-t^2} - e^{-0} \right) \right\}$

$= \lim_{t \to -\infty} \left\{ \frac{-1}{2} \left(1 - e^{-t^2} \right) \right\} + \lim_{t \to \infty} \left\{ \frac{-1}{2} \left(e^{-t^2} - 1 \right) \right\}$

$= \frac{-1}{2} \left(1 - e^{-\infty} \right) + \frac{1}{2} \left(e^{-\infty} - 1 \right)$

$= \frac{-1}{2} + \frac{1}{2}$

$= 0$ //

4. $\int_{1}^{\infty} \frac{\log x}{x} dx$ (Jan 2015)

Soln:

Take $I = \int_{1}^{\infty} \frac{\log x}{x} dx$

$u = \log x$ | $dv = \frac{dx}{x}$

$du = \frac{1}{x} dx$ | $v = \int \frac{dx}{x} = \log x$

$\int u\,dv = uv - \int v\,du$

$I = (\log x)(\log x) - \int \log x \cdot \frac{1}{x} dx$

$= (\log x)^2 - \int \frac{\log x}{x} dx$

$= (\log x)^2 - I$

$2I = (\log x)^2$

$I = \frac{1}{2} (\log x)^2$

$\therefore \int_{1}^{\infty} \frac{\log x}{x} dx = \lim_{t \to \infty} \int_{1}^{t} \frac{\log x}{x} dx$

$= \lim_{t \to \infty} \left[\frac{1}{2} (\log x)^2 \right]_1^t$

$= \lim_{t \to \infty} \frac{1}{2} \left[(\log t)^2 - (\log 1)^2 \right]$

$= \frac{1}{2} \lim_{t \to \infty} (\log t)^2$

$= \frac{1}{2} \cdot \infty$

$= \infty$

Hence $\int_{1}^{\infty} \frac{\log x}{x} dx$ is divergent //

MOHAMED SATHAK A.J. COLLEGE OF ENGINEERING

UNIT –V MULTIPLE INTEGRALS

NOTES

Double Integral

It is denoted by $\iint_R f(x, y)\ dx\,dy$ where R is the region of integration corresponding to interval of integration.

Note:

1) The order of integration is denoted by $\int_{y_1}^{y_2} \int_{f_1(y)}^{f_2(y)} f(x, y)\ dx\,dy$ (or) $\int_{x_1}^{x_2} \int_{f_1(x)}^{f_2(x)} f(x, y)\ dy\,dx$
2) Area of the region = $\iint dx\,dy$
3) For constant limits it does not matter whether we first integrate w.r.t. x and then w.r.t. y or vice versa.

Problems

1. Evaluate $\int_0^1 \int_{x^2}^{x} x^3 y\ dx\,dy$

Sol.
$$Let\ \ I = \int_0^1 \int_{x^2}^{x} x^3 y\ dy\,dx$$
$$= \int_0^1 x^3 \left[\frac{y^2}{2}\right]_{x^2}^{x} dx$$
$$= \frac{1}{2}\int_0^1 x^3\,(x^2 - x^4)\ dx$$
$$= \frac{1}{2}\int_0^1 (x^5 - x^7)\ dx$$
$$= \frac{1}{2}\left[\frac{x^6}{6} - \frac{x^8}{8}\right]_0^1$$
$$= \frac{1}{2}\left[\left(\frac{1}{6} - \frac{1}{8}\right) - (0 - 0)\right]$$
$$= \frac{1}{2}\left(\frac{4-3}{24}\right)$$
$$= \frac{1}{48}$$

2. Evaluate $\int_0^3 \int_1^2 xy(x+y)\,dy\,dx$

Sol. $Let\ \ I = \int_0^3 \int_1^2 (x^2y + xy^2)\,dy\,dx$

$$= \int_0^3 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_1^2 dx$$

$$= \int_0^3 \left[\left(\frac{4x^2}{2} + \frac{8x}{3}\right) - \left(\frac{x^2}{2} + \frac{x}{3}\right)\right] dx$$

$$= \int_0^3 \left(\frac{3x^2}{2} + \frac{7x}{3}\right) dx$$

$$= \left[\frac{3x^3}{6} + \frac{7x^2}{6}\right]_0^3$$

$$= \left[\left(\frac{27}{2} + \frac{21}{2}\right) - (0-0)\right] = \frac{48}{2} = 24$$

3. Evaluate $\int_0^1 \int_0^x e^{y/x}\,dy\,dx$

Sol. $Let\ \ I = \int_0^1 \int_0^x e^{y/x}\,dy\,dx$

$$= \int_0^1 \left[\frac{e^{y/x}}{1/x}\right]_0^x dx$$

$$= \int_0^1 x(e^1 - e^0)\,dx$$

$$= (e-1)\int_0^1 x\,dx \quad = (e-1)\left[\frac{x^2}{2}\right]_0^1$$

$$= (e-1)\left(\frac{1}{2} - 0\right)$$

$$= \frac{e-1}{2}$$

4. Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

Sol. $\displaystyle Let\;\; I = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy}{(1+x^2)+y^2}\;\; dx$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}} \right) \right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1}(1) - \tan^{-1}(0) \right] dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\frac{\pi}{4} - 0 \right] dx$$

$$= \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{1+x^2}} \qquad = \frac{\pi}{4} \left[\log\left(\frac{x+\sqrt{1+x^2}}{1} \right) \right]_0^1$$

$$= \frac{\pi}{4} \left[\log(1+\sqrt{2}) - \log(0+1) \right]$$

$$= \frac{\pi}{4} \log(1+\sqrt{2}) \qquad (\sin ce\;\; \log 1 = 0)$$

5. Evaluate $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$

Sol. $\displaystyle Let\;\; I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{dy}{\sqrt{(a^2-x^2)-y^2}} dx$

$$= \int_0^a \left[\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}} \right) \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left[\sin^{-1}(1) - \sin^{-1}(0) \right] dx$$

$$= \int_0^a \left[\frac{\pi}{2} - 0 \right] dx = \frac{\pi}{2} \int_0^a dx \quad = \frac{\pi}{2} [x]_0^a = \frac{\pi a}{2}$$

6. Evaluate $\int_2^3 \int_1^2 \frac{dx\,dy}{xy}$

Sol. $$\int_2^3 \int_1^2 \frac{dx\,dy}{xy} = \int_2^3 [\log x]_1^2 \frac{dy}{y}$$

$$= \int_2^3 [\log 2 - \log 1] \frac{dy}{y}$$

$$= \int_2^3 [\log 2 - 0] \frac{dy}{y}$$

$$= \log 2 \int_2^3 \frac{dy}{y}$$

$$= \log 2 \, [\log y]_2^3$$

$$= \log 2 \, (\log 3 - \log 2)$$

$$= \log 2 \, \log\left(\frac{3}{2}\right)$$

7. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} dx\,dy$

Sol. $$\int_0^a \int_0^{\sqrt{a^2-x^2}} dx\,dy = \int_0^a [y]_0^{\sqrt{a^2-x^2}} \; dx$$

$$= \int_0^a [y]_0^{\sqrt{a^2-x^2}} \; dx$$

$$= \int_0^a \left[\sqrt{a^2-x^2} - 0\right] dx$$

$$= \int_0^a \sqrt{a^2-x^2} \; dx$$

$$= \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

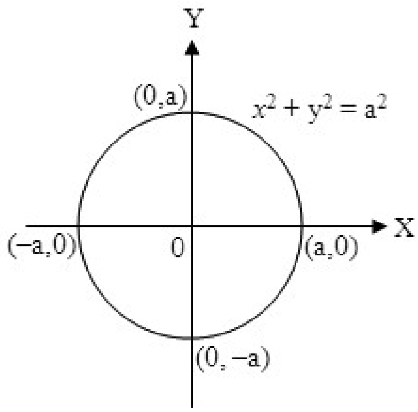
$$= \left\{0 + \frac{a^2}{2}\sin^{-1}(1)\right\} - \{0+0\} = \frac{a^2}{2}\left(\frac{\pi}{2}\right) = \frac{\pi a^2}{4}$$

Some Rough diagrams for standard equation

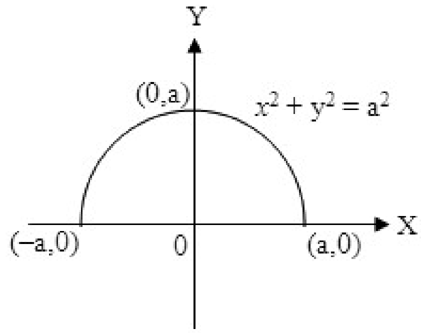
Circle

$x^2 + y^2 = a^2$

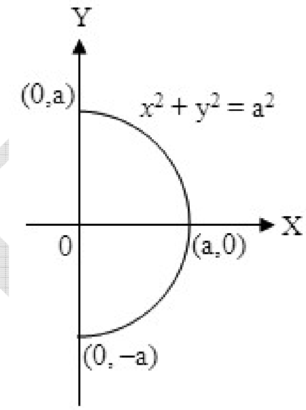
Centre = (0,0), Radius = a



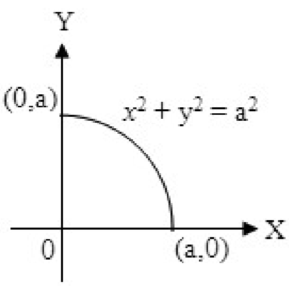
$x^2 + y^2 = a^2, \ y \geq 0$



$x^2 + y^2 = a^2, \ x \geq 0$

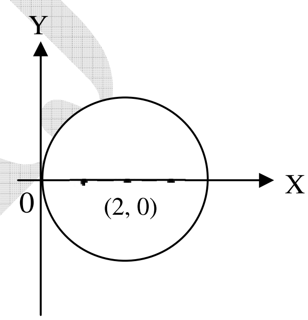


First quadrant of the circle $x^2 + y^2 = a^2$



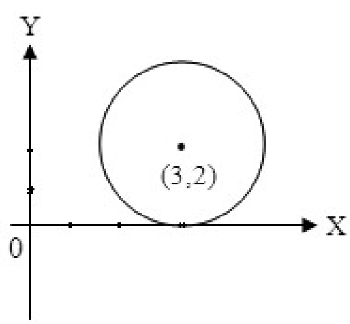
$(x - 2)^2 + y^2 = 4$

Centre = (2,0), Radius = 2



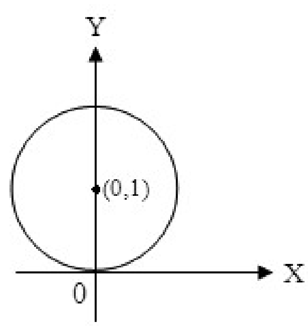
$(x - 3)^2 + (y - 2)^2 = 4$

Centre = (3,2), Radius = 2



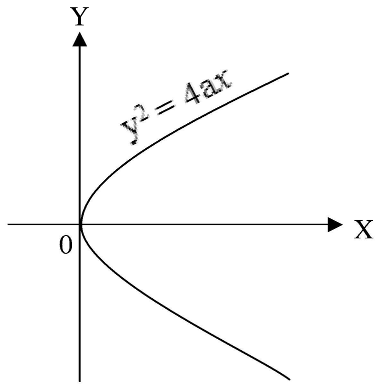
$x^2 + (y - 1)^2 = 1$

Centre = (0,1), Radius = 1

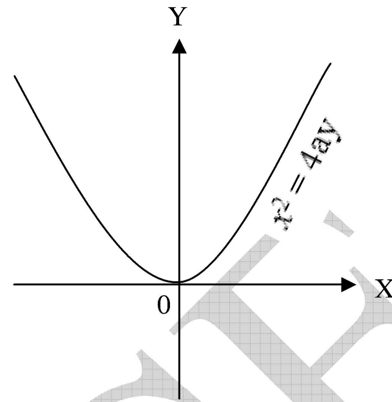


Parabola

$y^2 = 4ax$



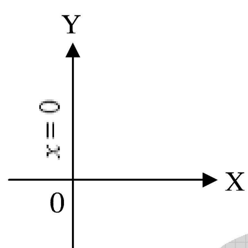
$x^2 = 4ay$



Straight Lines

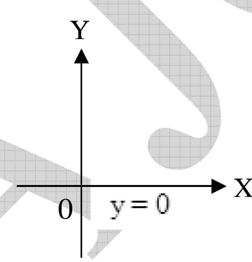
$x = 0$

$x = 0$ is the equation of y-axis.



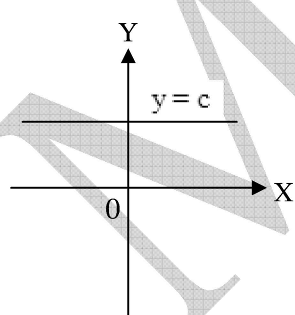
$y = 0$

$y = 0$ is the equation of x-axis.



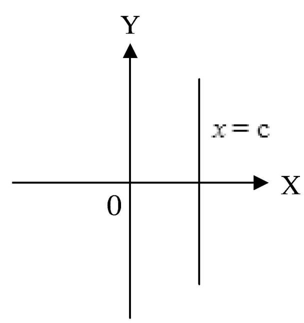
$y = c$

$y = c$ is the straight line parallel to x-axis.



$x = c$

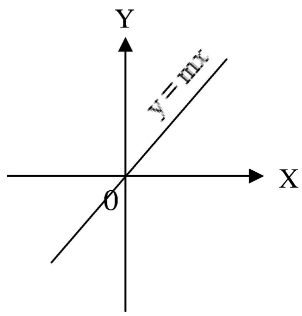
$x = c$ is the straight line parallel to y-axis.

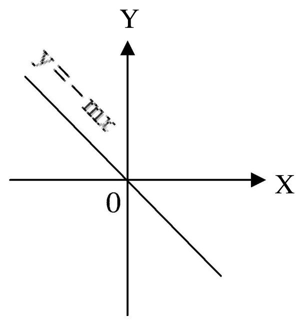


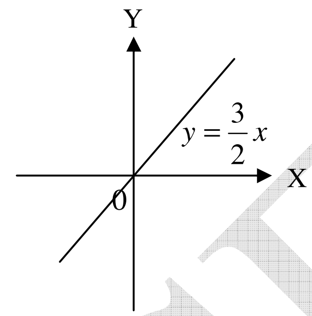
Straight lines passing through the origin

$y = mx$ $\qquad$ $y = -mx$ $\qquad$ $3x = 2y$

$$y = \frac{3}{2}x$$







$4x + 8y = 16$

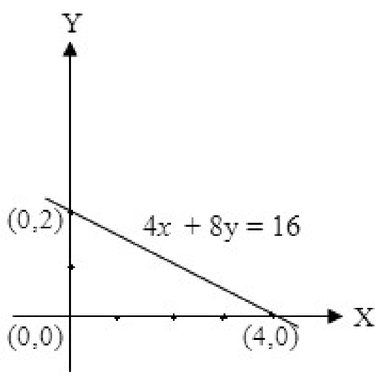
$$\frac{x}{4} + \frac{y}{2} = 1$$

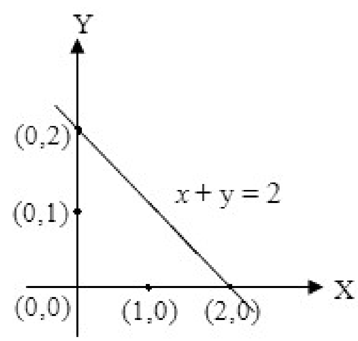
$x + y = 2$

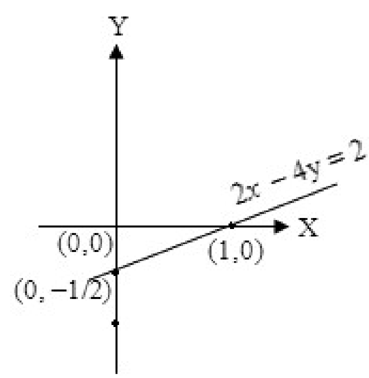
$$\frac{x}{2} + \frac{y}{2} = 1$$

$2x - 4y = 2$

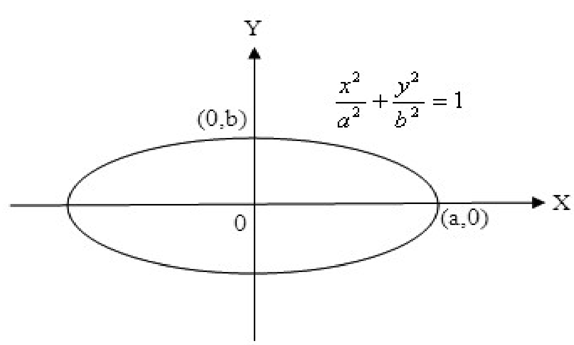
$$\frac{x}{1} + \frac{y}{-1/2} = 1$$







Ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Unknown curve: $y = 4x - x^2$

If we do not know how to draw the curve for the given equation, plot the points for the given equation and draw the curve.

$y = 4x - x^2$

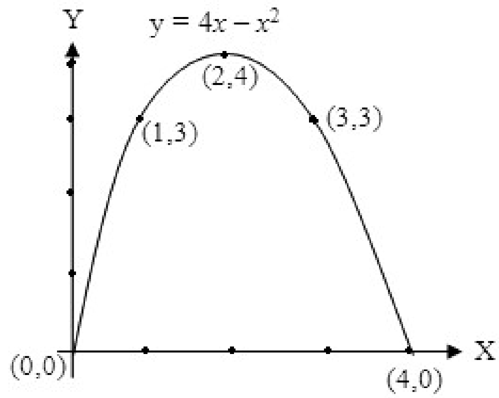
$x = 0 \Rightarrow y = 0$

$x = 1 \Rightarrow y = 3$

$x = 2 \Rightarrow y = 4$

$x = 3 \Rightarrow y = 3$

$x = 4 \Rightarrow y = 0$



$y^2 = 4 - x$

Plot the points for the given equation and draw the curve.

$y^2 = 4 - x$

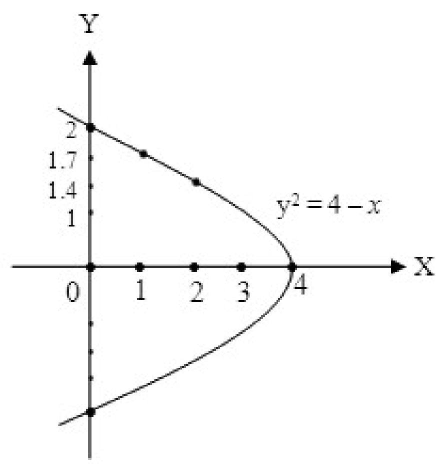
When $x = 0$, $y = \pm 2$

When $x = 1$, $y = \sqrt{3} = \pm 1.7$

When $x = 2$, $y = \sqrt{2} = \pm 1.4$

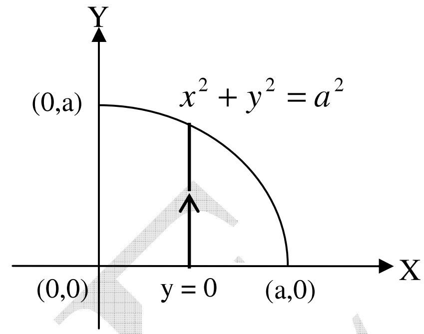
When $x = 3$, $y = \pm 1$

When $x = 4$, $y = 0$



8. Evaluate $\iint xy\, dx\, dy$ taken over the positive quadrant of the circle $x^2 + y^2 = a^2$.

Sol. $$\iint xy\, dx\, dy = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} xy\, dy\, dx$$



$$= \int_0^a x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a x[(a^2 - x^2) - 0]\ dx$$

$$= \frac{1}{2}\int_0^a (a^2 x - x^3)\ dx$$

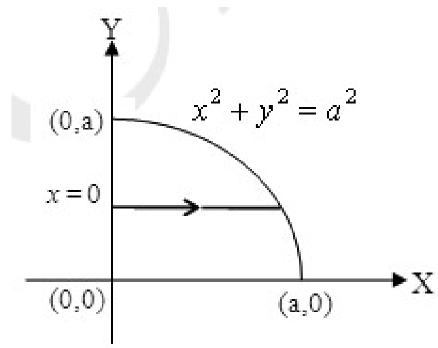
$$= \frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\left(\frac{a^4}{2} - \frac{a^4}{4}\right) - (0 - 0)\right]$$

$$= \frac{1}{2}\left(\frac{2a^4 - a^4}{4}\right) = \frac{a^4}{8}$$

<u>Aliter (Another method)</u>

$$\iint xy\, dx\, dy = \int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} xy\, dx\, dy$$



$$= \int_0^a y\left[\frac{x^2}{2}\right]_0^{\sqrt{a^2-y^2}} dx$$

$$= \frac{1}{2}\int_0^a y[(a^2 - y^2) - 0]\ dx$$

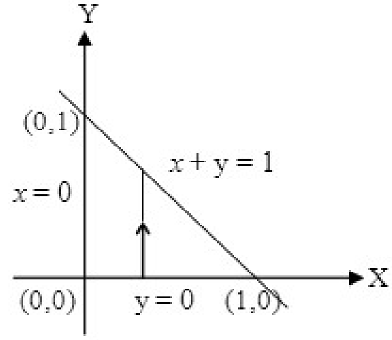
$$= \frac{1}{2}\int_0^a (a^2 y - y^3)\ dx$$

$$= \frac{1}{2}\left[\frac{a^2y^2}{2} - \frac{y^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\left(\frac{a^4}{2} - \frac{a^4}{4}\right) - (0 - 0)\right] = \frac{1}{2}\left(\frac{2a^4 - a^4}{4}\right) = \frac{a^4}{8}$$

9. Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the region for which x, y are each ≥ 0 and x + y ≤ 1.

Sol. $$\iint (x^2+y^2)\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1-x}(x^2+y^2)\,dy\,dx$$



$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1 \left[\left\{x^2(1-x) + \frac{(1-x)^3}{3}\right\} - (0+0)\right] dx$$

$$= \int_0^1 \left\{x^2 - x^3 + \frac{(1-x)^3}{3}\right\} dx$$

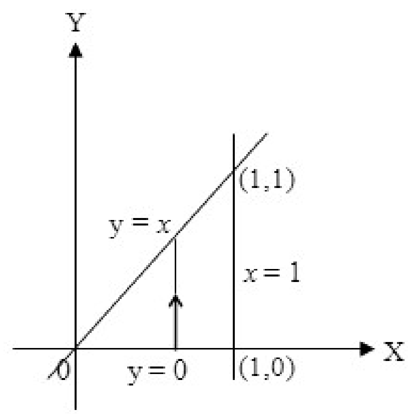
$$= \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{(1-x)^4}{-12}\right]_0^1$$

$$= \left[\left(\frac{1}{3} - \frac{1}{4} + 0\right) - \left(0 - 0 - \frac{1}{12}\right)\right]$$

$$= \frac{1}{12} + \frac{1}{12} = \frac{2}{12} = \frac{1}{6}$$

10. Evaluate $\iint (2xy - x)\,dx\,dy$ over the region formed by the lines y = 0, x = 1 and y = x.

Sol. $$\iint (2xy-x)\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{x}(2xy-x)\,dy\,dx$$



$$= \int_0^1 \left[2x\frac{y^2}{2} - xy\right]_0^x dx$$

$$= \int_0^1 [(x^3 - x^2) - (0-0)]\,dx$$

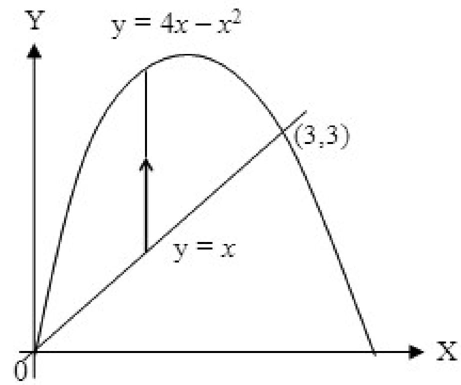
$$= \left[\frac{x^4}{4} - \frac{x^3}{3}\right]_0^1$$

$$= \left[\left(\frac{1}{4} - \frac{1}{3}\right) - (0-0)\right]$$

$$= \frac{3-4}{12} = -\frac{1}{12}$$

11. Evaluate $\iint y\,dx\,dy$ over the region formed by the lines y = x and y = $4x - x^2$.

Sol. $$\iint y\,dx\,dy = \int_{x=0}^{3}\int_{y=x}^{4x-x^2} y\,dy\,dx$$



$y = 4x - x^2$		
$x = 0$	$\Rightarrow$	y = 0
$x = 1$	$\Rightarrow$	y = 3
$x = 2$	$\Rightarrow$	y = 4
$x = 3$	$\Rightarrow$	y = 3
$x = 4$	$\Rightarrow$	y = 0

$$= \int_0^3 \left[\frac{y^2}{2}\right]_x^{4x-x^2} dx$$

$$= \frac{1}{2}\int_0^3 [(4x - x^2)^2 - x^2]\,dx$$

$$= \frac{1}{2}\int_0^3 (x^4 - 8x^3 + 15x^2]\,dx$$

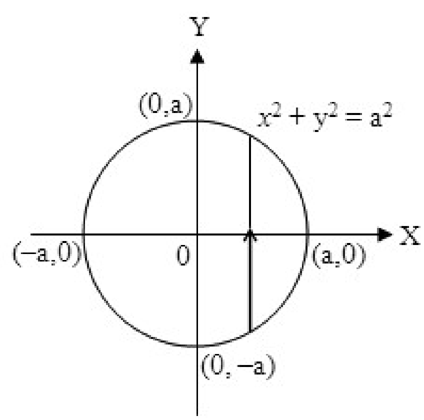
$$= \frac{1}{2}\left[\frac{x^5}{5} - \frac{8x^4}{4} + \frac{15x^3}{3}\right]_0^3$$

$$= \frac{1}{2}\left[\left(\frac{243}{5} - 2(81) + 5(27)\right) - (0 - 0 + 0)\right]$$

$$= \frac{1}{2}\left(\frac{243}{5} - 27\right) = \frac{108}{10} = \frac{54}{5}$$

12. Evaluate $\iint xy\,dx\,dy$ taken over the area of the circle x^2 + y^2 = a^2.

Sol. $$\iint xy\,dx\,dy = \int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} xy\,dy\,dx$$



$$= \int_{-a}^{a} x\left[\frac{y^2}{2}\right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

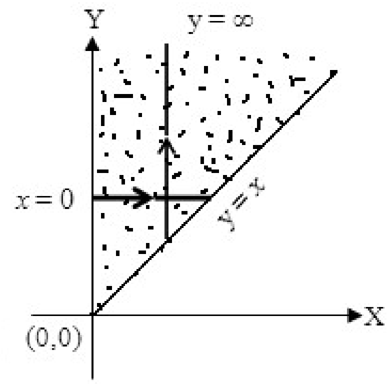
$$= \frac{1}{2}\int_{-a}^{a} x[(a^2 - x^2) - (a^2 - x^2)]\,dx$$

$$= \frac{1}{2}\int_{-a}^{a} x(0)\,dx$$

$$= 0$$

13. Evaluate $\iint\limits_R \dfrac{e^{-y}}{y}\,dx\,dy$ given that R is the region between the lines $x = 0$, $x = \text{y}$ and y = ∞.

Sol. $$\iint\limits_R \frac{e^{-y}}{y}\,dx\,dy = \int\limits_{x=0}^{\infty}\int\limits_{y=x}^{\infty}\frac{e^{-y}}{y}\,dy\,dx$$



We note that the choice of order of integration is wrong, as the inner integration cannot be performed. Hence we try to integrate w.r.t. *x* first.

$$\iint\limits_R \frac{e^{-y}}{y}\,dx\,dy = \int\limits_{y=0}^{\infty}\int\limits_{x=0}^{y}\frac{e^{-y}}{y}\,dx\,dy$$

$$= \int\limits_{y=0}^{\infty}\frac{e^{-y}}{y}\left[x\right]_0^y\,dy$$

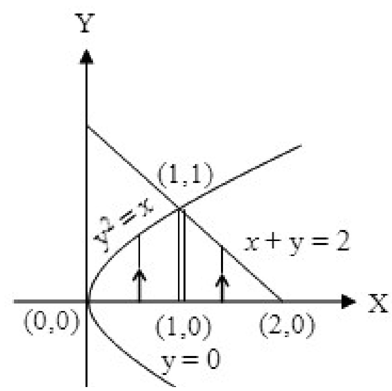
$$= \int\limits_{y=0}^{\infty}\frac{e^{-y}}{y}(y-0)\,dy = \int\limits_{y=0}^{\infty}e^{-y}\,dy$$

$$= \left[\frac{e^{-y}}{-1}\right]_0^{\infty}$$

$$= -(e^{-\infty} - e^{0}) = -(0-1) = 1$$

14. Evaluate $\iint\limits_R xy\,dx\,dy$ where R is the region bounded by the parabola $\text{y}^2 = x$ and the lines y = 0 and x + y = 2 lying in the first quadrant.

Sol. $$\iint\limits_R xy\,dx\,dy = \int\limits_{x=0}^{1}\int\limits_{y=0}^{\sqrt{x}} xy\,dy\,dx + \int\limits_{x=1}^{2}\int\limits_{y=0}^{2-x} xy\,dy\,dx$$

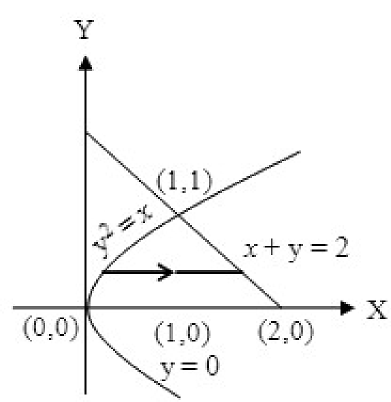


(OR) $$\iint\limits_R xy\,dx\,dy = \int\limits_{y=0}^{1}\int\limits_{x=y^2}^{2-y} xy\,dx\,dy$$

$$= \int_0^1 y\left[\frac{x^2}{2}\right]_{y^2}^{2-y} dy$$

$$= \frac{1}{2}\int_0^1 y[(2-y)^2 - y^4]\,dy$$

$$= \frac{1}{2}\int_0^1 y(y^2 - 4y + 4 - y^4)\,dy$$



$$= \frac{1}{2}\int_0^1 (y^3 - 4y^2 + 4y - y^5)\ dy$$

$$= \frac{1}{2}\left[\frac{y^4}{4} - \frac{4y^3}{3} + \frac{4y^2}{2} - \frac{y^6}{6}\right]_0^1$$

$$= \frac{1}{2}\left[\left\{\frac{1}{4} - \frac{4}{3} + 2 - \frac{1}{6}\right\} - (0)\right]$$

$$= \frac{1}{2}\left[\frac{6 - 32 + 48 - 4}{24}\right] = \frac{1}{2}\left(\frac{18}{24}\right) = \frac{3}{8}$$

15. Evaluate $\iint_A x^2\ dx\,dy$ where A is the region in the first quadrant bounded by the hyperbola xy = 16 and the lines y = x, y = 0 and x = 8.

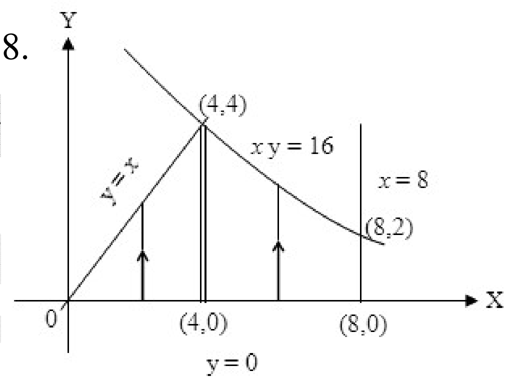
Sol. $\iint_A xy\ dx\,dy = \int_{x=0}^{4}\int_{y=0}^{x} x^2\,dy\,dx + \int_{x=4}^{8}\int_{y=0}^{16/x} x^2\,dy\,dx$

= ……….

= ……….

= 64 + 384

= 448



16. Evaluate $\iint_R (x^2 + y^2)\ dx\,dy$ where R is the area of the parallelogram whose vertices are (1,0), (3,1), (2,2) and (0,1).

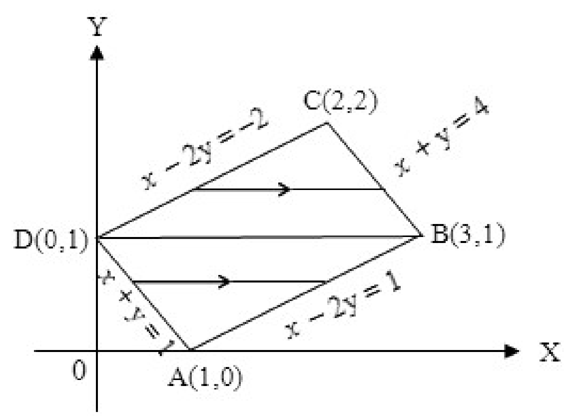
Sol. $\iint_R (x^2 + y^2)\ dx\,dy = \int_{y=0}^{1}\int_{x=1-y}^{1+2y} (x^2 + y^2)\,dx\,dy + \int_{y=1}^{2}\int_{x=2y-2}^{4-y} (x^2 + y^2)\,dx\,dy$

= ……….

= ……….

$= 4 + \frac{15}{2}$

$= \frac{23}{2}$



Equation of AB is

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$$

$$\frac{y - 0}{0 - 1} = \frac{x - 1}{1 - 3}$$

$$\frac{y}{-1} = \frac{x - 1}{-2}$$

$$\Rightarrow x - 2y = 1$$

Equation of BC is $x + y = 4$

Equation of CD is $x - 2y = -2$

Equation of DA is $x + y = 1$

17. Find the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

Sol. Area of the region $= \iint dx\,dy$

$$= \int_{x=0}^{4a} \int_{y=x^2/4a}^{2\sqrt{ax}} dy\,dx$$

$$= \int_{x=0}^{4a} \left[y \right]_{x^2/4a}^{2\sqrt{ax}} dx$$

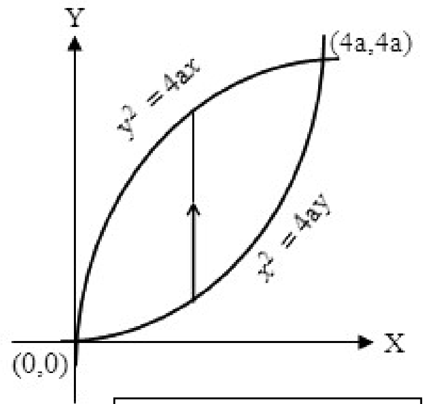
$$= \int_{x=0}^{4a} \left[2\sqrt{ax} - \frac{x^2}{4a} \right] dx$$

$$= \left[\frac{2\sqrt{a}\ x^{3/2}}{3/2} - \frac{x^3}{12a} \right]_0^{4a}$$

$$= \left(\frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{64a^3}{12a} \right) - 0$$

$$= \frac{4\sqrt{a}\ 4\sqrt{4}\ a\sqrt{a}}{3} - \frac{16a^2}{3}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}$$



$$y^2 = 4ax$$
$$\left(\frac{x^2}{4a} \right)^2 = 4ax$$
$$x^4 = 16a^2(4ax)$$
$$x^4 - 64a^3 x = 0$$
$$x(x^3 - 64a^3) = 0$$
$$x = 0 \ or\ x^3 = 64a^3$$
$$x = 4a$$

When $x = 0$, y = 0

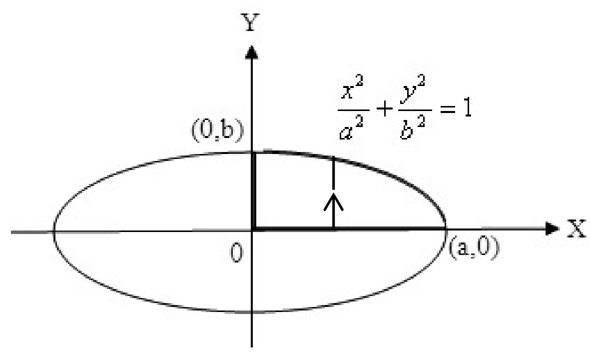
When $x = 4a$, y = 4a

18. Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol. Area of the region = 4[Area enclosed in the first quadrant]

$$= 4 \iint dx\,dy$$

$$= 4 \int_{x=0}^{a} \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} dy\,dx$$



$$= 4 \int_{x=0}^{a} \left[y \right]_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= 4 \int_{x=0}^{a} \left[b\sqrt{1 - \frac{x^2}{a^2}} - 0 \right] dx$$

$$= \frac{4b}{a} \int_{x=0}^{a} \sqrt{a^2 - x^2}\ dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

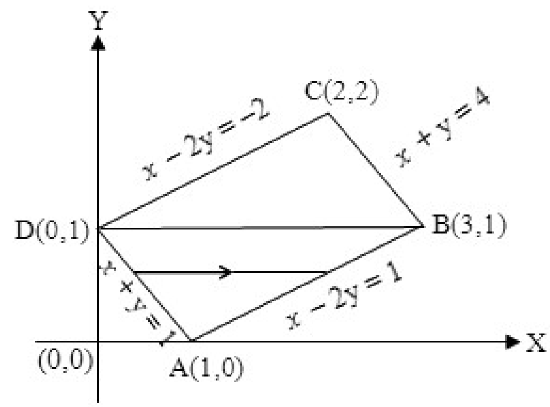
$$= \frac{4b}{a}\left[\left\{0 + \frac{a^2}{2}\sin^{-1}(1)\right\} - (0 - 0)\right]$$

$$= \frac{4b}{a}\left[\frac{a^2}{2}\cdot\frac{\pi}{2}\right]$$

$$= \pi ab$$

19. Find the area of the parallelogram whose vertices are A(1,0), B(3,1), C(2,2) and D(0,1) by using double integration.

Sol. Area of the parallelogram = 2(Area of ABD)



$$= 2\iint dx\, dy$$

$$= 2\int_{y=0}^{1}\int_{x=1-y}^{1+2y} dx\, dy$$

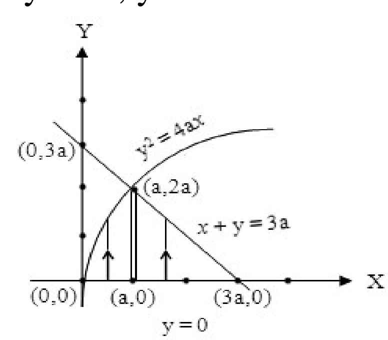
$$= 2\int_{y=0}^{1}\left[x\right]_{1-y}^{1+2y} dy$$

$$= 2\int_{y=0}^{1}\left[(1+2y)-(1-y)\right]dy$$

$$= 2\int_{y=0}^{1} 3y\, dy = 6\left[\frac{y^2}{2}\right]_0^1 = 3(1-0) = 3$$

20. Find the area enclosed by the curve $y^2 = 4ax$ and the lines $x + y = 3a$, $y = 0$.

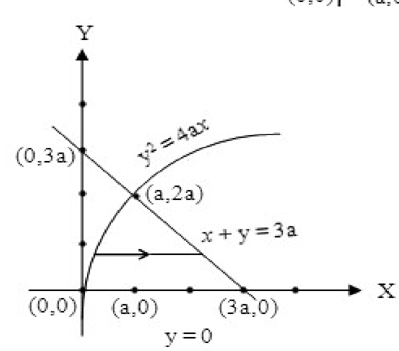
Sol. $$Area = \int_{x=0}^{a}\int_{y=0}^{2\sqrt{ax}} dy\, dx + \int_{x=a}^{3a}\int_{y=0}^{3a-x} dy\, dx$$



(OR) $$Area = \int_{y=0}^{2a}\int_{x=y^2/4a}^{3a-y} dx\, dy$$

$$= \int_{y=0}^{2a}\left[x\right]_{y^2/4a}^{3a-y} dy$$

$$= \int_{y=0}^{2a}\left[(3a-y) - \frac{y^2}{4a}\right]dy$$

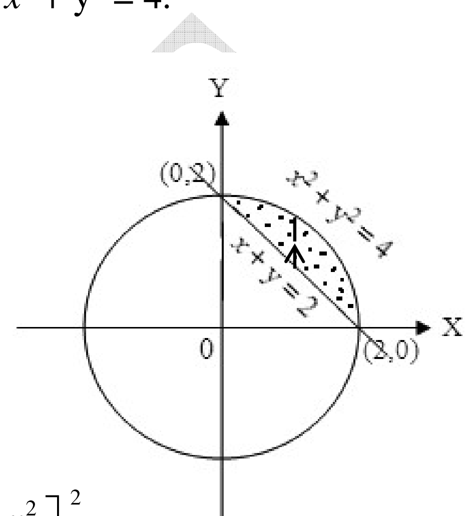


$$= \left[3ay - \frac{y^2}{2} - \frac{y^3}{12a}\right]_0^{2a}$$

$$= \left(6a^2 - 2a^2 - \frac{8a^3}{12a}\right) - 0 = 4a^2 - \frac{2a^2}{3} = \frac{10a^2}{3}$$

21. Find the smaller of the area bounded by y = 2 – x and x^2 + y^2 = 4.

Sol. Area of the region = $\iint dx\,dy$



$$= \int_{x=0}^{2} \int_{y=2-x}^{\sqrt{4-x^2}} dy\,dx$$

$$= \int_{x=0}^{2} [y]_{2-x}^{\sqrt{4-x^2}}\,dx$$

$$= \int_{x=0}^{2} \left[\sqrt{4-x^2} - (2-x)\right] dx$$

$$= \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right) - 2x + \frac{x^2}{2}\right]_0^2$$

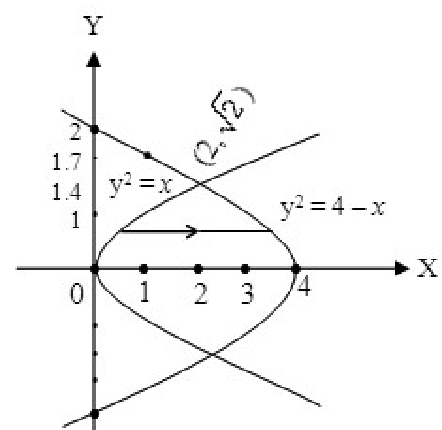
$$= [0 + 2\sin^{-1}(1) - 4 + 2] - (0 + 0 - 0 + 0)$$

$$= 2\left(\frac{\pi}{2}\right) - 2 = \pi - 2$$

22. Find the area bounded by the parabolas y^2 = 4 – x and y^2 = x by double integration.

Sol. Area of the region = 2 (Upper area)

$$= 2\iint dx\,dy$$



$$= 2\int_{y=0}^{\sqrt{2}} \int_{x=y^2}^{4-y^2} dx\,dy$$

$$= 2\int_{y=0}^{\sqrt{2}} [x]_{y^2}^{4-y^2}\,dy$$

$$= 2\int_{y=0}^{\sqrt{2}} \left[(4-y^2) - y^2\right] dy$$

$$= 2\int_{y=0}^{\sqrt{2}} (4-2y^2)\,dy$$

$y^2 = 4 - x$
When x = 0, y = ± 2
When x = 1, y = $\sqrt{3}$ = ± 1.7
When x = 2, y = $\sqrt{2}$ = ±1.4
When x = 3, y = ±1
When x = 4, y = 0

$$= 2\left[4y - \frac{2y^3}{3}\right]_0^{\sqrt{2}} = 2\left[\left\{4\sqrt{2} - \frac{2(2\sqrt{2})}{3}\right\} - 0\right] = 2\left[\frac{12\sqrt{2} - 4\sqrt{2}}{3}\right]$$

$$= 2\left(\frac{8\sqrt{2}}{3}\right) = \frac{16\sqrt{2}}{3}$$

Home Work

1. Evaluate the following:

i) $\int_0^a \int_0^b (x^2 + y^2)\, dx\, dy$ ii) $\int_0^{\pi/2} \int_0^{\pi/2} \sin(x+2y)\, dy\, dx$ iii) $\int_1^2 \int_1^x xy^2\, dx\, dy$

iv) $\int_0^1 \int_{\sqrt{y}}^{2-y} x^2\, dx\, dy$ v) $\int_1^b \int_1^a \frac{dx\, dy}{xy}$

2. Find the limits in $\iint_R f(x, y)\, dx\, dy$ where R is the region in the 1st quadrant bounded by $x = 1$, y = 0, $y^2 = 4x$.
3. Evaluate $\iint dx\, dy$ over the region bounded by $x = 0$, $x = 2$, y = 0 and y = 2.
4. Evaluate $\iint xy\, dx\, dy$ taken over the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
5. Evaluate $\iint (x - y)\, dx\, dy$ over the region between the line x = y and the parabola y = x^2.
6. Evaluate $\iint (x^2 + y^2)\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$.
7. Find the area in the 1st quadrant included between the parabola x^2 = 16y, the Y axis and the line y = 2.
8. Evaluate $\iint (1 + xy)\, dx\, dy$ in the region bounded by the line y = x – 1 and the parabola $y^2 = 2x + 6$.
9. Evaluate $\iint (x + y)\, dx\, dy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
10. Find the area of the triangle formed by the lines x = 0, y = 0, 2x + 3y = 6.
11. Evaluate $\iint_R x^2\, dx\, dy$ where R is the region bounded by the hyperbola xy = 4, y = 0, x = 1 and x = 2.

Answers

1) $(i)\ \frac{ab}{3}(a^2 + b^2)$ $(ii)\ 1$ $(iii)\ \frac{47}{30}$ $(iv)\ \frac{67}{60}$ $(v)\ \log a . \log b$

2) $\int_{x=0}^{1} \int_{y=0}^{2\sqrt{x}} f(x, y)\, dy\, dx$ 3) 4 4) $\frac{a^2 b^2}{8}$ 5) $\frac{1}{60}$ 6) $\frac{\pi a^4}{8}$ 7) $\frac{16\sqrt{2}}{3}$ 8) 54

9) $\frac{ab}{3}(a+b)$ 10) 3 11) 6

Transformation from Cartesian co-ordinates to Polar co-ordinates

In certain cases the evaluation of a double integral which is in terms of x and y is made simpler by changing the co-ordinates into Polar co-ordinates.

In two dimension, the Polar co-ordinates are r, θ.
The relation between x,y and r,θ are

$$x = r\cos\theta,\ \ y = r\sin\theta$$

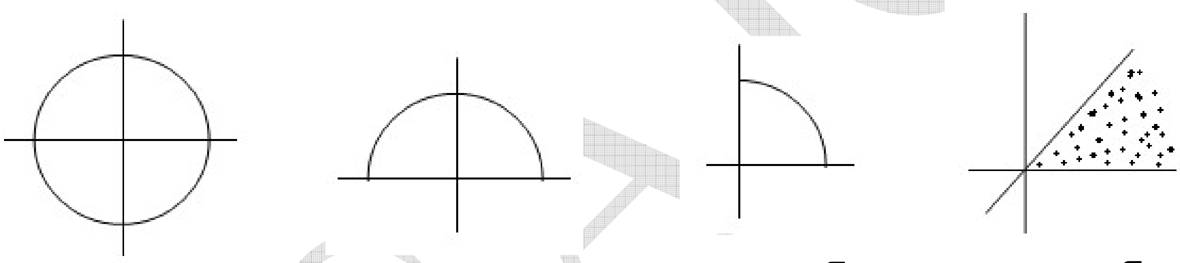
$$dx\,dy = \left|\frac{\partial(x, y)}{\partial(r,\theta)}\right| dr\,d\theta$$

$$= r\,dr\,d\theta$$

$$where\ \ 0 \le r < \infty,\ \ 0 \le \theta \le 2\pi$$

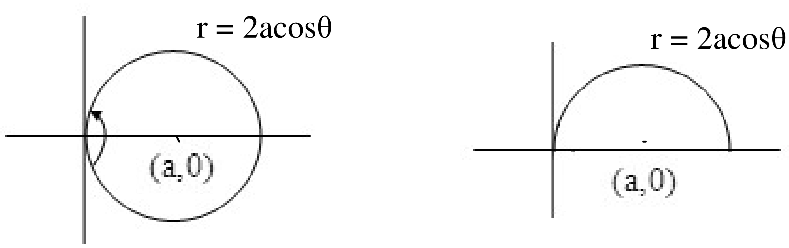
<u>Note:</u>
1) θ depending upon the given region.

<u>Example</u>:



$0 \le \theta \le 2\pi$ $0 \le \theta \le \pi$ $0 \le \theta \le \dfrac{\pi}{2}$ $0 \le \theta \le \dfrac{\pi}{4}$

2) The equation of circle in Polar co-ordinates is r = 2acosθ whose centre is (a, 0) and radius is 'a'.

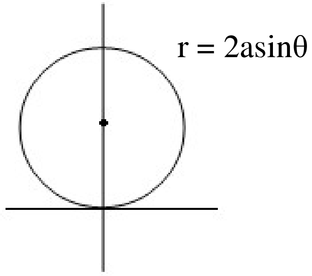


Here $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$ $0 \le \theta \le \dfrac{\pi}{2}$

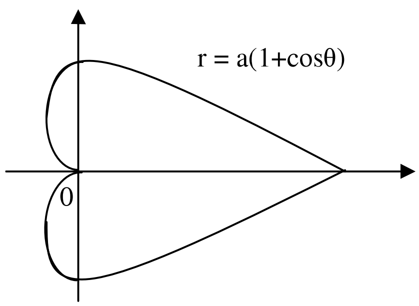
Also the equation r = acosθ represents the circle whose centre is $\left(\dfrac{a}{2}, 0\right)$ and radius is $\dfrac{a}{2}$.

r = a is the equation of the circle whose centre is (0,0) and radius is 'a'.

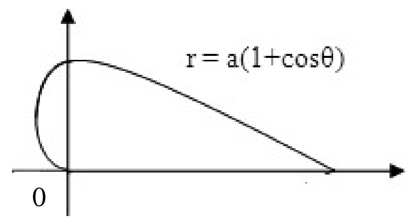
3) The equation $r = 2a\sin\theta$ also represents the circle whose centre is (0, a) and radius is ‘a’.



4) (i) The equation $r = a(1 + \cos\theta)$ represents the cardioid.

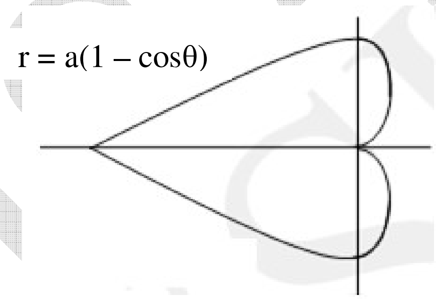


$0 \le \theta \le 2\pi$



$0 \le \theta \le \pi$

(ii) The equation $r = a(1 - \cos\theta)$ also represents the cardioid.



$0 \le \theta \le 2\pi$

Problems

1. Evaluate $\int_0^{\pi}\int_0^{\sin\theta} r\,dr\,d\theta$

Sol. $$\int_0^{\pi}\int_0^{\sin\theta} r\,dr\,d\theta = \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{\sin\theta} d\theta$$

$$=\frac{1}{2}\int_0^{\pi}\left[\sin^2\theta - 0\right] d\theta$$

$$=\frac{1}{2}\times 2\int_0^{\pi/2}\sin^2\theta\,d\theta$$

$$=\left[\frac{2-1}{2}.\frac{\pi}{2}\right]=\frac{\pi}{4}$$

2. Evaluate $\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta$

Sol. $$\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left[\frac{r^3}{3}\right]_0^{2\cos\theta} d\theta$$

$$=\frac{1}{3}\int_{-\pi/2}^{\pi/2}\left[8\cos^3\theta - 0\right] d\theta$$

$$=\frac{1}{3}\times 2\int_0^{\pi/2} 8\cos^3\theta\,d\theta$$

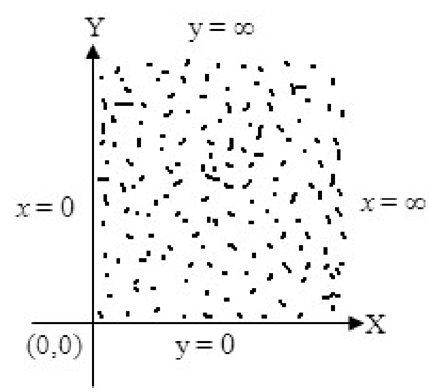
$$=\frac{16}{3}\left[\frac{3-1}{3}\right]=\frac{32}{9}$$

3. Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy$. Hence evaluate $\int_0^{\infty} e^{-x^2}\,dx.$

Sol. Given $x = 0$ to $x = \infty$

and y = 0 to y = ∞

$$\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\,r\,dr\,d\theta$$



Put $r^2 = t$
$2rdr = dt$

$$= \int_{\theta=0}^{\pi/2}\int_{t=0}^{\infty} e^{-t}\,\frac{dt}{2}\,d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2}\left[\frac{e^{-t}}{-1}\right]_0^{\infty} d\theta$$

$$= -\frac{1}{2}\int_{\theta=0}^{\pi/2}[e^{-\infty} - e^{0}]\ d\theta$$

$$= -\frac{1}{2}\int_{\theta=0}^{\pi/2}[0-1]\ d\theta$$

$$= \frac{1}{2}\ [\theta]_0^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{4}$$

Now, we have $\displaystyle\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} dx\,dy = \frac{\pi}{4}$

$$\int_0^{\infty}\int_0^{\infty} e^{-x^2}\ e^{-y^2}\ dx\,dy = \frac{\pi}{4}$$

$$\int_0^{\infty} e^{-x^2}\ dx\ \int_0^{\infty} e^{-y^2}\ dy = \frac{\pi}{4}$$

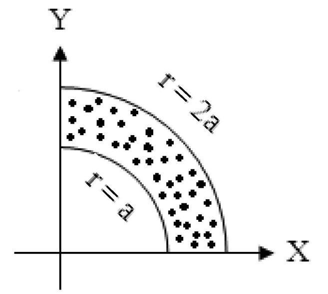
$$\int_0^{\infty} e^{-x^2}\ dx\ \int_0^{\infty} e^{-x^2}\ dx\ = \frac{\pi}{4}$$

$$\left[\int_0^{\infty} e^{-x^2}\ dx\right]^2 = \frac{\pi}{4}$$

$$(i.e.)\ \int_0^{\infty} e^{-x^2}\ dx\ = \frac{\sqrt{\pi}}{2}$$

4. Evaluate $\iint\limits_D \frac{x\,y}{\sqrt{x^2+y^2}}\,dx\,dy$ by transforming to Polar co-ordinates where D is the region enclosed by the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = 4a^2$ in the first quadrant.

Sol. $$\iint\limits_D \frac{x\,y}{\sqrt{x^2+y^2}}\,dx\,dy = \int\limits_{\theta=0}^{\frac{\pi}{2}} \int\limits_{r=a}^{2a} \frac{r\cos\theta . r\sin\theta}{r}\, r\,dr\,d\theta$$



$x^2 + y^2 = a^2$
(or) r = a
$x^2 + y^2 = 4a^2$
(or) r = 2a

$$= \int\limits_{\theta=0}^{\frac{\pi}{2}} \int\limits_{r=a}^{2a} \sin\theta\cos\theta\; r^2\,dr\,d\theta$$

$$= \int\limits_{\theta=0}^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_a^{2a} \frac{\sin 2\theta}{2}\,d\theta$$

$$= \frac{1}{6}\int\limits_{\theta=0}^{\frac{\pi}{2}} (8a^3 - a^3)\ \sin 2\theta\, d\theta$$

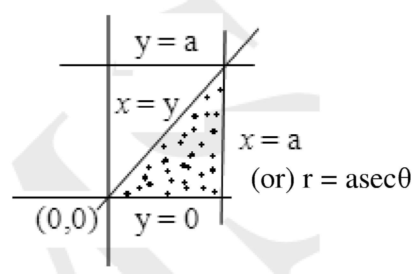
$$= \frac{7a^3}{6}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= -\frac{7a^3}{12}[-1-1] \ = \frac{7a^3}{6}$$

5. Evaluate by changing to Polar co-ordinates the integral

(i) $\int\limits_0^a\int\limits_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy$ $\qquad$ (ii) $\int\limits_0^a\int\limits_y^a \frac{x^2}{(x^2+y^2)^{3/2}}\,dx\,dy$

Sol. Given $x = y$ to $x = a$
and y = 0 to y = a



$x = a$
(or) rcosθ = a
r = asecθ

$$(i)\ \int\limits_0^a\int\limits_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy = \int\limits_{\theta=0}^{\pi/4}\int\limits_{r=0}^{a\sec\theta} \frac{r^2\cos^2\theta}{r}\, r\,dr\,d\theta$$

$$= \int\limits_{\theta=0}^{\frac{\pi}{4}}\int\limits_{r=0}^{a\sec\theta} \cos^2\theta\; r^2\,dr\,d\theta$$

$$= \int\limits_{\theta=0}^{\frac{\pi}{4}} \left[\frac{r^3}{3}\right]_a^{a\sec\theta} \cos^2\theta\, d\theta$$

$$= \frac{1}{3} \int_{\theta=0}^{\frac{\pi}{4}} (a^3 \sec^3 \theta - 0) \; \frac{1}{\sec^2 \theta} d\theta$$

$$= \frac{a^3}{3} \int_{\theta=0}^{\frac{\pi}{4}} \sec\theta \; d\theta$$

$$= \frac{a^3}{3} \left[\log(\sec\theta + \tan\theta)\right]_0^{\pi/4}$$

$$= \frac{a^3}{3} \left[\log(\sqrt{2} + 1) - \log(1+0)\right]$$

$$= \frac{a^3}{3} \log(\sqrt{2}+1) \qquad (\because \log 1 = 0)$$

$$(ii) \quad \int_0^a \int_y^a \frac{x^2}{(x^2+y^2)^{3/2}} \; dx \; dy = \int_{\theta=0}^{\pi/4} \int_{r=0}^{a\sec\theta} \frac{r^2 \cos^2\theta}{(r^2)^{3/2}} \; r \, dr \, d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{4}} \int_{r=0}^{a\sec\theta} \cos^2\theta \; dr \, d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{4}} [r]_a^{a\sec\theta} \cos^2\theta \, d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{4}} (a\sec\theta - 0) \; \frac{1}{\sec^2\theta} d\theta$$

$$= a \int_{\theta=0}^{\frac{\pi}{4}} \cos\theta \; d\theta$$

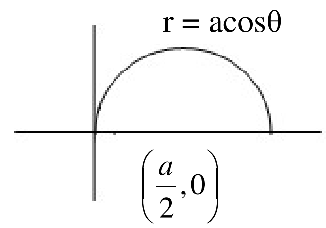
$$= a\left[\sin\theta\right]_0^{\pi/4}$$

$$= a\left[\frac{1}{\sqrt{2}} - 0\right]$$

$$= \frac{a}{\sqrt{2}}$$

6. Evaluate $\iint r\sqrt{a^2-r^2}\;dr\,d\theta$ over the upper half of the circle r = acosθ.

Sol. $$\iint r\sqrt{a^2-r^2}\;dr\,d\theta = \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{a\cos\theta} r\sqrt{a^2-r^2}\;dr\,d\theta$$



$$= \int_{\theta=0}^{\frac{\pi}{2}}\int_{t=a}^{a\sin\theta} t(-t\,dt)\,d\theta$$

Put $\sqrt{a^2-r^2}=t$

$a^2-r^2=t^2$

$-2r\,dr = 2t\,dt$

$r\,dr = -t\,dt$

$$= -\int_{\theta=0}^{\frac{\pi}{2}}\left[\frac{t^3}{3}\right]_a^{a\sin\theta} d\theta$$

$$= -\frac{1}{3}\int_{\theta=0}^{\frac{\pi}{2}}(a^3\sin^3\theta - a^3)\,d\theta$$

$\sqrt{a^2-r^2}=t$

When r = 0, t = a

When r = acosθ,

t = $\sqrt{a^2-a^2\cos^2\theta}$

t = a$\sqrt{1-\cos^2\theta}$

t = asinθ

$$= -\frac{a^3}{3}\int_{\theta=0}^{\frac{\pi}{2}}\sin^3\theta\;d\theta + \frac{a^3}{3}\int_{\theta=0}^{\frac{\pi}{2}}d\theta$$

$$= -\frac{a^3}{3}\left[\frac{3-1}{3}.1\right] + \frac{a^3}{3}[\theta]_0^{\pi/2}$$

$$= -\frac{a^3}{3}\left(\frac{2}{3}\right) + \frac{a^3}{3}\left(\frac{\pi}{2}-0\right)$$

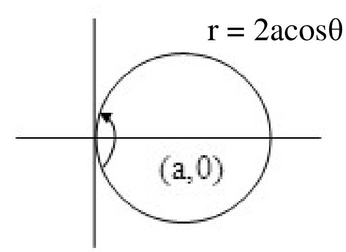
$$= \frac{a^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right)$$

7. Find the area of a circle of radius 'a' in Polar co-ordinates.

Sol. The equation of the circle of radius 'a' is r = 2acosθ.

Area of a circle = 2 (Upper Area)

$$= 2\iint r\,dr\,d\theta$$



$$= 2\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{\pi/2}\left[\frac{r^2}{2}\right]_0^{2a\cos\theta} d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[4a^2 \cos^2 \theta - 0\right] d\theta$$

$$= 4a^2 \int_{\theta=0}^{\pi/2} \cos^2 \theta \; d\theta$$

$$= 4a^2 \left(\frac{2-1}{2} . \frac{\pi}{2}\right)$$

$$= \pi a^2$$

8. Evaluate $\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} (x^2 + y^2)\, dx\, dy$ by changing into polar coordinates.

Sol. *Given* $y = 0$ *to* $y = \sqrt{2ax - x^2}$

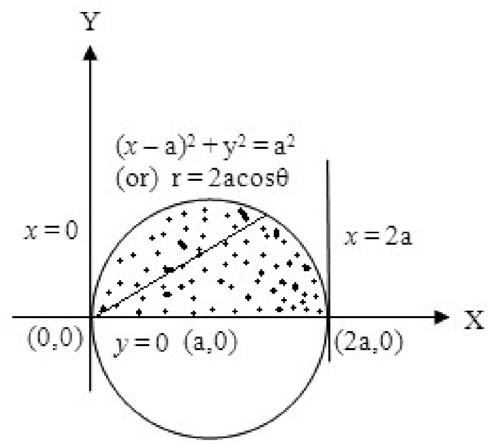
$$y^2 = 2ax - x^2$$

$$x^2 - 2ax + y^2 = 0$$

$$(x-a)^2 + y^2 = a^2$$

and $x = 0$ to $x = 2a$

$$x^2 + y^2 - 2ax = 0$$
$$x^2 + y^2 = 2ax$$
$$r^2 = 2ar\cos\theta$$
$$r = 2a\cos\theta$$



$$\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} (x^2 + y^2)\, dx\, dy = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{2a\cos\theta} r^2 \, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{16a^4 \cos^4 \theta}{4} - 0\right] d\theta$$

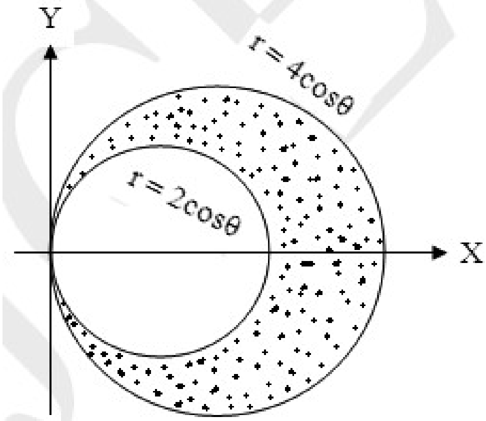
$$= 4a^4 \int_{\theta=0}^{\frac{\pi}{2}} \cos^4 \theta\, d\theta$$

$$= 4a^4 \left[\frac{4-1}{4} . \frac{4-3}{4-2} . \frac{\pi}{2}\right]$$

$$= 4a^4 \left[\frac{3}{4} . \frac{1}{2} . \frac{\pi}{2}\right]$$

$$= \frac{3\pi a^4}{4}$$

9. Find the area of the region outside the inner circle r = 2cosθ and inside the outer circle r = 4cosθ.



Sol. Area of the region = 2 (Upper area)

$$= 2 \iint r\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{\pi/2}\int_{r=2\cos\theta}^{4\cos\theta} r\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{\pi/2}\left[\frac{r^2}{2}\right]_{2\cos\theta}^{4\cos\theta} d\theta$$

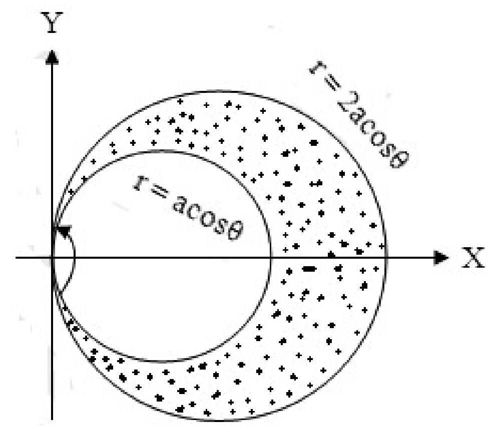
$$= \int_{\theta=0}^{\pi/2}\left[16\cos^2\theta - 4\cos^2\theta\right] d\theta$$

$$= 12\int_{\theta=0}^{\pi/2} \cos^2\theta\ d\theta$$

$$= 12\left(\frac{2-1}{2}.\frac{\pi}{2}\right)$$

$$= 3\pi$$

10. Evaluate $\iint_A r^2\,dr\,d\theta$, A is the area between the circle r = acosθ and r = 2acosθ.



Sol. $$\iint_A r^2\,dr\,d\theta = \int_{\theta=-\pi/2}^{\pi/2}\int_{r=a\cos\theta}^{2a\cos\theta} r^2\,dr\,d\theta$$

$$= \int_{\theta=-\pi/2}^{\pi/2}\left[\frac{r^3}{3}\right]_{a\cos\theta}^{2a\cos\theta} d\theta$$

$$= \frac{1}{3}\int_{\theta=-\pi/2}^{\pi/2}\left[8a^3\cos^3\theta - a^3\cos^3\theta\right] d\theta$$

$$= \frac{7a^3}{3}\int_{\theta=-\pi/2}^{\pi/2}\cos^3\theta\ d\theta$$

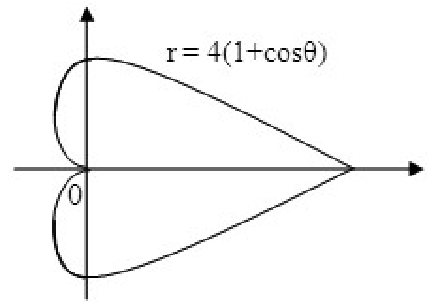
$$= \frac{7a^3}{3}\,2\int_{\theta=0}^{\pi/2}\cos^3\theta\ d\theta$$

$$= \frac{14a^3}{3}\left(\frac{3-1}{3}.1\right)$$

$$= \frac{28a^3}{9}$$

11. Find the area of the cardioid r = 4(1+cosθ).

Sol. Area of the cardioid = 2 (Upper area)



$$= 2 \iint r\,dr\,d\theta$$

$$= 2 \int_{\theta=0}^{\pi} \int_{r=0}^{4(1+\cos\theta)} r\,dr\,d\theta$$

$$= 2 \int_{\theta=0}^{\pi} \left[\frac{r^2}{2}\right]_0^{4(1+\cos\theta)} d\theta$$

$$= \int_{\theta=0}^{\pi} \left[16(1+\cos\theta)^2 - 0\right] d\theta$$

$$= 16 \int_{\theta=0}^{\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= 16 \int_{\theta=0}^{\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

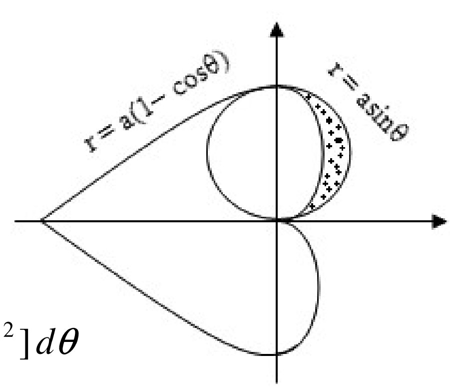
$$= 16 \left[\theta + 2\sin\theta + \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_0^{\pi}$$

$$= 16 \left[\left\{\pi + 0 + \frac{1}{2}(\pi + 0)\right\} - \{0\}\right]$$

$$= 16\left(\frac{3\pi}{2}\right)$$

$$= 24\pi$$

12. Find the area lying inside the circle r = asinθ and outside the cardioid r = a(1– cosθ).



$$Sol.\ Area\ of\ the\ region = \int_{\theta=0}^{\pi/2} \int_{r=a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi/2} [a^2 \sin^2\theta - a^2(1-\cos\theta)^2]\, d\theta$$

$$= \frac{a^2}{2} \int_{\theta=0}^{\pi/2} \left[\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta \right] d\theta$$

$$= \frac{a^2}{2} \int_{\theta=0}^{\pi/2} \left[2\cos\theta - 1 - (\cos^2\theta - \sin^2\theta) \right] d\theta$$

$$= \frac{a^2}{2} \int_{\theta=0}^{\pi/2} \left[2\cos\theta - 1 - \cos 2\theta \right] d\theta$$

$$= \frac{a^2}{2} \left[2\sin\theta - \theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/2}$$

$$= \frac{a^2}{2} \left[\left\{ 2 - \frac{\pi}{2} - 0 \right\} - \{0\} \right]$$

$$= a^2 \left(1 - \frac{\pi}{4} \right)$$

13. Transform $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$ into Polar co-ordinates.

Sol. $Given \;\; y = \sqrt{ax-x^2} \;\; to \;\; y = \sqrt{a^2-x^2}$

$$y^2 = ax - x^2 \;\; to \;\; y^2 = a^2 - x^2$$

$$x^2 - ax + y^2 = 0 \;\; to \;\; x^2 + y^2 = a^2$$

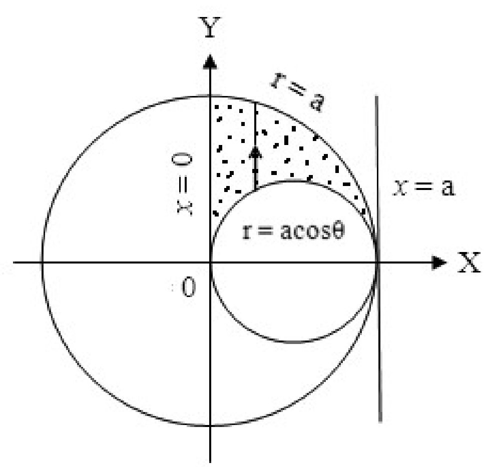
$$\left(x - \frac{a}{2} \right)^2 + y^2 = \left(\frac{a}{2} \right)^2 \;\; to \;\; x^2 + y^2 = a^2$$

and $x = 0$ to $x = a$

$x^2 + y^2 - ax = 0$
$x^2 + y^2 = ax$
$r^2 = ar\cos\theta$
$r = a\cos\theta$

$x^2 + y^2 = a^2$
$r^2 = a^2$
$r = a$

$$\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dx\,dy}{\sqrt{a^2-x^2-y^2}} = \int_{\theta=0}^{\pi/2} \int_{r=a\cos\theta}^{a} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}$$



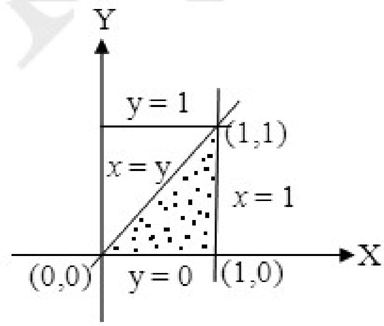
Changing the order of integration

On changing the order of integration the limits of integration change. To find the new limits, we draw the rough sketch of the region of integration. From the sketch, the limits of x and y should determined as usual.

Note: For constant limits the order of integration is immaterial.

Problems

1. Change the order of integration in $\int_0^1 \int_y^1 f(x,y)\,dx\,dy$

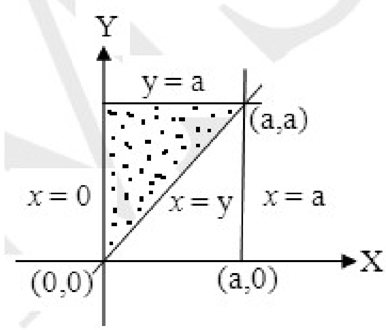


Sol. Given $x = y$ to $x = 1$

and y = 0 to y = 1

$$\int_{y=0}^{1} \int_{x=y}^{1} f(x,y)\,dx\,dy = \int_{x=0}^{1} \int_{y=0}^{x} f(x,y)\,dy\,dx$$

2. Evaluate by changing the order of integration in $\int_0^a \int_x^a (x^2+y^2)\,dx\,dy$



Sol. Given y = x to y = a

and $x = 0$ to $x = a$

$$\int_{x=0}^{a} \int_{y=x}^{a} (x^2+y^2)\,dx\,dy = \int_{y=0}^{a} \int_{x=0}^{y} (x^2+y^2)\,dx\,dy$$

$$= \int_{y=0}^{a} \left[\frac{x^3}{3} + x\,y^2\right]_0^y dy$$

$$= \int_{y=0}^{a} \left[\left\{\frac{y^3}{3} + y^3\right\} - \{0\}\right] dy$$

$$= \int_{y=0}^{a} \frac{4y^3}{3}\,dy$$

$$= \frac{4}{3}\left[\frac{y^4}{4}\right]_0^a$$

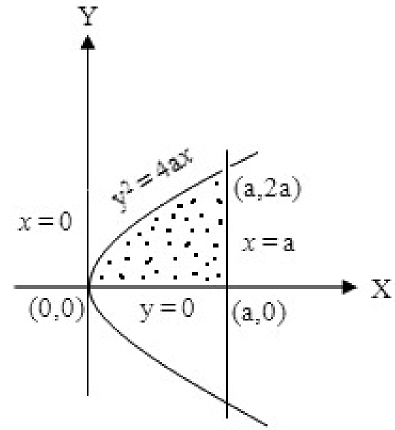
$$= \frac{1}{3}(a^4 - 0) = \frac{a^4}{3}$$

3. Evaluate by changing the order of integration in $\int_0^a \int_0^{2\sqrt{ax}} x^2 \, dx \, dy$

Sol. Given y = 0 to y = $2\sqrt{ax}$

$$y^2 = 4ax$$

and $x = 0$ to $x = a$



$$\int_{x=0}^{a} \int_{y=0}^{2\sqrt{ax}} x^2 \, dx \, dy = \int_{y=0}^{2a} \int_{x=\frac{y^2}{4a}}^{a} x^2 \, dx \, dy$$

$$= \int_{y=0}^{2a} \left[\frac{x^3}{3}\right]_{\frac{y^2}{4a}}^{a} dy$$

$$= \frac{1}{3} \int_{y=0}^{2a} \left[a^3 - \frac{y^6}{64a^3}\right] dy$$

$$= \frac{1}{3}\left[a^3 y - \frac{y^7}{7(64a^3)}\right]_0^{2a}$$

$$= \frac{1}{3}\left[\left\{2a^4 - \frac{2^7 a^7}{7(2^6 a^3)}\right\} - \{0\}\right]$$

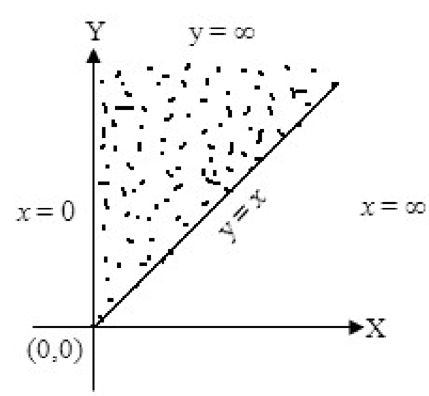
$$= \frac{1}{3}\left[2a^4 - \frac{2a^4}{7}\right]$$

$$= \frac{1}{3}\left(\frac{12a^4}{7}\right) = \frac{4a^4}{7}$$

4. Evaluate by changing the order of integration in $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dx \, dy$

Sol. Given y = x to y = ∞

and $x = 0$ to $x = \infty$



$$\int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{e^{-y}}{y} \, dy \, dx = \int_{y=0}^{\infty} \int_{x=0}^{y} \frac{e^{-y}}{y} \, dx \, dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y} [x]_0^y \, dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y}(y-0)\,dy = \int_{y=0}^{\infty} e^{-y}\,dy$$

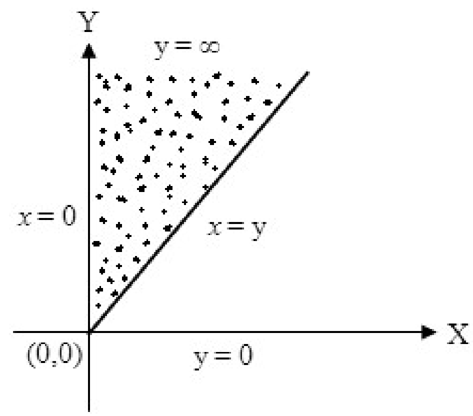
$$= \left[\frac{e^{-y}}{-1}\right]_0^{\infty}$$

$$= -(e^{-\infty} - e^{0}) = -(0-1) = 1$$

5. Change the order of integration $\int_0^{\infty}\int_0^{y} y\,e^{-y^2/x}\,dx\,dy$ and hence evaluate it.

Sol. Given $x = 0$ to $x =$ y

and y = 0 to y = ∞



$$\int_{y=0}^{\infty}\int_{x=0}^{y} y\,e^{-y^2/x}\,dx\,dy = \int_{x=0}^{\infty}\int_{y=x}^{\infty} y\,e^{-y^2/x}\,dy\,dx$$

$$= \int_{x=0}^{\infty}\int_{t=x}^{\infty} e^{-t}\,\frac{x}{2}\,dt\;\,dx$$

Put $\frac{y^2}{x} = t$

$2y\,dy = x\,dt$

$y\,dy = \frac{x}{2}dt$

When $y = x \Rightarrow t = x$

$y = \infty \Rightarrow t = \infty$

$$= \frac{1}{2}\int_{x=0}^{\infty}\left[\frac{e^{-t}}{-1}\right]_x^{\infty} x\,dx$$

$$= -\frac{1}{2}\int_{x=0}^{\infty}[0 - e^{-x}]\;x\,dx$$

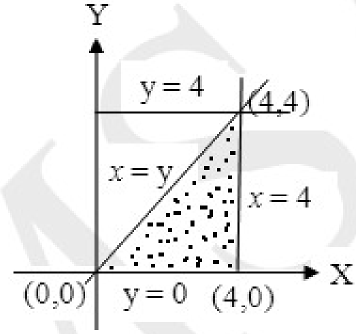
$$= \frac{1}{2}\int_{x=0}^{\infty} x\,e^{-x}\,dx$$

$$= \frac{1}{2}\left[x\left(\frac{e^{-x}}{-1}\right) - (1)\left(\frac{e^{-x}}{1}\right)\right]_0^{\infty}$$

$$= \frac{1}{2}[\{0\} - \{0-1\}]$$

$$= \frac{1}{2}$$

6. Evaluate by changing the order of integration in $\int_0^4 \int_y^4 \frac{x}{x^2+y^2}\,dx\,dy$



Sol. $$\int_{y=0}^{4} \int_{x=y}^{4} \frac{x}{x^2+y^2}\,dx\,dy = \int_{x=0}^{4} \int_{y=0}^{x} \frac{dy}{x^2+y^2}\; x\,dx$$

$$= \int_{x=0}^{4} \left[\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right]_0^x x\,dx$$

$$= \int_{x=0}^{4} \left[\tan^{-1}(1) - \tan^{-1}(0)\right] dx$$

$$= \int_{x=0}^{4} \left[\frac{\pi}{4} - 0\right] dx$$

$$= \frac{\pi}{4}\left[x\right]_0^4$$

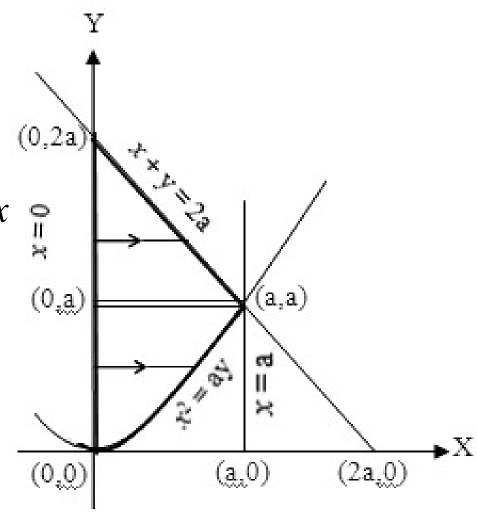
$$= \frac{\pi}{4}(4-0) = \pi$$

7. By changing the order of integration, evaluate $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\,dy\,dx$

Sol. Given $y = x^2/a$ to $y = 2a - x$

(i.e.) $ay = x^2$ to $x + y = 2a$

and $x = 0$ to $x = a$



$$\int_{x=0}^{a} \int_{y=\frac{x^2}{a}}^{2a-x} xy\,dy\,dx = \int_{y=a}^{2a} \int_{x=0}^{2a-y} xy\,dy\,dx + \int_{y=0}^{a} \int_{x=0}^{\sqrt{ay}} xy\,dy\,dx$$

$$\int_{y=a}^{2a} \int_{x=0}^{2a-y} xy\,dy\,dx = \int_{y=a}^{2a} y\left[\frac{x^2}{2}\right]_0^{2a-y} dy$$

$$= \frac{1}{2}\int_{y=a}^{2a} y[(2a-y)^2 - 0]\,dy$$

$$= \frac{1}{2}\int_{y=a}^{2a} (y^3 - 4ay^2 + 4a^2 y)\,dy$$

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$$= \frac{1}{2}\left[\frac{y^4}{4} - 4a\frac{y^3}{3} + 4a^2\frac{y^2}{2}\right]_a^{2a}$$

$$= \frac{1}{2}\left[\left\{4a^4 - \frac{32a^4}{3} + 8a^4\right\} - \left\{\frac{a^4}{4} - \frac{4a^4}{3} + 2a^4\right\}\right]$$

$$= \frac{1}{2}\left[10a^4 - \frac{a^4}{4} - \frac{28a^4}{3}\right] = \frac{a^4}{2}\left[\frac{120-3-112}{12}\right] = \frac{5a^4}{24}$$

$$\int_{y=0}^{a}\int_{x=0}^{\sqrt{ay}} xy\,dy\,dx = \int_{y=0}^{a} y\left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} dy$$

$$= \frac{1}{2}\int_{y=0}^{a} y[ay-0]\,dy = \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a = \frac{a}{6}(a^3-0) = \frac{a^4}{6}$$

$$\therefore \int_{x=0}^{a}\int_{y=\frac{x^2}{a}}^{2a-x} xy\,dy\,dx = \frac{5a^4}{24} + \frac{a^4}{6}$$

$$= \frac{5a^4 + 4a^4}{24} = \frac{9a^4}{24} = \frac{3a^4}{8}$$

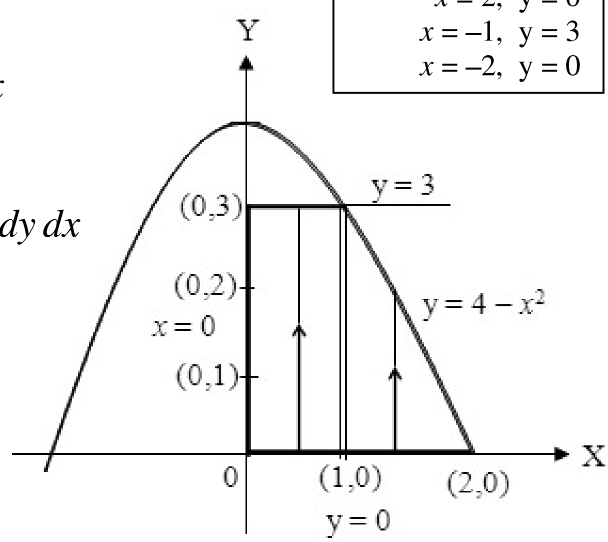
8. Evaluate by changing the order of integration in $\int_0^3 \int_0^{\sqrt{4-y}} (x+y)\,dx\,dy$

Sol. Given $x = 0$ to $x = \sqrt{4-y}$

$$x^2 = 4 - y \ (or) \ y = 4 - x^2$$

and y = 0 to y = 3

$y = 4 - x^2$
When $x = 0$, $y = 4$
$x = 1$, $y = 3$
$x = 2$, $y = 0$
$x = -1$, $y = 3$
$x = -2$, $y = 0$



$$\int_{y=0}^{3}\int_{x=0}^{\sqrt{4-y}} (x+y)\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{3} (x+y)\,dy\,dx$$

$$+ \int_{x=1}^{2}\int_{y=0}^{4-x^2} (x+y)\,dy\,dx$$

$$\int_{x=0}^{1}\int_{y=0}^{3} (x+y)\,dx\,dy = \int_{x=0}^{1}\left[xy + \frac{y^2}{2}\right]_0^3 dx$$

$$= \int_{x=0}^{1}\left[\left\{3x + \frac{9}{2}\right\} - 0\right] dx$$

$$= \int_{x=0}^{1} \left(3x + \frac{9}{2}\right) dx$$

$$= \left[\frac{3x^2}{2} + \frac{9}{2}x\right]_0^1$$

$$= \left(\frac{3}{2} + \frac{9}{2}\right) - 0 = \frac{12}{2} = 6$$

$$\int_{x=1}^{2} \int_{y=0}^{4-x^2} (x+y)\,dx\,dy = \int_{x=1}^{2} \left[xy + \frac{y^2}{2}\right]_0^{4-x^2} dx$$

$$= \int_{x=1}^{2} \left[\left\{x(4-x^2) + \frac{(4-x^2)^2}{2}\right\} - \{0\}\right] dx$$

$$= \int_{x=1}^{2} \left[4x - x^3 + \frac{16 - 8x^2 + x^4}{2}\right] dx$$

$$= \left[\frac{4x^2}{2} - \frac{x^4}{4} + \frac{1}{2}\left(16x - \frac{8x^3}{3} + \frac{x^5}{5}\right)\right]_1^2$$

$$= \left\{8 - 4 + \frac{1}{2}\left(32 - \frac{64}{3} + \frac{32}{5}\right)\right\} - \left\{2 - \frac{1}{4} + \frac{1}{2}\left(16 - \frac{8}{3} + \frac{1}{5}\right)\right\}$$

$$= 4 + \frac{1}{2}\left(\frac{480 - 320 + 96}{15}\right) - 2 + \frac{1}{4} - \frac{1}{2}\left(\frac{240 - 40 + 3}{15}\right)$$

$$= 2 + \frac{1}{4} + \frac{1}{2}\left(\frac{256}{15}\right) - \frac{1}{2}\left(\frac{203}{15}\right)$$

$$= \frac{9}{4} + \frac{256}{30} - \frac{203}{30}$$

$$= \frac{135 + 512 - 406}{60} = \frac{241}{60}$$

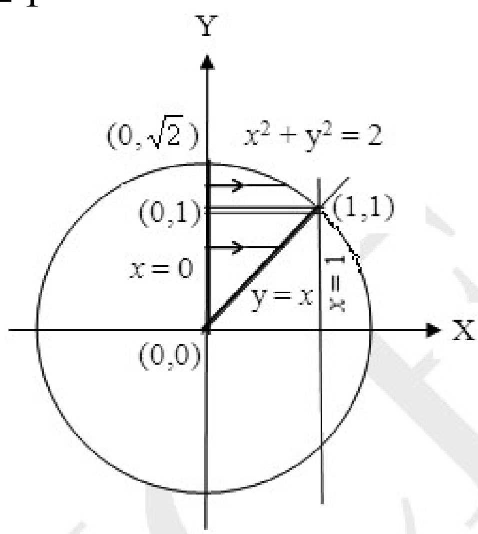
$$\therefore \int_{y=0}^{3} \int_{x=0}^{\sqrt{4-y}} (x+y)\,dx\,dy = 6 + \frac{241}{60} = \frac{360 + 241}{60} = \frac{601}{60}$$

9. Evaluate by changing the order of integration in $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$

Sol. Given y = x to y = $\sqrt{2-x^2}$

$y^2 = 2 - x^2$ (or) $x^2 + y^2 = 2$

and $x = 0$ to $x = 1$



$$\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx = \int_{y=0}^{1} \int_{x=0}^{y} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy + \int_{y=1}^{\sqrt{2}} \int_{x=0}^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy$$

$$\int_{y=0}^{1} \int_{x=0}^{y} \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy = \int_{y=0}^{1} \int_{t=y}^{y\sqrt{2}} \frac{t\, dt}{t}\, dy$$

Put $x^2 + y^2 = t^2$
$2x\, dx = 2t\, dt$
$x\, dx = t\, dt$

$x^2 + y^2 = t^2$
When $x = 0$, $t^2 = y^2$
$t = y$
When $x = y$, $t^2 = 2y^2$
$t = y\sqrt{2}$

$$= \int_{y=0}^{1} [t]_y^{y\sqrt{2}}\, dy$$

$$= \int_{y=0}^{1} \left[y\sqrt{2} - y\right] dy$$

$$= \int_{y=0}^{1} (\sqrt{2}-1)\, y\, dy$$

$$= (\sqrt{2}-1)\left[\frac{y^2}{2}\right]_0^1$$

$$= (\sqrt{2}-1)\left(\frac{1}{2} - 0\right) = \frac{\sqrt{2}-1}{2}$$

$$\int_{y=1}^{\sqrt{2}}\int_{x=0}^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy = \int_{y=1}^{\sqrt{2}}\int_{t=y}^{\sqrt{2}} \frac{t\,dt}{t}\,dy$$

Put $x^2 + y^2 = t^2$
$2x\,dx = 2t\,dt$
$x\,dx = t\,dt$

$$= \int_{y=1}^{\sqrt{2}} \left[t\right]_y^{\sqrt{2}} dy$$

$x^2 + y^2 = t^2$
When $x = 0$, $t^2 = y^2$
$t = y$
When $x = \sqrt{2-y^2}$, $t^2 = 2 - y^2 + y^2$
$t = \sqrt{2}$

$$= \int_{y=1}^{\sqrt{2}} \left[\sqrt{2} - y\right] dy$$

$$= \left[\sqrt{2}\,y - \frac{y^2}{2}\right]_1^{\sqrt{2}}$$

$$= \left(2 - \frac{2}{2}\right) - \left(\sqrt{2} - \frac{1}{2}\right)$$

$$= 1 - \sqrt{2} + \frac{1}{2}$$

$$= \frac{3}{2} - \sqrt{2} = \frac{3 - 2\sqrt{2}}{2}$$

$$\therefore \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\,dy\,dx = \frac{\sqrt{2}-1}{2} + \frac{3-2\sqrt{2}}{2}$$

$$= \frac{\sqrt{2} - 1 + 3 - 2\sqrt{2}}{2}$$

$$= \frac{2-\sqrt{2}}{2}$$

$$= 1 - \frac{\sqrt{2}}{2}$$

$$= 1 - \frac{1}{\sqrt{2}}$$

Triple Integral

It is denoted by $\iiint f(x,y,z)\,dx\,dy\,dz$

<u>Note</u>:

1) The order of integration is denoted by

$$\int_{x_1}^{x_2}\int_{f_1(x)}^{f_2(x)}\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,dz\,dy\,dx \qquad (or) \qquad \int_{z_1}^{z_2}\int_{f_1(z)}^{f_2(z)}\int_{\phi_1(y,z)}^{\phi_2(y,z)} f(x,y,z)\,dx\,dy\,dz$$

2) Volume of the region = $\iiint dx\,dy\,dz$

<u>Problems</u>

1. Evaluate $\displaystyle\int_0^1\int_0^{1-x}\int_0^{(x+y)^2} x\;dx\,dy\,dz$

Sol. $$Let\;\; I = \int_0^1\int_0^{1-x}\int_0^{(x+y)^2} x\;dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} x\left[z\right]_0^{(x+y)^2} dy\,dx$$

$$= \int_0^1\int_0^{1-x} x\left[(x+y)^2 - 0\right] dy\,dx$$

$$= \int_0^1 x\left[\frac{(x+y)^3}{3}\right]_0^{1-x} dx$$

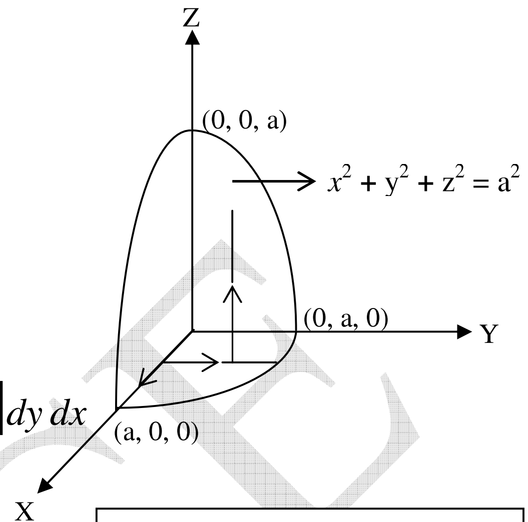
$$= \frac{1}{3}\int_0^1 x\left[(x+1-x)^3 - (x+0)^3\right]dx$$

$$= \frac{1}{3}\int_0^1 x\left[1-x^3\right]dx$$

$$= \frac{1}{3}\int_0^1 (x-x^4)\,dx$$

$$= \frac{1}{3}\left[\frac{x^2}{2}-\frac{x^5}{5}\right]_0^1 = \frac{1}{3}\left[\left(\frac{1}{2}-\frac{1}{5}\right)-0\right] = \frac{1}{3}\left(\frac{3}{10}\right) = \frac{1}{10}$$

2. Evaluate $\iiint xyz\,dx\,dy\,dz$ taken through the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$.



$x = 0$ is the equation of yoz plane
y = 0 is the equation of $x$oz plane
z = 0 is the equation of $x$oy plane

Sol. $$\iiint xyz\,dx\,dy\,dz = \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} xyz\,dz\,dy\,dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy\left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= \frac{1}{2}\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy\left[(a^2 - x^2 - y^2) - 0\right] dy\,dx$$

$$= \frac{1}{2}\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} (a^2xy - x^3y - xy^3)\,dy\,dx$$

$$= \frac{1}{2}\int_{x=0}^{a}\left[\frac{a^2xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_{x=0}^{a}\left[\left\{\frac{a^2x(a^2-x^2)}{2} - \frac{x^3(a^2-x^2)}{2} - \frac{x(a^2-x^2)^2}{4}\right\} - \{0\}\right] dx$$

$$= \frac{1}{2}\int_{x=0}^{a}(a^2-x^2)\left[\frac{a^2x}{2} - \frac{x^3}{2} - \frac{x(a^2-x^2)}{4}\right] dx$$

$$= \frac{1}{2}\int_{x=0}^{a}(a^2-x^2)\left[\frac{2a^2x - 2x^3 - a^2x + x^3}{4}\right] dx$$

$$= \frac{1}{8}\int_{x=0}^{a}(a^2-x^2)(a^2x - x^3)\,dx$$

$$= \frac{1}{8}\int_{x=0}^{a}(a^4x - a^2x^3 - a^2x^3 + x^5)\,dx$$

$$= \frac{1}{8}\int_{x=0}^{a}(a^4x - 2a^2x^3 + x^5)\,dx$$

$$= \frac{1}{8}\left[\frac{a^4x^2}{2} - \frac{2a^2x^4}{4} + \frac{x^6}{6}\right]_0^a = \frac{1}{8}\left[\left\{\frac{a^6}{2} - \frac{a^6}{2} + \frac{a^6}{6}\right\} - 0\right] = \frac{a^6}{48}$$

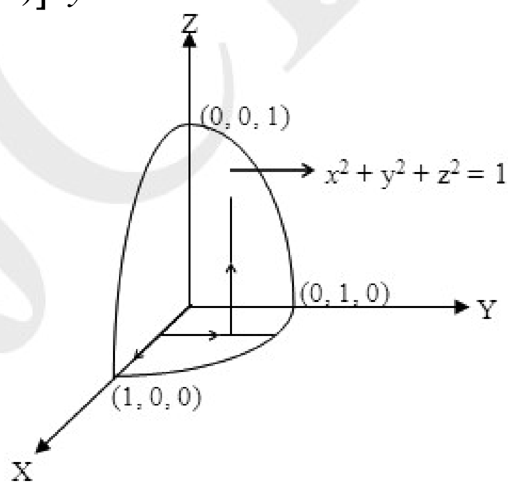
3. Evaluate $\iiint \dfrac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$, $x, y, z \geq 0$, $x^2+y^2+z^2 \leq 1$.

Sol. $$\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}} = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} \frac{dz}{\sqrt{(1-x^2-y^2)-z^2}}\;dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}}\right)\right]_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \left[\sin^{-1}(1)-\sin^{-1}(0)\right] dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \left[\frac{\pi}{2}-0\right] dy\,dx$$



$$= \frac{\pi}{2}\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} dy\,dx$$

$$= \frac{\pi}{2}\int_{x=0}^{1}\left[y\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{\pi}{2}\int_{x=0}^{1}\left[\sqrt{1-x^2}-0\right] dx$$

$$= \frac{\pi}{2}\int_{x=0}^{1}\sqrt{1-x^2}\;dx$$

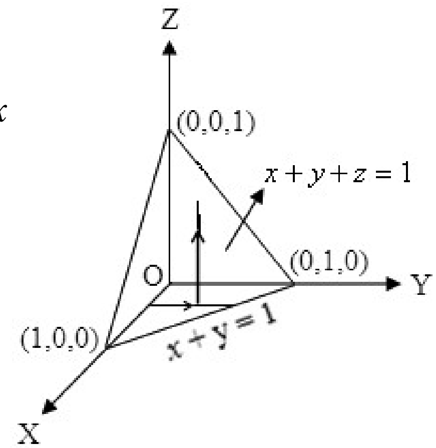
$$= \frac{\pi}{2}\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right)\right]_0^1$$

$$= \frac{\pi}{2}\left[\left\{0+\frac{1}{2}\sin^{-1}(1)\right\}-\{0+0\}\right]$$

$$= \frac{\pi}{2}\left[\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= \frac{\pi^2}{8}$$

4. Evaluate $\iiint\limits_V \dfrac{dz\,dy\,dx}{(x+y+z+1)^3}$ where V is the region bounded by $x = 0$, y = 0, z = 0 and x + y + z = 1.



Sol. $$\iiint\limits_V \frac{dz\,dy\,dx}{(x+y+z+1)^3} = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} \frac{dz}{(x+y+z+1)^3}\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\left[\frac{(x+y+z+1)^{-2}}{-2}\right]_0^{1-x-y} dy\,dx$$

$$= \frac{1}{-2}\int_{x=0}^{1}\int_{y=0}^{1-x}\left[\frac{1}{(x+y+z+1)^2}\right]_0^{1-x-y} dy\,dx$$

$$= \frac{1}{-2}\int_{x=0}^{1}\int_{y=0}^{1-x}\left[\left\{\frac{1}{(x+y+1-x-y+1)^2}\right\}-\left\{\frac{1}{(x+y+0+1)^2}\right\}\right] dy\,dx$$

$$= \frac{1}{-2}\int_{x=0}^{1}\int_{y=0}^{1-x}\left[\frac{1}{4}-(x+y+1)^{-2}\right] dy\,dx$$

$$= \frac{1}{-2}\int_{x=0}^{1}\left[\frac{y}{4}-\frac{(x+y+1)^{-1}}{-1}\right]_0^{1-x} dx$$

$$= \frac{1}{-2}\int_{x=0}^{1}\left[\left\{\frac{1-x}{4}+\frac{1}{x+1-x+1}\right\}-\left\{0+\frac{1}{x+0+1}\right\}\right] dx$$

$$= \frac{1}{-2}\int_{x=0}^{1}\left[\frac{1-x}{4}+\frac{1}{2}-\frac{1}{x+1}\right] dx$$

$$= \frac{1}{-2}\left[\frac{(1-x)^2}{-8}+\frac{x}{2}-\log(x+1)\right]_0^1$$

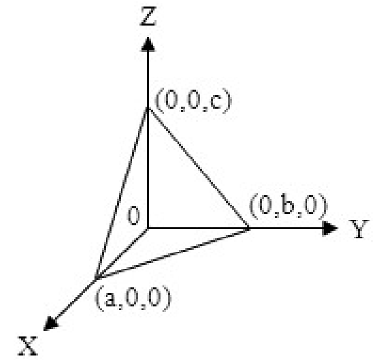
$$= \frac{1}{-2}\left[\left\{0+\frac{1}{2}-\log 2\right\}-\left\{\frac{1}{-8}+0-\log 1\right\}\right]$$

$$= \frac{1}{-2}\left[\frac{1}{2}+\frac{1}{8}-\log 2\right]$$

$$= \frac{1}{-2}\left[\frac{5}{8}-\log 2\right]$$

$$= \frac{1}{2}\log 2-\frac{5}{16}$$

5. Find the volume of the tetrahedron bounded by the coordinate planes and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$



Sol. Volume of the tetrahedron = $\iiint dx\,dy\,dz$

$$= \int_{x=0}^{a} \int_{y=0}^{b\left(1-\frac{x}{a}\right)} \int_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\,dy\,dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{b\left(1-\frac{x}{a}\right)} \left[z\right]_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dy\,dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{b\left(1-\frac{x}{a}\right)} \left[c\left(1-\frac{x}{a}-\frac{y}{b}\right)-0\right] dy\,dx$$

$$= c\int_{x=0}^{a} \int_{y=0}^{b\left(1-\frac{x}{a}\right)} \left(1-\frac{x}{a}-\frac{y}{b}\right) dy\,dx$$

$$= c\int_{x=0}^{a} \left[y-\frac{xy}{a}-\frac{y^2}{2b}\right]_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$= c\int_{x=0}^{a} \left[\left\{b\left(1-\frac{x}{a}\right)-\frac{x}{a}b\left(1-\frac{x}{a}\right)-\frac{b^2}{2b}\left(1-\frac{x}{a}\right)^2\right\}-\{0\}\right] dx$$

$$= bc\int_{x=0}^{a} \left[\left(1-\frac{x}{a}\right)-\frac{x}{a}\left(1-\frac{x}{a}\right)-\frac{1}{2}\left(1-\frac{x}{a}\right)^2\right] dx$$

$$= bc\int_{x=0}^{a} \left[\left(1-\frac{x}{a}\right)\left(1-\frac{x}{a}\right)-\frac{1}{2}\left(1-\frac{x}{a}\right)^2\right] dx$$

$$= bc\int_{x=0}^{a} \left[\left(1-\frac{x}{a}\right)^2-\frac{1}{2}\left(1-\frac{x}{a}\right)^2\right] dx$$

$$= bc\int_{x=0}^{a} \frac{1}{2}\left(1-\frac{x}{a}\right)^2 dx$$

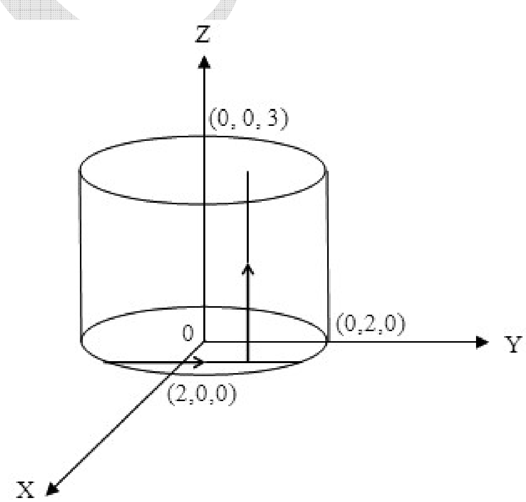
$$= \frac{bc}{2}\left[\frac{\left(1-\frac{x}{a}\right)^3}{-\frac{3}{a}}\right]_0^a$$

$$= -\frac{abc}{6}[0-1]$$

$$= \frac{abc}{6}$$

6. Evaluate $\iiint\limits_V dx\,dy\,dz$ where V is the region of space inside the cylinder $x^2 + y^2 = 4$, that is bounded by the planes z = 0 and z = 3.

Sol. $$\iiint\limits_V dx\,dy\,dz = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3} dz\,dy\,dx$$



$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[z\right]_0^3 dy\,dx$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (3-0)\,dy\,dx$$

$$= 3\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\,dx$$

$$= 3\int_{\theta=0}^{2\pi}\int_{r=0}^{2} r\,dr\,d\theta \quad (by\ \ Polar\ \ coordinates)$$

$$= 3\int_{\theta=0}^{2\pi}\left[\frac{r^2}{2}\right]_0^2 d\theta$$

$$= 3\int_{\theta=0}^{2\pi}(2-0)\,d\theta$$

$$= 6\int_{\theta=0}^{2\pi} d\theta = 6\left[\theta\right]_0^{2\pi} = 6(2\pi-0) = 12\pi.$$

7. Find the volume of the region $x^2 + y^2 + z^2 = r^2$, using triple integral.

Sol. Volume of the region = 8 (Volume in the 1st octant)

$$= 8 \iiint dx\,dy\,dz$$

$$= 8 \int_{x=0}^{r} \int_{y=0}^{\sqrt{r^2-x^2}} \int_{z=0}^{\sqrt{r^2-x^2-y^2}} dz\,dy\,dx$$

$$= 8 \int_{x=0}^{r} \int_{y=0}^{\sqrt{r^2-x^2}} [z]_0^{\sqrt{r^2-x^2-y^2}}\,dy\,dx$$

$$= 8 \int_{x=0}^{r} \int_{y=0}^{\sqrt{r^2-x^2}} \left[\sqrt{r^2-x^2-y^2} - 0\right] dy\,dx$$

$$= 8 \int_{x=0}^{r} \int_{y=0}^{\sqrt{r^2-x^2}} \sqrt{(r^2-x^2)-y^2}\,dy\,dx$$

$$= 8 \int_{x=0}^{r} \left[\frac{y}{2}\sqrt{(r^2-x^2)-y^2} + \frac{r^2-x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{r^2-x^2}}\right)\right]_0^{\sqrt{r^2-x^2}} dx$$

$$= 8 \int_{x=0}^{r} \left[\left\{0 + \frac{r^2-x^2}{2}\sin^{-1}(1)\right\} - \{0+0\}\right] dx$$

$$= 8 \int_{x=0}^{r} \frac{r^2-x^2}{2} \cdot \frac{\pi}{2}\,dx$$

$$= 2\pi \int_{x=0}^{r} (r^2-x^2)\,dx$$

$$= 2\pi \left[r^2 x - \frac{x^3}{3}\right]_0^r$$

$$= 2\pi \left[\left\{r^3 - \frac{r^3}{3}\right\} - \{0-0\}\right]$$

$$= 2\pi \left(\frac{2r^3}{3}\right)$$

$$= \frac{4}{3}\pi r^3$$

8. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Sol. Volume of the ellipsoid = 8 (Volume in the 1st octant)

$$= 8 \iiint dx\,dy\,dz$$

$$= 8 \int_{x=0}^{a} \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} \int_{z=0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$$

$$= 8 \int_{x=0}^{a} \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} \left[z\right]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dy\,dx$$

$$= 8 \int_{x=0}^{a} \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} \left[c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} - 0\right] dy\,dx$$

$$= 8c \int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} \sqrt{\frac{a^2-x^2}{a^2}-\frac{y^2}{b^2}}\; dy\,dx$$

$$= 8c \int_{x=0}^{a} \int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} \frac{1}{b}\sqrt{\frac{b^2(a^2-x^2)}{a^2}-y^2}\; dy\,dx$$

$$\boxed{\text{where } k^2 = \frac{b^2(a^2-x^2)}{a^2} \qquad k = \frac{b}{a}\sqrt{a^2-x^2}}$$

$$= \frac{8c}{b} \int_{x=0}^{a} \int_{y=0}^{k} \sqrt{k^2-y^2}\; dy\,dx$$

$$= \frac{8c}{b} \int_{x=0}^{a} \left[\frac{y}{2}\sqrt{k^2-y^2} + \frac{k^2}{2}\sin^{-1}\left(\frac{y}{k}\right)\right]_0^k dx$$

$$= \frac{8c}{b} \int_{x=0}^{a} \left[\left\{0+\frac{k^2}{2}\sin^{-1}(1)\right\} - \{0+0\}\right] dx$$

$$= \frac{8c}{b} \int_{x=0}^{a} \frac{k^2}{2}.\frac{\pi}{2}\; dx$$

$$= \frac{2c\pi}{b} \int_{x=0}^{a} \frac{b^2(a^2-x^2)}{a^2}\; dx$$

$$= \frac{2bc\pi}{a^2} \int_{x=0}^{a} (a^2 - x^2)\ dx$$

$$= \frac{2bc\pi}{a^2} \left[a^2 x - \frac{x^3}{3} \right]_0^a$$

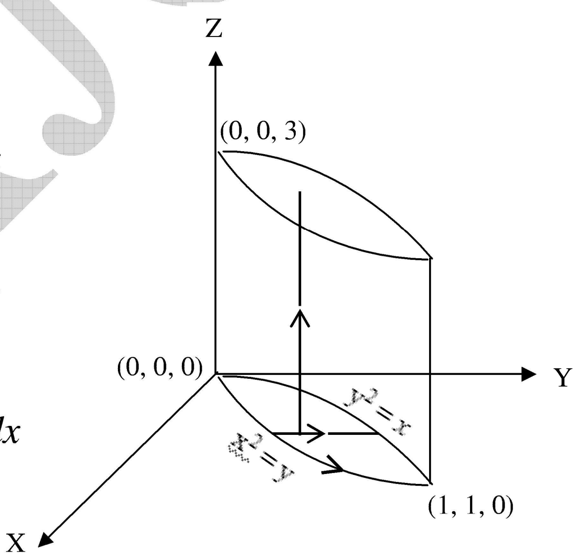
$$= \frac{2bc\pi}{a^2} \left[\left\{ a^3 - \frac{a^3}{3} \right\} - 0 \right]$$

$$= \frac{2bc\pi}{a^2} \left(\frac{2a^3}{3} \right)$$

$$= \frac{4}{3} \pi a b c.$$

9. Find the volume of the region bounded by the surface $y = x^2$, $x = y^2$ and the planes $z = 0$, $z = 3$.

Sol. Volume of the region $= \iiint dx\, dy\, dz$



$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} \int_{z=0}^{3} dz\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} [z]_0^3\ dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (3-0)\ dy\, dx$$

$$= 3 \int_{x=0}^{1} [y]_{x^2}^{\sqrt{x}}\ dx$$

$$= 3 \int_{x=0}^{1} \left[\sqrt{x} - x^2 \right] dx$$

$$= 3 \left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3} \right]_0^1$$

$$= 3 \left[\left(\frac{2}{3} - \frac{1}{3} \right) - 0 \right] = 3 \left(\frac{1}{3} \right) = 1.$$